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Uniqueness and value-sharing of differential polynomials of meromorphic functions

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ABSTRACT

In this paper, we investigate uniqueness problems of meromorphic functions concerning differential polynomials sharing non-zero finite value and give some results. As particular cases of our results we deduce several earlier results of Fang and Hong (2001) [2], Fang and Hua (1996) [1], Lin and Yi (2004) [5], Yang and Hua (1997) [6] and others.

1. Introduction and main results

Let $f(z)$ be a non-constant meromorphic function in the whole complex plane. We shall use the following standard notations of the value distribution theory:

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \ldots .$$

See Hayman [4], Yang [8] and Yi and Yang [7]. We denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = o\big(T(r,f)\big),$$

as $r \to +\infty$, possibly outside of a set with finite measure. For any constant 'a' we define

$$\Theta(a,f) = 1 - \varlimsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r,f)},$$

where $\overline{N}(r, \frac{1}{f-a})$ is the counting function which counts zeros of $f-a$ in $|z| \leqslant r$, counted only once.

Let $g(z)$ be a meromorphic function. If $f(z)-a$ and $g(z)-a$ assume the same zeros with the same multiplicities then we say that $f(z)$ and $g(z)$ share the value 'a', CM, where 'a' is any constant.

It is assumed that the reader is familiar with the notations of the Nevanlinna theory that can be found in [4,8,7].

In 1996, Fang and Hua [1] obtained the following theorem.

Theorem A. *Let f and g be two transcendental entire functions, $n \geqslant 6$ an integer. If $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f = dg$ for some $(n+1)$-th root of unity d or $f^n f' g^n g' = 1$.*

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In 1997, Yang and Hua [6] proved the following result.

Theorem B. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, $n \geqslant 11$ an integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = dg$ for some $(n+1)$-th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$.*

In 2001, Fang and Hong [2] proved the following result.

Theorem C. *Let $f(z)$ and $g(z)$ be two transcendental entire functions, $n \geqslant 11$ be a positive integer. If $f^n(z)(f(z)-1)f'(z)$ and $g^n(z)(g(z)-1)g'(z)$ share 1 CM, then $f(z) \equiv g(z)$.*

In 2004, Lin and Yi [5] proved the following three theorems.

Theorem D. *Let f and g be two transcendental entire functions, $n \geqslant 7$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f(z) \equiv g(z)$.*

Theorem E. *Let f and g be two non-constant meromorphic functions, $n \geqslant 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then*

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \qquad f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})},$$

where h is a non-constant meromorphic function.

Theorem F. *Let f and g be two non-constant meromorphic functions, $n \geqslant 13$ an integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share the value 1 CM, then $f(z) \equiv g(z)$.*

In this paper, by introducing the notion of multiplicity, we reduce and improve Theorems A, B, C, D, E, F by obtaining the following results.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let $n \geqslant 2$ be an integer satisfying $(n+1)s \geqslant 12$. If $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f = dg$, for some $(n+1)$-th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

Remark 1.1. If $s = 1$ in Theorem 1.1, then Theorem 1.1 reduces to Theorem B.

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let n be an integer satisfying $(n-2)s \geqslant 10$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then*

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \qquad f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})},$$

where h is a non-constant meromorphic function.

Remark 1.2. If $s = 1$ in Theorem 1.2, then Theorem 1.2 reduces to Theorem E.

Theorem 1.3. *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let n be an integer satisfying $(n-3)s \geqslant 10$. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share the value 1 CM, then $f(z) \equiv g(z)$.*

Remark 1.3. If $s = 1$ in Theorem 1.3, then Theorem 1.3 reduces Theorem F.

Theorem 1.4. *Let $f(z)$ and $g(z)$ be two transcendental entire functions, whose zeros are of multiplicities at least s, where s is a positive integer. Let n be an integer satisfying $(n+1)s \geqslant 7$. If $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f = dg$, for some $(n+1)$-th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

Remark 1.4. If $s = 1$ in Theorem 1.4, then Theorem 1.4 reduces to Theorem A.

Theorem 1.5. *Let f and g be two transcendental entire functions, whose zeros are of multiplicities at least s, where s is a positive integer. Let n be an integer satisfying $(n-2)s \geqslant 5$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f(z) \equiv g(z)$.*

Remark 1.5. If $s=1$ in Theorem 1.5, then Theorem 1.5 reduces to Theorem D.

Remark 1.6. In Theorem 1.1, giving specific values for s in Theorem 1.1, we get the following interesting cases:

(i) If $s=1$, then $n \geqslant 11$.
(ii) If $s=2$, then $n \geqslant 5$.
(iii) If $s=3$, then $n \geqslant 3$.
(iv) If $s \geqslant 4$, then $n \geqslant 2$.

We can conclude that if f and g have zeros and poles of higher order multiplicity, then we can reduce the value of n.

2. Some lemmas

Lemma 2.1. *(See [8,7].) Let $f(z)$ be a non-constant meromorphic function, k a positive integer and let c be a non-zero finite complex number. Then*

$$\begin{aligned} T(r,f) &\leqslant \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-c}\right) - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &\leqslant \overline{N}(r,f) + (k+1)\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-c}\right) - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f), \end{aligned} \tag{2.1}$$

here $N_0(r,\frac{1}{f^{(k+1)}})$ is the counting function which only counts those points such that $f^{(k+1)}=0$ but $f(f^{(k)}-c)\neq 0$.

In order to prove our theorems we shall first prove the following lemmas:

Lemma 2.2. *Let $f(z)$ and $g(z)$ be two non-constant transcendental meromorphic functions, k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 CM and if*

$$\Delta = (k+2)\Theta(\infty,f) + 2\Theta(\infty,g) + (k+2)\big[\Theta(0,f)+\Theta(0,g)\big] > 3k+7$$

then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.

Proof. Let

$$\Phi(z) = \frac{f^{(k+2)}}{f^{(k+1)}} - 2\frac{f^{(k+1)}}{f^{(k)}-1} - \frac{g^{(k+2)}}{g^{(k+1)}} + 2\frac{g^{(k+1)}}{g^{(k)}-1}. \tag{2.2}$$

Clearly $m(r,\Phi) = S(r,f)+S(r,g)$. We consider the cases $\Phi(z) \not\equiv 0$ and $\Phi(z) \equiv 0$.

Let $\Phi(z) \not\equiv 0$. Then if z_0 is a common simple 1-point of $f^{(k)}$ and $g^{(k)}$, substituting their Taylor series at z_0 into (2.2), we see that z_0 is a zero of $\Phi(z)$. Thus, we have

$$N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right) = N_{1)}\left(r,\frac{1}{g^{(k)}-1}\right) \leqslant \overline{N}\left(r,\frac{1}{\Phi}\right) \leqslant T(r,\Phi)+O(1) \leqslant N(r,\Phi)+S(r,f)+S(r,g), \tag{2.3}$$

here $N_{1)}(r,\frac{1}{f^{(k)}-1})$ is the counting function which only counts those points such that $f^{(k)}-1=0$ but $f^{(k+1)}\neq 0$.

Our assumptions are that $\Phi(z)$ has poles, all simple only at zeros of $f^{(k+1)}$ and $g^{(k+1)}$ and poles of f and g. Thus, we deduce from (2.2) that

$$N(r,\Phi) \leqslant \overline{N}(r,f) + \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right) + N_0\left(r,\frac{1}{f^{(k+1)}}\right) + N_0\left(r,\frac{1}{f^{(k+1)}}\right), \tag{2.4}$$

here $N_0(r,\frac{1}{f^{(k+1)}})$ has the same meaning as in Lemma 2.1. Obviously,

$$\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) + \overline{N}\left(r,\frac{1}{g^{(k)}-1}\right) = 2\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) \leqslant N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right) + N\left(r,\frac{1}{f^{(k)}-1}\right). \tag{2.5}$$

From Lemma 2.1, we have

$$T(r,f) \leqslant \overline{N}(r,f) + (k+1)\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f), \tag{2.6}$$

$$T(r,g) \leqslant \overline{N}(r,g) + (k+1)\overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g^{(k)}-1}\right) - N_0\left(r,\frac{1}{g^{(k+1)}}\right) + S(r,g). \tag{2.7}$$

Since

$$N\left(r, \frac{1}{f^{(k)}-1}\right) \leqslant T(r,f) + k\overline{N}(r,f) + S(r,f). \tag{2.8}$$

Thus we deduce from (2.3)–(2.8) that

$$T(r,f)+T(r,g) \leqslant 2\overline{N}(r,f) + 2\overline{N}(r,g) + (k+2)\left[\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right)\right] \\ + k\overline{N}(r,f) + T(r,f) + S(r,f) + S(r,g).$$

Hence

$$T(r,g) \leqslant (k+2)\overline{N}(r,f) + 2\overline{N}(r,g) + (k+2)\left[\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right)\right] + S(r,f) + S(r,g).$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leqslant T(r,g)$ for $r \in I$. Hence

$$T(r,g) \leqslant \left\{(k+2)\left[1-\Theta(\infty,f)\right] + 2\left[1-\Theta(\infty,g)\right] + (k+2)\left[2-\left(\Theta(0,f)+\Theta(0,g)\right)\right] + \epsilon\right\}T(r,g) + S(r,f)$$

for $r \in I$ and $0 < \epsilon < \Delta - (3k+7)$.

Therefore,

$$T(r,g) \leqslant \left\{(3k+8) - \Delta + \epsilon\right\}T(r,g) + S(r,g)$$

for $r \in I$. This gives

$$\Delta - (3k+7) \leqslant 0 \quad \text{i.e.,} \quad \Delta \leqslant 3k+7$$

which is a contradiction to our hypothesis $\Delta > 3k+7$. Hence, we get $\Phi(z) \equiv 0$. Therefore by (2.2), we have

$$\frac{f^{(k+2)}}{f^{(k+1)}} - \frac{2f^{(k+1)}}{f^{(k)}-1} \equiv \frac{g^{(k+2)}}{g^{(k+1)}} - \frac{2g^{(k+1)}}{g^{(k)}-1}.$$

By solving this, we obtain

$$\frac{1}{f^{(k)}-1} = \frac{bg^{(k)}+a-b}{g^{(k)}-1}, \tag{2.9}$$

where a and b are two constants and $a \neq 0$. Next, we consider three cases:

Case 1. $a = b$.

(i) If $b = -1$, then from (2.9), we obtain that

$$g^{(k)}f^{(k)} \equiv 1.$$

(ii) If $b \neq -1$, then from (2.9), we obtain that

$$\frac{1}{f^{(k)}-1} = \frac{bg^{(k)}}{g^{(k)}-1}.$$

Since

$$\frac{1}{f^{(k)}} = \frac{bg^{(k)}}{(1+b)g^{(k)}-1}, \tag{2.10}$$

we can write

$$\overline{N}\left[r, \frac{1}{g^{(k)}-\frac{1}{1+b}}\right] \leqslant \overline{N}\left[r, \frac{g^{(k)}}{g^{(k)}-\frac{1}{1+b}}\right]. \tag{2.11}$$

From (2.10) and (2.11), we have

$$\overline{N}\left(r, \frac{1}{g^{(k)}-\frac{1}{1+b}}\right) \leqslant \overline{N}\left(r, \frac{1}{f^{(k)}}\right). \tag{2.12}$$

By the first fundamental theorem, we obtain the following inequality

$$\begin{aligned}\overline{N}\left(r,\frac{1}{f^{(k)}}\right) &\leqslant \overline{N}\left(r,\frac{f}{f^{(k)}}\right)+\overline{N}\left(r,\frac{1}{f}\right)\\ &\leqslant T\left(r,\frac{f}{f^{(k)}}\right)+\overline{N}\left(r,\frac{1}{f}\right)\\ &\leqslant N\left(r,\frac{f^{(k)}}{f}\right)+\overline{N}\left(r,\frac{1}{f}\right)+S(r,f).\end{aligned}$$

Clearly, any zero or pole of f of order m is a pole of $\frac{f^{(k)}}{f}$ of order at most k. Hence,

$$N\left(r,\frac{f^{(k)}}{f}\right)\leqslant k\left[\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\right].$$

Therefore

$$\begin{aligned}\overline{N}\left(r,\frac{1}{f^{(k)}}\right) &\leqslant k\left[\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)\right]+\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)\\ &\leqslant k\overline{N}(r,f)+(k+1)\overline{N}\left(r,\frac{1}{f}\right)+S(r,f).\end{aligned}\tag{2.13}$$

Therefore, from (2.12) and (2.13), we have

$$\overline{N}\left(r,\frac{1}{g^{(k)}-\frac{1}{1+b}}\right)\leqslant k\overline{N}(r,f)+(k+1)\overline{N}\left(r,\frac{1}{f}\right)+S(r,f).\tag{2.14}$$

From (2.14) and by Lemma 2.1, we have

$$\begin{aligned}T(r,g) &\leqslant \overline{N}(r,g)+(k+1)\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-\frac{1}{1+b}}\right)-N_0\left(r,\frac{1}{g^{(k+1)}}\right)+S(r,g)\\ &\leqslant \overline{N}(r,g)+(k+1)\overline{N}\left(r,\frac{1}{g}\right)+k\overline{N}(r,f)+(k+1)\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)+S(r,g)\\ &\leqslant 2\overline{N}(r,g)+(k+2)\overline{N}(r,f)+(k+2)\left[\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\right]+S(r,f)+S(r,g).\end{aligned}$$

That is,

$$T(r,g)\leqslant\big[(3k+8)-\Delta\big]T(r,g)+S(r,g)$$

for $r\in I$ and r is sufficiently large. That is, $\Delta\leqslant 3k+7$, which is contradiction to our hypothesis $\Delta>3k+7$.

Case 2. $b\neq 0$ and $a\neq b$.

Then from (2.9), we obtain

$$f^{(k)}-\left(1+\frac{1}{b}\right)=\frac{-a}{b^2[g^{(k)}+\frac{a-b}{b}]}.$$

This implies

$$\overline{N}\left[r,\frac{1}{g^{(k)}+(\frac{a-b}{b})}\right]=\overline{N}\left[r,f^{(k)}-\left(1+\frac{1}{b}\right)\right]=\overline{N}\big(r,f^{(k)}\big)=\overline{N}(r,f).\tag{2.15}$$

From Lemma 2.1 and from (2.15), we have

$$\begin{aligned}T(r,g) &\leqslant \overline{N}(r,g)+(k+1)\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}+\frac{a-b}{b}}\right)+S(r,g)\\ &\leqslant \overline{N}(r,g)+(k+1)\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}(r,f)+S(r,g)\\ &\leqslant 2\overline{N}(r,g)+(k+2)\overline{N}(r,f)+(k+2)\left[\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{f}\right)\right]+S(r,g).\end{aligned}$$

Using the argument as in Case 1, we get a contradiction.

Case 3. $b=0$. From (2.9), we obtain

$$f=\frac{1}{a}g+p(z), \tag{2.16}$$

where $p(z)$ is a polynomial. If $p(z)\not\equiv 0$, then by second fundamental theorem for small functions, we have

$$\begin{aligned} T(r,g) &\leqslant \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g+ap(z)}\right)+S(r,f)\\ &\leqslant \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{af}\right)+S(r,f)\\ &\leqslant \overline{N}(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{f}\right)+S(r,f). \end{aligned} \tag{2.17}$$

Using the argument as in Case 2, we get a contradiction. Therefore, we get $p(z)\equiv 0$, that is,

$$f=\frac{1}{a}g. \tag{2.18}$$

If $a\neq 1$, then $f^{(k)}$ and $g^{(k)}$ sharing the value 1 CM, we deduce from (2.18) that $g^{(k)}\neq 1$, that is,

$$\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)=0.$$

We can deduce a contradiction as in Case 2. Thus we get that $a=1$, that is, $f\equiv g$.

Thus the proof of Lemma 2.2 is completed. □

Lemma 2.3. *Let $f(z)$ and $g(z)$ be two non-constant transcendental entire functions, k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 CM and if $\Delta=(k+2)[\Theta(0,f)+\Theta(0,g)]>2k+3$, then either $f^{(k)}g^{(k)}\equiv 1$ or $f\equiv g$.*

Proof. Since f and g are entire functions, we have $\overline{N}(r,f)=0$ and $\overline{N}(r,g)=0$. Proceeding as in the proof of Lemma 2.2, we shall obtain conclusion of Lemma 2.3. □

Lemma 2.4. *(See [6].) Let f and g be two non-constant entire functions, $n\geqslant 1$. If $f^nf'g^ng'=1$, then $g(z)=c_1e^{cz}$ and $f(z)=c_2e^{-cz}$, where c,c_1,c_2 are constants satisfying $(c_1c_2)^{n+2}c^2=-1$.*

Lemma 2.5. *(See [3,7].) Let $Q(w)=(n-1)^2(w^n-1)(w^{n-2}-1)-n(n-2)(w^{n-1}-1)^2$, then $Q(w)=(w-1)^4(w-\beta_1)\times(w-\beta_2)\cdots(w-\beta_{2n-6})$, where $\beta_j\in\mathbb{C}-\{0,1\}$ $(j=1,2,\ldots,2n-6)$, which are distinct respectively.*

In 1997, Yang and Hua proved the following lemma:

Lemma 2.6. *(See [6].) Let f and g be two non-constant meromorphic functions. If f and g share 1 CM, one of the following three cases holds*:

(i)
$$\begin{aligned} T(r,f) \leqslant{}& \overline{N}(r,f)+\overline{N}_{(2}(r,f)+\overline{N}(r,g)+\overline{N}_{(2}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}_{(2}\left(r,\frac{1}{f}\right)\\ &+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}_{(2}\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g), \end{aligned}$$

the same inequality holding for $T(r,g)$,

(ii) $f\equiv g$,

(iii) $fg\equiv 1$,

where $\overline{N}_{(2}(r,\frac{1}{f})=\overline{N}(r,\frac{1}{f})-N_{1)}(r,\frac{1}{f})$ and $N_{1)}(r,\frac{1}{f})$ is the counting function of the simple zeros of f in $\{z;\ |z|\leqslant r\}$.

3. Proof of theorems

3.1. Proof of Theorem 1.1

Let $F = \frac{f^{n+1}}{n+1}$ and $G = \frac{g^{n+1}}{n+1}$. Then $F' = f^n f'$ and $G' = g^n g'$.
Consider

$$\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{f^{n+1}}\right) \leqslant \frac{1}{s(n+1)} N\left(r, \frac{1}{F}\right) \leqslant \frac{1}{s(n+1)}\bigl[T(r,F) + O(1)\bigr].$$

Therefore

$$\Theta(0,F) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, \frac{1}{F})}{T(r,F)} \geqslant 1 - \frac{1}{s(n+1)}.$$

Similarly

$$\Theta(0,G) \geqslant 1 - \frac{1}{s(n+1)}, \qquad \Theta(\infty,F) \geqslant 1 - \frac{1}{s(n+1)}, \qquad \Theta(\infty,G) \geqslant 1 - \frac{1}{s(n+1)}.$$

Therefore

$$\Delta = (k+2)\Theta(\infty,F) + 2\Theta(\infty,G) + (k+2)\bigl[\Theta(0,F) + \Theta(0,G)\bigr] \geqslant (3k+8) - \frac{3k+8}{s(n+1)}. \tag{3.1}$$

For $k = 1$, we obtain $\Delta > 10$.

Here $F' = f^n f'$ and $G' = g^n g'$ share the value 1 and $\Delta > 10$. Then by Lemma 2.2, we get either

$$F'G' \equiv 1 \quad \text{or} \quad F \equiv G. \tag{3.2}$$

Consider the case $F'G' \equiv 1$, that is,

$$f^n f' g^n g' \equiv 1. \tag{3.3}$$

Suppose that f has a pole z_0 (with order $p \geqslant s$ say). Then z_0 is a zero of g (with order $m \geqslant s$ say). By (3.3), we get

$$nm + m - 1 = np + p + 1.$$

That is, $(m-p)(n+1) = 2$, which is impossible since $n \geqslant 2$ and m, p are positive integers. Therefore, we conclude that f and g are entire functions. From Lemma 2.6, we get $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.

Next we consider another case $F \equiv G$. This gives

$$\frac{f^{n+1}}{n+1} \equiv \frac{g^{n+1}}{n+1}, \quad \text{i.e.,} \quad f^{n+1} = g^{n+1}.$$

Hence $f = dg$ for some $(n+1)$-th root of unity d.

3.2. Proof of Theorem 1.2

Let

$$F = \frac{1}{n+2} f^{n+2} - \frac{1}{n+1} f^{n+1}, \qquad G = \frac{1}{n+2} g^{n+2} - \frac{1}{n+1} g^{n+1},$$

then $F' = f^n(f-1)f'$ and $G' = g^n(g-1)g'$. By hypothesis F' and G' share the value 1 CM. Since

$$m\left(r, \frac{1}{F}\right) \leqslant m\left(r, \frac{F'}{F}\right) + m\left(r, \frac{1}{F'}\right) \leqslant m\left(r, \frac{1}{F'}\right) + S(r,f)$$

and by the first fundamental theorem, we get

$$T(r,F) \leqslant T\bigl(r,F'\bigr) + N\left(r, \frac{1}{F}\right) - N\left(r, \frac{1}{F'}\right) + S(r,f). \tag{3.4}$$

We have

$$N\left(r,\frac{1}{F}\right)=(n+1)N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-\frac{n+2}{n+1}}\right), \tag{3.5}$$

$$N\left(r,\frac{1}{F'}\right)=nN\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f'}\right). \tag{3.6}$$

From (3.4)–(3.6), we deduce that

$$T(r,F)\leqslant T\left(r,F'\right)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-\frac{n+2}{n+1}}\right)-N\left(r,\frac{1}{f-1}\right)-N\left(r,\frac{1}{f'}\right)+S(r,f). \tag{3.7}$$

Since F' and G' share the value 1, we suppose that (i) of Lemma 2.6 holds, that is,

$$\begin{aligned}T\left(r,F'\right)\leqslant{}& \overline{N}\left(r,F'\right)+\overline{N}_{(2}\left(r,F'\right)+\overline{N}\left(r,G'\right)+\overline{N}_{(2}\left(r,G'\right)+\overline{N}\left(r,\frac{1}{F'}\right)+\overline{N}_{(2}\left(r,\frac{1}{F'}\right)\\ &+\overline{N}\left(r,\frac{1}{G'}\right)+\overline{N}_{(2}\left(r,\frac{1}{G'}\right)+S\left(r,F'\right)+S\left(r,G'\right)\\ \leqslant{}& 2\overline{N}(r,f)+2\overline{N}(r,g)+2\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f'}\right)\\ &+2\overline{N}\left(r,\frac{1}{g}\right)+N\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g'}\right)+S(r,f)+S(r,g).\end{aligned}$$

Using $N(r,\frac{1}{g'})\leqslant N(r,\frac{1}{g})+\overline{N}(r,g)$, and by our assumption, zeros and poles of f and g are of multiplicities at least s, that is $\overline{N}(r,g)\leqslant\frac{1}{s}N(r,g)\leqslant\frac{1}{s}T(r,g)$ and $\overline{N}(r,\frac{1}{g})\leqslant\frac{1}{s}N(r,\frac{1}{g})\leqslant\frac{1}{s}T(r,g)$, we deduce above inequality as

$$T\left(r,F'\right)\leqslant\frac{4}{s}T(r,f)+N\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f'}\right)+\left(\frac{5}{s}+2\right)T(r,g)+S(r,f)+S(r,g). \tag{3.8}$$

By (3.7) and (3.8), we obtain

$$T(r,F)\leqslant\left(\frac{4}{s}+2\right)T(r,f)+\left(\frac{5}{s}+2\right)T(r,g)+S(r,f)+S(r,g).$$

Thus

$$\left(n-\frac{4}{s}\right)T(r,f)\leqslant\left(\frac{5}{s}+2\right)T(r,g)+S(r,f)+S(r,g). \tag{3.9}$$

Similarly,

$$\left(n-\frac{4}{s}\right)T(r,g)\leqslant\left(\frac{5}{s}+2\right)T(r,f)+S(r,f)+S(r,g). \tag{3.10}$$

From (3.9) and (3.10), we deduce that $(n-2)s\leqslant 9$, which contradicts $(n-2)s\geqslant 10$. Therefore, by Lemma 2.6, we get either $F'G'\equiv 1$ or $F'\equiv G'$. Consider the case $F'G'\equiv 1$, that is,

$$f^n(f-1)f'g^n(g-1)g'\equiv 1. \tag{3.11}$$

Let z_0 be a zero of f of order p_0. From (3.11) we know that z_0 is a pole of g. Suppose that z_0 is a pole of g of order q_0. Again by (3.11), we obtain

$$np_0+p_0-1=nq_0+2q_0+1,$$

that is, $(n+1)(p_0-q_0)=q_0+2$, which implies

$$p_0\geqslant q_0+1,\quad\text{and}\quad q_0+2\geqslant n+1.\quad\text{Hence}\quad p_0\geqslant n. \tag{3.12}$$

Let z_1 be a zero of $f-1$ of order p_1, then from (3.11) z_1 is a pole of g of order q_1. Again by (3.11), we get

$$p_1+p_1-1=nq_1+2q_1+1$$

i.e.,

$$p_1\geqslant\frac{ns+2s+2}{2}=\frac{(n-2)s+4s+2}{2}\geqslant\frac{12+4s}{2}. \tag{3.13}$$

Let z_2 be a zero of f' of order p_2, that is not zero of $f(f-1)$, then from (3.11) z_2 is a pole of g of order q_2. Again by (3.11), we get

$$p_2 = nq_2 + 2q_1 + 1 \quad \text{or} \quad p_2 \geqslant ns + 2s + 1 = (n-2)s + 4s + 1 \geqslant 11 + 4s. \tag{3.14}$$

In the same manner as above, we have the similar results for the zeros of $g(g-1)g'$. From (3.11), we can write

$$\overline{N}\big(r, f^n(f-1)f'\big) = \overline{N}\left(r, \frac{1}{g^n(g-1)g'}\right),$$

i.e.,

$$\overline{N}(r, f) = \overline{N}\left(r, \frac{1}{g^n(g-1)g'}\right) = \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g-1}\right) + \overline{N}_0\left(r, \frac{1}{g'}\right).$$

From (3.12)–(3.14), we obtain

$$\begin{aligned}\overline{N}(r, f) &\leqslant \frac{1}{p_0}N\left(r, \frac{1}{g}\right) + \frac{1}{p_1}N\left(r, \frac{1}{g-1}\right) + \frac{1}{p_2}N\left(r, \frac{1}{g'}\right)\\ &\leqslant \frac{1}{n}N\left(r, \frac{1}{g}\right) + \frac{2}{12+4s}N\left(r, \frac{1}{g-1}\right) + \frac{1}{11+4s}N\left(r, \frac{1}{g'}\right)\\ &\leqslant \left(\frac{1}{n} + \frac{2}{12+4s} + \frac{2}{11+4s}\right)T(r, g) + S(r, g). \end{aligned}\tag{3.15}$$

By the second fundamental theorem and (3.15), we have

$$\begin{aligned}T(r, f) &\leqslant \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}(r, f) + S(r, f)\\ &\leqslant \left(\frac{1}{n} + \frac{2}{12+4s}\right)T(r, f) + \left(\frac{1}{n} + \frac{2}{12+4s} + \frac{2}{11+4s}\right)T(r, g) + S(r, f) + S(r, g).\end{aligned}\tag{3.16}$$

Similarly, we have

$$T(r, g) \leqslant \left(\frac{1}{n} + \frac{2}{12+4s}\right)T(r, g) + \left(\frac{1}{n} + \frac{2}{12+4s} + \frac{2}{11+4s}\right)T(r, f) + S(r, f) + S(r, g). \tag{3.17}$$

From (3.16) and (3.17), we have

$$T(r, f) + T(r, g) \leqslant \left(\frac{2}{n} + \frac{4}{12+4s} + \frac{2}{11+s}\right)\big[T(r, f) + T(r, g)\big] + S(r, f) + S(r, g).$$

Giving specific values for s and n from our assumption $(n-2)s \geqslant 10$, we deduce that

$$T(r, f) + T(r, g) \leqslant (0.8)\big[T(r, f) + T(r, g)\big] + S(r, f) + S(r, g),$$

which is a contradiction.

Thus we have

$$F' \equiv G', \quad \text{that is,} \quad F \equiv G + c, \tag{3.18}$$

where c is a constant.

It follows that

$$T(r, f) = T(r, g) + S(r, f). \tag{3.19}$$

Suppose that $c \neq 0$. By the second fundamental theorem,

$$\begin{aligned}T(r, G) &\leqslant \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G+c}\right) + \overline{N}(r, G) + S(r, g),\\ (n+2)T(r, g) &\leqslant \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g - \frac{n+2}{n+1}}\right) + \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f - \frac{n+2}{n+1}}\right) + S(r, f)\\ &\leqslant \frac{1}{s}N\left(r, \frac{1}{g}\right) + \frac{1}{s}N(r, g) + \frac{1}{s}N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g - \frac{n+2}{n+1}}\right) + \overline{N}\left(r, \frac{1}{g - \frac{n+2}{n+1}}\right) + S(r, f),\\ (n+2)T(r, g) &\leqslant \left(2 + \frac{3}{s}\right)T(r, g) + S(r, g),\end{aligned}$$

which contradicts the assumption $(n-2)s \geqslant 10$. Hence $F \equiv G$, that is,

$$f^{n+1}\left(f-\frac{n+2}{n+1}\right)=g^{n+1}\left(g-\frac{n+2}{n+1}\right). \tag{3.20}$$

Let $h=\frac{f}{g}$. Since $f \not\equiv g$, we have $h \not\equiv 1$, and hence we deduce that

$$g=\frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \qquad f=\frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})},$$

where h is a non-constant meromorphic function. This completes the proof of Theorem 1.2.

3.3. *Proof of Theorem 1.3*

Let

$$F=\frac{1}{n+3}f^{n+3}-\frac{2}{n+2}f^{n+2}+\frac{1}{n+1}f^{n+1} \quad \text{and} \quad G=\frac{1}{n+3}g^{n+3}-\frac{2}{n+2}g^{n+2}+\frac{1}{n+1}g^{n+1},$$

then

$$F'=f^n(f-1)^2f' \quad \text{and} \quad G'=g^n(g-1)^2g'.$$

We have

$$N\left(r,\frac{1}{F}\right)=(n+1)N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-a_1}\right)+N\left(r,\frac{1}{f-a_2}\right),$$
$$N\left(r,\frac{1}{F'}\right)=nN\left(r,\frac{1}{f}\right)+2N\left(r,\frac{1}{f-1}\right)+N\left(r,\frac{1}{f'}\right),$$

where a_1, a_2 are distinct roots of the algebraic equation $\frac{z^2}{n+3}-\frac{2}{n+2}z+\frac{1}{n+1}=0$.

Proceeding as in the proof of Theorem 1.2, we have

$$\begin{aligned}
T(r,F) &\leqslant T(r,F')+N\left(r,\frac{1}{F}\right)-N\left(r,\frac{1}{F'}\right)+S(r,F') \\
&\leqslant T(r,F')+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-a_1}\right)+N\left(r,\frac{1}{f-a_2}\right)-2N\left(r,\frac{1}{f-1}\right)-N\left(r,\frac{1}{f'}\right)+S(r,f).
\end{aligned} \tag{3.21}$$

Since F' and G' share 1 CM, we suppose that (i) of Lemma 2.6 holds, that is,

$$\begin{aligned}
T(r,F') &\leqslant \overline{N}(r,F')+\overline{N}_{(2}(r,F')+\overline{N}(r,G')+\overline{N}_{(2}(r,G')+\overline{N}\left(r,\frac{1}{F'}\right) \\
&\quad +\overline{N}_{(2}\left(r,\frac{1}{F'}\right)+\overline{N}\left(r,\frac{1}{G'}\right)+\overline{N}_{(2}\left(r,\frac{1}{G'}\right)+S(r,F')+S(r,G') \\
&\leqslant 2\overline{N}(r,f)+2\overline{N}(r,g)+2\overline{N}\left(r,\frac{1}{f}\right)+2N\left(r,\frac{1}{f-1}\right) \\
&\quad +N\left(r,\frac{1}{f'}\right)+2\overline{N}\left(r,\frac{1}{g}\right)+2N\left(r,\frac{1}{g-1}\right)+N\left(r,\frac{1}{g'}\right)+S(r,f)+S(r,g).
\end{aligned} \tag{3.22}$$

Since $N(r,\frac{1}{g'}) \leqslant N(r,\frac{1}{g})+\overline{N}(r,g) \leqslant T(r,g)+\frac{1}{s}T(r,g)$ and from (3.21) and (3.22), we deduce that

$$(n+3)T(r,f) \leqslant \left(\frac{4}{s}+3\right)T(r,f)+\left(\frac{5}{s}+3\right)T(r,g)+S(r,f)+S(r,g). \tag{3.23}$$

Similarly, we get

$$(n+3)T(r,g) \leqslant \left(\frac{4}{s}+3\right)T(r,g)+\left(\frac{5}{s}+3\right)T(r,f)+S(r,f)+S(r,g). \tag{3.24}$$

From (3.23) and (3.24), we obtain $(n-3)s \leqslant 9$, which is a contradiction. Hence by Lemma 2.6, we get either $F'G' \equiv 1$ or $F' \equiv G'$.

In the same manner as in the proof of Theorem 1.2, we obtain $F \equiv G$, that is,

$$\frac{f^{n+3}}{n+3} - \frac{2f^{n+2}}{n+2} + \frac{f^{n+1}}{n+1} \equiv \frac{g^{n+3}}{n+3} - \frac{2}{n+2}g^{n+2} + \frac{1}{n+1}g^{n+1}. \tag{3.25}$$

Setting $h = \frac{f}{g}$, we substitute $f = hg$ in (3.25). It follows that

$$(n+2)(n+1)g^2\big(h^{n+3} - 1\big) - 2(n+3)(n+1)g\big(h^{n+2} - 1\big) + (n+2)(n+3)\big(h^{n+1} - 1\big) = 0. \tag{3.26}$$

First suppose that h is not constant. Making use of (3.26) and Lemma 2.5 (with $n+3$ instead of n), we obtain

$$\big[(n+1)(n+2)\big(h^{n+3} - 1\big)g - (n+3)(n+1)\big(h^{n+2} - 1\big)\big]^2 = -(n+3)(n+1)Q(h),$$

where $Q(h) = (h-1)^4(h-\beta_1)(h-\beta_2)\cdots(h-\beta_{2n})$, $\beta_j \in \mathbb{C} - \{0, 1\}$ $(j = 1, 2, \ldots, 2n)$, which are pairwise distinct.

This implies that every zero of $h - \beta_j$ $(j = 1, 2, \ldots, 2n)$ has a multiplicity of at least 2. By the second fundamental theorem we obtain that $n \leqslant 2$, which is again a contradiction to $(n-3)s \geqslant 10$. Therefore, h is a constant.

If $h \neq 1$, then by (3.26) h has to be bounded in the plane, a contradiction. Hence $f \equiv g$.

3.4. Proof of Theorem 1.4

Since f and g are entire functions, we have $N(r, f) = N(r, g) = 0$. Proceeding as in the proof of Theorem 1.1 and applying Lemma 2.3 we shall obtain that Theorem 1.4 holds.

3.5. Proof of Theorem 1.5

Since f and g are entire functions, we have $N(r, f) = N(r, g) = 0$. Proceeding as in the proof of Theorem 1.2 and applying Lemma 2.6, we can easily prove Theorem 1.5.

4. Open problems

Question 4.1. Can 1 point shared value in Theorems 1.1–1.5 be replaced by fixed point?

Question 4.2. Are the conditions $(n+1)s \geqslant 11$ in Theorem 1.1, $(n-2)s \geqslant 10$ in Theorem 1.2 and $(n-3)s \geqslant 10$ in Theorem 1.3 sharp?

Question 4.3. Can CM shared value be replaced by an IM shared value in Theorems 1.1–1.5?

Question 4.4. Can the differential polynomials in Theorems 1.1–1.5 be replaced by differential polynomials of the form $(f^n)^{(k)}$ and $[f^n(f-1)]^{(k)}$?

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