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L^1 well posedness of Euler equations with dynamic phase boundaries

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ABSTRACT

We discuss the well posedness of the initial value problem to Euler equations related to phase transition. The solution contains two phase boundaries moving in opposite directions. Entropy condition and kinetic relation are used as the main admissibility criteria to select the physically relevant solution. We show the existence of the entropy solution under a suitable Finiteness Condition and a Stability Condition guarantees the stability of the problem in $L^1 \cap BV$ and the existence of a Lipschitz semigroup of solutions. We also discuss the well posedness of the problem given that the wave speeds do not differ significantly between different phases.

1. Introduction

We study the well posedness of the Cauchy problem to Euler equations with two phase boundaries moving in opposite directions. The system is given by

$$\begin{aligned} &v_t - u_x = 0,\\ &u_t - f_x = 0,\\ &E_t - (uf)_x = 0, \end{aligned} \tag{1.1}$$

where v, u and E are strain, velocity and total energy, respectively. $f = -p$ is stress where p is pressure. The total energy E is given by $E = e + \frac{1}{2}u^2$, where e is the internal energy and $u^2/2$ is the kinetic energy. We take strain and entropy s as state variables. Therefore, e and f are functions of v and s. The solution to the system is written as $U = (v, u, s)$. We assume that e is a smooth function of v and s, $e_s = T > 0$, and $e_{vs} = T_v < 0$, where T is temperature. We have the following thermodynamic relation

$$de = f\,dv + T\,ds. \tag{1.2}$$

The initial value of (1.1) is given by

$$U(0, x) = \bar{U} = (\bar{v}, \bar{u}, \bar{s}) = \begin{cases} \bar{U}_1, & x < x_1,\\ \bar{U}_2, & x_1 < x < x_2,\\ \bar{U}_3, & x > x_2, \end{cases} \tag{1.3}$$

where $\bar{U}_i$ $(i = 1, 2, 3)$ are perturbed constant states. Specifically, there exists a function U^c that takes constant value on each of the intervals $(-\infty, x_1)$, (x_1, x_2) and $(x_2, +\infty)$ satisfying $\bar{U} - U^c \in L^1 \cap BV$ with total variation sufficiently small. We

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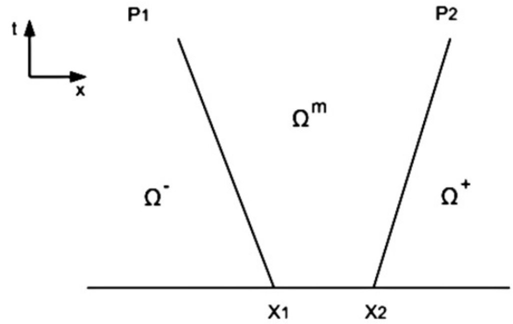

Fig. 1.1. The solution to the initial value problem (1.1), (1.3). The upper xt-plane is divided into three regions Ω^-, Ω^m and Ω^+ by the phase boundaries P_1 and P_2.

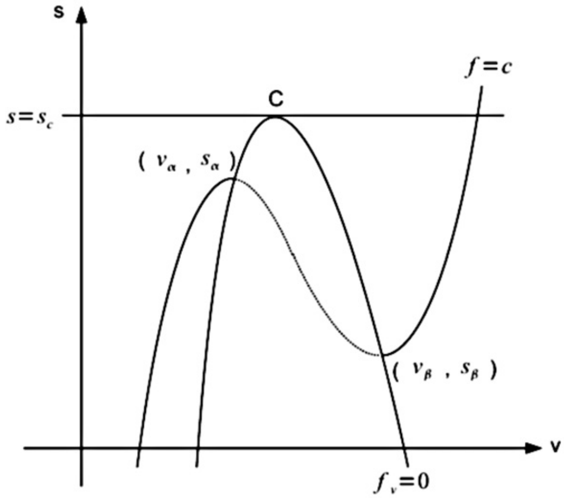

Fig. 1.2. An example of a level curve $f(v,s)=c$.

assume that the solution contains two phase boundaries, denoted by P_1 and P_2, respectively, moving in opposite directions (see Fig. 1.1). We consider one interesting case in physics [9] where

> *the speed of a phase boundary is close to zero such that a phase boundary moves much slower than any shocks* (*except for contact discontinuity*) *or rarefaction waves.*

We also assume that there exists a constant c_0 such that every level curve $f(v,s)=c$, where c is a constant satisfying $c>c_0$, is non-monotone in the vs-plane (see Fig. 1.2, where the curve $f_v=0$ is also sketched). Note that $f_v<0$ is inside the curve $f_v=0$. This region where $f_v<0$ is called the spinodal region and the states with the values of (v,s) in this region are physically unstable and are not observable. For $s<s_c$, the region where $f_v>0$ is separated into two sub-regions. If v is on the left and right of $f_v=0$ and $f_v(v,s)>0$, v is said to be in the α-phase and β-phase, respectively. In the region where $s<s_c$ and $f_v>0$, we assume that $f_{vv}\neq 0$ so that the system is genuinely nonlinear. A typical material satisfying the above assumptions at least locally in the α- and β-phase is the van der Waals fluid where

$$f=-\frac{8e^{3s/(8a)}}{(3v-1)^{1+1/a}}+\frac{3}{v^2}$$

with the positive constant a.

As the weak solutions to the initial value problem with phase transition are not unique, we use the entropy condition and the kinetic relation as the admissibility criteria to select a physically relevant solution. The entropy condition imposes that the entropy increases across jump discontinuities. The rate of decay of the entropy is given by

$$E(v_-,s_-,v_+,s_+)=\sigma(v_-,v_+)(s_+-s_-),$$

where $\sigma(v_-, v_+) = \pm\sqrt{\frac{f_+ - f_-}{v_+ - v_-}}$ is the speed of the jump discontinuity and the subscripts $-$ and $+$ denote the states to the left and right of the discontinuity, respectively. The entropy condition requires that $E(v_-, s_-, v_+, s_+) \leqslant 0$ holds across each discontinuity.

The kinetic relation is proposed by Abeyaratne and Knowles [1,2]. It postulates that there exists a non-decreasing function $\phi(g)$ satisfying $\phi(0) = 0$, where g is called the driving traction, such that the speed of discontinuity is given by

$$\sigma = \phi(g).$$

In order that this relation is consistent with the entropy condition, we require that $\phi' > 0$ so that $\sigma g > 0$ holds. In this paper we choose $g = -(s_+ - s_-)$ for the driving traction. In particular, we use the following kinetic condition

$$\sigma_p = \epsilon(s_- - s_+), \tag{1.4}$$

where ϵ is a small positive constant. This relation is applied to the solutions satisfying the entropy condition.

We also use the initiation criterion which has been used in [1,2,10]. This criterion imposes that no new phase boundary occurs from any point unless no solution exists without the creation of a new phase. This ensures that the spontaneous initiation of a new phase cannot occur from two nearby initial states in the same phase.

In the absence of phase change, J. Glimm proved the existence of the weak solution in the space of bounded variations in his classical paper [7] for $n \times n$ hyperbolic system when the initial data are sufficiently small in *BV*. Bressan, Crasta and Piccoli [5] proved that this problem is well posed when the total variation of the initial data $u_0 \in L^1 \cap BV$ is sufficiently small and they showed that the entropy solutions constitute a semigroup which is Lipschitz continuous with respect to time and initial data. Their analysis of stability was then simplified by Bressan, Liu and Yang in [6] where they introduced a functional that is equivalent to the L^1 distance between two different solutions and they showed that this functional is almost decreasing with respective to time. In the case of $n \times n$ hyperbolic system with large initial data, Lewicka and Trivisa [14] considered the initial value problem which is a Riemann problem solved by two large shocks. They showed the existence of the weak solutions under suitable Finiteness Condition and the stability under the Stability Condition. Lewicka [11] considered the general case with m large shocks, $2 < m \leqslant n$, and showed existence and L^1 stability of the problem under similar Finiteness and Stability Conditions. Moreover, the general case of $n \times n$ system of conservation laws with large non-interacting shocks, contact discontinuities and rarefaction waves was analyzed in [13] with appropriate Finiteness and Stability Conditions.

In the case of conservation laws involving phase transition, Hattori [8] discussed the existence of weak solutions with moving phase boundaries. He considered the case where there are two non-interacting phase boundaries moving in the opposite directions and obtained the existence in *BV* provided that the wave speeds do not differ significantly between different phases. The case where the two phase boundaries collide was also mentioned.

The goal of this paper is to show the existence and L^1 stability of initial value problem (1.1), (1.3). As in [6,14] we introduce a functional which is equivalent to the L^1 distance between two different solutions. We formulate a Finiteness Condition and a Stability Condition that are similar to those in [14]. We show that the Finiteness Condition guarantees the existence of the weak solution and the Stability Condition implies the stability and yields the existence of a Lipschitz semigroup of entropy solutions.

Hattori [8] obtained the existence of the solution in *BV* to the initial value problem (1.1), (1.3) given that the wave speeds do not differ significantly between different phases. We show that both the Finiteness Condition and Stability Condition hold in this case, such that the weak solution not only exists but also is stable.

This paper consists of five sections. Section 2 is the preliminary where we summarize the solutions of the Riemann problems discussed in [9] and introduce the Finiteness Condition and Stability Condition. In Section 3, we introduce the front tracking approximation of the initial value problem and state the main theorem on existence. The L^1 Lyapunov functional is stated in Section 4 whose derivative with respective to time will be analyzed in Section 5.

2. Preliminaries

In this section, we firstly summarize the results in [9] concerning the Riemann problems with dynamic phase transitions. The configuration of Riemann problems is essential in the front tracking approximations when a phase boundary collides with a small physical wave. Then we introduce the Finiteness Condition and Stability Condition that will play an important role in derivation of the existence and stability, respectively, of the initial value problem (1.1), (1.3).

2.1. Phase boundaries

A phase boundary is a line of discontinuity in the xt-plane across which the phase changes. Similar to a shock, the phase boundary satisfies the Rankine–Hugoniot condition

$$\sigma_P(v - v_0) = -(u - u_0),$$
$$\sigma_P(u - u_0) = -(f - f_0),$$
$$\sigma_P(E - E_0) = -(fu - f_0u_0),$$

where σ_P is the speed of discontinuity. However, a phase boundary does not belong to any characteristic family. The phase boundary curve $P(U_0)$ is the set of all possible states $U = (v, u, s)$ connected to $U_0 = (v_0, u_0, s_0)$ by a phase boundary. The projection of $P(U_0)$ onto the *vs*-plane is called the Hugoniot locus and denoted by $H(v_0, s_0)$. As v_0 and v are in the different phases, $H(v_0, s_0)$ is a semi-infinite curve in *vs*-plane with an end (v_0^*, s_0^*) in the phase other than the one that (v_0, s_0) lies in. In the following lemmas, we discuss the location of (v_0^*, s_0^*) given (v_0, s_0) and the relations between s_0 and s_0^*. This end point plays an important role in the Riemann problems. In what follows, we assume that (v_0, s_0) and (v_0^*, s_0^*) are in the α-phase and β-phase, respectively. Integrating both sides of (1.2) along a level curve $C : f = f_0$ in the *vs*-plane from (v_0, s_0) to (v_0^*, s_0^*) gives

$$e - e_0 = f_0 + \int_C e_s \, ds.$$

Therefore, we have the following lemma (see [9] for more details of the proof).

Lemma 2.1. *If $e(v_\beta, s_\beta) - e_0 \leqslant f_0(v_\beta - v_0)$, where (v_β, s_β) is the state in the vs-plane at which $f_v = 0$ along a level curve $f = f_0$ (Fig. 1.2), then in the vs-plane the Hugoniot locus $H(v_0, s_0)$ of the phase boundary curve $P(U_0)$ is a semi-infinite curve starting from the point (v_0^*, s_0^*), with v_0^* in the other phase, satisfying*

$$f\left(v_0^*, s_0^*\right) = f_0, \qquad e_0^* - e_0 = f_0\left(v_0^* - v_0\right).$$

Furthermore, $\int_C e_s \, ds = 0$ holds, where the integral is the path integral along $f = f_0$ from (v_0, s_0) to (v_0^*, s_0^*).

Next lemma can be regarded as an extension of the Maxwell equal area rule from the isothermal case to non-isothermal case.

Lemma 2.2. *In the vs-plane, if the level curve $f = f_0$ is not monotone, there exists a unique state (v_0^m, s_0^m) on $f(v_0^m, s_0^m) = f_0$ with v_0^m in the α-phase such that $(v_0^*, s_0^*) = (v_0^*, s_0^m)$, i.e., the Hugoniot locus of the phase boundary curve $P(U_0)$ starts from the same value of entropy. This also implies that if $s_0 > s_0^m$, $s_0 < s_0^*$ and if $s_0 < s_0^m$, $s_0 > s_0^*$. The corresponding result holds if v_0^m is given in the β-phase.*

We call the states (v, s) in the α-phase (or β-phase) "stable" if it satisfies that $s \leqslant s_0^m$ (or $s \geqslant s_0^m$) and the states (v, s) in the α-phase (or β-phase) "metastable" if $s > s_0^m$ (or $s < s_0^m$).

2.2. The Riemann problems

The initial data of a Riemann problem is given by

$$U(0, x) = (v, u, s)(0, x) = \begin{cases} U_l = (v_., u_l, s_l), & x < 0, \\ U_r = (v_r, u_r, s_r), & x > 0. \end{cases}$$

We seek a self-similar solution consisting of constant states separated by the backward and forward wave, the phase boundaries, and contact discontinuity or the stationary phase boundary. The backward and forward waves are shocks and rarefaction waves. In the case where the speeds of phase boundaries are much smaller than those of the forward and backward waves, we have the configuration of the phase boundaries given the following theorem. For the discussion of the cases where v_l and v_r in the same phase, refer to [9].

Theorem 2.3. *If v_l and v_r are specified in different phases, there are four different solution configurations near $\sigma_p = 0$ depending on the values of s_1 and s_4 at $\sigma_p = 0$.*

(1) *If $s_1 \leqslant s_1^m$ and $s_4 \geqslant s_4^m$ at $\sigma_P = 0$, then the solution with the stationary phase boundary is the only solution satisfying the entropy condition.*
(2) *If $s_1 > s_1^m$ and $s_4 \geqslant s_4^m$ at $\sigma_P = 0$, there is a one-parameter family of solutions with the backward phase boundary.*
(3) *If $s_1 \leqslant s_1^m$ and $s_4 < s_4^m$ at $\sigma_P = 0$, there is a one-parameter family of solutions with the forward phase boundary.*
(4) *If $s_1 > s_1^m$ and $s_4 < s_4^m$ at $\sigma_P = 0$, there are three solution configurations; there are two one-parameter families of solutions, one with the backward phase boundary and another with forward phase boundary. Also it is possible to construct the solutions with three phase boundaries where the left phase boundary moves backward, the middle one is stationary and the right one moves froward. In this case we have a two-parameter family of solutions.*

Furthermore, except case (4) there is a unique solution satisfying the kinetic relation (1.4) provided that ϵ is sufficiently small. The details of the proof are available in [9]. In what follows, we choose the initial data such that cases (2) and (3) in Theorem 2.2 will occur.

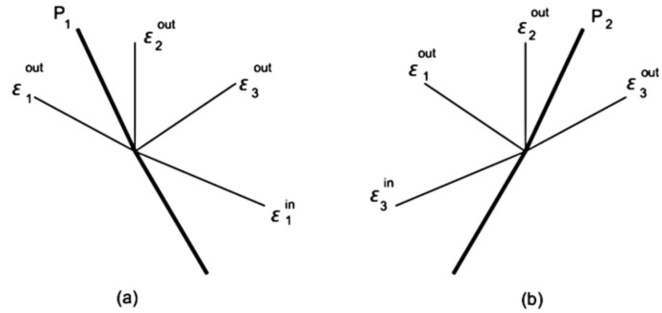

Fig. 2.1. The definition of $m_{13}^{P_1}$ (a) and $m_{31}^{P_2}$ (b).

2.3. The Finiteness Condition and Stability Condition

Consider the wave interaction pattern in Fig. 2.1(a) where a 1-family wave impinges a backward phase boundary, denoted by P_1, from the right. As in [14], we define the number

$$m_{13}^{P_1} = \frac{\partial \varepsilon_3^{out}}{\partial \varepsilon_1^{in}}.$$

Similarly, if a 3-family wave interacts with a forward phase boundary, denoted by P_2, from the left (see Fig. 2.1(b)), we define

$$m_{31}^{P_2} = \frac{\partial \varepsilon_1^{out}}{\partial \varepsilon_3^{in}}.$$

We introduce the following Finiteness Condition and Stability Condition that can be regarded as extensions of those in [14] where the large shocks are replaced by phase boundaries.

Finiteness Condition. There exist positive weights w_1, w_3 and a number $\theta \in (0, 1)$ such that

$$\frac{w_3}{w_1} \cdot \left| m_{13}^{P_1} \right| < \theta \tag{2.1}$$

and

$$\frac{w_1}{w_3} \cdot \left| m_{31}^{P_2} \right| < \theta, \tag{2.2}$$

where w_2 is a small fixed constant.

Remark 1. In the present situation with two non-colliding phase boundaries, the above Finiteness Condition turns out to be equivalent to

$$\left| m_{13}^{P_1} m_{31}^{P_2} \right| < 1. \tag{2.3}$$

On one hand, multiplying (2.1) with (2.2) gives (2.3). (The weights w_1 and w_3 in (2.1) and (2.2) are not exactly the same but close to each other due to the fact that the perturbations are very small.) On the other hand, (2.3) implies (2.1) and (2.2) if we choose $w_1 = 1$ and $w_3 = |m_{31}^{P_2}| + \delta$ with some $\delta > 0$ sufficiently small.

Stability Condition. There exist positive weights $\tilde{w}_1, \tilde{w}_3$ and a number $\Theta \in (0, 1)$ such that

$$\frac{\tilde{w}_3}{\tilde{w}_1} \cdot \left| m_{13}^{P_1} \right| \cdot \left| \frac{\lambda_3(U_0^m) - \sigma^{P_1}}{\lambda_1(U_0^m) - \sigma^{P_1}} \right| < \Theta \tag{2.4}$$

and

$$\frac{\tilde{w}_1}{\tilde{w}_3} \cdot \left| m_{31}^{P_2} \right| \cdot \left| \frac{\lambda_1(U_0^m) - \sigma^{P_2}}{\lambda_3(U_0^m) - \sigma^{P_2}} \right| < \Theta, \tag{2.5}$$

where $\tilde{w}_2$ is a small fixed constant and U_0^m is a constant state in Ω^m defined in (3.3).

Remark 2. Similarly to (2.3), an equivalence of the Stability Condition is given by

$$\left| m_{13}^{P_1} m_{31}^{P_2} \cdot \frac{\lambda_3(U_0^m) - \sigma^{P_1}}{\lambda_1(U_0^m) - \sigma^{P_1}} \cdot \frac{\lambda_1(U_0^m) - \sigma^{P_2}}{\lambda_3(U_0^m) - \sigma^{P_2}} \right| < 1. \tag{2.6}$$

This inequality reduces to (2.3) if $|\sigma^{P_1} - \sigma^{P_2}|$ is sufficiently small.

We will show in Section 3 the existence of the weak solutions of (1.1) and (1.3) under the Finiteness Condition. As $\lambda_1(U_0^m) = -\sqrt{f_v(U_0^m)}$, $\lambda_3(U_0^m) = \sqrt{f_v(U_0^m)}$ and $\sigma^{P_1} < 0 < \sigma^{P_2}$, we have

$$\left| \frac{\lambda_3(U_0^m) - \sigma^{P_1}}{\lambda_1(U_0^m) - \sigma^{P_1}} \right| > 1$$

and

$$\left| \frac{\lambda_1(U_0^m) - \sigma^{P_2}}{\lambda_3(U_0^m) - \sigma^{P_2}} \right| > 1.$$

Therefore, the Stability Condition is stronger than the Finiteness Condition (for the general $n \times n$ hyperbolic system, see [12]). The Stability Condition is essential in the proof of stability in Section 4 and Section 5.

3. Front tracking approximations

Given the Cauchy problem (1.1), (1.3), we employ the strategy of [3,4,14] to obtain the existence of its solution as follows:

(i) Approximate the initial data $\bar{U}$ by piecewise constant data $\bar{U}_\varepsilon$.
(ii) Construct an "approximate solution" U_ε to (1.1) with $U_\varepsilon(0,\cdot) = \bar{U}_\varepsilon$. The approximating function U_ε is piecewise constant with finitely many jumps occurring along straight discontinuity lines. For example, the rarefaction wave will be approximated by finitely many small discontinuities.
(iii) Show that for some parameter sequence $\varepsilon_n \to 0$, the sequence U_{ε_n} has a limit in L^1_{loc}, and that this limit is a solution to (1.1) and (1.3).

As a phase boundary moves much slower than a forward or backward wave, for convenience, we call a backward phase boundary (P_1) a $1\frac{1}{2}$-family wave and a forward phase boundary (P_2) a $2\frac{1}{2}$-family wave. Note that a $1\frac{1}{2}$-family wave or a $2\frac{1}{2}$-family wave is just a notation indicating a slow backward or forward phase boundary. Therefore, we have in total 6 families of waves:

1-family: backward shock or rarefaction wave,
$1\frac{1}{2}$-family: backward phase boundary,
2-family: contact discontinuity,
$2\frac{1}{2}$-family: forward phase boundary,
3-family: forward shock or rarefaction wave,
4-family: non-physical wave [4,3,14].

We denote by λ_k the characteristic speed of a k-family wave for $k = 1, 2, 3$. $\lambda_{1\frac{1}{2}} = \sigma_{P_1}$ and $\lambda_{2\frac{1}{2}} = \sigma_{P_2}$ represent the speeds of the backward and forward phase boundaries, respectively. The speed of a non-physical wave is usually written as $\hat{\lambda}$. Then we have $\lambda_1 < \lambda_{1\frac{1}{2}} < \lambda_2 = 0 < \lambda_{2\frac{1}{2}} < \lambda_3 < \hat{\lambda}$. In our problem, the strengths of all waves are very small except for the phase boundaries.

In the construction of the wave front tracking method, we assume that at most two waves interact with each other at any moment. We solve the Riemann problem (U_l, U_r) when interaction occurs. One problem in constructing the front tracking approximation is to keep the number of wave fronts finite for all times $t > 0$ (see [3]). Therefore, we choose a threshold number ε_T and solve a Riemann problem by the Accurate Riemann Solver when the product of the strengths of the colliding waves is greater than ε_T. When the product is less than ε_T, a Riemann problem is solved by the so-called Simplified Riemann Solver where we let the incoming waves pass through each other, changing their speeds slightly, and collect the remaining waves into the non-physical wave. If both states U_l and U_r are in the same set Ω^-, Ω^m or Ω^+, the wave interaction occurs in the same phase and we solve the Riemann problem as [3]. If a small wave interacts with a phase boundary, we solve the problem as follows.

(i) *Accurate Riemann Solver.*

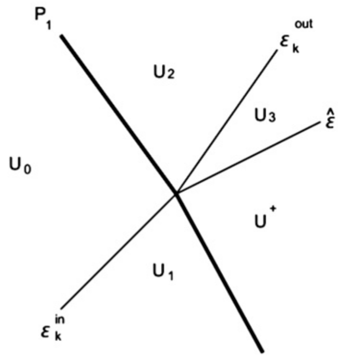

Fig. 3.1. Approximate Riemann Solver.

This is a self-similar solution with the rarefaction wave replaced by a piecewise constant rarefaction fan [14]. In this paper, we choose the initial value such that case (2) and case (3) in Theorem 2.3 occur. As the incoming wave is very small, the configuration of the phase boundary in Theorem 2.3 will not change after interaction.

(ii) *Approximate Riemann Solver.*

Case 1. A physical wave of family k $(k = 2, 3)$ impinges a backward phase boundary from the left. The Riemann problem is solved as follows (see Fig. 3.1, where the case of $k = 3$ is shown):

$$\begin{cases} U_0 & \text{for } x/t < \sigma_{P_1}, \\ U_2 & \text{for } x/t \in \big(\sigma_{P_1}, \lambda_k(U_2, U_3)\big), \\ U_3 & \text{for } x/t \in \big(\lambda_k(U_2, U_3), \hat{\lambda}\big), \\ U^+ & \text{for } x/t > \hat{\lambda}, \end{cases}$$

where $U_3 = \Psi_k(U_2, \varepsilon)$ is the state connected to U_2 through a 3-family shock-rarefaction curve (when $k = 3$) or a contact discontinuity (when $k = 2$). If $k = 2$, $\lambda_k = 0$. If $k = 3$,

$$\lambda_3(U_2, U_3) = \begin{cases} \lambda_3(U_2) & \text{if } \varepsilon > 0, \\ \frac{u_2 - u_0}{v_0 - v_2} & \text{if } \varepsilon < 0. \end{cases}$$

The middle state U_2 is defined as follows. Let $\Psi_{1\frac{1}{2}}(U_l, U_r) = 0$ be the backward phase boundary curve connecting the two states U_l and U_r. Ψ_k $(k = 1, 3)$ is the k-family shock-rarefaction curve. In this case, we have $U_1 = \Psi_3(U_0, \varepsilon^{in})$. We also use an equivalent expression $U_0 = \tilde{\Psi}_3(U_1, -\varepsilon^{in})$. Ψ_4 is the non-physical wave curve and $U^+ = \Psi_4(U_3, \hat{\varepsilon})$ or $U_3 = \tilde{\Psi}_4(U^+, -\hat{\varepsilon})$. Let

$$\begin{aligned} F\big(U_1, U^+, \varepsilon^{in}, \varepsilon^{out}, \hat{\varepsilon}\big) &= \Psi_{1\frac{1}{2}}\big(\tilde{\Psi}_3\big(U_1, -\varepsilon^{in}\big), \tilde{\Psi}_3\big(\tilde{\Psi}_4\big(U^+, -\hat{\varepsilon}\big), -\varepsilon^{out}\big)\big) \\ &= \Psi_{1\frac{1}{2}}(U_0, U_2) = 0, \end{aligned} \tag{3.1}$$

then

$$\frac{\partial F}{\partial(\varepsilon^{out}, \hat{\varepsilon})}\big(U_1, U^+, 0, 0, 0\big) = \frac{\partial \Psi_{1\frac{1}{2}}}{\partial(U_l, U_r)}\big(r_3(U_3), r_4\big(U^+\big)\big),$$

where r_k are the right eigenvectors of corresponding characteristic families. As r_3 and r_4 are independent, by the Implicit Function Theorem, there exits a unique solution for (3.1) and U_2 is given by $\tilde{\Psi}_3(\tilde{\Psi}_4(U^+, -\hat{\varepsilon}), -\varepsilon^{out})$.

We define the strength of a non-physical wave as the distance between its right and left states. Moreover,

we define the strength a phase boundary to be a fixed number D which is bigger than all strengths of small waves.

The strength of a phase boundary will change slightly after it collides with a small physical wave such that the actual strength of a phase boundary should be D plus an error term $O(\varepsilon)$ where ε is the strength of the small wave. In what follows, one can see that this error term will be overwhelmed with D in our analysis.

Definition 3.1 *(Approaching waves).*

(i) We say that two small (possibly non-physical) fronts α and β, located at $x_\alpha < x_\beta$ and belonging to the characteristic family $k_\alpha, k_\beta \in \{1, 2, 3, 4\}$, respectively, approach each other if and only if the two conditions hold:
- x_α and x_β lie in the same set Ω^-, Ω^m or Ω^+.
- Either $k_\alpha > k_\beta$ or $k_\alpha = k_\beta$ and at least one of the waves is a genuinely nonlinear shock.

This case is represented by $(\alpha, \beta) \in \mathcal{A}$.

(ii) We say that a small wave α located at x_α is approaching a phase boundary at x_β if and only if $k_\alpha < k_\beta$ and $x_\alpha > x_\beta$ or $k_\alpha > k_\beta$ and $x_\alpha < x_\beta$. This case is written as $\alpha \in \mathcal{A}_b$ if the k_β-wave is a backward phase boundary and $\alpha \in \mathcal{A}_f$ if the k_β-wave is a forward phase boundary.

Notice that the 2-family is linearly degenerate, such that a 2-wave (contact discontinuity) does not approach other 2-waves.

Let $J = \{1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 4\}$ be the set of the indices and assume that $\{w_k^-\}_{k\in J}$, $\{w_k^m\}_{k\in J}$ and $\{w_k^+\}_{k\in J}$ are three sets of positive numbers. For a small wave of family $k \in J$ and strength ε_k, that connects two states v_1 and v_2, we follow the notation in [14] to define the weighted strength of the wave as

$$b_k = \begin{cases} w_k^- \varepsilon_k & \text{if } v_1, v_2 \in \Omega^-, \\ w_k^m \varepsilon_k & \text{if } v_1, v_2 \in \Omega^m, \\ w_k^- \varepsilon_k & \text{if } v_1, v_2 \in \Omega^+. \end{cases}$$

Definition. Let $t > 0$. The total weighted strength of waves in $U(t, \cdot)$ is defined by

$$V(t) = \sum_\alpha |b_\alpha|,$$

where the summation ranges over all small wave fronts. The (weighted) wave interaction potentials are defined as

$$Q_{\mathcal{A}}(t) = \sum_{(\alpha,\beta)\in\mathcal{A}} |b_\alpha b_\beta|,$$

$$Q_b(t) = \sum_{\alpha\in\mathcal{A}_b} |b_\alpha|,$$

$$Q_f(t) = \sum_{\alpha\in\mathcal{A}_f} |b_\alpha|,$$

and

$$Q(t) = \kappa Q_{\mathcal{A}}(t) + Q_b(t) + Q_f(t).$$

The Glimm functional is

$$\Gamma(t) = V(t) + \tilde{\kappa} Q(t) + \left|U^*(t) - U_0^m\right|,$$

where $\kappa, \tilde{\kappa} > 0$ are constants to be specified later. The vector $U^*(t)$ is the right state of the backward phase boundary at time t.

In order to prove the existence of the solution, we need the following interaction estimate.

Lemma 3.1. *If a small wave b_α ($\alpha = 1, 2, 3$) interacts with a phase boundary, producing outgoing waves c_1, c_2 and c_3. Then*

$$\sum_{k=1}^{3} |c_k| = O(1)|b_\alpha|. \tag{3.2}$$

Proof. When $\alpha = 1$ (or $\alpha = 2$) and the phase boundary moves forward, this estimate is a direct consequence of Lemma 3.2 in [8] where we take $a_1 = a_2 = a_3 = 0$ and $b_2 = 0$ (or $b_1 = 0$, respectively). When $\alpha = 1$ and the phase boundary moves backward, this estimate is implied in Lemma 3.3 in [8] if we let $a_1 = a_2 = a_3 = 0$. Using Taylor expansions as in Lemma 3.2 and Lemma 3.3 in [8], we can show that (3.2) also holds for other cases. $\square$

We define for a given $\delta_0 > 0$ the domain

$$\tilde{\mathcal{D}}_{\delta_0} = cl\Big\{ U : R \to R^3;\ \text{there exist two points } x^b < x^f \text{ in } R \\ \text{such that letting } \tilde{U}(x) = \begin{cases} U_0^-, & x < x^b, \\ U_0^m, & x^b < x < x^f, \\ U_0^+, & x > x^f \end{cases} \\ \text{we have } U - \tilde{U} \in L^1\big(R, R^3\big) \text{ and } T.V.(U - \tilde{U}) \leqslant \delta_0 \Big\}. \tag{3.3}$$

Lemma 3.2. *We assume that the Finiteness Condition holds. There exist* $\{w_k^-\}$, $\{w_k^m\}$, $\{w_k^+\}$, *constants* $\kappa, \tilde{\kappa} > 0$ *and* $\delta > 0$ *such that the following holds*

$$\lim_{x \to -\infty} U(0, x) = U_0^-, \qquad \lim_{x \to \infty} U(0, x) = U_0^+.$$

There exist points $x^b < x^f$ *in* R *such that*

$$U(x, 0) \in \begin{cases} \Omega^- & \text{for } x < x^b, \\ \Omega^m & \text{for } x^b < x < x^f, \\ \Omega^+ & \text{for } x > x^f. \end{cases}$$

If $T.V.(U(x, 0) - \tilde{U}) < \delta$, *then for any* $t > 0$ *when two wave fronts* b_α *and* b_β *interact we have*

(i)
$$\Delta Q(t) = Q(t+) - Q(t-) \\ \leqslant \begin{cases} -c|b_\alpha b_\beta| & \text{if both waves are small}, \\ -c|b_\alpha| & \text{if } \alpha \text{ wave is small and } \beta \text{ wave is a phase boundary}, \end{cases}$$

where the number c *is some small positive, uniform constant.*

(ii) *The same estimate holds for* $\Delta\Gamma(t) = \Gamma(t+) - \Gamma(t-)$.

Proof. (i) Let $t > 0$ be fixed time of interaction of two waves one of which could be a phase boundary or a non-physical wave.

- *Case I.* Two small waves interact with each other.

By the standard estimates in [17], we have

$$\Delta Q_b = O(1)|b_\alpha b_\beta|,$$

$$\Delta Q_f = O(1)|b_\alpha b_\beta|,$$

$$\Delta Q_{\mathcal{A}} = -|b_\alpha b_\beta| + O(1)V(t-)|b_\alpha b_\beta|.$$

This is exactly Case I of Proposition 3.4 in [14]. Let C denote the largest uniform constant in the estimates above. If $\kappa \geqslant 4C$ and $V(t) \leqslant 1/\kappa$, one sees that (i) holds for $c = C$.

- *Case II.* A small wave interacts with the backward phase boundary from the left.

Suppose that the interaction is solved by the Accurate Riemann Solver and the outgoing waves are denoted by c_i $(i = 1, 2, 3)$. If the small wave belongs to the 3rd characteristic family, we have

$$\Delta Q_b = -|b_3|,$$

$$\Delta Q_f = |c_3|,$$

$$\Delta Q_{\mathcal{A}} = O(1)V(t-)|b_3|.$$

When $V(t-) < 1/4\kappa C$ and we choose a small weight w_3^m for the transpassing wave c_3 such that $|c_3| \leqslant |b_3|/4$, (i) holds for the constant $c = 1/2$.

If the interaction is solved by the Approximate Riemann Solver, we have

$$\Delta Q_b = -|b_3|,$$
$$\Delta Q_f = |c_3| + |c_4|,$$
$$\Delta Q_{\mathcal{A}} = O(1)V(t-)|b_3|.$$

As the total strength of non-physical waves remains uniformly small [3], i.e. $|c_4| < |b_3|/4$, (i) holds if we choose $c = 1/4$.

Similar assertion holds when the incoming small wave is a 2-family wave and we need to choose a small weight w_2^m for the transpassing wave c_2.

- *Case III.* A small wave interacts with the forward phase boundary from the right.

This case is similar to Case II while we need to choose the weight w_1^m (or w_2^m) sufficiently small if the incoming wave is a 1-wave (or 2-wave).

- *Case IV.* A small wave interacts with a backward phase boundary from the right.

We need the *Finiteness Condition* in this case. Under the assumption that phase boundaries move much slower than a 1- or 3-wave, the small physical wave must be a 1-wave and

$$\Delta Q_b = -|b_1|,$$
$$\Delta Q_f = |c_3|,$$
$$\Delta Q_{\mathcal{A}} = O(1)V(t-)|b_1|.$$

By the *Finiteness Condition* (2.1),

$$\Delta Q(t) \leqslant \big(C\kappa V(t-) - 1 + \theta\big)|b_1|.$$

Therefore, (i) holds for $c = (1-\theta)/2$ if we choose $V(t-) \leqslant (1-\theta)/2\kappa C$.

- *Case V.* A small wave interacts with a forward phase boundary from the left.

This case is similar to Case IV and we need the *Finiteness Condition* (2.2).

(ii) Note that

$$\Delta V(t) = V(t+) - V(t-) \leqslant \begin{cases} C|b_\alpha b_\beta| & \text{in Case I,} \\ C|b_\alpha| & \text{in Case II and Case III,} \end{cases}$$

by [4] and Lemma 3.1. In Cases I and III $U^*(t-) = U^*(t+)$, so $|U^*(t) - U_0^m|$ does not change across the interaction time t. In Case II $|U^*(t-) - U^*(t+)| = O(1)|b_\alpha|$ by Lemma 3.1. Thus, if $\tilde{\kappa}$ is large enough, we get (ii) provided that

$$V(t-) \leqslant \tilde{\delta} = \min\left\{\frac{1}{\kappa}, \frac{1}{4C}, \frac{1-\theta}{2\kappa C}\right\}.$$

Notice that

$$\begin{aligned} V(t-) \leqslant \Gamma(t-) \leqslant \Gamma(0) &= V(0) + \tilde{\kappa}Q(0) + \big|U^*(0) - U_0^m\big| \\ &\leqslant C_1 \cdot T.V.\big(U(0,\cdot) - \tilde{U}\big) + \tilde{\kappa}\big\{\kappa C_1 \cdot \big[T.V.\big(U(0,\cdot) - \tilde{U}\big)\big]^2 + 2C_1 \cdot T.V.\big(U(0,\cdot) - \tilde{U}\big)\big\}, \end{aligned}$$

where C_1 is a uniform positive constant. If the constant δ is small enough, the inequality $T.V.(U(0,\cdot) - \tilde{U}) < \delta$ implies $V(t-) < \tilde{\delta}$ and the result follows. □

As in the case without the presence of large waves [3], Lemma 3.2 results in the following assertions. If $U(0,\cdot)$ satisfies the assumption of Lemma 3.2, then our wave front tracking algorithm generates a piecewise constant approximate solution that has finitely many discontinuity lines for all $t \in [0,\infty)$. Moreover, the functional Γ is nonincreasing in time, and we have

$$\Gamma(t) \leqslant \Gamma(0),$$
$$T.V.\big(U(t,\cdot) - \hat{U}\big) = O(1) \cdot \Gamma(t) = O(1) \cdot T.V.\big(U(t,\cdot) - \tilde{U}\big)$$

for some $\hat{U}$ in (3.3) as $\tilde{U}$. The total strength of all non-physical waves occurring at any fixed time $t > 0$ is of the order $O(1)(\delta)$.

Following [6] and [14], we gather the main properties of the wave front tracking approximate solutions.

Theorem 3.3. *Assume that a piecewise constant function $U(0,\cdot)$ satisfies the assumption of Lemma* 3.2. *Given $\varepsilon > 0$, for some parameters $\delta > 0$ the corresponding wave front tracking algorithm produces the function $U[0,+\infty) \mapsto L^1(R;R^3)$ such that*:

(i) *As a function of two variables, $U = U(t,x)$ is piecewise constant, with discontinuities occurring along finitely many lines in the t–x plane. Only finitely many wave front interactions occur, each involving exactly two incoming fronts. Jumps can be of four types: shocks (or contact discontinuities), rarefactions, non-physical waves and phase boundaries denoted as $\mathcal{F} = \mathcal{S} \cup \mathcal{R} \cup \mathcal{NP} \cup \mathcal{PB}$.*

(ii) *Along each shock (or contact discontinuity) $x = x_\alpha(t)$, $\alpha \in \mathcal{S}$, the values $U^- \doteq U(t,x_\alpha-)$ and $U^+ \doteq U(t,x_\alpha+)$ are related by*

$$U^+ = S_{k_\alpha}(\sigma_\alpha)\big(U^-\big), \tag{3.4}$$

for some $k_\alpha \in \{1,2,3\}$ and some wave size σ_α. If the k_α-family is genuinely nonlinear, then the entropy admissibility condition $\sigma_\alpha < 0$ also holds. Moreover, the speed of the shock front satisfies

$$\big|\dot{x}_\alpha - \lambda_{k_\alpha}\big(U^+,U^-\big)\big| \leqslant \varepsilon. \tag{3.5}$$

(iii) *Along each rarefaction front $x = x_\alpha(t)$, $\alpha \in \mathcal{R}$, one has*

$$U^+ = R_{k_\alpha}(\sigma_\alpha)\big(U^-\big), \quad \sigma_\alpha \in (0,\varepsilon] \tag{3.6}$$

for some genuinely nonlinear family k_α. Moreover,

$$\big|\dot{x}_\alpha - \lambda_{k_\alpha}\big(U^+\big)\big| \leqslant \varepsilon. \tag{3.7}$$

(iv) *All non-physical fronts $x = x_\alpha(t)$, $\alpha \in \mathcal{NP}$ have the same speed*:

$$\dot{x}_\alpha(t) \equiv \hat{\lambda}, \tag{3.8}$$

where $\hat{\lambda}$ is a fixed constant strictly greater than all characteristic speeds. The total strength of all non-physical fronts in $U(t,\cdot)$ remains uniformly small, namely,

$$\sum_{\alpha\in\mathcal{NP}} \big|U(t,x_\alpha+) - U(t,x_\alpha-)\big| \leqslant \varepsilon \quad \textit{for all } t \geqslant 0. \tag{3.9}$$

(v) *The backward and forward phase boundaries, denoted by P_1 and P_2 respectively, are determined by Theorem* 2.3.

The function U will be called an ε-approximate solution of (1.1) *and* (1.3).

Now we can obtain the existence of the weak solution to (1.1) and (1.3). The following theorem is similar to Theorem A in [14].

Theorem 3.4. *If the Finiteness Condition is satisfied, then there exists $\delta_0 > 0$ such that for every $\bar{U} \in \tilde{\mathcal{D}}_{\delta_0}$ there exists a weak solution to* (1.1) *and* (1.3) *defined for all $t > 0$.*

Proof. Take $\bar{U} \in \tilde{\mathcal{D}}_{\delta_0}$, for some δ_0 smaller than δ in Lemma 3.2. Given $\varepsilon > 0$, fix a piecewise constant $\bar{U}_\varepsilon \in \tilde{\mathcal{D}}_{\delta_0}$, such that

$$\|\bar{U} - \bar{U}_\varepsilon\|_{L^1(R,R^3)} < \varepsilon.$$

Let U_ε be the ε-approximation of (1.1) with $U_\varepsilon(0,\cdot) = \bar{U}_\varepsilon$, as in Theorem 3.3. Let $\varepsilon \to 0$, we can extract a sequence U_{ε_n} converging in L^1_{loc} to a function $U(t,x)$. By the inequalities in Theorem 3.3, U must be a solution to (1.1) and (1.3). □

We can obtain the existence of the weak solutions by Theorem 3.4 under the Finiteness Condition. However, whether this condition holds has not been shown for any given system. In what follows we discuss a case where the Finiteness Condition is satisfied. Let us consider the initial value problem (1.1) and (1.3) given that the wave speeds $\sqrt{f_v}$ do not differ significantly between α- and β-phase, i.e. $|\lambda_1| \approx |\lambda_3|$. This case is important in physics and the existence of the weak solutions is obtained in [8]. In the following theorem, we will show that the Finiteness Condition holds therefore the existence of weak solutions can be derived from Theorem 3.4.

Corollary 3.5. *Suppose that the wave speeds $\sqrt{f_v}$ do not differ significantly between α- and β-phase, the conclusion in Theorem* 3.4 *holds.*

Proof. We only need to show that the Finiteness Condition holds in this case.

Let us consider the wave interaction pattern shown in Fig. 2.1(a). Obviously, this is a special case of Lemma 3.3 in [8] with $a_1 = a_2 = a_3 = 0$ such that we have

$$c_3 = \delta b_1,$$

where $c_3 = \varepsilon_3^{out}$ and $b_1 = \varepsilon_1^{in}$ for some constant δ satisfying $0 < \delta < 1$. When the incoming wave strength ε_1^{in} is very small,

$$\frac{\varepsilon_3^{out}}{\varepsilon_1^{in}} \approx \frac{\partial \varepsilon_3^{out}}{\partial \varepsilon_1^{in}} = \left|m_{13}^{P_1}\right|.$$

Choosing $w_1 = w_3$, we have

$$\frac{w_3}{w_1}\left|m_{13}^{P_1}\right| \approx \delta < 1.$$

Therefore, we have (2.1) for some number θ satisfying $\delta < \theta < 1$.

Similarly, considering the wave interaction pattern shown in Fig. 2.1(b), we can see that (2.2) holds for $w_1 = w_3$. □

4. The Lyapunov functional and stability

In order to show the L^1 stability of the weak solutions, we follow [15,16,6,14] to introduce the Lyapunov functional $\Phi(U, V)$ satisfying

$$\frac{1}{C} \cdot \left\|U(t,\cdot) - V(t,\cdot)\right\|_{L^1} \leqslant \Phi\big(U(t,\cdot), V(t,\cdot)\big) \leqslant C \cdot \left\|U(t,\cdot) - V(t,\cdot)\right\|_{L^1} \tag{4.1}$$

and

$$\Phi\big(U(t,\cdot), V(t,\cdot)\big) - \Phi\big(U(s,\cdot), V(s,\cdot)\big) \leqslant C \cdot \varepsilon \cdot (t-s), \quad \forall t > s \geqslant 0, \tag{4.2}$$

for any two ε-approximate solutions U and V. The functional is equivalent to the L^1 distance between U and V and is "almost decreasing" in time.

We define

$$\Phi(U, V) \doteq \sum_{k \in I} \int_{-\infty}^{+\infty} \left|q_k(x)\right| W_k(x)\, dx,$$

where $I = \{1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3\}$ is set of the indices without non-physical waves and q_k is the size of the k-th shock. The weights W_k are defined by

$$W_k(x) \doteq 1 + \kappa_1 A_k(x) + \kappa_2\big[Q(u) + Q(v)\big]. \tag{4.3}$$

The constants κ_1 and κ_2 are to be defined later. Q is the Glimm's interaction functional. When $k \in \{1\frac{1}{2}, 2, 2\frac{1}{2}\}$ we simply take

$$A_k \doteq \Bigg[\sum_{x_\alpha < x,\ k < k_\alpha \leqslant 3} + \sum_{x_\alpha > x,\ 1 \leqslant k_\alpha < k}\Bigg]|\varepsilon_\alpha|.$$

The summations here extend to waves both of U and V. This is similar to the linearly degenerate case in [14]. When $k \in \{1, 3\}$, $A_k = B_k + C_k$ where

$$B_k \doteq \Bigg[\sum_{\substack{\alpha \in \mathcal{F}(U) \cup \mathcal{F}(V) \\ x_\alpha < x,\ k < k_\alpha \leqslant 3}} + \sum_{\substack{\alpha \in \mathcal{F}(U) \cup \mathcal{F}(V) \\ x_\alpha > x,\ 1 \leqslant k_\alpha < k}}\Bigg]|\varepsilon_\alpha|,$$

$$C_k \doteq \begin{cases} \big[\sum_{\substack{\alpha \in \mathcal{F}(U) \setminus \mathcal{PB} \\ x_\alpha < x,\ k_\alpha = k}} + \sum_{\substack{\alpha \in \mathcal{F}(V) \setminus \mathcal{PB} \\ x_\alpha > x,\ k_\alpha = k}}\big]|\varepsilon_\alpha| & \text{if } q_k(x) < 0, \\ \big[\sum_{\substack{\alpha \in \mathcal{F}(V) \setminus \mathcal{PB} \\ x_\alpha < x,\ k_\alpha = k}} + \sum_{\substack{\alpha \in \mathcal{F}(U) \setminus \mathcal{PB} \\ x_\alpha > x, k_\alpha = k}}\big]|\varepsilon_\alpha| & \text{if } q_k(x) > 0. \end{cases}$$

We can always assume that

$$1 \leqslant W_k(x) \leqslant 2 \tag{4.4}$$

given that $W_k(x)$ does not contain any phase boundaries and

$$1 + m\kappa_1 D \leqslant W_k(x) \leqslant 2 + m\kappa_1 D \tag{4.5}$$

if $W_k(x)$ contains m phase boundaries. As there are in total 4 phase boundaries in both U and V, we have $m \leqslant 4$ and

$$1 \leqslant W_k(x) \leqslant 2 + 4\kappa_1 D. \tag{4.6}$$

Thus,

$$\frac{1}{2 + 4\kappa_1 D}\big|U(x) - V(x)\big| \leqslant \sum_{k\in I} |q_k(x)| \leqslant (2 + 4\kappa_1 D)\big|U(x) - V(x)\big|$$

and (4.1) holds.

Theorem 4.1. *If the Stability Condition is satisfied, there exist $\delta_0 > 0$, $L > 0$, a closed domain $\mathcal{D}_{\delta_0} \subset L^1_{loc}(R, R^3)$ containing $\tilde{\mathcal{D}}_{\delta_0}$, and a continuous semigroup $S : [0, +\infty) \times \mathcal{D}_{\delta_0} \to \mathcal{D}_{\delta_0}$ such that*

(i) $S(0, \bar{U}) = \bar{U}$, $S(t + s) = S(t, S(s, \bar{U}))$, $\forall t, s \geqslant 0$, $\forall \bar{U} \in \mathcal{D}_{\delta_0}$.
(ii) $\|S(t, \bar{U}) - S(s, \bar{V})\|_{L^1} \leqslant L(|t - s|) + \|\bar{U} - \bar{V}\|_{L^1}$, $\forall t, s \geqslant 0$, $\forall \bar{U}, \bar{V} \in \mathcal{D}_{\delta_0}$.
(iii) *Each trajectory $t \mapsto S(t, \bar{U})$ is a weak solution of* (1.1), (1.3).

Proof. This is a standard proof in [6,14] given that (4.1) and (4.2) hold. □

Theorem 4.1 shows the stability of the weak solution under the Stability Condition, while it remains unknown whether this condition holds in any given system. Similar to Corollary 3.5, we consider the case where the wave speeds $\sqrt{f_v}$ do not differ significantly between α- and β-phase. We will show that the Stability Condition also holds in this case and the L^1 stability of the weak solution can be obtained consequently.

Corollary 4.2. *Suppose that the wave speeds $\sqrt{f_v}$ do not differ significantly between α- and β-phase, i.e. $|\lambda_1| \approx |\lambda_3|$, the conclusions in Theorem* 4.1 *hold.*

Proof. We only need to verify that the Stability Condition holds in this case.

Firstly, we consider the wave interaction pattern shown in Fig. 2.1(a). We know from the assumption that

$$\lambda_3 = -\lambda_1 + \varepsilon_0,$$

where $\lambda_1 < 0$ and $\lambda_3 > 0$ are the characteristic speeds of the 1- and 3-wave, respectively, and ε_0 is constant satisfying that $|\varepsilon_0| \ll |\lambda_i|$ $(i = 1, 3)$. Hence,

$$\begin{aligned}\left|\frac{\lambda_3 - \sigma_{P_1}}{\lambda_1 - \sigma_{P_1}}\right| &= \left|\frac{-\lambda_1 + \varepsilon_0 - \sigma_{P_1}}{\lambda_1 - \sigma_{P_1}}\right| = \left|\frac{\lambda_1 - \sigma_{P_1} - \varepsilon + 2\sigma_{P_1}}{\lambda_1 - \sigma_{P_1}}\right| \\ &= \left|1 + \frac{2\sigma_{P_1} - \varepsilon}{\lambda_1 - \sigma_{P_1}}\right| \leqslant 1 + \left|\frac{2\sigma_{P_1} - \varepsilon}{\lambda_1 - \sigma_{P_1}}\right| \leqslant 1 + O\big(|\sigma_{P_1}| + |\varepsilon|\big),\end{aligned}$$

where the last inequality holds because the phase boundary moves much slower than a 1- or 3-wave, i.e. $|\lambda_1| \gg |\sigma_{P_1}|$, in our problem.

Choosing the weights $\tilde{w}_1 = \tilde{w}_3$, we have

$$\frac{\tilde{w}_3}{\tilde{w}_1}\big|m_{13}^{P_1}\big|\left|\frac{\lambda_3 - \sigma_{P_1}}{\lambda_1 - \sigma_{P_1}}\right| = \big|m_{13}^{P_1}\big|\big(1 + O\big(|\sigma_{P_1}| + |\varepsilon|\big)\big) \approx \frac{\partial\varepsilon_3^{out}}{\partial\varepsilon_1^{in}} \approx \frac{\varepsilon_3^{out}}{\varepsilon_1^{in}} = \delta < 1,$$

where δ is the same as in Corollary 3.5 and last inequality is derived from Lemma 3.3 in [8]. Therefore, (2.4) holds if we choose some number Θ satisfying $\delta < \Theta < 1$. Similarly, (2.5) holds with $\tilde{w}_1 = \tilde{w}_3$ if we consider the wave interaction pattern shown in Fig. 2.1(b). □

In order to prove (4.2), we differentiate the functional Φ at a time t which is not the interaction time of the waves in $U(t, \cdot)$ or $V(t, \cdot)$ to show that

$$\begin{aligned}\frac{d}{dt}\Phi\big(U(t), V(t)\big) &= \sum_{\alpha\in\mathcal{F}}\sum_{k\in I}\big\{\big|q_k(x_\alpha-)\big|W_k(x_\alpha-) - \big|q_k(x_\alpha+)\big|W(x_\alpha+)\big\}\dot{x}_\alpha \\ &= \sum_{\alpha\in\mathcal{F}}\sum_{k\in I}\big\{\big|q_k(x_\alpha+)\big|W(x_\alpha+)\big(\lambda_k(x_\alpha+) - \dot{x}_\alpha\big) - \big|q_k(x_\alpha-)\big|W(x_\alpha-)\big(\lambda_k(x_\alpha-) - \dot{x}_\alpha\big)\big\}\dot{x}_\alpha,\end{aligned} \tag{4.7}$$

where $\dot{x}_\alpha$ is the speed of the discontinuity at the α wave. Let

$$E_{\alpha,k} = |q_k^{\alpha+}| W_k^{\alpha+}(\lambda_k^{\alpha+} - \dot{x}_\alpha) - |q_k^{\alpha-}| W_k^{\alpha-}(\lambda_k^{\alpha-} - \dot{x}_\alpha),$$

where $q_k^{\alpha+} = q_k(x_\alpha+)$, $\lambda_k^{\alpha+} = \lambda_k(x_\alpha+)$ and so on. Then (4.7) becomes

$$\frac{d}{dt}\Phi\big(U(t), V(t)\big) = \sum_{\alpha\in\mathcal{F}}\sum_{k\in I} E_{\alpha,k}.$$

Our main goal will be to establish

$$\sum_{k\in I} E_{\alpha,k} = O(1)\cdot|\varepsilon_\alpha|, \quad \forall\alpha\in\mathcal{NP}, \tag{4.8}$$

$$\sum_{k\in I} E_{\alpha,k} \leqslant 0, \quad \forall\alpha\in\mathcal{PB}, \tag{4.9}$$

$$\sum_{k\in I} E_{\alpha,k} = O(1)\cdot\varepsilon|\varepsilon_\alpha|, \quad \forall\alpha\in\mathcal{S}\cup\mathcal{R}. \tag{4.10}$$

Also we need that all weights $W_k(x)$ decrease after an interaction of wave fronts in U or V. Recalling Lemma 3.1 and Lemma 3.2(i). One sees this statement holds if $\kappa_2 \gg \kappa_1$ in (4.3).

The proof of (4.8) is the same as [6,14] and thus omitted. In next section, we will show that (4.9) and (4.10) hold in different wave interaction patterns. Combining (4.8), (4.9) and (4.10), recalling (3.9) and the uniform bound on the total strengths of waves, denoted by the functional $\Gamma(t)$, we have

$$\frac{d}{dt}\Phi\big(U(t), V(t)\big) \leqslant C\cdot\varepsilon.$$

Integrating this inequality gives (4.2).

5. Stability of approximate solution

In this section, we will prove (4.9) and (4.10). In what follows, we drop the notation α to write $E_{\alpha,k}$ as

$$E_k = |q_k^+| W_k^+(\lambda_k^+ - \dot{x}_\alpha) - |q_k^-| W_k^-(\lambda_k^- - \dot{x}_\alpha)$$

without any ambiguity. We will need to choose different weights $\tilde{w}_k$ ($k\in\{1,2,3\}$) in proving (4.9) and (4.10). In general, *all weights will be chosen very small* while some should be relatively larger than others. We summarize the sizes of the weights as follows:

(i) In the domain Ω^-, $\tilde{w}_1^-$ is small relative to $\tilde{w}_2^-$ and $\tilde{w}_3^-$.
(ii) In the domain Ω^+, $\tilde{w}_3^+$ is small relative to $\tilde{w}_1^+$ and $\tilde{w}_2^+$.
(iii) All the weights in the domain Ω^m are small. Moreover, $\tilde{w}_1^m$ and $\tilde{w}_3^m$ need to satisfy (2.4) and (2.5).

5.1. Cases of phase boundaries – the estimate (4.9)

We consider the cases where the discontinuities in U or V are phase boundaries. Under the assumption that a phase boundary moves much slower than a 1- or 3-wave, we have in total four different cases.

Case 1. See Fig. 5.1.

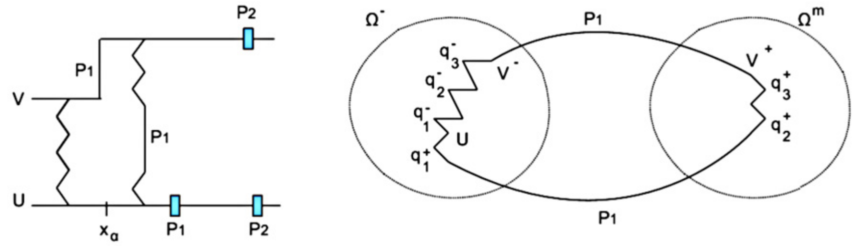

Fig. 5.1. Case 1.

As there is no phase boundary in the solution of the Riemann problem (U, V^-), $q^-_{1\frac{1}{2}} = 0$ and

$$E_{1\frac{1}{2}} = \left|q^+_{1\frac{1}{2}}\right| W^+_{1\frac{1}{2}}\left(\lambda^+_{1\frac{1}{2}} - \dot{x}_\alpha\right).$$

Therefore, (4.4) implies that

$$E_{1\frac{1}{2}} \leqslant D \cdot 4|\sigma_{P_1}|,$$

where $|q^+_{1\frac{1}{2}}| = D$ is the strength of the phase boundary.

Notice that $\lambda_1^\pm < 0$, we can always assume that $\lambda_1^\pm - \dot{x}_\alpha < -c < 0$ due to the fact that the speed of a phase boundary $|\dot{x}_\alpha| = |\sigma_{P_1}|$ is very small compared with characteristic speeds $|\lambda_k^\pm|$ $(k = 1, 3)$. Thus,

$$\begin{aligned} E_1 &= \left|q_1^+\right|\left(W_1^+ - W_1^-\right)\left(\lambda_1^+ - \dot{x}_\alpha\right) + W_1^-\left[\left|q_1^+\right|\left(\lambda_1^+ - \dot{x}_\alpha\right) - \left|q_1^-\right|\left(\lambda_1^- - \dot{x}_\alpha\right)\right] \\ &\leqslant -\left|q_1^+\right| \cdot \kappa_1 D \cdot \left|\lambda_1^+ - \dot{x}_\alpha\right| + 2\left|q_1^-\right| \cdot \left|\lambda_1^- - \dot{x}_\alpha\right|, \end{aligned} \tag{5.1}$$

where $W_1^+ - W_1^- = \kappa_1 D$. The 2-family wave is a contact discontinuity whose characteristic speed, $\lambda_2^\pm$, is always 0 in Lagrange coordinates. Such that

$$\begin{aligned} E_2 &= \left(\left|q_2^-\right| W_2^- - \left|q_2^+\right| W_2^+\right)\dot{x}_\alpha \\ &\leqslant -2\kappa_1 D\left|q_2^-\right| \cdot |\dot{x}_\alpha| + (\kappa_1 D + 2) \cdot \left|q_2^+\right| \cdot |\dot{x}_\alpha|, \end{aligned} \tag{5.2}$$

where the inequality holds because $1 + 2\kappa_1 D \leqslant W_2^- \leqslant 2 + 2\kappa_1 D$ and $1 + \kappa_1 D \leqslant W_2^+ \leqslant 2 + \kappa_1 D$ by (4.5). Notice that $\lambda_3^\pm - \dot{x}_\alpha > 0$, we have

$$\begin{aligned} E_3 &= \left|q_3^+\right| W_3^+\left(\lambda_3^+ - \dot{x}_\alpha\right) - \left|q_3^-\right| W_3^-\left(\lambda_3^- - \dot{x}_\alpha\right) \\ &\leqslant (3\kappa_1 D + 2) \cdot \left|q_3^+\right| \cdot \left|\lambda_3^+ - \dot{x}_\alpha\right| - 4\kappa_1 D \cdot \left|q_3^-\right| \cdot \left|\lambda_3^- - \dot{x}_\alpha\right|. \end{aligned} \tag{5.3}$$

Summing (5.1), (5.2) and (5.3) gives

$$\begin{aligned} \sum_{k=1,1\frac{1}{2},2,3} E_k \leqslant{}& 4D \cdot |\sigma_{P_1}| - \kappa_1 D \cdot \left|q_1^+\right| \cdot \left|\lambda_1^+ - \dot{x}_\alpha\right| - 2\kappa_1 D\left|q_2^-\right| \cdot |\dot{x}_\alpha| + \kappa_1 D \cdot \left|q_2^+\right| \cdot |\dot{x}_\alpha| + 3\kappa_1 D\left|q_3^+\right| \cdot \left|\lambda_3^+ - \dot{x}_\alpha\right| \\ & - 4\kappa_1 D \cdot \left|q_3^-\right| \cdot \left|\lambda_3^- - \dot{x}_\alpha\right| + 2\left|q_1^-\right| \cdot \left|\lambda_1^- - \dot{x}_\alpha\right| + 2\left|q_3^+\right| \cdot |\lambda - \dot{x}_\alpha| + 2\left|q_2^+\right| \cdot |\dot{x}_\alpha|. \end{aligned} \tag{5.4}$$

Similar to Lemma 5.1(iv) in [14], we have the following lemma.

Lemma 5.1. *In Case* 1, *we have*

$$\left|\varepsilon_2^+\right| + \left|\varepsilon_3^+\right| \leqslant O(1)\left(\left|\varepsilon_2^-\right| + \left|\varepsilon_3^-\right|\right),$$

where $\varepsilon_k^\pm$ are the unweighted strengths of corresponding waves.

Proof. If $\varepsilon_2^- = \varepsilon_3^- = 0$, by the uniqueness of the solution to the Riemann problem (U, V^+), $\varepsilon_2^+ = \varepsilon_3^+ = 0$. Then the result follows from the Lipschitz continuity of the problem. □

If we choose the constant κ_1 very large and the weights $\tilde{w}_2^-$ and $\tilde{w}_3^-$ big enough, then (4.9) holds by Lemma 5.1.

Case 2. See Fig. 5.2.

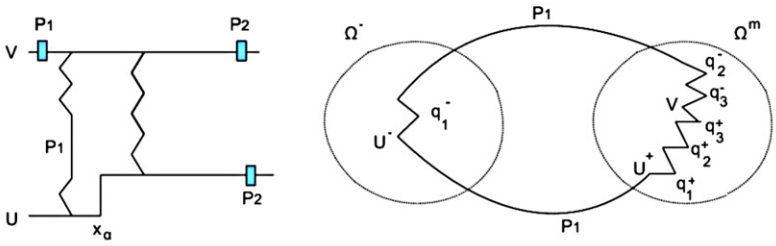

Fig. 5.2. Case 2.

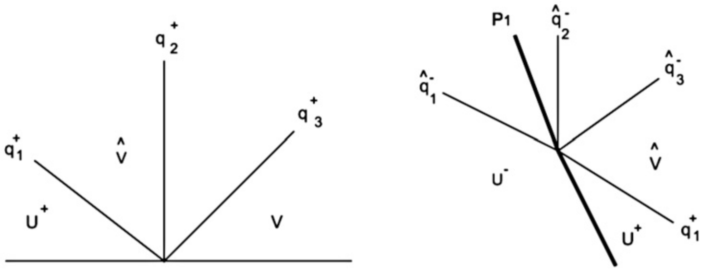

Fig. 5.3. The definition of $\hat{V}$.

As there is no phase boundary in the solution of the Riemann problem (U^+, V), $q^+_{1\frac{1}{2}} = 0$ and

$$E_{1\frac{1}{2}} = -\left|q^-_{1\frac{1}{2}}\right| W^-_{1\frac{1}{2}}\left(\lambda^-_{1\frac{1}{2}} - \dot{x}_\alpha\right) \leqslant D \cdot 4|\sigma_{P_1}|. \tag{5.5}$$

Moreover, we have

$$\begin{aligned} E_1 &= \left|q_1^+\right| W_1^+\left(\lambda_1^+ - \dot{x}_\alpha\right) - \left|q_1^-\right| W_1^-\left(\lambda_1^- - \dot{x}_\alpha\right) \\ &\leqslant -2\kappa_1 D \cdot \left|q_1^+\right| \cdot \left|\lambda_1^+ - \dot{x}_\alpha\right| + (\kappa_1 D + 2) \cdot \left|q_1^-\right| \cdot \left|\lambda_1^- - \dot{x}_\alpha\right|, \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} E_2 &= \left(\left|q_2^-\right| W_2^- - \left|q_2^+\right| W_2^+\right)\dot{x}_\alpha \\ &= \left[\left|q_2^-\right|\left(W_2^- - W_2^+\right) + W_2^+\left(\left|q_2^-\right| - \left|q_2^+\right|\right)\right]\dot{x}_\alpha \\ &\leqslant -\kappa_1 D\left|q_2^-\right| \cdot |\dot{x}_\alpha| + 2\left|q_2^+\right| \cdot |\dot{x}_\alpha|, \end{aligned} \tag{5.7}$$

where $1 \leqslant W_2^+ \leqslant 2$ by (4.4) as W_2^+ does not contain any phase boundaries. We also have

$$\begin{aligned} E_3 &= \left|q_3^+\right| W_3^+\left(\lambda_3^+ - \dot{x}_\alpha\right) - \left|q_3^-\right| W_3^-\left(\lambda_3^- - \dot{x}_\alpha\right) \\ &\leqslant (2\kappa_1 D + 2)\left|q_3^+\right| \cdot \left|\lambda_3^+ - \dot{x}_\alpha\right| - 3\kappa_1 D \cdot \left|q_3^-\right| \cdot \left|\lambda_3^- - \dot{x}_\alpha\right|. \end{aligned} \tag{5.8}$$

Summing up (5.5) through (5.8) gives

$$\begin{aligned} \sum_{k=1,1\frac{1}{2},2,3} E_k &\leqslant -2\kappa_1 D \cdot \left|q_1^+\right| \cdot \left|\lambda_1^+ - \dot{x}_\alpha\right| + \kappa_1 D \cdot \left|q_1^-\right| \cdot \left|\lambda_1^- - \dot{x}_\alpha\right| - \kappa_1 D\left|q_2^-\right| \cdot |\dot{x}_\alpha| + 2\kappa_1 D\left|q_3^+\right| \cdot \left|\lambda_3^+ - \dot{x}_\alpha\right| \\ &\quad - 3\kappa_1 D \cdot \left|q_3^-\right| \cdot \left|\lambda_3^- - \dot{x}_\alpha\right| + 4D|\sigma_{P_1}| + 2\left|q_1^-\right| \cdot \left|\lambda_1^- - \dot{x}_\alpha\right| + 2\left|q_2^+\right| \cdot |\dot{x}_\alpha| + 2\left|q_3^+\right| \cdot \left|\lambda_3^+ - \dot{x}_\alpha\right|. \end{aligned} \tag{5.9}$$

When $q_2^+ = q_3^+ = 0$, (4.9) holds for a large κ_1 and a small weight $\tilde{w}_1^-$. When $q_2^+, q_3^+ \neq 0$, we define $\hat{V} = S_1(U^+, q_1^+)$ to be the state connected to U on the right through a 1-shock (see Fig. 5.3).

Lemma 5.2. *For the states $U^-, U^+, V, \hat{V}$ and the waves q_i^+ and $\hat{q}_i^-$ $(i = 1, 2, 3)$ defined in Fig. 5.3, we have the following estimates*

$$\begin{aligned} q_1^- &= \hat{q}_1^- + R_2, \\ q_2^- &= \hat{q}_2^- + q_2^+ + R_2, \\ q_3^- &= \hat{q}_3^- + q_3^+ + R_2, \end{aligned}$$

where $R_2 = O(|\hat{q}_3^-|(|q_2^+| + |q_3^+|))$ is a second order error term.

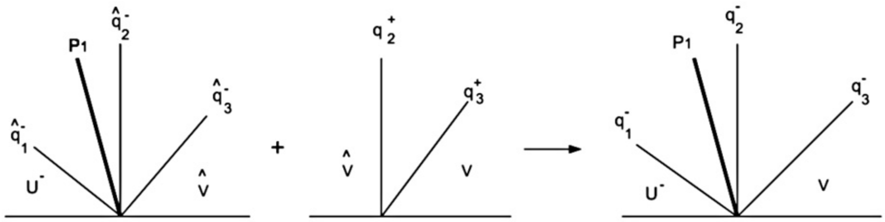

Fig. 5.4. The wave interaction pattern $(U^-, \hat{V}) + (\hat{V}, V) \to (U^-, V)$.

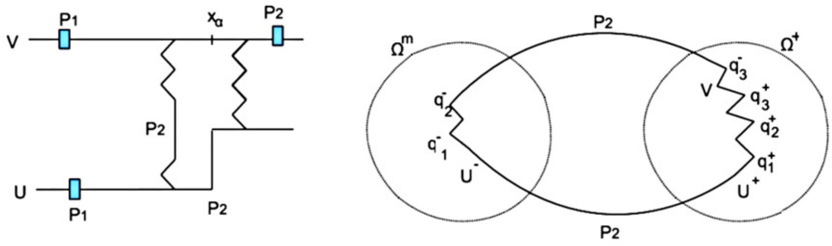

Fig. 5.5. Case 3.

Proof. In the wave interaction pattern $(U^-, \hat{V}) + (\hat{V}, V) \to (U^-, V)$ in Fig. 5.4, $\hat{q}_3^-$ approaches q_2^+ and q_3^+. Such that the lemma holds by Theorem 19.2 in [17]. □

Notice that

$$
\begin{aligned}
&2\kappa_1 D\left|q_3^- - \hat{q}_3^-\right| \cdot \left|\lambda_3^+ - \dot{x}_\alpha\right| - 3\kappa_1 D \cdot \left|q_3^-\right| \cdot \left|\lambda_3^- - \dot{x}_\alpha\right| \\
&\quad \leqslant 2\kappa_1 D\left(\left|q_3^-\right| + \left|\hat{q}_3^-\right|\right) \cdot \left|\lambda_3^+ - \dot{x}_\alpha\right| - 3\kappa_1 D \cdot \left|q_3^-\right| \cdot \left|\lambda_3^- - \dot{x}_\alpha\right| \\
&\quad = -\kappa_1 D \cdot \left|q_3^-\right| \cdot \left|\lambda_3^- - \dot{x}_\alpha\right| + 2\kappa_1 D\left|\hat{q}_3^-\right| \cdot \left|\lambda_3^+ - \dot{x}_\alpha\right| + 2\kappa_1 D\left|q_3^-\right| \cdot \left(\left|\lambda_3^+ - \dot{x}_\alpha\right| - \left|\lambda_3^- - \dot{x}_\alpha\right|\right) \\
&\quad \leqslant -\kappa_1 D \cdot \left|q_3^-\right| \cdot \left|\lambda_3^- - \dot{x}_\alpha\right| + 2\kappa_1 D\left|\hat{q}_3^-\right| \cdot \left|\lambda_3^+ - \dot{x}_\alpha\right| + 2\kappa_1 D\left|q_3^-\right| \cdot \left|\lambda_3^+ - \lambda_3^-\right| \\
&\quad \leqslant -\kappa_1 D \cdot \left|q_3^-\right| \cdot \left|\lambda_3^- - \dot{x}_\alpha\right| + 2\kappa_1 D\left|\hat{q}_3^-\right| \cdot \left|\lambda_3^+ - \dot{x}_\alpha\right| + 2\kappa_1 D\left|q_3^-\right| \cdot O\left(\left|\hat{q}_3^-\right|\right),
\end{aligned}
$$

where the last inequality holds because $|\lambda_3^+ - \lambda_3^-| = O(|q_3^+ - q_3^-|) = O(|\hat{q}_3^-|)$. Thus, (5.9) becomes

$$
\begin{aligned}
\sum_{k=1,1\frac{1}{2},2,3} E_k \leqslant{}& -2\kappa_1 D \cdot \left|q_1^+\right| \cdot \left|\lambda_1^+ - \dot{x}_\alpha\right| + \kappa_1 D \cdot \left|\hat{q}_1^-\right| \cdot \left|\lambda_1^- - \dot{x}_\alpha\right| - \kappa_1 D\left|q_2^-\right| \cdot |\dot{x}_\alpha| - \kappa_1 D \cdot \left|q_3^-\right| \cdot \left|\lambda_3^- - \dot{x}_\alpha\right| \\
&+ 2\kappa_1 D\left|\hat{q}_3^-\right| \cdot \left|\lambda_3^+ - \dot{x}_\alpha\right| + 2\kappa_1 D \cdot O(1)\left(\left|q_3^-\right| \cdot \left|\hat{q}_3^-\right|\right) + O\left(|\sigma_{P_1}| + \left|q_1^-\right| + \left|q_2^+\right| + \left|q_3^+\right|\right) + R_2. \qquad (5.10)
\end{aligned}
$$

By the Stability Condition (2.4), we have in the wave interaction pattern shown in Fig. 5.3(b) that there exist weights $\tilde{w}_1^+$ and $\hat{w}_3^-$ such that $|\hat{q}_3^-| \cdot |\lambda_3^+ - \dot{x}_\alpha| \leqslant \Theta|q_1^+| \cdot |\lambda_1^+ - \dot{x}_\alpha|$. Thus,

$$
\begin{aligned}
\sum_{k=1,1\frac{1}{2},2,3} E_k \leqslant{}& -2(1-\Theta)\kappa_1 D \cdot \left|q_1^+\right| \cdot \left|\lambda_1^+ - \dot{x}_\alpha\right| + \kappa_1 D \cdot \left|\hat{q}_1^-\right| \cdot \left|\lambda_1^- - \dot{x}_\alpha\right| \\
&- \kappa_1 D\left|q_2^-\right| \cdot |\dot{x}_\alpha| - \kappa_1 D \cdot \left|q_3^-\right| \cdot \left|\lambda_3^- - \dot{x}_\alpha\right| + O(1),
\end{aligned}
$$

where $O(1) = 2\kappa_1 D \cdot O(1)(|q_3^-| \cdot |\hat{q}_3^-|) + O(|\sigma_{P_1}| + |q_1^-| + |q_2^+| + |q_3^+|) + R_2$ which will be overwhelmed by other terms if we choose κ_1 very large. Then (4.9) holds if we choose κ_1 large and $\tilde{w}_1^-$ very small relative to other weights.

Case 3. See Fig. 5.5.

This case is similar to Case 1. (4.9) holds if we choose κ_1 and the weights $\tilde{w}_1^+, \tilde{w}_2^+$ large enough.

Case 4. See Fig. 5.6.

This case is similar to Case 2. If we choose κ_1 large and the weight $\tilde{w}_3^+$ small enough, (4.9) holds under the Stability Condition.

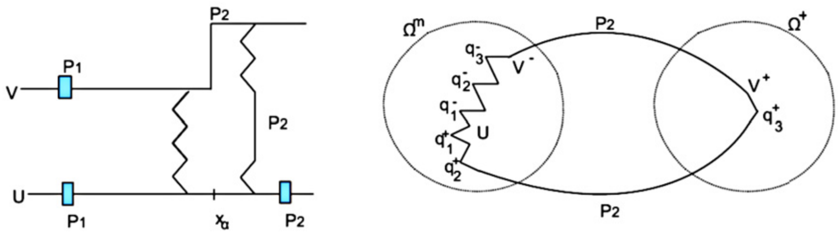

Fig. 5.6. Case 4.

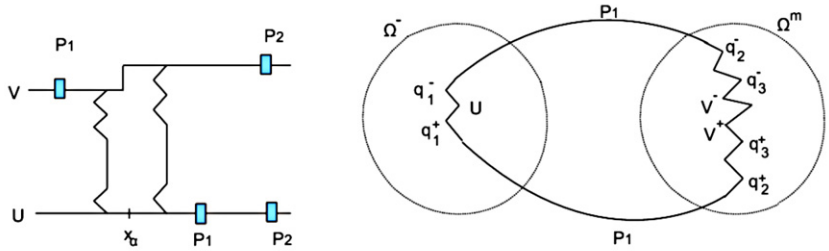

Fig. 5.7. Case A.

5.2. Cases of small physical waves – the estimate (4.10)

We consider the cases where the discontinuities in U or V are physical waves. (4.10) can be obtained in the same way as [6], if both U and V lie in the same region, stable or metastable, near $x = x_\alpha$. Such that, under the assumption that a phase boundary moves slower than any 1- or 3-waves, we only need to consider the following four cases.

Case A. See Fig. 5.7.

When $k_\alpha = 1$, we have

$$\begin{aligned} E_{1\frac{1}{2}} &= \left|q^+_{\frac{1}{2}}\right|\left(W^+_{1\frac{1}{2}} - W^-_{1\frac{1}{2}}\right)\left(\lambda^+_{1\frac{1}{2}} - \dot{x}_\alpha\right) + W^-_{1\frac{1}{2}}\left[\left|q^+_{1\frac{1}{2}}\right|\left(\lambda^+_{1\frac{1}{2}} - \dot{x}_\alpha\right) - \left|q^-_{1\frac{1}{2}}\right|\left(\lambda^-_{1\frac{1}{2}} - \dot{x}_\alpha\right)\right] \\ &\leqslant D\left(-\kappa_1|\varepsilon_\alpha|\right)\left|\lambda^+_{1\frac{1}{2}} - \dot{x}_\alpha\right| + O(1)\left|\lambda^+_{1\frac{1}{2}} - \lambda^-_{1\frac{1}{2}}\right| \\ &\leqslant -\kappa_1 D\cdot|\varepsilon_\alpha|\cdot\left|\lambda^+_{1\frac{1}{2}} - \dot{x}_\alpha\right| + O(1)|\varepsilon_\alpha|. \end{aligned} \tag{5.11}$$

The estimation of E_1 can be divided into two cases. If $q_1^- q_1^+ > 0$, we have

$$\begin{aligned} E_1 &= \left|q_1^\pm\right|\left(W_1^+ - W_1^-\right)\left(\lambda_1^\pm - \dot{x}_\alpha\right) + W_1^\mp\left[\left|q_1^+\right|\left(\lambda_1^+ - \dot{x}_\alpha\right) - \left|q_1^-\right|\left(\lambda_1^- - \dot{x}_\alpha\right)\right] \\ &\leqslant -\left|q_1^\pm\right|\cdot\kappa_1|\varepsilon_\alpha|\cdot\left|\lambda_1^\pm - \dot{x}_\alpha\right| + \left(O(1) + \kappa_1 D\right)\left[\left(\left|q_1^+\right| - \left|q_1^-\right|\right)\left(\lambda_1^+ - \dot{x}_\alpha\right) + \left|q_1^-\right|\left(\lambda_1^+ - \lambda_1^-\right)\right] \\ &\leqslant \left(O(1) + \kappa_1 D\right)\left[O(1)\left|q_1^+ - q_1^-\right| + O(1)\left|q_1^-\right|\cdot|\varepsilon_\alpha|\right]. \end{aligned} \tag{5.12}$$

Otherwise, if $q_1^- q_1^+ < 0$, then $W_1^+ = W_1^-$ and

$$E_1 \leqslant \left(O(1) + \kappa_1 D\right)\left[O(1)\left|q_1^+ - q_1^-\right| + O(1)\left|q_1^-\right|\cdot|\varepsilon_\alpha|\right]. \tag{5.13}$$

For E_2, we have

$$\begin{aligned} E_2 &= \left[\left|q_2^-\right|\left(W_2^- - W_2^+\right) + W_2^+\left(\left|q_2^-\right| - \left|q_2^+\right|\right)\right]\dot{x}_\alpha \\ &\leqslant -\kappa_1|\varepsilon_\alpha|\cdot\left|q_2^-\right|\cdot|\dot{x}_\alpha| + \left(O(1) + \kappa_1 D\right)\cdot\left|q_2^+ - q_2^-\right|\cdot|\dot{x}_\alpha| \\ &\leqslant \left(O(1) + \kappa_1 D\right)\cdot\left|q_2^+ - q_2^-\right|\cdot|\dot{x}_\alpha|. \end{aligned} \tag{5.14}$$

In addition, as $W_3^+ = W_3^-$, we have

$$\begin{aligned} E_3 &= \left|q_3^+\right|\left(W_3^+ - W_3^-\right)\left(\lambda_3^+ - \dot{x}_\alpha\right) + W_3^-\left[\left|q_3^+\right|\left(\lambda_3^+ - \dot{x}_\alpha\right) - \left|q_3^-\right|\left(\lambda_3^- - \dot{x}_\alpha\right)\right] \\ &\leqslant \left(O(1) + \kappa_1 D\right)\left[\left(\left|q_3^+\right| - \left|q_3^-\right|\right)\left(\lambda_3^+ - \dot{x}_\alpha\right) + \left|q_3^-\right|\left(\lambda_3^+ - \lambda_3^-\right)\right] \\ &\leqslant \left(O(1) + \kappa_1 D\right)\left[O(1)\left|q_3^+ - q_3^-\right| + O(1)\left|q_3^-\right|\cdot|\varepsilon_\alpha|\right]. \end{aligned} \tag{5.15}$$

Summing (5.11), (5.12) (or (5.13)), (5.14) and (5.15) gives

$$\sum_{k=1,1\frac{1}{2},2,3} E_k \leqslant -\kappa_1 D\cdot|\varepsilon_\alpha|\cdot\left|\lambda^+_{1\frac{1}{2}}-\dot{x}_\alpha\right|+O(1)|\varepsilon_\alpha|+O(1)(1+\kappa_1 D)$$
$$\cdot\left[\left|q_1^+-q_1^-\right|+\left|q_2^+-q_2^-\right|+\left|q_3^+-q_3^-\right|+\left(\left|q_1^-\right|+\left|q_3^-\right|\right)|\varepsilon_\alpha|\right].$$

Such that (4.9) holds if we choose κ_1 large enough and all the weights $\tilde{w}_k^\pm$ very small. (Notice that $|\varepsilon_\alpha|$ will not be changed.)

When $k_\alpha=2$, $\dot{x}_\alpha=0$. Hence

$$\begin{aligned}E_{1\frac{1}{2}} &= \left|q^+_{\frac{1}{2}}\right|W^+_{1\frac{1}{2}}\lambda^+_{1\frac{1}{2}}-\left|q^-_{1\frac{1}{2}}\right|W^-_{1\frac{1}{2}}\lambda^-_{1\frac{1}{2}}\\ &= D\left[\left(W^+_{1\frac{1}{2}}-W^-_{1\frac{1}{2}}\right)\lambda^+_{1\frac{1}{2}}+W^-_{1\frac{1}{2}}\left(\lambda^+_{1\frac{1}{2}}-\lambda^-_{1\frac{1}{2}}\right)\right]\\ &\leqslant -D\kappa_1|\varepsilon_\alpha|\cdot|\sigma_{P_1}|+O(1)D|\varepsilon_\alpha| \end{aligned}\tag{5.16}$$

and

$$\begin{aligned}E_1 &= \left|q_1^+\right|\left(W_1^+-W_1^-\right)\lambda_1^+ + W_1^-\left[\left|q_1^+\right|\lambda_1^+-\left|q_1^-\right|\lambda_1^-\right]\\ &\leqslant -\left|q_1^+\right|\cdot\kappa_1|\varepsilon_\alpha|\cdot\left|\lambda_1^+\right|+\left(O(1)+\kappa_1 D\right)\cdot\left[\left(\left|q_1^+\right|-\left|q_1^-\right|\right)\lambda_1^+ + \left|q_1^-\right|\left(\lambda_1^+-\lambda_1^-\right)\right]\\ &\leqslant \left(O(1)+\kappa_1 D\right)\left[O(1)\left|q_1^+-q_1^-\right|+O(1)\left|q_1^-\right|\cdot|\varepsilon_\alpha|\right].\end{aligned}\tag{5.17}$$

We can see by the definition that $E_2=0$. Moreover,

$$\begin{aligned}E_3 &= \left|q_3^+\right|\left(W_3^+-W_3^-\right)\lambda_3^+ + W_3^-\left[\left|q_3^+\right|\lambda_3^+-\left|q_3^-\right|\lambda_3^-\right]\\ &\leqslant -\kappa_1|\varepsilon_\alpha|\cdot\left|q_3^+\right|\cdot\left|\lambda_3^+\right|+\left(O(1)+\kappa_1 D\right)\cdot\left[\left(\left|q_3^+\right|-\left|q_3^-\right|\right)\left(\lambda_3^+ + \left|q_3^-\right|\left(\lambda_3^+-\lambda_3^-\right)\right)\right]\\ &\leqslant \left(O(1)+\kappa_1 D\right)\left[O(1)\left|q_3^+-q_3^-\right|+O(1)\left|q_3^-\right|\cdot|\varepsilon_\alpha|\right].\end{aligned}\tag{5.18}$$

Summing (5.16), (5.17) and (5.18) gives

$$\sum_{k=1,1\frac{1}{2},2,3} E_k \leqslant -D\kappa_1|\varepsilon_\alpha|\cdot|\sigma_{P_1}|+O(1)D|\varepsilon_\alpha|+\left(O(1)+\kappa_1 D\right)$$
$$\cdot\left[O(1)\left|q_1^+-q_1^-\right|+O(1)\left|q_3^+-q_3^-\right|+O(1)\left|q_1^-\right|\cdot|\varepsilon_\alpha|\right]$$

and (4.9) holds if we choose κ_1 large and all the weights $\tilde{w}_k^\pm$ small enough.

The case where $k_\alpha=3$ is similar to that where $k_\alpha=1$. We have

$$E_{1\frac{1}{2}} \leqslant -\kappa_1 D\cdot|\varepsilon_\alpha|\cdot\left|\lambda^+_{1\frac{1}{2}}-\dot{x}_\alpha\right|+O(1)|\varepsilon_\alpha|.$$

Since $W_1^+=W_1^-$,

$$E_1 \leqslant \left(O(1)+\kappa_1 D\right)\left[O(1)\left|q_1^+-q_1^-\right|+O(1)\left|q_1^-\right|\cdot|\varepsilon_\alpha|\right].$$

We have

$$E_2 \leqslant \left(O(1)+\kappa_1 D\right)\cdot\left|q_2^+-q_2^-\right|\cdot|\dot{x}_\alpha|$$

and

$$E_3 \leqslant \left(O(1)+3\kappa_1 D\right)\left[O(1)\left|q_3^+-q_3^-\right|+O(1)\left|q_3^-\right|\cdot|\varepsilon_\alpha|\right].$$

Thus,

$$\sum_{k=1,1\frac{1}{2},2,3} E_k \leqslant -\kappa_1 D\cdot|\varepsilon_\alpha|\cdot\left|\lambda^+_{1\frac{1}{2}}-\dot{x}_\alpha\right|+O(1)|\varepsilon_\alpha|+O(1)(1+\kappa_1 D)$$
$$\cdot\left[\left|q_1^+-q_1^-\right|+\left|q_2^+-q_2^-\right|+\left|q_3^+-q_3^-\right|+\left(\left|q_1^-\right|+\left|q_3^-\right|\right)|\varepsilon_\alpha|\right].$$

Also (4.9) holds if we choose κ_1 large and all the weights $\tilde{w}_k^\pm$ small enough.

Case B. See Fig. 5.8.

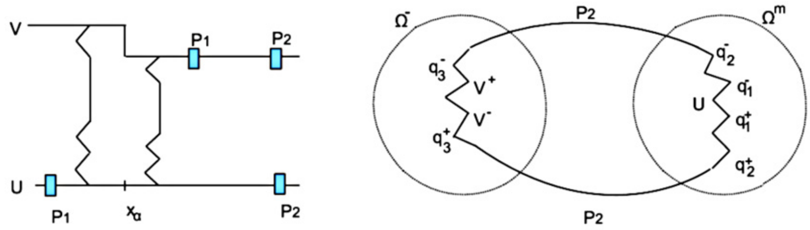

Fig. 5.8. Case B.

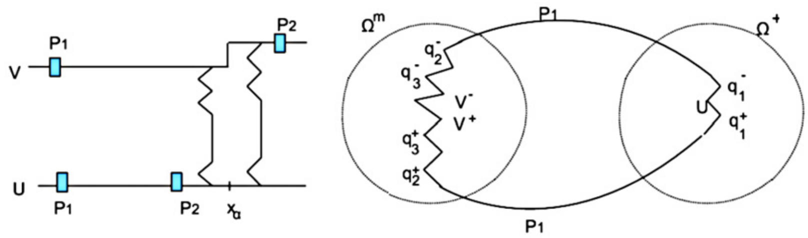

Fig. 5.9. Case C.

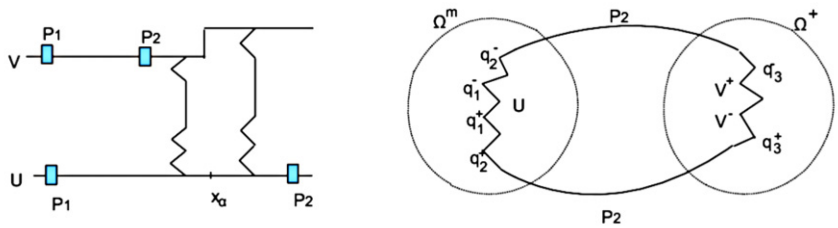

Fig. 5.10. Case D.

In this case, we have $U \in \Omega^m$ which is stable and $V \in \Omega^-$ which is metastable, such that the solution to the Riemann problem (U, V) contains a forward phase boundary P_2. The analysis of this case is similar to Case A with the large negative term given by $E_{2\frac{1}{2}}$ instead of $E_{1\frac{1}{2}}$. We also have (4.10) by choosing all the weights $\tilde{w}_k^{\pm}$ small enough.

Case C. See Fig. 5.9.

As $U \in \Omega^+$ is metastable and $V \in \Omega^m$ is stable, the solution to the Riemann problem (U, V) contains a backward phase boundary. Therefore, this case is similar to Case A.

Case D. See Fig. 5.10.

This case is similar to Case B and the large negative term is given by $E_{2\frac{1}{2}}$.

6. Discussion

We have obtained the existence of solutions of the initial value problem of Euler equations ((1.1), (1.3)) involving two phase boundaries moving in opposite directions (Fig. 1.1) under the Finiteness Condition (Theorem 3.4). The Finiteness Condition ((2.1), (2.2)) says that when a small 1-wave hits a backward phase boundary from the right, the weighted strength of the outgoing 3-wave is smaller than that of the incoming wave (Fig. 2.1(a)). Symmetrically, when a small 3-wave hits a forward phase boundary from the left, the weighted strength of the outgoing 1-wave is smaller than that of the incoming wave (Fig. 2.1(b)). In Theorem 4.1, we show that the Stability Condition ((2.4), (2.5)) guarantees the L^1 stability of solutions and the existence of a semigroup of solutions. The Stability Condition is stronger than the Finiteness Condition [12] and has similar physical interpretations.

The initial value problem discussed in this paper ((1.1), (1.3)) is not a perturbed Riemann problem as the initial value contains three constant states with perturbations. It is trivial to extend our conclusions to a Riemann problem involving two phase boundaries (other waves have zero strengths) under the configuration given by Theorem 3.4(4) in [9]. Similar to (1.1), (1.3), the left and right states are metastable and middle state is stable.

Lewicka [11] also studies the well posedness of the initial value problem of an $n \times n$ hyperbolic system which is a perturbed Riemann problem solved by M $(2 \leqslant M \leqslant n)$ large shocks. The Finiteness Condition and Stability Condition in

[11] imply that when a small wave hits a large shock, *the TOTAL weighted strength of the outgoing small waves are smaller than the weighted strength of the incoming wave*. For Euler equations with phase transition, a general Riemann problem in the configuration given by Theorem 3.4(4) in [9] contains three large physical waves, including a 1-wave, a 3-wave and a contact discontinuity, and two phase boundaries. We can show that when both 1- and 3-waves are shocks, the L^1 well posedness holds under a Finiteness Condition and a Stability Condition that are similar to those in [11]. However, it remains unknown whether the Finiteness Condition and Stability Condition hold for Euler equations. In other words, we do not know whether there exist such weights w_i in the Finiteness Condition (or $\tilde{w}_i$ in the Stability Condition) that the *total* weighted strength of the outgoing small waves are smaller than the weighted strength of the incoming wave when a small wave hits a large shock or a phase boundary.

Another open problem is the well posedness of the initial value problem with two colliding phase boundaries. Hattori [8] showed the existence of the weak solution in *BV* when the left and right states are close to each other. It should be interesting to discuss the well posedness of this case under some suitable Finiteness Condition and Stability Condition.

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