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Blow-up of solutions to semilinear wave equations with variable coefficients and boundary

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ABSTRACT

This paper is devoted to studying initial–boundary value problems for semilinear wave equations and derivative semilinear wave equations with variable coefficients on exterior domain with subcritical exponents in n space dimensions. We will establish blow-up results for the initial–boundary value problems. It is proved that there can be no global solutions no matter how small the initial data are, and also we give the life span estimate of solutions for the problems.

1. Introduction and main results

In this paper, we will consider the blow-up of solutions of the initial–boundary value problems for the following two semilinear wave equations on exterior domain

$$\begin{cases} u_{tt} - \partial_i\big(a_{ij}(x)\partial_j u\big) = |u|^p, \quad (x,t) \in \Omega^c \times (0,+\infty),\ n \geqslant 3, \\ u(0,x) = \varepsilon f(x), \qquad u_t(0,x) = \varepsilon g(x), \quad x \in \Omega^c, \\ u(t,x)|_{\partial\Omega} = 0, \quad \text{for } t \geqslant 0, \end{cases} \tag{1.1}$$

and

$$\begin{cases} u_{tt} - \partial_i\big(a_{ij}(x)\partial_j u\big) = |u_t|^p, \quad (x,t) \in \Omega^c \times (0,+\infty),\ n \geqslant 1, \\ u(0,x) = \varepsilon f(x), \qquad u_t(0,x) = \varepsilon g(x), \quad x \in \Omega^c, \\ u(t,x)|_{\partial\Omega} = 0, \quad \text{for } t \geqslant 0, \end{cases} \tag{1.2}$$

where $A(x) = \{a_{ij}(x)\}_{i,j=1}^n$ denotes a matrix valued smooth function of the variable $x \in \Omega^c$, which takes values in the real, symmetric, $n \times n$ matrices, such that for some $C > 0$,

$$C^{-1}|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant C|\xi|^2, \quad \forall \xi \in R^n,\ x \in \Omega^c,$$

here and in the sequence, a repeated sum on an index is never indicated, and

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$$a_{ij}(x)=\delta_{ij}, \quad \text{when } |x|\geqslant R,$$

where δ_{ij} stands for the Kronecker delta function. Ω is a smooth compact obstacle in R^n, Ω^c is its complement, $n\geqslant 3$ for (1.1) and $n\geqslant 1$ for (1.2). Without loss of generality, we assume that $0\in\Omega\Subset B_R$, where B_R is a ball of radius R centered at the origin and $\operatorname{supp}\{f,g\}\subset B_R$. We consider dimensions $n\geqslant 3$ and exponents $p\in(1,p_1(n))$ for problem (1.1), and dimensions $n\geqslant 1$ and exponents $p\leqslant p_2(n)$ for problem (1.2), where $p_1(n)$ is the larger root of the quadratic equation $(n-1)p^2-(n+1)p-2=0$, and $p_2(n)=\frac{2}{n-1}+1$, respectively. The number $p_1(n)$ is known as the critical exponent of the semilinear wave equation (1.1) (see, e.g., [24]) and the number $p_2(n)$ is known as the critical exponent of the semilinear wave equation (1.2) (see, e.g., [34]). And we consider compactly supported nonnegative data $(f,g)\in H^1(\Omega^c)\times L^2(\Omega^c)$ for problem (1.1) and $f,g\in C_0^\infty(\Omega^c)$ for problem (1.2).

If $a_{ij}=\delta_{ij}$, we say problems (1.1), (1.2) are of constant coefficients. In the case of Cauchy problems of subcritical semilinear wave equation with constant coefficients, there is an extensive literature which we shall review briefly, for details, see [4–6,9,11–13,17,20,23–26,28,30–34].

For the problem (1.1) with constant coefficients, the case $n=3$ was first done by F. John [9] in 1979, he showed that when $n=3$ global solutions always exist if $p>p_1(3)=1+\sqrt{2}$ and initial data are suitably small, and moreover, the global solutions do not exist if $1<p<p_1(3)=1+\sqrt{2}$ for any nontrivial choice of f and g. The number $p_1(3)=1+\sqrt{2}$ appears to have first arisen in Strauss' work on low energy scattering for the nonlinear Klein–Gordon equation [23]. This led him to conjecture that when $n\geqslant 2$ global solutions of (1.1) should always exist if initial data are sufficiently small and p is greater than a critical power $p_1(n)$. The conjecture was verified when $n=2$ by R.T. Glassey [6]. In higher space dimensions, the case $n=4$ was proved by Y. Zhou [33] and V. Georgiev, H. Lindblad and C. Sogge [4] showed that when $n\geqslant 4$ and $p_1(n)<p\leqslant\frac{n+3}{n-1}$, (1.1) has global solutions for small initial values (see also [14]). Later, a simple proof was given by D. Tataru [27] in the case $p>p_1(n)$ and $n\geqslant 4$. R.T. Glassey [5] and T.C. Sideris [20] showed the blow-up result of $1<p<p_1(n)$ for $n=2$ and all $n\geqslant 4$, respectively. Sideris' proof of the blow-up result is quite delicate, using sophisticated computation involving spherical harmonics and other special functions. His proof was simplified by M.A. Rammaha [16]. In 2005, the proof was further simplified by B. Yordanov and Q.S. Zhang [28] by using a simple test function, also, more importantly they use their method to establish blow-up phenomenon for wave equations (1.1) with constant coefficients and a potential. On the other hand, for the critical case $p=p_1(n)$, it was shown by J. Schaeffer [17] that the critical power also belongs to the blow-up case for small data when $n=2,3$ (see also [25,31,32]). B. Yordanov, and Q.S. Zhang [29] and Y. Zhou [35] independently have extended Sideris' blow-up result to $p=p_1(n)$ for all $n\geqslant 4$ by different methods respectively.

For the problem (1.2) with constant coefficients, the blow-up part was first proved by F. John [10] and the global existence part was first obtained by T.C. Sideris [21] in the case $n=3$, and both by J. Schaeffer [18] in the case $n=5$. The blow-up part in the case $n=2$ was proved by J. Schaeffer [19] for $p=p_2(2)$. Later, R. Agemi [1] proved it for $1<p\leqslant p_2(2)$ by different method from [19]. The case $n=1$ is essentially due to K. Masuda [15] who proved the blow-up result in the case $n=1,2,3$ and $p=2$. A simple proof of blow-up part was later given by Y. Zhou [34].

Recently, K. Hidano et al. [7] has established global existence for problem (1.1) with $p>p_1(n)$ and $n=3,4$. For related result, one can see C.D. Sogge and C.B. Wang's work [22]. However, to the best of our knowledge, there are no blow-up results concerning initial–boundary value problems for semilinear wave equations with variable coefficients on exterior domain. In this paper, we shall establish blow-up results for the initial–boundary value problem for subcritical values of p. We shall also estimate the life span $T(\varepsilon)$ for small initial data of size ε. Our result is complement to the global existence result of K. Hidano et al. [7]. For the problem (1.1), we obtain our result by constructing two test functions ϕ_0 and ψ_1 (see Section 2), which is motivated by the work of B. Yordanov and Q.S. Zhang [28]. For the problem (1.2), we still use the test function ψ_1 and by introducing an auxiliary function $G_0(t)$ (see Section 4), we reduced the problem to a Riccati equation. This proof is new even in the constant coefficients case.

We are interested in showing the "blow-up" of solutions to problems (1.1) and (1.2). For that, we require

$$1<p<p_1(n) \quad \text{for (1.1)}, \quad \text{and} \quad p\leqslant p_2(n) \quad \text{for (1.2)}, \tag{1.3}$$

where $p_1(n)$ is the larger root of the quadratic equation $(n-1)p^2-(n+1)p-2=0$, and $p_2(n)=\frac{2}{n-1}+1$. We are also interested in estimating the time when "blow-up" occurs. For initial data of the form

$$u(0,x)=\varepsilon f(x), \qquad u_t(0,x)=\varepsilon g(x), \tag{1.4}$$

with constant $0<\varepsilon\leqslant 1$, smallness can be measured conveniently by the size of ε for fixed f, g. We define "life span" $T(\varepsilon)$ of the solutions of (1.1) or (1.2) to be the largest value such that solutions exist for $x\in\Omega^c$, $0\leqslant t<T(\varepsilon)$.

For problem (1.1), we consider compactly supported nonnegative data $(f,g)\in H^1(\Omega^c)\times L^2(\Omega^c)$, $n\geqslant 3$ and satisfy

$$f(x)\geqslant 0, \qquad g(x)\geqslant 0, \quad \text{a.e.}, \quad f(x)=g(x)=0, \quad \text{for } |x|>R, \quad \text{and} \quad f(x)\not\equiv 0. \tag{1.5}$$

We establish the following theorem for (1.1):

Theorem 1.1. *Let $(f,g) \in H^1(\Omega^c) \times L^2(\Omega^c)$ and satisfy (1.5), $\partial\Omega$ is smooth, and Ω satisfies the exterior ball conditions, space dimensions $n \geqslant 3$. Suppose that problem (1.1) has a solution $(u, u_t) \in C([0,T), H^1(\Omega^c) \times L^2(\Omega^c))$ such that*

$$\operatorname{supp}(u, u_t) \subset \{(x,t)\colon |x| \leqslant t + R\} \cap (\Omega^c \times R^+).$$

If $1 < p < p_1(n)$, then $T < \infty$, and there exists a positive constant A_1 which is independent of ε such that

$$T(\varepsilon) \leqslant A_1 \varepsilon^{-\frac{2p(p-1)}{2+(n+1)p-(n-1)p^2}}. \tag{1.6}$$

Remark 1.1. Exterior ball condition may not be necessary, but in certain point of our proof, we use strong maximum principle for the elliptic equation, so this condition is needed technically.

For problem (1.2), we consider compactly supported nonnegative data $f, g \in C_0^\infty(\Omega^c)$, $n \geqslant 1$ and satisfy

$$f(x) \geqslant 0, \qquad g(x) \geqslant 0, \qquad f(x) = g(x) = 0, \quad \text{for } |x| > R \quad \text{and} \quad g(x) \not\equiv 0. \tag{1.7}$$

Similarly, we establish the following theorem for (1.2):

Theorem 1.2. *Let f, g are smooth functions with compact support $f, g \in C_0^\infty(\Omega^c)$ and satisfy (1.7), space dimensions $n \geqslant 1$. Suppose that problem (1.2) has a solution $(u, u_t) \in C([0,T), H^1(\Omega^c) \times L^q(\Omega^c))$, where $q = \max(2, p)$ such that*

$$\operatorname{supp}(u, u_t) \subset \{(x,t)\colon |x| \leqslant t + R\} \cap (\Omega^c \times R^+).$$

If $p \leqslant p_2(n)$, then $T < \infty$, moreover, we have the following estimates for the life span $T(\varepsilon)$ of solutions of (1.2):

(i) *If $(n-1)(p-1) < 2$, then there exists a positive constant A_2 which is independent of ε such that*

$$T(\varepsilon) \leqslant A_2 \varepsilon^{-\frac{p-1}{1-(n-1)(p-1)/2}}. \tag{1.8}$$

(ii) *If $(n-1)(p-1) = 2$, then there exists a positive constant B_2 which is independent of ε such that*

$$T(\varepsilon) \leqslant \exp\bigl(B_2 \varepsilon^{-(p-1)}\bigr). \tag{1.9}$$

The rest of the paper is arranged as follows. We state several preliminary propositions in Section 2, Section 3 is devoted to the blow-up proof for our Theorem 1.1 and we prove Theorem 1.2 in Section 4.

2. Preliminaries

To prove the main results in this paper, we will employ the following important ODE result:

Lemma 2.1. *(See [20].) Let $p > 1$, $a \geqslant 1$, and $(p-1)a > q - 2$. If $F \in C^2([0,T))$ satisfies*

(1) $F(t) \geqslant \delta(t+R)^a$,
(2) $\frac{d^2F(t)}{dt^2} \geqslant k(t+R)^{-q}[F(t)]^p$,

with some positive constants δ, k, and R, then $F(t)$ will blow up in finite time, $T < \infty$. Furthermore, we have the following estimate for the life span $T(\delta)$ of $F(t)$:

$$T(\delta) \leqslant c\delta^{-\frac{(p-1)}{(p-1)a-q+2}}, \tag{2.1}$$

where c is a positive constant depending on k and R but independent of δ.

Proof. For the proof of blow-up result part see Sideris [20]. We only prove the estimate of the life span of $F(t)$ as following:

Let us make a translation $\tau = t\delta^{\frac{(p-1)}{(p-1)a-q+2}}$ and define

$$H(\tau) = \delta^{\frac{(q-2)}{(p-1)a-q+2}} F(t) = \delta^{\frac{(q-2)}{(p-1)a-q+2}} F\bigl(\tau\delta^{\frac{-(p-1)}{(p-1)a-q+2}}\bigr),$$

then we have

$$\begin{cases} H(\tau) \geqslant \bigl(\delta^{\frac{(p-1)}{(p-1)a-q+2}} R + \tau\bigr)^a, \\ H''(\tau) \geqslant k\bigl(\delta^{\frac{(p-1)}{(p-1)a-q+2}} R + \tau\bigr)^{-q} H^p(\tau). \end{cases} \tag{2.2}$$

So when $\delta \leqslant 1$, easy computation shows that

$$\begin{cases} H(\tau) \geqslant \tau^a, \\ H''(\tau) \geqslant k(R+\tau)^{-q} H^p(\tau). \end{cases} \tag{2.3}$$

So $H(\tau)$ will blow up in finite time and the life span of $F(t)$ satisfies (2.1). This completes the proof. □

Lemma 2.2. *There exists function $\phi_0(x) \in C^2(\Omega^c)$, space dimensions $n \geqslant 3$, satisfying the following boundary value problem:*

$$\begin{cases} \partial_i\big(a_{ij}\partial_j\phi_0(x)\big) = 0, \quad \text{in } \Omega^c,\ n \geqslant 3, \\ \phi_0|_{\partial\Omega} = 0, \\ |x| \to \infty, \qquad \phi_0(x) \to 1. \end{cases} \tag{2.4}$$

Moreover, $\phi_0(x)$ satisfies: for $\forall x \in \Omega^c$, $0 < \phi_0(x) < 1$.

Proof. To solve $\phi_0(x)$, let $\tilde{\phi_0}$ be solution for the following boundary value problem on exterior domain:

$$\begin{cases} \partial_i\big(a_{ij}\partial_j\tilde{\phi_0}(x)\big) = 0, \quad \text{in } \Omega^c,\ n \geqslant 3, \\ \tilde{\phi_0}|_{\partial\Omega} = -1, \\ |x| \to \infty, \qquad \tilde{\phi_0}(x) \to 0, \end{cases} \tag{2.5}$$

by the theory of second order elliptic partial differential equation [2] (see also [3]), this problem is well-posed, it has unique solution $\tilde{\phi_0}(x)$, and by maximum principle, we can easily obtain $-1 < \tilde{\phi_0}(x) < 0$, for $\forall x \in \Omega^c$, then we can easily check that $\phi_0(x) = 1 + \tilde{\phi_0}(x)$ satisfy the boundary value problem (2.4). This proves the existence of ϕ_0 in (2.4) and satisfies $0 < \phi_0(x) < 1$ for $\forall x \in \Omega^c$, $n \geqslant 3$. The proof is complete. □

Similarly, we have the following:

Lemma 2.3. *There exists a function $\phi_1(x) \in C^2(\Omega^c)$, space dimensions $n \geqslant 1$, satisfying the following boundary value problem:*

$$\begin{cases} \partial_i\big(a_{ij}\partial_j\phi_1(x)\big) = \phi_1, \quad \text{in } \Omega^c,\ n \geqslant 1, \\ \phi_1|_{\partial\Omega} = 0, \\ |x| \to \infty, \qquad \phi_1(x) \to \displaystyle\int_{S^{n-1}} e^{x\cdot\omega}\, d\omega. \end{cases} \tag{2.6}$$

Moreover, $\phi_1(x)$ satisfies: there exists positive constant C_1, for $\forall x \in \Omega^c$, $0 < \phi_1(x) \leqslant C_1(1+|x|)^{-(n-1)/2} \cdot e^{|x|}$.

Proof. To solve $\phi_1(x)$, let $\tilde{\phi_1}$ be solution for the following boundary value problem on exterior domain:

$$\begin{cases} \partial_i\big(a_{ij}\partial_j\tilde{\phi_1}(x)\big) = \tilde{\phi_1}(x) - w(x), \quad \text{in } \Omega^c,\ n \geqslant 1, \\ \tilde{\phi_1}|_{\partial\Omega} = -h(x)|_{\partial\Omega}, \\ |x| \to \infty, \qquad \tilde{\phi_1}(x) \to 0, \end{cases} \tag{2.7}$$

where $h(x) = \int_{S^{n-1}} e^{x\cdot\omega}\, d\omega$, $w(x) = \partial_i((a_{ij} - \delta_{ij})\partial_j h)$, since the function h satisfies $\Delta h = h$, so by the condition of $a_{ij}(x)$, we get $w(x) \in C_c^\infty(\Omega^c)$, so by the theory of second order elliptic partial differential equation [2] (see also [3]), the problem (2.7) is well-posed, it has unique solution $\tilde{\phi_1}(x)$, then we can easily check that $\phi_1(x) = h(x) + \tilde{\phi_1}(x)$ satisfies the boundary value problem (2.6), this proves the existence of ϕ_1 in (2.6). To derive the estimate of $\phi_1(x)$ in Ω^c, we rewrite the boundary value problem (2.6) as the following form:

$$\begin{cases} -\partial_i\big(a_{ij}\partial_j\phi_1(x)\big) + \phi_1(x) = 0, \quad \text{in } \Omega^c,\ n \geqslant 1, \\ \phi_1|_{\partial\Omega} = 0, \\ |x| \to \infty, \qquad \phi_1(x) \to h(x). \end{cases} \tag{2.8}$$

So by maximum principle, we can easily get

$$\phi_1(x) > 0, \quad \text{for } \forall x \in \Omega^c. \tag{2.9}$$

Next we analyze $\tilde{\phi_1}(x)$ in order to get the estimation of $\phi_1(x)$, we will prove that $\tilde{\phi_1}(x)$ is bounded by some positive constant C, that is, $|\tilde{\phi_1}(x)| \leqslant C$ for $\forall x \in \Omega^c$. Here and hereafter, we shall denote by C (or c) a positive constant in the estimates, and the meaning of C (or c) may change from line to line.

For this purpose, we rewrite problem (2.7) as follows

$$\begin{cases} -\partial_i\big(a_{ij}\partial_j\tilde{\phi_1}(x)\big)+\tilde{\phi_1}(x)=w(x), & \text{in } \Omega^c,\ n\geqslant 1,\\ \tilde{\phi_1}|_{\partial\Omega}=-h(x)|_{\partial\Omega}, \\ |x|\to\infty, & \tilde{\phi_1}(x)\to 0. \end{cases} \tag{2.10}$$

For the purpose of employing the maximum principle, we denote $C=\max_{x\in\partial\Omega}|h(x)|+\max_{x\in\Omega^c}|w(x)|>0$, because the function $w(x)$ is compactly supported function in Ω^c, so the above expression C is well defined. By the maximum principle, we can get the upper bound of $\tilde{\phi_1}(x)$ as follows:

We rewrite the equation of $\tilde{\phi_1}(x)$ as following

$$\begin{cases} -\partial_i\big(a_{ij}\partial_j\big(\tilde{\phi_1}(x)-C\big)\big)+\big(\tilde{\phi_1}(x)-C\big)=w(x)-C\leqslant 0, & \text{in } \Omega^c,\ n\geqslant 1,\\ (\tilde{\phi_1}-C)|_{\partial\Omega}=\big(-h(x)-C\big)\big|_{\partial\Omega}\leqslant 0, \\ |x|\to\infty, & \big(\tilde{\phi_1}(x)-C\big)\to -C\leqslant 0. \end{cases} \tag{2.11}$$

So we apply maximum principle to $(\tilde{\phi_1}(x)-C)$, we can obtain for $\forall x\in\Omega^c$, $\tilde{\phi_1}(x)-C\leqslant 0$, that is, $\tilde{\phi_1}(x)\leqslant C$, in Ω^c.

In a similar way, we can get $-\tilde{\phi_1}(x)\leqslant C$, in Ω^c.

Thus we conclude that $|\tilde{\phi_1}(x)|\leqslant C$ for any $x\in\Omega^c$.

Hence we have for $\forall x\in\Omega^c$,

$$\phi_1(x)=\tilde{\phi_1}(x)+h(x)\leqslant C+h(x)\leqslant C'h(x)\leqslant C_1\big(1+|x|\big)^{-(n-1)/2}\cdot e^{|x|}, \tag{2.12}$$

for the estimate of $h(x)$, see F. John's book [8].

This together with (2.9) implies that $\phi_1(x)$ satisfies

$$0<\phi_1(x)\leqslant C_1\big(1+|x|\big)^{-(n-1)/2}\cdot e^{|x|}, \quad \text{in } \Omega^c,\ n\geqslant 1. \tag{2.13}$$

This proves Lemma 2.3. □

In order to describe the following lemmas, we define the following test function

$$\psi_1(x,t)=\phi_1(x)e^{-t}, \quad \forall x\in\Omega^c,\ t\geqslant 0. \tag{2.14}$$

We have

Lemma 2.4. *Let $p>1$. Assume that ϕ_1 satisfies the conditions in Lemma 2.3, $\psi_1(x,t)$ is as in (2.14). Then for $\forall t\geqslant 0$,*

$$\int\limits_{\Omega^c\cap\{|x|\leqslant t+R\}}\big[\psi_1(x,t)\big]^{p/(p-1)}\,dx\leqslant C(t+R)^{n-1-(n-1)p'/2},$$

where $p'=p/(p-1)$ and C is a positive constant.

Proof. Let $I(t)$ be the integral in Lemma 2.4, by the property of $\phi_1(x)$, we have

$$\begin{aligned} I(t)&=\int\limits_{\Omega^c\cap\{|x|\leqslant t+R\}}\big[\psi_1(x,t)\big]^{p/(p-1)}\,dx=\int\limits_{\Omega^c\cap\{|x|\leqslant t+R\}}\big[\phi_1(x)e^{-t}\big]^{p/(p-1)}\,dx\\ &\leqslant\int\limits_{\Omega^c\cap\{|x|\leqslant t+R\}}\big[C_1\big(1+|x|\big)^{-(n-1)/2}\cdot e^{|x|}\big]^{p/(p-1)}\cdot e^{-tp'}\,dx\\ &\leqslant\int\limits_{\{|x|\leqslant t+R\}}\big[C_1\big(1+|x|\big)^{-(n-1)/2}\cdot e^{|x|}\big]^{p/(p-1)}\cdot e^{-tp'}\,dx\\ &=\operatorname{area}\big(S^{n-1}\big)C_1^{p/(p-1)}\int\limits_0^{t+R}(1+r)^{-(n-1)p'/2}\cdot e^{p'r}r^{n-1}e^{-tp'}\,dr, \end{aligned} \tag{2.15}$$

where $p'=p/(p-1)$ and S^{n-1} is the unit sphere in R^n. It is sufficient to show that

$$I(t)\leqslant Ce^{-tp'}\int\limits_0^{t+R}(1+r)^{n-1-(n-1)p'/2}\cdot e^{p'r}\,dr\leqslant C(t+R)^{n-1-(n-1)p'/2}. \tag{2.16}$$

This estimate is evident after splitting the last integral into two parts, that is,

$$\int_0^{t+R} (1+r)^{n-1-(n-1)p'/2} \cdot e^{p'r}\,dr = \left[\int_0^{(t+R)/2} + \int_{(t+R)/2}^{t+R}\right](1+r)^{n-1-(n-1)p'/2} \cdot e^{p'r}\,dr,$$

$$\begin{aligned}\int_0^{(t+R)/2} (1+r)^{n-1-(n-1)p'/2} \cdot e^{p'r}\,dr &\leqslant (1+t+R)^{q_1} \int_0^{(t+R)/2} e^{p'r}\,dr \\ &= (1+t+R)^{q_1} \cdot \frac{1}{p'}\big(e^{p'(t+R)/2}-1\big) \\ &\leqslant (1+t+R)^{q_1} \cdot \frac{1}{p'} e^{p'(t+R)/2} = \frac{e^{p'R/2}}{p'}(1+t+R)^{q_1} e^{p't/2},\end{aligned} \tag{2.17}$$

where $q_1 = \max(0, n-1-(n-1)p'/2)$, and

$$\begin{aligned}\int_{(t+R)/2}^{t+R} (1+r)^{n-1-(n-1)p'/2} \cdot e^{p'r}\,dr &\leqslant 2^{-q_2}(1+t+R)^{n-1-(n-1)p'/2} \int_{(t+R)/2}^{t+R} e^{p'r}\,dr \\ &= 2^{-q_2}(1+t+R)^{n-1-(n-1)p'/2} \cdot \frac{1}{p'}\big(e^{p'(t+R)} - e^{p'(t+R)/2}\big) \\ &\leqslant \frac{2^{-q_2} e^{p'R}}{p'} \cdot (1+t+R)^{n-1-(n-1)p'/2} e^{p't},\end{aligned}$$

where $q_2 = \min(0, n-1-(n-1)p'/2)$.

This proves Lemma 2.4. $\square$

Lemma 2.5. *Let $p > 1$. Assume that ϕ_0 and ϕ_1 satisfy the conditions in Lemma 2.2 and Lemma 2.3, respectively, $\psi_1(x,t)$ is as in (2.14), $\partial\Omega$ and Ω satisfy the conditions in Theorem 1.1. Then for $\forall t \geqslant 0$,*

$$\int_{\Omega^c \cap \{|x| \leqslant t+R\}} \big[\phi_0(x)\big]^{-1/(p-1)} \cdot \big[\psi_1(x,t)\big]^{p/(p-1)}\,dx \leqslant C(t+R)^{n-1-(n-1)p'/2}, \tag{2.18}$$

where $p' = p/(p-1)$ and C is a positive constant.

Proof. To estimate the integral in Lemma 2.5, we split it into two parts as follows

$$\begin{aligned}&\int_{\Omega^c \cap \{|x| \leqslant t+R\}} \big[\phi_0(x)\big]^{-1/(p-1)} \cdot \big[\psi_1(x,t)\big]^{p/(p-1)}\,dx \\ &\quad = \int_{\Omega^c \cap B_R} \big[\phi_0(x)\big]^{-1/(p-1)} \cdot \big[\psi_1(x,t)\big]^{p/(p-1)}\,dx + \int_{B_R^c \cap \{|x| \leqslant t+R\}} \big[\phi_0(x)\big]^{-1/(p-1)} \cdot \big[\psi_1(x,t)\big]^{p/(p-1)}\,dx \\ &\quad = I_1(t) + I_2(t).\end{aligned} \tag{2.19}$$

We will estimate $I_1(t)$ and $I_2(t)$ separately.

First let us estimate $I_2(t)$. Since for $\forall x \in \Omega^c$, $0 < \phi_0(x) < 1$, we remark that there exists a constant $c \in (0,1)$, such that when $x \in B_R^c \cap \{|x| \leqslant t+R\}$, $\phi_0(x) \geqslant c$. By Lemma 2.4, we have

$$\begin{aligned}I_2(t) &= \int_{B_R^c \cap \{|x| \leqslant t+R\}} \big[\phi_0(x)\big]^{-1/(p-1)} \cdot \big[\psi_1(x,t)\big]^{p/(p-1)}\,dx \\ &\leqslant \int_{B_R^c \cap \{|x| \leqslant t+R\}} c^{-1/(p-1)} \cdot \big[\psi_1(x,t)\big]^{p/(p-1)}\,dx \\ &\leqslant c^{-1/(p-1)} \int_{\Omega^c \cap \{|x| \leqslant t+R\}} \big[\psi_1(x,t)\big]^{p/(p-1)}\,dx\end{aligned}$$

$$\leqslant c^{-1/(p-1)}C(t+R)^{n-1-(n-1)p'/2}$$
$$= C_2(t+R)^{n-1-(n-1)p'/2}. \tag{2.20}$$

Next we estimate $I_1(t)$. On the one hand, because of regularity of $\phi_1(x)$, the first derivative of $\phi_1(x)$ is bounded in $\Omega^c \cap B_R$, this lead to $\phi_1(x) = \phi_1(x) - \phi_1(y) \leqslant C_3|x-y|$, for $\forall y \in \partial\Omega$. Therefore by taking the infimum on $\partial\Omega$ we have,

$$\left|\phi_1(x)\right| \leqslant C_3 \operatorname{dist}(x, \partial\Omega).$$

On the other hand, $\phi_0(x)$ obeys the maximum (minimum) principle, and assumes its minimum value (zero) on $\partial\Omega$, since Ω satisfies exterior ball condition, so by [3, Hopf's lemma, p. 330], it follows that, for any $y \in \partial\Omega$, there exists an open ball $B \subset \Omega^c$ with $y \in \partial B$, then we have, for any $y \in \partial\Omega$,

$$\frac{\partial\phi_0}{\partial\nu}(y) > 0, \tag{2.21}$$

where ν is the inner unit normal to Ω^c at y. By the compactness of $\partial\Omega$, we have, for $\forall y \in \partial\Omega$, we have

$$\frac{\partial\phi_0}{\partial\nu}(y) \geqslant C_* > 0,$$

where C_* is a positive constant.

For $\forall x \in \Omega^c \cap B_R$, there exists a $y \in \partial\Omega$ such that $(x-y) /\!\!/ \nu(y)$, i.e., $\frac{(x-y)}{|x-y|} = \nu(y)$, $\nu(y)$ is the outer unit normal to $\partial\Omega$ at y. So we have

$$\nabla\phi_0(y) \cdot \frac{(x-y)}{|x-y|} = \frac{\partial\phi_0(y)}{\partial\nu} \geqslant C_* > 0,$$

$$\phi_0(x) = \phi_0(x) - \phi_0(y) = \int_0^1 \nabla\phi_0\big(sx + (1-s)y\big)\, ds \cdot (x-y)$$
$$= \int_0^1 \nabla\phi_0\big(sx + (1-s)y\big)\, ds \cdot \frac{(x-y)}{|x-y|} \cdot |x-y|, \tag{2.22}$$

by the continuity, for $\forall x \in \Omega^c \cap B_R$ and $|x-y| \ll 1$, we know that $(sx + (1-s)y)$ is sufficiently close to y, so we can guarantee that

$$\nabla\phi_0\big(sx + (1-s)y\big) \cdot \frac{(x-y)}{|x-y|} \geqslant \frac{1}{2}C_* > 0.$$

So there exists a positive constant $\varepsilon_0 > 0$ such that the above expression holds for $\forall x \in \Omega^c \cap B_R$ and $\operatorname{dist}(x, \partial\Omega) < \varepsilon_0$.

We discuss in the following in two cases respectively:

One case is that for $x \in \Omega^c \cap B_R$, and $\operatorname{dist}(x, \partial\Omega) < \varepsilon_0$, we have

$$\left|\phi_0(x)\right| \geqslant \frac{1}{2}C_*|x-y| \geqslant \frac{1}{2}C_* \operatorname{dist}(x, \partial\Omega). \tag{2.23}$$

The other case is that when $x \in \Omega^c \cap B_R$, and $\operatorname{dist}(x, \partial\Omega) \geqslant \varepsilon_0$, on the one hand, by the property of the function $\phi_0(x)$, there is a positive constant $c_1 \in (0, 1)$, such that

$$\phi_0(x) \geqslant c_1 > 0,$$

on the other hand, for $x \in \Omega^c \cap B_R$, there definitely exists a positive constant $c' > 0$ such that $\operatorname{dist}(x, \partial\Omega) \leqslant c'$, so we have

$$\frac{\phi_0(x)}{\operatorname{dist}(x, \partial\Omega)} \geqslant \frac{\phi_0(x)}{c'} \geqslant \frac{c_1}{c'} = c'' > 0, \quad \text{for } x \in \Omega^c \cap B_R, \text{ and } \operatorname{dist}(x, \partial\Omega) \geqslant \varepsilon_0, \tag{2.24}$$

that is

$$\phi_0(x) \geqslant c'' \operatorname{dist}(x, \partial\Omega),$$

where c'' is a positive constant.

So combining the above two cases, for $\forall x \in \Omega^c \cap B_R$, we have

$$\phi_0(x) \geqslant C_{**} \operatorname{dist}(x, \partial\Omega),$$

where C_{**} is a positive constant.

Hence, we have

$$\begin{aligned}
I_1(t) &= \int_{\Omega^c\cap B_R} \big[\phi_0(x)\big]^{-1/(p-1)} \cdot \big[\psi_1(x,t)\big]^{p/(p-1)}\,dx \\
&\leqslant \int_{\Omega^c\cap B_R} [C_{**}]^{-1/(p-1)}\big[\operatorname{dist}(x,\partial\Omega)\big]^{-1/(p-1)} \cdot \big[\psi_1(x,t)\big]^{p/(p-1)}\,dx \\
&= \int_{\Omega^c\cap B_R} [C_{**}]^{-1/(p-1)}\big[\operatorname{dist}(x,\partial\Omega)\big]^{-1/(p-1)} \cdot e^{-tp'}\big[\phi_1(x)\big]^{p/(p-1)}\,dx \\
&\leqslant \int_{\Omega^c\cap B_R} [C_{**}]^{-1/(p-1)}\big[\operatorname{dist}(x,\partial\Omega)\big]^{-1/(p-1)} \cdot e^{-tp'}C_3^{p/(p-1)}\big[\operatorname{dist}(x,\partial\Omega)\big]^{p/(p-1)}\,dx \\
&= e^{-tp'}\int_{\Omega^c\cap B_R} [C_{**}]^{-1/(p-1)}C_3^{p/(p-1)}\operatorname{dist}(x,\partial\Omega)\,dx \\
&= Ce^{-tp'}\int_{\Omega^c\cap B_R} \operatorname{dist}(x,\partial\Omega)\,dx \leqslant C_4e^{-tp'},
\end{aligned} \tag{2.25}$$

where $p' = p/(p-1)$.

So we conclude that

$$\begin{aligned}
\int_{\Omega^c\cap\{|x|\leqslant t+R\}} \big[\phi_0(x)\big]^{-1/(p-1)} \cdot \big[\psi_1(x,t)\big]^{p/(p-1)}\,dx &= I_1(t)+I_2(t) \\
&\leqslant C_4e^{-tp'} + C_2(t+R)^{n-1-(n-1)p'/2} \\
&\leqslant C_5(t+R)^{n-1-(n-1)p'/2},
\end{aligned} \tag{2.26}$$

where C_5 is a positive constant. The proof is complete. □

Lemma 2.6. *Let $p>1$. Assume that ϕ_1 satisfies the conditions in Lemma 2.3, $\psi_1(x,t)$ is as in (2.14). Then for $\forall t \geqslant 0$,*

$$\int_{\Omega^c\cap\{|x|\leqslant t+R\}} \psi_1\,dx \leqslant C(t+R)^{(n-1)/2}, \tag{2.27}$$

where C is a positive constant.

Proof. We note that for $\forall t \geqslant 0$, $\psi_1(x,t) = e^{-t}\phi_1(x)$, and since for $\forall x \in \Omega^c$, $0 < \phi_1(x) \leqslant C_1(1+|x|)^{-(n-1)/2}e^{|x|}$, we can get that there exists a positive constant C_6 such that $0 < \phi_1(x) \leqslant C_6|x|^{-(n-1)/2}e^{|x|}$ for any $x \in \Omega^c$.

So we have

$$\begin{aligned}
\int_{\Omega^c\cap\{|x|\leqslant t+R\}} \psi_1\,dx &= \int_{\Omega^c\cap\{|x|\leqslant t+R\}} e^{-t}\phi_1(x)\,dx \\
&\leqslant \int_{\Omega^c\cap\{|x|\leqslant t+R\}} e^{-t}\cdot C_6|x|^{-(n-1)/2}e^{|x|}\,dx \\
&\leqslant \int_{\{|x|\leqslant t+R\}} e^{-t}\cdot C_6|x|^{-(n-1)/2}e^{|x|}\,dx \\
&= C_6e^{-t}\int_0^{t+R} r^{-(n-1)/2}e^r\cdot r^{n-1}\,dr \int_{S^{n-1}} d\omega = C_7e^{-t}\int_0^{t+R} e^r\cdot r^{(n-1)/2}\,dr \\
&= C_7e^{-t}\left[e^r r^{(n-1)/2}\Big|_0^{t+R} - \int_0^{t+R} e^r\left(\frac{n-1}{2}\right)r^{(n-3)/2}\,dr\right] \\
&\leqslant C_7e^{-t}e^{t+R}(t+R)^{(n-1)/2} = C_7e^R(t+R)^{(n-1)/2} = C_8(t+R)^{(n-1)/2}.
\end{aligned} \tag{2.28}$$

This completes the proof. □

3. The proof of Theorem 1.1

Theorem 1.1 is a consequence of the lower bound and the blow-up result about nonlinear differential inequalities in Lemma 2.1.

To outline the method, we will introduce the following functions:

$$\begin{cases} F_0(t) = \int\limits_{\Omega^c} u(x,t)\phi_0(x)\,dx, \\ F_1(t) = \int\limits_{\Omega^c} u(x,t)\psi_1(x,t)\,dx, \qquad \psi_1(x,t) = \phi_1(x)e^{-t}, \end{cases} \tag{3.1}$$

here $\phi_0(x)$ and $\phi_1(x)$ are as in Lemma 2.2 and Lemma 2.3. The assumptions on u imply that $F_0(t)$ and $F_1(t)$ are well-defined C^2-functions for all t. By a standard procedure, we derive a nonlinear differential inequality for $F_0(t)$. We also derive a linear differential inequality for $F_1(t)$ and combine these to obtain a polynomial lower bound on $F_0(t)$ as $t \to \infty$.

To this end, we first establish the following lemma:

Lemma 3.1. *Let (f,g) satisfy* (1.5)*. Suppose that problem* (1.1) *has a solution $(u,u_t) \in C([0,T), H^1(\Omega^c) \times L^2(\Omega^c))$, such that*

$$\operatorname{supp}(u,u_t) \subset \{(x,t)\colon |x| \leqslant t+R\} \cap (\Omega^c \times R^+).$$

Then for all $t \geqslant 0$,

$$F_1(t) \geqslant \frac{1}{2}\left(1-e^{-2t}\right)\varepsilon \int\limits_{\Omega^c} \left[f(x)+g(x)\right]\phi_1(x)\,dx + e^{-2t}\varepsilon \int\limits_{\Omega^c} f(x)\phi_1(x)\,dx \geqslant \varepsilon c_0 > 0.$$

Proof. We multiply (1.1) by the test function $\psi_1 \in C^2(\Omega^c \times R)$ and integrate over $\Omega^c \times [0,t]$, then we use integration by parts and Lemma 2.3.

First,

$$\int_0^t \int\limits_{\Omega^c} \psi_1\left(\partial_i\left(a_{ij}(x)\partial_j u\right) - u_{tt} + |u|^p\right)dx\,d\tau = 0.$$

By the expression $\psi_1(x,t) = \phi_1(x)e^{-t}$ and Lemma 2.3, we have

$$\begin{aligned} \int_0^t \int\limits_{\Omega^c} \psi_1 \partial_i\left(a_{ij}(x)\partial_j u\right)dx\,d\tau &= \int_0^t \Bigg[\int\limits_{\partial\Omega} \psi_1 a_{ij}(x)\partial_j u \cdot n_i\,dS - \int\limits_{\Omega^c} \left(a_{ij}(x)\partial_i\psi_1\right)\partial_j u\,dx\Bigg]d\tau \\ &= -\int_0^t \Bigg[\int\limits_{\partial\Omega} a_{ij}(x)\partial_i\psi_1 \cdot u \cdot n_j\,dS - \int\limits_{\Omega^c} \partial_j\left(a_{ij}(x)\partial_i\psi_1\right)u\,dx\Bigg]d\tau \\ &= \int_0^t \int\limits_{\Omega^c} \psi_1 u\,dx\,d\tau, \end{aligned}$$

by the expression of $\psi_1(x,t)$, we get $(\psi_1)_t = -\psi_1$, $(\psi_1)_{tt} = \psi_1$. So we have

$$\begin{aligned} \int_0^t \int\limits_{\Omega^c} \psi_1 u_{tt}\,dx\,d\tau &= \int_0^t \int\limits_{\Omega^c} \left[\partial_\tau(\psi_1 u_\tau) - (\psi_1)_\tau u_\tau\right]dx\,d\tau \\ &= \int\limits_{\Omega^c} \psi_1 u_\tau\,dx\bigg|_{\tau=t} - \int\limits_{\Omega^c} \psi_1 u_\tau\,dx\bigg|_{\tau=0} + \int_0^t \int\limits_{\Omega^c} \psi_1 u_\tau\,dx\,d\tau \\ &= \int\limits_{\Omega^c} \psi_1 u_\tau\,dx\bigg|_{\tau=t} - \int\limits_{\Omega^c} \psi_1 u_\tau\,dx\bigg|_{\tau=0} + \int_0^t \int\limits_{\Omega^c} \left[\partial_\tau(\psi_1 u) - (\psi_1)_\tau u\right]dx\,d\tau \end{aligned}$$

$$= \int_{\Omega^c} \psi_1 u_\tau\, dx\Big|_{\tau=t} - \int_{\Omega^c} \psi_1 u_\tau\, dx\Big|_{\tau=0} + \int_{\Omega^c} \psi_1 u\, dx\Big|_{\tau=t} - \int_{\Omega^c} \psi_1 u\, dx\Big|_{\tau=0} + \int_0^t \int_{\Omega^c} \psi_1 u\, dx\, d\tau$$

$$= \int_{\Omega^c} (\psi_1 u_t + u\psi_1)\, dx - \varepsilon \int_{\Omega^c} \phi_1(x) g(x)\, dx - \varepsilon \int_{\Omega^c} \phi_1(x) f(x)\, dx + \int_0^t \int_{\Omega^c} \psi_1 u\, dx\, d\tau.$$

Combining the above equalities, we have

$$\int_0^t \int_{\Omega^c} \psi_1 |u|^p\, dx\, d\tau = \int_{\Omega^c} (\psi_1 u_t + \psi_1 u)\, dx - \varepsilon \int_{\Omega^c} \phi_1(x)\big[f(x) + g(x)\big]\, dx.$$

We notice that

$$\begin{aligned} \int_{\Omega^c} (\psi_1 u_t + \psi_1 u)\, dx &= \frac{d}{dt} \int_{\Omega^c} (\psi_1 u)\, dx - \int_{\Omega^c} (\psi_1)_t u\, dx + \int_{\Omega^c} \psi_1 u\, dx \\ &= \frac{d}{dt} \int_{\Omega^c} (\psi_1 u)\, dx + 2 \int_{\Omega^c} \psi_1 u\, dx \\ &= \frac{dF_1(t)}{dt} + 2F_1(t). \end{aligned}$$

So by $\psi_1 > 0$, we have

$$\begin{aligned} \frac{dF_1(t)}{dt} + 2F_1(t) &= \int_0^t \int_{\Omega^c} |u|^p \psi_1(x,\tau)\, dx\, d\tau + \varepsilon \int_{\Omega^c} \phi_1(x)\big[f(x) + g(x)\big]\, dx \\ &\geqslant \varepsilon \int_{\Omega^c} \big[f(x) + g(x)\big]\phi_1(x)\, dx. \end{aligned}$$

Multiplying the above expression by e^{2t}, we obtain

$$\frac{d(e^{2t}F_1(t))}{dt} \geqslant e^{2t}\varepsilon \int_{\Omega^c} \big[f(x) + g(x)\big]\phi_1(x)\, dx,$$

and integrating the above differential inequality over $[0,t]$, we get

$$e^{2t}F_1(t) - F_1(0) \geqslant \frac{1}{2}\big(e^{2t} - 1\big)\varepsilon \int_{\Omega^c} \big[f(x) + g(x)\big]\phi_1(x)\, dx.$$

Observing $F_1(0) = \int_{\Omega^c} u(x,0)\psi_1(x,0)\, dx = \varepsilon \int_{\Omega^c} f(x)\phi_1(x)\, dx$. So, by the property of the function $f(x)$ and $\phi_1(x)$, we arrive at

$$F_1(t) \geqslant \frac{1}{2}\big(1 - e^{-2t}\big)\varepsilon \int_{\Omega^c} \big[f(x) + g(x)\big]\phi_1(x)\, dx + e^{-2t}\varepsilon \int_{\Omega^c} f(x)\phi_1(x)\, dx \geqslant \varepsilon c_0 > 0.$$

Thus we obtain the lower bound in Lemma 3.1. □

Next we shall show that $F_0(t)$ satisfies the differential inequalities in Lemma 2.1 for suitable a, q. For this purpose, we multiply (1.1) by ϕ_0 and integrate over Ω^c. We note that for a fixed t, $u(\cdot,t) \in H_0^1(D_t)$ where D_t is the support of $u(\cdot,t)$. Hence we can use integration by parts and Lemma 2.2.

First,

$$\int_{\Omega^c} \big[\phi_0 \partial_i\big(a_{ij}(x)\partial_j u\big) - \phi_0 u_{tt} + |u|^p \phi_0\big]\, dx = 0.$$

Since

$$\int_{\Omega^c} \phi_0 \partial_i\big(a_{ij}(x)\partial_j u\big)\,dx = \int_{\partial\Omega} \phi_0 a_{ij}(x)\partial_j u\cdot n_i\,dS - \int_{\Omega^c} \partial_i\phi_0 a_{ij}(x)\partial_j u\,dx$$

$$= -\Big(\int_{\partial\Omega} a_{ij}(x)\partial_i\phi_0 u\cdot n_j\,dS - \int_{\Omega^c} \partial_j\big(a_{ij}(x)\partial_i\phi_0\big)u\,dx\Big)$$

$$= \int_{\Omega^c} \partial_j\big(a_{ij}(x)\partial_i\phi_0\big)u\,dx = 0. \tag{3.2}$$

So we get

$$\frac{d^2F_0(t)}{dt^2} = \int_{\Omega^c} \big|u(x,t)\big|^p\phi_0(x)\,dx.$$

Estimating the right side of the above equality by the Hölder inequality, we have

$$\Big|\int_{\Omega^c} u(x,t)\phi_0(x)\,dx\Big| = \Big|\int_{\Omega^c\cap\{|x|\leqslant t+R\}} u(x,t)\big[\phi_0(x)\big]^{1/p}\big[\phi_0(x)\big]^{(p-1)/p}\,dx\Big|$$

$$\leqslant \Big(\int_{\Omega^c\cap\{|x|\leqslant t+R\}} \big|u(x,t)\big[\phi_0(x)\big]^{1/p}\big|^p\,dx\Big)^{1/p}\cdot\Big(\int_{\Omega^c\cap\{|x|\leqslant t+R\}} \big|\big[\phi_0(x)\big]^{(p-1)/p}\big|^{p'}\,dx\Big)^{1/p'}$$

where $p' = p/(p-1)$, this implies that

$$\Big|\int_{\Omega^c} u(x,t)\phi_0(x)\,dx\Big|^p \leqslant \Big(\int_{\{|x|\leqslant t+R\}\cap\Omega^c} \big|u(x,t)\big|^p\phi_0(x)\,dx\Big)\Big(\int_{\{|x|\leqslant t+R\}\cap\Omega^c} \phi_0(x)\,dx\Big)^{p-1}$$

$$\leqslant \Big(\int_{\Omega^c} \big|u(x,t)\big|^p\phi_0(x)\,dx\Big)\Big(\int_{\{|x|\leqslant t+R\}\cap\Omega^c} \phi_0(x)\,dx\Big)^{p-1}.$$

So we have

$$\int_{\Omega^c} \big|u(x,t)\big|^p\phi_0(x)\,dx \geqslant \frac{|\int_{\Omega^c} u(x,t)\phi_0(x)\,dx|^p}{(\int_{\{|x|\leqslant t+R\}\cap\Omega^c} \phi_0(x)\,dx)^{p-1}}.$$

By Lemma 2.2, we have

$$\int_{\{|x|\leqslant t+R\}\cap\Omega^c} \phi_0(x)\,dx \leqslant \int_{\{|x|\leqslant t+R\}} 1\,dx \leqslant \mathrm{Vol}\big\{x:\ |x|\leqslant t+R\big\} = \mathrm{Vol}\big(\mathbf{B}^n\big)(t+R)^n.$$

Therefore

$$\int_{\Omega^c} \big|u(x,t)\big|^p\phi_0(x)\,dx \geqslant \frac{|\int_{\Omega^c} u(x,t)\phi_0(x)\,dx|^p}{[\mathrm{Vol}(\mathbf{B}^n)(t+R)^n]^{p-1}} = \frac{|F_0(t)|^p}{[\mathrm{Vol}(\mathbf{B}^n)]^{p-1}\cdot(t+R)^{n(p-1)}}.$$

Thus

$$\frac{d^2F_0(t)}{dt^2} \geqslant k(t+R)^{-n(p-1)}\cdot\big|F_0(t)\big|^p, \tag{3.3}$$

where $k = [\mathrm{Vol}(\mathbf{B}^n)]^{-(p-1)} > 0$. So F_0 satisfies the differential inequality (2) in Lemma 2.1. To show that F_0 admits the lower bound (1) in Lemma 2.1, we relate $d^2F_0(t)/dt^2$ to F_1 using again (1.1) and the Hölder inequality.

Since

$$\begin{aligned}\left|\int_{\Omega^c} u(x,t)\psi_1(x,t)\,dx\right| &= \left|\int_{\Omega^c\cap\{|x|\leqslant t+R\}} u(x,t)\big[\phi_0(x)\big]^{1/p}\cdot\big[\phi_0(x)\big]^{-1/p}\cdot\psi_1(x,t)\,dx\right| \\ &\leqslant \left(\int_{\Omega^c\cap\{|x|\leqslant t+R\}} \big|u(x,t)\big|^p\cdot\phi_0(x)\,dx\right)^{1/p}\cdot\left(\int_{\Omega^c\cap\{|x|\leqslant t+R\}} \big|\big[\phi_0(x)\big]^{-1/p}\cdot\psi_1(x,t)\big|^{p'}\,dx\right)^{1/p'} \\ &\leqslant \left(\int_{\Omega^c} \big|u(x,t)\big|^p\cdot\phi_0(x)\,dx\right)^{1/p}\cdot\left(\int_{\Omega^c\cap\{|x|\leqslant t+R\}} \big[\phi_0(x)\big]^{-1/(p-1)}\cdot\big[\psi_1(x,t)\big]^{p/(p-1)}\,dx\right)^{(p-1)/p},\end{aligned}$$

where $p' = p/(p-1)$, this implies that

$$\left|\int_{\Omega^c} u(x,t)\psi_1(x,t)\,dx\right|^p \leqslant \left(\int_{\Omega^c} \big|u(x,t)\big|^p\cdot\phi_0(x)\,dx\right)\cdot\left(\int_{\Omega^c\cap\{|x|\leqslant t+R\}} \big[\phi_0(x)\big]^{-1/(p-1)}\cdot\big[\psi_1(x,t)\big]^{p/(p-1)}\,dx\right)^{p-1}.$$

By (3.1), the above becomes

$$\begin{aligned}\frac{d^2F_0(t)}{dt^2} &= \int_{\Omega^c} \big|u(x,t)\big|^p\phi_0(x)\,dx \\ &\geqslant \frac{|\int_{\Omega^c} u(x,t)\psi_1(x,t)\,dx|^p}{(\int_{\Omega^c\cap\{|x|\leqslant t+R\}}[\phi_0(x)]^{-1/(p-1)}\cdot[\psi_1(x,t)]^{p/(p-1)}\,dx)^{p-1}} \\ &= \frac{|F_1(t)|^p}{(\int_{\Omega^c\cap\{|x|\leqslant t+R\}}[\phi_0(x)]^{-1/(p-1)}\cdot[\psi_1(x,t)]^{p/(p-1)}\,dx)^{p-1}}.\end{aligned}$$

In the following, we will estimate the numerator and denominator, respectively, and provide a lower bound on d^2F_0/dt^2.

By Lemma 3.1, we have

$$\big|F_1(t)\big|^p \geqslant \varepsilon^p(c_0)^p > 0. \tag{3.4}$$

Also, by Lemma 2.5 we know that

$$\int_{\Omega^c\cap\{|x|\leqslant t+R\}} \big[\phi_0(x)\big]^{-1/(p-1)}\cdot\big[\psi_1(x,t)\big]^{p/(p-1)}\,dx \leqslant C_5(t+R)^{n-1-(n-1)p'/2}, \tag{3.5}$$

where $p' = p/(p-1)$ and C_5 is a positive constant.

So by combining (3.4) and (3.5), we obtain

$$\begin{aligned}\frac{d^2F_0(t)}{dt^2} &\geqslant \frac{|F_1(t)|^p}{(\int_{\Omega^c\cap\{|x|\leqslant t+R\}}[\phi_0(x)]^{-1/(p-1)}\cdot[\psi_1(x,t)]^{p/(p-1)}\,dx)^{p-1}} \\ &\geqslant \frac{\varepsilon^p c_0^p}{[C_5(t+R)^{n-1-(n-1)p'/2}]^{p-1}} \\ &\geqslant L(t+R)^{-(n-1)(p/2-1)},\end{aligned}$$

where $L = \varepsilon^p c_0^p C_5^{-(p-1)} > 0$. Integrating twice, we have the final estimate

$$F_0(t) \geqslant \delta(t+R)^{n+1-(n-1)p/2} + \frac{dF_0(0)}{dt}t + F_0(0),$$

with constant

$$\delta = \frac{L}{[n-\frac{1}{2}(n-1)p+1][n-\frac{1}{2}(n-1)p]} = \frac{\varepsilon^p c_0^p C_5^{-(p-1)}}{[n-\frac{1}{2}(n-1)p+1][n-\frac{1}{2}(n-1)p]} > 0.$$

When $1 < p < p_1(n)$, it is easy to check that $n+1-(n-1)p/2 > 1$. Hence the following estimate is valid when t is sufficiently large:

$$F_0(t) \geqslant \frac{1}{2}\delta(t+R)^{n+1-(n-1)p/2}. \tag{3.6}$$

Estimates (3.3) together with (3.6) and Lemma 2.1 with parameters

$$a \equiv n+1-(n-1)p/2, \quad \text{and} \quad q \equiv n(p-1)$$

imply Theorem 1.1 for all exponents p such that

$$(p-1)\big(n+1-(n-1)p/2\big) > n(p-1)-2 \quad \text{and} \quad p>1.$$

It is easy to see that the solution set is $p \in (1, p_1(n))$, so by Lemma 2.1, all solutions of problem (1.1) with nontrivial nonnegative initial values must blow up in finite time.

Also, recall from Lemma 2.1, we have the following estimate for the life span $T(\varepsilon)$ of solutions of (1.1) as follows

$$\begin{aligned} T(\varepsilon) &\leqslant c\left(\frac{1}{2}\delta\right)^{-\frac{(p-1)}{(p-1)a-q+2}} \\ &= A_1(\varepsilon^p)^{-\frac{(p-1)}{(p-1)(n+1-(n-1)p/2)-n(p-1)+2}} \\ &= A_1\varepsilon^{-\frac{p(p-1)}{(p-1)(1-(n-1)p/2)+2}} \\ &= A_1\varepsilon^{-\frac{2p(p-1)}{2+(n+1)p-(n-1)p^2}}, \end{aligned} \tag{3.7}$$

where A_1 is a positive constant which is independent of ε. The proof of Theorem 1.1 is complete.

4. The proof of Theorem 1.2

By the expression $\psi_1(x,t) = e^{-t}\phi_1(x) \geqslant 0$, we have $(\psi_1)_t = -\psi_1$, and $\partial_i(a_{ij}\partial_j\psi_1(x,t)) = \psi_1(x,t)$, in $\Omega^c \times (0,+\infty)$. So ψ_1 satisfies

$$\begin{cases} \partial_i\big(a_{ij}\partial_j\psi_1(x,t)\big) = \psi_1(x,t), & \text{in } \Omega^c \times (0,+\infty), \\ \psi_1|_{\partial\Omega\times(0,+\infty)} = 0, \\ |x|\to\infty, \qquad \psi_1(x,t) \to e^{-t}\displaystyle\int_{S^{n-1}} e^{x\cdot\omega}\,d\omega, & \text{for } t \geqslant 0. \end{cases} \tag{4.1}$$

We multiply (1.2) by function ψ_1, and integrate over Ω^c, then we use integration by parts and Lemma 2.3.

First,

$$\int_{\Omega^c} \psi_1\big(u_{tt} - \partial_i\big(a_{ij}(x)\partial_j u\big)\big)\,dx = \int_{\Omega^c} \psi_1|u_t|^p\,dx.$$

Note that for a fixed t, $u(\cdot,t) \in H_0^1(D_t)$, where D_t is the support of $u(\cdot,t)$. Hence by integration by parts and Lemma 2.3, we have

$$\begin{aligned} \int_{\Omega^c} \psi_1\partial_i\big(a_{ij}(x)\partial_j u\big)\,dx &= \int_{\Omega^c} \big(\partial_i\big[\psi_1 a_{ij}(x)\partial_j u\big] - \partial_i\psi_1 a_{ij}(x)\partial_j u\big)\,dx \\ &= \int_{\partial\Omega} \psi_1 a_{ij}(x)\partial_j u \cdot n_i\,dS - \int_{\Omega^c} \big(a_{ij}(x)\partial_i\psi_1\big)\cdot\partial_j u\,dx \\ &= -\int_{\partial\Omega} a_{ij}(x)\partial_i\psi_1\cdot u\cdot n_j\,dS + \int_{\Omega^c} \partial_j\big(a_{ij}(x)\partial_i\psi_1\big)u\,dx \\ &= \int_{\Omega^c} \partial_j\big(a_{ij}(x)\partial_i\psi_1\big)\cdot u\,dx = \int_{\Omega^c} \psi_1\cdot u\,dx. \end{aligned}$$

Combining the above two identities, we conclude

$$\int_{\Omega^c} \psi_1 u_{tt} - \int_{\Omega^c} \psi_1\cdot u\,dx = \int_{\Omega^c} \psi_1\cdot|u_t|^p\,dx. \tag{4.2}$$

Notice that

$$\frac{d}{dt}\int_{\Omega^c}\psi_1 u_t\,dx=\int_{\Omega^c}(\psi_1\cdot u_{tt}-u_t\psi_1)\,dx, \tag{4.3}$$

$$\frac{d}{dt}\int_{\Omega^c}(\psi_1 u)\,dx=\int_{\Omega^c}\big[(\psi_1)_t\cdot u+u_t\cdot\psi_1\big]\,dx=\int_{\Omega^c}[\psi_1\cdot u_t-u\psi_1]\,dx. \tag{4.4}$$

Adding up the above two expressions, we obtain the following

$$\frac{d}{dt}\int_{\Omega^c}(\psi_1 u_t+\psi_1 u)\,dx=\int_{\Omega^c}(\psi_1\cdot u_{tt}-u\cdot\psi_1)\,dx=\int_{\Omega^c}\psi_1\cdot|u_t|^p\,dx. \tag{4.5}$$

So we have

$$\begin{aligned}\int_{\Omega^c}(\psi_1 u_t+\psi_1 u)\,dx&=\int_{\Omega^c}(\psi_1 u_t+\psi_1 u)\,dx\bigg|_{t=0}+\int_0^t\int_{\Omega^c}\psi_1\cdot|u_\tau|^p\,dx\,d\tau\\&=\int_{\Omega^c}\varepsilon\phi_1(x)\big[f(x)+g(x)\big]\,dx+\int_0^t\int_{\Omega^c}\psi_1\cdot|u_\tau|^p\,dx\,d\tau\\&\geqslant\int_{\Omega^c}\varepsilon\phi_1(x)g(x)\,dx+\int_0^t\int_{\Omega^c}\psi_1\cdot|u_\tau|^p\,dx\,d\tau.\end{aligned} \tag{4.6}$$

Adding two expressions (4.2) and (4.6), we have

$$\int_{\Omega^c}(\psi_1 u_{tt}+\psi_1 u_t)\,dx\geqslant\int_{\Omega^c}\psi_1\cdot|u_t|^p\,dx+\int_0^t\int_{\Omega^c}\psi_1\cdot|u_\tau|^p\,dx\,d\tau+\varepsilon\int_{\Omega^c}\phi_1(x)g(x)\,dx. \tag{4.7}$$

Also, we know that

$$\begin{aligned}\frac{d}{dt}\int_{\Omega^c}\psi_1 u_t\,dx+2\int_{\Omega^c}\psi_1\cdot u_t\,dx&=\int_{\Omega^c}\big[\psi_1 u_{tt}+u_t(\psi_1)_t+2\psi_1 u_t\big]\,dx\\&=\int_{\Omega^c}(\psi_1 u_{tt}+\psi_1 u_t)\,dx.\end{aligned} \tag{4.8}$$

So we have

$$\frac{d}{dt}\int_{\Omega^c}\psi_1 u_t\,dx+2\int_{\Omega^c}\psi_1\cdot u_t\,dx\geqslant\int_{\Omega^c}\psi_1\cdot|u_t|^p\,dx+\int_0^t\int_{\Omega^c}\psi_1\cdot|u_\tau|^p\,dx\,d\tau+\varepsilon\int_{\Omega^c}\phi_1(x)g(x)\,dx. \tag{4.9}$$

To show the blow-up property, we define the following auxiliary function

$$G_0(t)=\int_{\Omega^c}\psi_1 u_t\,dx-\frac{1}{2}\int_0^t\int_{\Omega^c}\psi_1\cdot|u_\tau|^p\,dx\,d\tau-\frac{\varepsilon}{2}\int_{\Omega^c}\phi_1(x)g(x)\,dx. \tag{4.10}$$

We note that, when $t=0$,

$$G_0(0)=\varepsilon\int_{\Omega^c}\phi_1(x)g(x)\,dx-\frac{\varepsilon}{2}\int_{\Omega^c}\phi_1(x)g(x)\,dx=\frac{\varepsilon}{2}\int_{\Omega^c}\phi_1(x)g(x)\,dx\geqslant 0,$$

and we have

$$\frac{d}{dt}G_0(t)=\int_{\Omega^c}(\psi_1 u_{tt}-u_t\psi_1)\,dx-\frac{1}{2}\int_{\Omega^c}\psi_1\cdot|u_t|^p\,dx. \tag{4.11}$$

Hence, we conclude that

$$\begin{aligned}\frac{d}{dt}G_0(t)+2G_0(t)&=\int\limits_{\Omega^c}(\psi_1 u_{tt}+u_t\psi_1)\,dx-\frac{1}{2}\int\limits_{\Omega^c}\psi_1\cdot|u_t|^p\,dx-\int\limits_0^t\int\limits_{\Omega^c}\psi_1\cdot|u_\tau|^p\,dx\,d\tau-\varepsilon\int\limits_{\Omega^c}\phi_1(x)g(x)\,dx\\&\geqslant\int\limits_{\Omega^c}\psi_1\cdot|u_t|^p\,dx+\int\limits_0^t\int\limits_{\Omega^c}\psi_1\cdot|u_\tau|^p\,dx\,d\tau+\varepsilon\int\limits_{\Omega^c}\phi_1(x)g(x)\,dx\\&\quad-\frac{1}{2}\int\limits_{\Omega^c}\psi_1|u_t|^p\,dx-\int\limits_0^t\int\limits_{\Omega^c}\psi_1\cdot|u_\tau|^p\,dx\,d\tau-\varepsilon\int\limits_{\Omega^c}\phi_1(x)g(x)\,dx\\&=\frac{1}{2}\int\limits_{\Omega^c}\psi_1\cdot|u_t|^p\,dx\geqslant 0.\end{aligned}\tag{4.12}$$

Multiplying the above differential inequality by e^{2t}, we get the following expression

$$\frac{d}{dt}\left(e^{2t}G_0(t)\right)\geqslant 0.$$

So for $\forall t\geqslant 0$, we have $e^{2t}G_0(t)\geqslant G_0(0)$, that is $G_0(t)\geqslant e^{-2t}G_0(0)\geqslant 0$.

By (4.10), we have for $\forall t\geqslant 0$,

$$\int\limits_{\Omega^c}\psi_1 u_t\,dx\geqslant\frac{1}{2}\int\limits_0^t\int\limits_{\Omega^c}\psi_1\cdot|u_t|^p\,dx\,d\tau+\frac{\varepsilon}{2}\int\limits_{\Omega^c}\phi_1(x)g(x)\,dx.\tag{4.13}$$

Let

$$F(t)=\frac{1}{2}\int\limits_0^t\int\limits_{\Omega^c}\psi_1\cdot|u_t|^p\,dx\,d\tau+\frac{\varepsilon}{2}\int\limits_{\Omega^c}\phi_1(x)g(x)\,dx,\quad t\geqslant 0.\tag{4.14}$$

Then we have

$$\int\limits_{\Omega^c}\psi_1 u_t\,dx\geqslant F(t),\quad\text{for }\forall t\geqslant 0.\tag{4.15}$$

Next we only need to prove that $F(t)$ blow up.

From the expression of $F(t)$, we get for $\forall t\geqslant 0$, $F(t)\geqslant 0$, and $F'(t)=\frac{1}{2}\int_{\Omega^c}\psi_1\cdot|u_t|^p\,dx$. Estimating the right side of $F'(t)$ by the Hölder inequality, we have

$$\int\limits_{\Omega^c}\left|u_t(x,t)\right|^p\psi_1\,dx\geqslant\frac{|\int_{\Omega^c}u_t(x,t)\psi_1\,dx|^p}{(\int_{\Omega^c\cap\{|x|\leqslant t+R\}}\psi_1\,dx)^{p-1}}.$$

By Lemma 2.6, we know that for $\forall t\geqslant 0$,

$$\int\limits_{\Omega^c\cap\{|x|\leqslant t+R\}}\psi_1\,dx\leqslant C_8(t+R)^{(n-1)/2},\tag{4.16}$$

where C_8 is a positive constant.

Therefore we conclude that

$$\begin{aligned}F'(t)=\frac{1}{2}\int\limits_{\Omega^c}\psi_1\cdot|u_t|^p\,dx&\geqslant\frac{1}{2}\frac{|\int_{\Omega^c}u_t(x,t)\psi_1\,dx|^p}{(\int_{\Omega^c\cap\{|x|\leqslant t+R\}}\psi_1\,dx)^{p-1}}\\&\geqslant C_9\frac{|\int_{\Omega^c}u_t(x,t)\psi_1\,dx|^p}{(t+R)^{(n-1)(p-1)/2}}\geqslant C_9\frac{|F(t)|^p}{(t+R)^{(n-1)(p-1)/2}}.\end{aligned}\tag{4.17}$$

By the property of Riccati equation, we know that when $(n-1)(p-1)/2\leqslant 1$, the solution of the initial–boundary value problem (1.2) blow up.

In detail, let

$$M = \frac{1}{2}\int_{\Omega^c} \phi_1(x)g(x)\,dx.$$

Then $F(t)$ satisfies the following problem

$$\begin{cases} F'(t) \geqslant C_9 \dfrac{|F(t)|^p}{(t+R)^{(n-1)(p-1)/2}}, \\ F(0) = M\varepsilon. \end{cases} \tag{4.18}$$

Now we introduce a function $v(t)$ satisfying the following Riccati equation

$$\begin{cases} v'(t) = C_9 \dfrac{|v(t)|^p}{(t+R)^{(n-1)(p-1)/2}}, \\ v(0) = M\varepsilon. \end{cases} \tag{4.19}$$

So the life span of F is less than that of v which will be the upper bound of $T(\varepsilon)$.

Thus, in the case $(n-1)(p-1) < 2$, integrating (4.19), we get

$$v(t) = \left[(M\varepsilon)^{-(p-1)} + C'R^{1-(n-1)(p-1)/2} - C'(t+R)^{1-(n-1)(p-1)/2}\right]^{-\frac{1}{p-1}}, \tag{4.20}$$

where

$$C' = \frac{C_9(p-1)}{1-(n-1)(p-1)/2}.$$

Thus

$$T(\varepsilon) \leqslant A_2 \varepsilon^{-\frac{p-1}{1-(n-1)(p-1)/2}},$$

where A_2 is a positive constant which is independent of ε.

When $(n-1)(p-1) = 2$, integrating (4.19), we get

$$v(t) = \left[(M\varepsilon)^{-(p-1)} - C'' \ln\left(\frac{t+R}{R}\right)\right]^{-\frac{1}{p-1}}, \tag{4.21}$$

where

$$C'' = C_9(p-1),$$

$$T(\varepsilon) \leqslant \exp\bigl(B_2 \varepsilon^{-(p-1)}\bigr),$$

where B_2 is a positive constant which is independent of ε. This ends the proof of Theorem 1.2.

5. Conclusion

We have obtained the blow-up results for the initial–boundary value problem for the semilinear wave equation (1.1) on exterior domain with subcritical exponent p, that is $1 < p < p_1(n)$, and the space dimensions $n \geqslant 3$, and also we give the estimate of upper bound of life span solutions for the problem. For the space dimension $n = 2$ and $n = 1$, the blow-up results are open. For the case of the critical exponents, there is no results for (1.1). Moreover, the estimate of lower bound of life span for the subcritical or critical case, and global existence for the supercritical case are largely open despite the results in dimension $3 \leqslant n \leqslant 4$ by [7,22].

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