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Seminormal composition operators on L^2 spaces induced by matrices: The Laplace density case ☆

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ABSTRACT

Bounded composition operators with matrix symbols on $L^2(\mu)$, where μ is a positive Borel measure with a Laplace type density function on the d-dimensional Euclidean space, are studied. Hyponormality and subnormality of adjoints of such operators are characterized explicitly in terms of their symbols.

1. Introduction

The foundations of the theory of composition operators in abstract L^2-spaces are well developed. Many properties of such operators are described with the help of appropriate Radon–Nikodym derivatives (cf. [8,20,29,12,14,7,15,16,24,3,1,2]). However, when dealing with specific measure spaces, one has to make a further effort to adapt a general approach to a particular context. Specificity of measure spaces under consideration may lead to interesting questions in other branches of mathematics like matrix theory and the theory of moments (see e.g. [19,25,6]).

Our aim in this paper is to examine a class of composition operators $C_A : L^2(\mu) \to L^2(\mu)$ given by $C_A(f) = f \circ A$, where A is a nonsingular $d \times d$ matrix and μ is a positive Borel measure on the real d-dimensional Euclidean space $\mathbb{R}^d$ defined by

$$\mu(\sigma) = \int_\sigma \mathrm{e}^{-\|x\|}\,\mathrm{d}x, \quad \sigma\text{-Borel subset of } \mathbb{R}^d. \tag{1.1}$$

Here $\|\cdot\|$ is a norm on $\mathbb{R}^d$ that is given by an inner product. In other words, the measure μ is given by a density function of the Laplace type. The case of other classical densities including the Gaussian one has been investigated in [19,25,6]. In particular, the cosubnormality of composition operators on such L^2 spaces has been completely characterized by normality of their symbols A on the Hilbert space $(\mathbb{R}^d, \|\cdot\|)$ (cf. [25, Theorem 2.5]). Comparing the Laplace density case (cf. Theorem 4.4) with the Gaussian one, we see that it may happen that the composition operator C_A is cosubnormal in the Gaussian density case but not in the Laplace one. It follows from what is proved in Section 4 that in the Laplace density case there exist cohyponormal composition operators C_A which are not cosubnormal; however, the same operators, considered in the Gaussian setting, are still cosubnormal. Fortunately, in the Laplace density case there exist cosubnormal composition operators C_A on $L^2(\mu)$ which are not normal.

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It is worth mentioning that hyponormal composition operators on $L^2(\nu)$ are automatically unitary whenever the measure ν is finite (cf. [12, Lemma 7]). This is the main reason why only cohyponormal and cosubnormal composition operators are studied in this article. However, under some circumstances (which appear in our paper) the adjoints of composition operators are unitarily equivalent to scalar multiples of some other composition operators (see e.g. [25, (UE), p. 309]). This means that our results concerning cohyponormal and cosubnormal composition operators on $L^2(\mu)$ with μ as in (1.1) can be easily formulated for hyponormal and subnormal composition operators with matrix symbols on $L^2(\mathrm{e}^{\|x\|}\,\mathrm{d}x)$.

In Section 3 we discuss some questions concerning positive definiteness and conditional positive definiteness; both notions play a pivotal role in our paper.

2. Preliminaries

Given a complex Hilbert space $\mathcal{H}$, we denote by $\boldsymbol{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \boldsymbol{B}(\mathcal{H})$ is said to be

- *subnormal* if there exist a complex Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ (isometric embedding) and a normal operator $N \in \boldsymbol{B}(\mathcal{K})$ such that $Tf = Nf$ for all $f \in \mathcal{H}$ (cf. [10]);
- *quasinormal* if the factors U and $|T|$ in the polar decomposition $T = U|T|$ of T commute;
- *hyponormal* if $T^*T - TT^* \geqslant 0$.

It is well known that isometric operators are quasinormal, quasinormal operators are subnormal and subnormal operators are hyponormal. However, neither of the reverse implications is true (cf. [4,5]). The adjoint of a hyponormal operator is called a *cohyponormal* operator. The same terminological rule with the prefix “co” is applied to subnormal, quasinormal and isometric operators.

Suppose that (X, Σ, μ) is σ-finite measure space and $\phi : X \to X$ is a Σ-measurable transformation such that the measure $\mu \circ \phi^{-1}$ defined by $(\mu \circ \phi^{-1})(\sigma) = \mu(\phi^{-1}(\sigma))$ for $\sigma \in \Sigma$ is absolutely continuous with respect to μ. Then the linear operator $C_\phi : L^2(\mu) \supseteq \mathcal{D}(C_\phi) \to L^2(\mu)$ given by

$$\mathscr{D}(C_\phi) = \bigl\{ f \in L^2(\mu)\colon\ f \circ \phi \in L^2(\mu) \bigr\} \quad \text{and} \quad C_\phi f = f \circ \phi \quad \text{for } f \in \mathscr{D}(C_\phi)$$

is well defined. We call it a *composition operator* induced by ϕ and ϕ is called a *symbol* of C_ϕ. Since $\mu \circ (\phi^n)^{-1}$ is absolutely continuous with respect to μ for each integer $n \geqslant 0$, we may consider the Radon–Nikodym derivatives

$$h_n = h_n^\phi := \frac{\mathrm{d}\mu \circ (\phi^n)^{-1}}{\mathrm{d}\mu}, \quad n = 0, 1, 2, \ldots,$$

where ϕ^n denotes the n-fold composition of ϕ with itself for $n \geqslant 1$, and ϕ^0 is the identity transformation of X. Since $\mathscr{D}(C_\phi)$ equipped with the graph norm of C_ϕ is the Hilbert space $L^2((1 + h_1^\phi)\,\mathrm{d}\mu)$, we see that the operator C_ϕ is closed. Recall that C_ϕ is a bounded operator on $L^2(\mu)$ if and only if $h_1^\phi \in L^\infty(\mu)$. If ψ is a Σ-measurable transformation of X such that the mapping $L^2(\mu) \ni f \mapsto f \circ \psi \in L^2(\mu)$ is well defined, then $\mu \circ \psi^{-1}$ is absolutely continuous with respect to μ, C_ψ is bounded and

$$\|C_\psi\| = \bigl\|h_1^\psi\bigr\|_\infty^{1/2}. \tag{2.1}$$

The interested reader is referred to [8,20,24] for further information.

The following characterization of subnormality of composition operators is due to Lambert (cf. [15]; the assumptions that the symbol of the composition operator is onto and $h_1 > 0$ a.e. are unnecessary).

Theorem 2.1. *A composition operator $C_\phi \in \boldsymbol{B}(L^2(\mu))$ is subnormal if and only if $\{h_n^\phi(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence*[1] *for μ-almost every $x \in X$.*

Let us now concentrate on a particular class of bounded composition operators C_A on $L^2(\mu_\varrho)$ induced by invertible linear transformations A of $\mathbb{R}^d$, where $\varrho : \mathbb{R}^d \to [0, \infty)$ is a Borel function such that $\varrho(x) > 0$ for almost every $x \in \mathbb{R}^d$ with respect to the d-dimensional Lebesgue measure on $\mathbb{R}^d$, and μ_ϱ is a positive Borel measure on $\mathbb{R}^d$ given by

$$\mu_\varrho(\sigma) = \int_\sigma \varrho(x)\,\mathrm{d}x, \quad \sigma\text{-Borel subset of } \mathbb{R}^d.$$

It is easily seen that $\mu_\varrho \circ A^{-1}$ is absolutely continuous with respect to μ_ϱ and

[1] For the definition of a Stieltjes moment sequence we refer the reader to Section 3.

$$h_n = h_n^A = \frac{1}{|\det A|^n} \frac{\varrho \circ A^{-n}}{\varrho}, \quad n = 0, 1, 2, \ldots. \tag{2.2}$$

This means that the composition operator C_A is well defined in $L^2(\mu_\varrho)$.

In the case of continuous density functions ϱ, Theorem 2.1 takes more familiar form (cf. [25, Proposition 2.4]).

Proposition 2.2. *Assume that $\varrho : \mathbb{R}^d \to (0, \infty)$ is a continuous function and A is an invertible linear transformations of $\mathbb{R}^d$ such that $C_A \in \boldsymbol{B}(L^2(\mu_\varrho))$. Then C_A is subnormal if and only if $\{\varrho(A^{-n}x)\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $x \in \mathbb{R}^d$.*

3. Positive definiteness

In what follows, the fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively; $\mathbb{R}_+$ stands for the set of all non-negative real numbers, while $\mathbb{Z}_+$ for the set of all non-negative integers.

Let X be a nonempty set. We say that a function $b : X \times X \to \mathbb{C}$ is *positive definite* if

$$\sum_{x,y \in X} b(x, y)\lambda(x)\overline{\lambda(y)} \geqslant 0 \tag{3.1}$$

for every function $\lambda : X \to \mathbb{C}$ of finite support $\{x \in X\colon\ \lambda(x) \neq 0\}$. Each positive definite function $b : X \times X \to \mathbb{C}$ satisfies the Cauchy–Schwarz inequality (cf. [17, p. 19]):

$$\bigl|b(x, y)\bigr|^2 \leqslant b(x, x)b(y, y), \quad x, y \in X. \tag{3.2}$$

A function $b : X \times X \to \mathbb{C}$ is said to be *conditionally positive definite* if it satisfies the inequality (3.1) for every function $\lambda : X \to \mathbb{C}$ of finite support such that $\sum_{x \in X} \lambda(x) = 0$. Conditional positive definiteness can be characterized by means of positive definiteness as follows (cf. [21, Lemma 1.7]; see also [26, Lemma 5.2] for the assertion (i) below).

Lemma 3.1. *For an arbitrary $b : X \times X \to \mathbb{C}$, the following assertions hold*:

(i) *if e^{tb} is positive definite for every positive real number t, then b is conditionally positive definite*;
(ii) *if b is conditionally positive definite and $\overline{b(x, y)} = b(y, x)$ for all $x, y \in X$, then e^{tb} is positive definite for every positive real number t.*

Clearly, positive definiteness implies conditional positive definiteness (see [21,18] for wide-ranging surveys on the subject). The converse implication does not hold in general because the function b is conditionally positive definite if and only if $b + \alpha$ is conditionally positive definite for all $\alpha \in \mathbb{C}$. However, as shown below, under some circumstances conditional positive definiteness may imply positive definiteness (see [28, Proposition 7] for a related result for functions defined on $*$-semigroups).

Lemma 3.2. *If $b : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C}$ is a conditionally positive definite function such that $\overline{b(k, l)} = b(l, k)$ for all $k, l \in \mathbb{Z}_+$, $\alpha := \lim_{n\to\infty} b(n, n) \in \mathbb{R}_+$ and $\lim_{n\to\infty} b(k, n) = \alpha$ for all $k \in \mathbb{Z}_+$, then b is positive definite.*

Proof. Considering $b - \alpha$ instead of b, we can assume without loss of generality that $\alpha = 0$. Take a function $\lambda : \mathbb{Z}_+ \to \mathbb{C}$ of finite support. Fix $n \in \mathbb{Z}_+$ such that the support of λ is contained in $\{0, \ldots, n\}$. For an integer $m > n$, we define the function $\lambda_m : \mathbb{Z}_+ \to \mathbb{C}$ of finite support via

$$\lambda_m(k) = \begin{cases} \lambda(k), & k = 0, \ldots, n, \\ -\sum_{j=0}^n \lambda(j), & k = m, \\ 0, & \text{otherwise}. \end{cases}$$

Then, by the Hermitian symmetry and the conditional positive definiteness of b, we have

$$0 \leqslant \sum_{j,k \geqslant 0} b(j, k)\lambda_m(j)\overline{\lambda_m(k)} = \sum_{j,k=0}^n b(j, k)\lambda(j)\overline{\lambda(k)} - 2\mathrm{Re} \sum_{j,k=0}^n b(j, m)\lambda(j)\overline{\lambda(k)} + \sum_{j,k=0}^n b(m, m)\lambda(j)\overline{\lambda(k)}.$$

Passing with m to ∞, we get the conclusion. $\square$

A sequence $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{R}_+$ is said to be a *Stieltjes moment* sequence if there exists a positive Borel measure[2] μ on $\mathbb{R}_+$ such that $a_n = \int_0^{\infty} x^n \,\mathrm{d}\mu(x)$ for all $n = 0, 1, 2, \ldots$; such μ is called a *representing measure* of $\{a_n\}_{n=0}^{\infty}$. If a Stieltjes moment sequence has a unique representing measure, we call it *determinate*. Recall that (cf. [9] for instance)

$$\text{each Stieltjes moment sequence which has a compactly supported representing measure is determinate.} \tag{3.3}$$

By the Stieltjes theorem (cf. [23, Theorem 1.3]),

$$\begin{gathered}\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{R}_+ \text{ is a \textit{Stieltjes moment} sequence if and only if the functions} \\ \mathbb{Z}_+ \times \mathbb{Z}_+ \ni (m,n) \mapsto a_{m+n} \text{ and } \mathbb{Z}_+ \times \mathbb{Z}_+ \ni (m,n) \mapsto a_{m+n+1} \text{ are positive definite.}\end{gathered} \tag{3.4}$$

We now state a result that will be of use in Section 4. It is a particular case of more general investigations contained in [27]. For the sake of completeness, we include its proof. It states that the square root of a Stieltjes moment sequence whose representing measure is concentrated on a two-point subset of $(0, \infty)$ is never a Stieltjes moment sequence.

Lemma 3.3. *If α_1, α_2, θ_1 and θ_2 are positive real numbers, then the sequence $\{\sqrt{\alpha_1\theta_1^n + \alpha_2\theta_2^n}\}_{n=0}^{\infty}$ is a Stieltjes moment sequence if and only if $\theta_1 = \theta_2$.*

Proof. Suppose that, contrary to our claim, $\{\sqrt{\alpha_1\theta_1^n + \alpha_2\theta_2^n}\}_{n=0}^{\infty}$ is a Stieltjes moment sequence for some $\theta_1 < \theta_2$, and μ is its representing measure. Then, by the measure transport theorem (cf. [11, Theorem C, p. 163]), we see that

$$\alpha_1\theta_1^n + \alpha_2\theta_2^n = \int_0^{\infty}\int_0^{\infty} (xy)^n \,\mathrm{d}(\mu \otimes \mu)(x, y) = \int_0^{\infty} z^n \,\mathrm{d}\big((\mu \otimes \mu) \circ \varphi^{-1}\big)(z) \tag{3.5}$$

for all $n \in \mathbb{Z}_+$, where $\mu \otimes \mu$ is the product of the measure μ by itself and $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is the function given by $\varphi(x, y) = xy$ for all $x, y \in \mathbb{R}_+$. Denote by δ_θ the Borel probability measure on $\mathbb{R}$ concentrated on $\{\theta\}$, $\theta \in \mathbb{R}$. Since

$$\alpha_1\theta_1^n + \alpha_2\theta_2^n = \int_0^{\infty} x^n \,\mathrm{d}(\alpha_1\delta_{\theta_1} + \alpha_2\delta_{\theta_2})(x), \quad n \in \mathbb{Z}_+,$$

we infer from (3.3) and (3.5) that

$$(\mu \otimes \mu) \circ \varphi^{-1} = \alpha_1\delta_{\theta_1} + \alpha_2\delta_{\theta_2}. \tag{3.6}$$

Set $\Omega_\zeta := \varphi^{-1}(\{\zeta\})$ for $\zeta > 0$. By (3.6), $\mu \otimes \mu(\varphi^{-1}(\mathbb{R}_+ \setminus \{\theta_1, \theta_2\})) = 0$, and so[3]

$$\operatorname{supp} \mu \otimes \mu \subseteq \Omega_{\theta_1} \sqcup \Omega_{\theta_2}. \tag{3.7}$$

Note now that

$$\Omega_{\theta_i} \cap \operatorname{supp} \mu \otimes \mu \neq \emptyset, \quad i = 1, 2. \tag{3.8}$$

Indeed, otherwise $\Omega_{\theta_i} \cap \operatorname{supp} \mu \otimes \mu = \emptyset$ for some $i \in \{1, 2\}$, and consequently

$$0 = (\mu \otimes \mu)\big(\mathbb{R}_+^2 \setminus \operatorname{supp} \mu \otimes \mu\big) \geqslant (\mu \otimes \mu)(\Omega_{\theta_i}) \overset{(3.6)}{>} 0,$$

which is a contradiction. Let us recall the following well-known identity

$$\operatorname{supp} \mu \otimes \mu = \operatorname{supp} \mu \times \operatorname{supp} \mu. \tag{3.9}$$

It follows from (3.8) that $\operatorname{supp} \mu \otimes \mu$ contains two distinct points $(x_1, y_1) \in \Omega_{\theta_1}$ and $(x_2, y_2) \in \Omega_{\theta_2}$. By (3.9), $\Xi := \{x_1, x_2, y_1, y_2\} \subseteq \operatorname{supp} \mu$, and so by (3.7) and (3.9),

$$u \cdot v \in \{\theta_1, \theta_2\}, \quad u, v \in \Xi. \tag{3.10}$$

Set $a = \min \Xi$ and $b = \max \Xi$. Since $\theta_1 < \theta_2$ and $a^2 \leqslant x_i y_i = \theta_i \leqslant b^2$ for $i = 1, 2$, we deduce from (3.10) that $a = \sqrt{\theta_1}$ and $b = \sqrt{\theta_2}$. Hence, $\theta_1 < \sqrt{\theta_1}\sqrt{\theta_2} = ab < \theta_2$, and so $ab \notin \{\theta_1, \theta_2\}$, which contradicts (3.10). This completes the proof. □

Regarding Lemma 3.3, it is natural to ask the question whether for a given integer $k \geqslant 2$, the square root of the Stieltjes moment sequence $\{\alpha_1\theta_1^n + \cdots + \alpha_k\theta_k^n\}_{n=0}^{\infty}$ is not a Stieltjes moment sequence for all real numbers $\alpha_1, \ldots, \alpha_k > 0$ and $0 < \theta_1 < \cdots < \theta_k$. The answer to this question is in the affirmative for $k = 2$ and $k = 4$, and in the negative for $k = 3$ and for all integers $k \geqslant 5$ (cf. [27]).

[2] Such μ being finite is automatically regular (see e.g., [22, Theorem 2.18]).

[3] As usual $\operatorname{supp} \nu$ stands for the closed support of a regular positive Borel measure ν.

4. The Laplace density case

In this section we deal with the measure μ given by (1.1). Clearly, $\mu = \mu_\varrho$ with the Laplace density function ϱ defined by

$$\varrho(x) = \mathrm{e}^{-\|x\|}, \quad x \in \mathbb{R}^d, \tag{4.1}$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^d$ that is given by an inner product. We begin by characterizing the boundedness of the composition operator C_A on $L^2(\mu)$. If A is a linear transformation of $\mathbb{R}^d$, then we write $\|A\|$ for the norm of A counted with respect to $(\mathbb{R}^d, \|\cdot\|)$.

Proposition 4.1. *Let μ be as in* (1.1) *and let A be an invertible linear transformation of $\mathbb{R}^d$. Then C_A is a bounded operator on $L^2(\mu)$ if and only if $\|A\| \leqslant 1$. Moreover, if C_A is bounded, then $\|C_A\| = \frac{1}{|\det A|^{1/2}}$.*

Proof. By (2.2) and (4.1), we have

$$h_1^A(x) = |\det A|^{-1} \exp\bigl(\|x\| - \|A^{-1}x\|\bigr), \quad x \in \mathbb{R}^d. \tag{4.2}$$

Since the measure μ is mutually absolutely continuous with respect to the d-dimensional Lebesgue measure on $\mathbb{R}^d$ and the function h_1^A is continuous, we see that $h_1^A \in L^\infty(\mu)$ if and only if $\sup_{x\in\mathbb{R}^d} h_1^A(x) < \infty$. If $\sup_{x\in\mathbb{R}^d} h_1^A(x) < \infty$, then by considering the functions

$$(0,\infty) \ni t \to h_1^A(tx) = |\det A|^{-1} \exp\bigl(t\bigl(\|x\| - \|A^{-1}x\|\bigr)\bigr), \quad x \in \mathbb{R}^d,$$

we deduce that $\|x\| - \|A^{-1}x\| \leqslant 0$ for all $x \in \mathbb{R}^d$, which is equivalent to $\|A\| \leqslant 1$. The reverse implication is certainly true. The "moreover" part of the conclusion is a direct consequence of (2.1) and (4.2). $\square$

As shown below, only unitary symbols A can induce hyponormal composition operators C_A on $L^2(\mu)$.

Proposition 4.2. *Let μ be as in* (1.1) *and let A be an invertible linear transformation of $\mathbb{R}^d$ such that $C_A \in \boldsymbol{B}(L^2(\mu))$. Then the following conditions are equivalent*:

(i) *C_A is hyponormal,*
(ii) *C_A is coquasinormal,*
(iii) *C_A is normal,*
(iv) *C_A is unitary,*
(v) *the transformation A is unitary on $(\mathbb{R}^d, \|\cdot\|)$.*

Proof. (i) $\Rightarrow$ (iv) Since the measure μ is finite, we infer from [12, Lemma 7] that C_A is unitary.

(iv) $\Leftrightarrow$ (v) By [25, (UN), p. 311], C_A is unitary if and only if

$$\exp\bigl(\|x\| - \|A^{-1}x\|\bigr) = |\det A|, \quad x \in \mathbb{R}^d. \tag{4.3}$$

Considering the functions $(0,1) \ni t \mapsto h_1^A(tx)$, we conclude that (4.3) holds if and only if $\|x\| - \|A^{-1}x\| = 0$ for all $x \in \mathbb{R}^d$, or equivalently if A is unitary on $(\mathbb{R}^d, \|\cdot\|)$.

(ii) $\Rightarrow$ (iii) Since by the measure transport theorem $\|C_A f\|^2 = \int_{\mathbb{R}^d} h_1^A |f|^2 \,\mathrm{d}\mu$ for all $f \in L^2(\mu)$ and $h_1^A > 0$, we deduce that C_A is injective. In turn, by [25, (UE), p. 309], the operator $|\det A| C_A^*$ is unitarily equivalent to the composition operator $C_{A^{-1}}$ acting on $L^2(\mu_\theta)$, where $\theta(x) = \exp(\|x\|)$ for $x \in \mathbb{R}^d$. Hence, arguing as above, we see that C_A^* is injective. This means that C_A^* is a quasinormal operator whose kernel and cokernel are trivial. Thus, as easily seen, C_A is normal.

Since the implications (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii) are obvious, the proof is complete. $\square$

Proposition 4.2 may suggest that there is no cohyponormal non-unitary composition operator C_A on $L^2(\mu)$. However, this is not the case. It turns out that there are cohyponormal composition operators C_A on $L^2(\mu)$ which are not cosubnormal (cf. Example 4.5) and cosubnormal composition operators C_A on $L^2(\mu)$ which are not normal (cf. Theorem 4.4 and Proposition 4.2).

Our next aim is to characterize the cohyponormality of composition operators on $L^2(\mu)$ which are induced by invertible linear transformations of $\mathbb{R}^d$.

Proposition 4.3. *Let μ be as in* (1.1) *and let A be an invertible linear transformation of $\mathbb{R}^d$ such that $C_A \in \boldsymbol{B}(L^2(\mu))$. Then C_A is cohyponormal if and only if $2\|x\| \leqslant \|Ax\| + \|A^{-1}x\|$ for all $x \in \mathbb{R}^d$. Moreover, if A is a normal operator on $(\mathbb{R}^d, \|\cdot\|)$, then C_A is cohyponormal.*

Proof. The characterization of cohyponormality follows from [25, (HY), p. 311]. If A is a normal operator on $(\mathbb{R}^d, \|\cdot\|)$, then A^{-1} is a normal operator on $(\mathbb{R}^d, \|\cdot\|)$, and consequently by the Schwarz inequality we have

$$\begin{aligned}2\|x\| &= 2\sqrt{\langle Ax, (A^{-1})^* x\rangle} \leqslant 2\sqrt{\|Ax\|}\sqrt{\|(A^{-1})^* x\|} \\ &= 2\sqrt{\|Ax\|}\sqrt{\|A^{-1}x\|} \leqslant \|Ax\| + \|A^{-1}x\|, \quad x \in \mathbb{R}^d,\end{aligned}$$

which implies that C_A is cohyponormal. □

We are now in a position to prove our main result which characterizes cosubnormal composition operators on $L^2(\mu)$ induced by invertible linear transformations of $\mathbb{R}^d$.

Theorem 4.4. *Let μ be as in* (1.1) *and let A be an invertible linear transformation of $\mathbb{R}^d$ such that $C_A \in \boldsymbol{B}(L^2(\mu))$. Then C_A is cosubnormal if and only if there exist a unitary operator U on $(\mathbb{R}^d, \|\cdot\|)$ and $\lambda \in (0,1]$ such that $A = \lambda U$.*

Proof. We split the proof into four steps. First, note that by Proposition 4.1 A is a contraction.

Step 1. C_A is cosubnormal if and only if $\{\|A^n x\|\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $x \in \mathbb{R}^d$.

Indeed, by [25, (UE), p. 309], the operator $|\det A| C_A^*$ is unitarily equivalent to the composition operator $C_{A^{-1}}$ acting on $L^2(\mu_\theta)$, where $\theta(x) = \exp(\|x\|)$ for $x \in \mathbb{R}^d$. Hence, the proof of Step 1 reduces to considering the subnormality of $C_{A^{-1}}$ on $L^2(\mu_\theta)$. Applying Proposition 2.2 to (θ, A^{-1}) in place of (ϱ, A), we see that C_A is cosubnormal if and only if $\{\exp(\|A^n x\|)\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $x \in \mathbb{R}^d$. This is equivalent to requiring that $\{\exp(t\|A^n x\|)\}_{n=0}^\infty$ is a Stieltjes moment sequence for every real $t > 0$ and for all $x \in \mathbb{R}^d$. By (3.4) and Lemma 3.1, the latter is equivalent to requiring that the functions $\mathbb{Z}_+ \times \mathbb{Z}_+ \ni (m,n) \mapsto \|A^{m+n}x\|$ and $\mathbb{Z}_+ \times \mathbb{Z}_+ \ni (m,n) \mapsto \|A^{m+n+1}x\|$ are conditionally positive definite for every $x \in \mathbb{R}^d$. Since the sequence $\{\|A^n x\|\}_{n=0}^\infty$ is monotonically decreasing for every $x \in \mathbb{R}^d$ (because $\|A\| \leqslant 1$), we can apply Lemma 3.2 and (3.4). This completes the proof of Step 1.

Step 2. If $\{\|A^n x\|\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $x \in \mathbb{R}^d$, then A is a normal operator on $(\mathbb{R}^d, \|\cdot\|)$.

Indeed, applying (3.2) to the positive definite function $\mathbb{Z}_+ \times \mathbb{Z}_+ \ni (m,n) \mapsto \|A^{m+n}x\|$ we get $\|Ax\|^2 \leqslant \|A^2 x\|\|x\|$ for all $x \in \mathbb{R}^d$, which means that A is a paranormal operator on a finite dimensional Hilbert space $(\mathbb{R}^d, \|\cdot\|)$. Hence, A is a normal operator on $(\mathbb{R}^d, \|\cdot\|)$ (see e.g., [13, Theorem 2.2]).

Step 3. If C_A is cosubnormal, then there exist $\lambda \in (0,1]$ and a unitary operator U on $(\mathbb{R}^d, \|\cdot\|)$ such that $A = \lambda U$.

Indeed, by Steps 1 and 2, A is a normal operator on $(\mathbb{R}^d, \|\cdot\|)$. Without loss of generality we can assume that $d > 1$. Let $A = U|A|$ be the polar decomposition of A. Since A is normal and invertible, we deduce that U is a unitary operator on $(\mathbb{R}^d, \|\cdot\|)$ and $U|A| = |A|U$. This implies that

$$\|A^n x\| = \|U^n |A|^n x\| = \||A|^n x\|, \quad n \in \mathbb{Z}_+,\ x \in \mathbb{R}^d. \tag{4.4}$$

Since $|A|$ is positive, selfadjoint and invertible, there exist an orthonormal basis $\{e_n\}_{n=1}^d$ of $(\mathbb{R}^d, \|\cdot\|)$ and a sequence $\{\lambda_n\}_{n=1}^d \subseteq (0,\infty)$ such that $|A|e_j = \lambda_j e_j$ for $j = 1, \dots, d$. Hence, by (4.4), we have

$$\left\| A^n \left(\sum_{j=1}^d x_j e_j \right) \right\| = \left\| \sum_{j=1}^d x_j \lambda_j^n e_j \right\| = \sqrt{\sum_{j=1}^d x_j^2 \lambda_j^{2n}}, \quad n \in \mathbb{Z}_+,$$

for all $x_1, \dots, x_d \in \mathbb{R}^d$. Substituting $x_i = 1$ for $i \in \{1, j\}$ and $x_i = 0$ otherwise, we deduce from Step 1 and Lemma 3.3 that $\lambda := \lambda_1 = \lambda_j$ for all $j = 2, \dots, d$. This implies that $|A| = \lambda I$ and consequently that $A = \lambda U$, where I is the identity mapping on $\mathbb{R}^d$. Since A is a contraction, we get $\lambda \in (0,1]$.

Step 4. If $A = \lambda U$, where U is a unitary operator on $(\mathbb{R}^d, \|\cdot\|)$ and $\lambda \in (0,1]$, then C_A is cosubnormal.

Indeed, since $\{\|A^n x\|\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $x \in \mathbb{R}^d$, we can apply Step 1. This completes the proof. □

It follows from Proposition 4.3 and Theorem 4.4 that for $d \geqslant 2$ the class of cohyponormal composition operators on $L^2(\mu)$ induced by invertible linear transformations of $\mathbb{R}^d$ is essentially larger than that of cosubnormal ones (because there exists a normal transformation A on $(\mathbb{R}^d, \|\cdot\|)$ whose spectrum is a two-point set). We now show that there are cohyponormal composition operators on $L^2(\mu)$ induced by invertible linear transformations of $\mathbb{R}^d$ which are not normal on $(\mathbb{R}^d, \|\cdot\|)$. The following example is an adaption of [25, Example 2.6].

Example 4.5. Suppose that $d \geqslant 2$. Let $a_1, \dots, a_d$ be a sequence of nonzero real numbers, and let A be the invertible linear transformation of $\mathbb{R}^d$ determined by the requirement that $Ae_j = a_j e_{j+1}$ for $j = 1, \dots, d$, where $e_1 = (1, 0, \dots, 0), \dots, e_d = (0, \dots, 0, 1)$ and $e_{d+1} := e_1$. Set $\|x\|^2 = \sum_{j=1}^d x_j^2$ for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. As usual, μ is defined by (1.1). Owing to Proposition 4.1, C_A is a bounded operator on $L^2(\mu)$ if and only if $|a_j| \leqslant 1$ for all $j = 1, \dots, d$.

Assume that C_A is bounded. In view of Proposition 4.3, C_A is cohyponormal if and only if

$$2\sqrt{\sum_{j=1}^{d} x_j^2} \leqslant \sqrt{\sum_{j=1}^{d} (a_j x_j)^2} + \sqrt{\sum_{j=1}^{d}\left(\frac{x_j}{a_{j-1}}\right)^2}, \quad x=(x_1,\ldots,x_d)\in\mathbb{R}^d, \tag{4.5}$$

where $a_0 := a_d$. We now show that

$$C_A \text{ is cohyponormal if and only if } 2 \leqslant |a_j| + \tfrac{1}{|a_{j-1}|} \text{ for all } j\in\{1,\ldots,d\}. \tag{4.6}$$

Indeed, the "only if" part can be deduced from (4.5) by substituting $x = e_j$. To prove the "if" part, note that by the Minkowski inequality

$$2\sqrt{\sum_{j=1}^{d} x_j^2} \leqslant \sqrt{\sum_{j=1}^{d}\left(|a_j x_j| + \left|\frac{x_j}{a_{j-1}}\right|\right)^2} \leqslant \sqrt{\sum_{j=1}^{d} (a_j x_j)^2} + \sqrt{\sum_{j=1}^{d}\left(\frac{x_j}{a_{j-1}}\right)^2},$$

for all $x\in\mathbb{R}^d$, which means that (4.5) is valid. Hence, C_A is cohyponormal.

It is easily seen that A is normal on $(\mathbb{R}^d, \|\cdot\|)$ if and only if $|a_1| = \cdots = |a_d|$. If $a_1,\ldots,a_d$ are such that $0 < |a_j| \leqslant \frac{1}{2}$ for all $j = 1,\ldots,d$, and $|a_k| \neq |a_l|$ for some distinct $k,l\in\{1,\ldots,d\}$, then A is not normal in $(\mathbb{R}^d, \|\cdot\|)$. Hence, by Theorem 4.4, C_A is not cosubnormal. However, $2 \leqslant |a_j| + \frac{1}{|a_{j-1}|}$ for all $j\in\{1,\ldots,d\}$, which in view of (4.6) implies that C_A is cohyponormal.

We now restrict the discussion to the case of $d = 2$. We assume that C_A is bounded on $L^2(\mu)$ (equivalently: $0 < |a_1|, |a_2| \leqslant 1$). By (4.6), C_A is cohyponormal if and only if $2 \leqslant |a_1| + \frac{1}{|a_2|}$ and $2 \leqslant |a_2| + \frac{1}{|a_1|}$. This is equivalent to

$$L(a_1) := \frac{2|a_1|-1}{|a_1|} \leqslant |a_2| \leqslant \frac{1}{2-|a_1|} =: R(a_1).$$

It is clear that $L(a_1) \leqslant R(a_1)$. Summarizing, we see that C_A is cohyponormal if and only if $|a_1| \in (0,1]$ and $|a_2| \in [L(a_1), R(a_1)] \cap (0,1]$, or equivalently if

$$|a_2| \in \begin{cases} (0, R(a_1)] & \text{if } 0 < |a_1| \leqslant \frac{1}{2}, \\ [L(a_1), R(a_1)] & \text{if } \frac{1}{2} < |a_1| \leqslant 1. \end{cases}$$

In particular, if $1 \geqslant |a_1| > \frac{2}{3}$ and $|a_2| \in [L(a_1), R(a_1)]$, then $|a_2| \geqslant L(a_1) > \frac{1}{2}$, which means that it may happen that C_A is cohyponormal though $|a_1|, |a_2| > \frac{1}{2}$.

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