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On a refinement of Heisenberg uncertainty relation by means of quantum Fisher information

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ABSTRACT

We show that an uncertainty relation proved by Luo and Yanagi and related to Wigner–Yanase–Dyson information cannot hold for an arbitrary quantum Fisher information. We show that, by changing a costant, a similar trace inequality holds in full generality.

1. Introduction

Starting from the Wigner–Yanase information $I_\rho(A) := \frac{1}{2}\mathrm{Tr}((i[\rho, A])^2)$ and the variance $V_\rho(A) := \mathrm{Tr}(\rho A^2) - (\mathrm{Tr}(\rho A))^2$, Luo introduced in [7] the quantity

$$U_\rho(A) := \sqrt{V_\rho(A)^2 - \left(V_\rho(A) - I_\rho(A)\right)^2},$$

and presented there some arguments in favour of considering $U_\rho(A)$ as a measure of quantum uncertainty. In this respect, Luo was able to prove an uncertainty principle for U in the form

$$U_\rho(A) \cdot U_\rho(B) \geqslant \frac{1}{4}\left|\mathrm{Tr}\left(\rho[A, B]\right)\right|^2. \tag{1.1}$$

The above inequality is a refinement of the Heisenberg uncertainty principle

$$V_\rho(A) \cdot V_\rho(B) \geqslant \frac{1}{4}\left|\mathrm{Tr}\left(\rho[A, B]\right)\right|^2$$

(see [8] at p. 13). There have been some attempts to generalize Luo's result [1,6] but without success, as explained in [8, Remarks 3.2 and 3.3].

Recently, a successful approach has been proposed by Yanagi. Consider the Wigner–Yanase–Dyson (WYD) information defined by $I_\rho^\alpha(A) := \frac{1}{2}\mathrm{Tr}(i[\rho^\alpha, A] \cdot i[\rho^{1-\alpha}, A])$ where $\alpha \in [0, 1]$. Introducing

$$U_\rho^\alpha(A) := \sqrt{V_\rho(A)^2 - \left(V_\rho(A) - I_\rho^\alpha(A)\right)^2},$$

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Yanagi was able to prove a generalization of inequality (1.1) in the following form

$$U_\rho^\alpha(A)\cdot U_\rho^\alpha(B)\geqslant\alpha(1-\alpha)\bigl|\mathrm{Tr}\bigl(\rho[A,B]\bigr)\bigr|^2,\quad \alpha\in[0,1]. \tag{1.2}$$

The constant $\alpha(1-\alpha)$ immediately suggests a further significant generalization. Indeed the WYD information is connected to special choices of quantum Fisher information [5], as the family of all quantum Fisher informations is parametrized by a certain class of operator monotone functions $\mathcal{F}_{op}$ (see below), and to the WYD informations correspond the functions

$$f_\alpha(x):=\alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha-1)(x^{1-\alpha}-1)},\quad \alpha\in(0,1/2].$$

To each function f of the class $\mathcal{F}_{op}$ one may associate the metric adjusted skew information $I_\rho^f(A)$, a generalization of WYD information, and therefore one can define also a generalized quantum uncertainty U^f by the formula

$$U_\rho^f(A):=\sqrt{V_\rho(A)^2-\bigl(V_\rho(A)-I_\rho^f(A)\bigr)^2}.$$

Since $f_\alpha(0)=\alpha(1-\alpha)$, it is natural to conjecture that inequality (1.2) is a particular case of a general inequality

$$U_\rho^f(A)\cdot U_\rho^f(B)\geqslant f(0)\bigl|\mathrm{Tr}\bigl(\rho[A,B]\bigr)\bigr|^2,\quad f\in\mathcal{F}_{op}^r. \tag{1.3}$$

Actually, this is not the case, and we show that inequality (1.3) does not hold in general. To have a general uncertainty relation for U^f one has to use the smaller constant $f(0)^2$. Indeed the main result of this paper is the following inequality

$$U_\rho^f(A)\cdot U_\rho^f(B)\geqslant f(0)^2\bigl|\mathrm{Tr}\bigl(\rho[A,B]\bigr)\bigr|^2,\quad f\in\mathcal{F}_{op}^r. \tag{1.4}$$

In Sections 2 and 3 we recall some notions on operator monotone functions, matrix means and quantum Fisher information. In Section 4 we describe a counterexample to inequality (1.3). In Section 5 we prove inequality (1.4).

2. Operator monotone functions

Let $M_n:=M_n(\mathbb{C})$ (resp. $M_{n,sa}:=M_{n,sa}(\mathbb{C})$) be the set of all $n\times n$ complex matrices (resp. all $n\times n$ self-adjoint matrices), endowed with the Hilbert–Schmidt scalar product $\langle A,B\rangle=\mathrm{Tr}(A^*B)$. Let $\mathcal{D}_n$ be the set of strictly positive elements of M_n and $\mathcal{D}_n^1\subset\mathcal{D}_n$ be the set of strictly positive density matrices, namely $\mathcal{D}_n^1=\{\rho\in M_n\mid \mathrm{Tr}\,\rho=1,\ \rho>0\}$. If it is not otherwise specified, from now on we shall treat the case of faithful states, namely $\rho>0$.

A function $f:(0,+\infty)\to\mathbb{R}$ is called *operator monotone* (*increasing*) if, for any $n\in\mathbb{N}$, and $A,B\in M_n$ such that $0\leqslant A\leqslant B$, the inequalities $0\leqslant f(A)\leqslant f(B)$ hold. An operator monotone function is called *symmetric* if $f(x)=xf(x^{-1})$ and *normalized* if $f(1)=1$.

Definition 2.1. $\mathcal{F}_{op}$ is the class of functions $f:(0,+\infty)\to(0,+\infty)$ such that

(i) $f(1)=1$,
(ii) $tf(t^{-1})=f(t)$,
(iii) f is operator monotone.

Example 2.2. Examples of elements of $\mathcal{F}_{op}$ are given by the following list

$$f_{RLD}(x):=\frac{2x}{x+1},\qquad f_{WY}(x):=\left(\frac{1+\sqrt{x}}{2}\right)^2,$$
$$f_{SLD}(x):=\frac{1+x}{2},\qquad f_\alpha(x):=\alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha-1)(x^{1-\alpha}-1)},\quad \alpha\in\left(0,\frac{1}{2}\right].$$

Remark 2.3. Any $f\in\mathcal{F}_{op}$ satisfies

$$\frac{2x}{1+x}\leqslant f(x)\leqslant\frac{1+x}{2},\quad \forall x>0.$$

For $f\in\mathcal{F}_{op}$ define $f(0):=\lim_{x\to0}f(x)$. We introduce the sets of regular and non-regular functions

$$\mathcal{F}_{op}^r:=\bigl\{f\in\mathcal{F}_{op}\bigm| f(0)\neq0\bigr\},\qquad \mathcal{F}_{op}^n:=\bigl\{f\in\mathcal{F}_{op}\bigm| f(0)=0\bigr\}$$

and notice that trivially $\mathcal{F}_{op}=\mathcal{F}_{op}^r\,\dot\cup\,\mathcal{F}_{op}^n$.

Definition 2.4. For $f \in \mathcal{F}_{op}^r$ we set

$$\tilde{f}(x) = \frac{1}{2}\left[(x+1) - (x-1)^2\frac{f(0)}{f(x)}\right], \quad x > 0.$$

Theorem 2.5. *(See [2,5].) The correspondence $f \to \tilde{f}$ is a bijection between $\mathcal{F}_{op}^r$ and $\mathcal{F}_{op}^n$.*

Remark 2.6. Note that for any f one has $f(0) \leqslant \frac{1}{2}$ and therefore

$$\frac{x+1}{4} - f(0)f(x) \geqslant 0.$$

The following new inequality is fundamental for our purposes.

Proposition 2.7. *For any $f \in \mathcal{F}_{op}^r$ and $x > 0$ one has*

$$\tilde{f}(x)^2 \leqslant \frac{1}{4}(x+1)^2 - f(0)^2(x-1)^2. \tag{2.1}$$

Proof. We have

$$\begin{aligned}
&\frac{1}{4}(x+1)^2 - f(0)^2(x-1)^2 - \tilde{f}(x)^2 \\
&\quad = \frac{(x+1)^2}{4} - f(0)^2(x-1)^2 - \frac{(x+1)^2}{4} - \frac{1}{4}\left(\frac{f(0)}{f(x)}\right)^2(x-1)^4 + \frac{1}{2}\frac{f(0)}{f(x)}(x+1)(x-1)^2 \\
&\quad = \frac{f(0)}{f(x)}(x-1)^2\left[-\frac{1}{4}\frac{f(0)}{f(x)}(x-1)^2 + \frac{1}{2}(x+1) - f(0)f(x)\right] \\
&\quad = \frac{f(0)}{f(x)}(x-1)^2\left[\frac{1}{4}(x+1) - \frac{1}{4}\frac{f(0)}{f(x)}(x-1)^2 + \frac{1}{4}(x+1) - f(0)f(x)\right] \\
&\quad = \frac{f(0)}{f(x)}(x-1)^2\left[\frac{1}{2}\tilde{f}(x) + \left(\frac{1}{4}(x+1) - f(0)f(x)\right)\right] \geqslant 0
\end{aligned}$$

because of Remark 2.6 and the fact that $f, \tilde{f} \geqslant 0$. The thesis follows from the last inequality. □

Let us recall that for a function $f \in \mathcal{F}_{op}$ the associated (numerical) mean is defined as

$$m_f(x, y) := xf\left(\frac{y}{x}\right), \quad x, y > 0.$$

From Proposition 2.7 one may deduce the following corollary:

Corollary 2.8.

$$f(0)^2(\lambda - \mu)^2 \leqslant \frac{1}{4}(\lambda + \mu)^2 - m_{\tilde{f}}(\lambda, \mu)^2.$$

Proof. Set $t = \frac{\mu}{\lambda}$ in the inequality of the above proposition and multiply it by λ^2. We get

$$\lambda^2\tilde{f}\left(\frac{\mu}{\lambda}\right)^2 \leqslant \frac{\lambda^2}{4}\left(1 + \frac{\mu}{\lambda}\right)^2 - f(0)^2\lambda^2\left(1 - \frac{\mu}{\lambda}\right)^2$$

and this proves the thesis. □

3. Means, Fisher information and metric adjusted skew information

In Kubo–Ando theory of matrix means one associates a mean to each operator monotone function $f \in \mathcal{F}_{op}$ by the formula

$$m_f(A, B) := A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}},$$

where $A, B \in \mathcal{D}_n$. Using the notion of matrix means one may define the class of monotone metrics (also called quantum Fisher informations) by the following formula

$$\langle A, B\rangle_{\rho,f} := \mathrm{Tr}\big(A \cdot m_f(L_\rho, R_\rho)^{-1}(B)\big),$$

where $L(A) := \rho A$, $R_\rho(A) := A\rho$. In this case one has to think of A, B as tangent vectors to the manifold $\mathcal{D}_n^1$ at the point ρ (see [5] for a more complete set of references).

Definition 3.1. For $A \in M_{n,sa}$, define

$$I_\rho^f(A) := \frac{f(0)}{2}\big\langle i[\rho, A], i[\rho, A]\big\rangle_{\rho,f},$$
$$C_\rho^f(A) := \mathrm{Tr}\big(m_f(L_\rho, R_\rho)(A) \cdot A\big),$$
$$U_\rho^f(A) := \sqrt{V_\rho(A)^2 - \big(V_\rho(A) - I_\rho^f(A)\big)^2}.$$

The quantity $I_\rho^f(A)$ is known as *metric adjusted skew information*. Set $A_0 := A - \mathrm{Tr}(\rho A)I$.

Proposition 3.2.

(1) $I_\rho^f(A) = \mathrm{Tr}(\rho A_0^2) - \mathrm{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)(A_0) \cdot A_0) = V_\rho(A) - C_\rho^{\tilde{f}}(A_0)$.
(2) $J_\rho^f(A) := \mathrm{Tr}(\rho A_0^2) + \mathrm{Tr}(m_{\tilde{f}}(L_\rho, R_\rho)(A_0) \cdot A_0) = V_\rho(A) + C_\rho^{\tilde{f}}(A_0)$.
(3) $0 \leqslant I_\rho^f(A) \leqslant U_\rho^f(A) \leqslant V_\rho(A)$.
(4) $U_\rho^f(A) = \sqrt{I_\rho^f(A) \cdot J_\rho^f(A)}$.

Proof. (1) See Proposition 6.1, Corollary 7.1 and Proposition 7.2 in [2].
(2) Since $J_\rho^f(A) = 2V_\rho(A) - I_\rho^f(A)$, it is clear.
(3), (4) follow from direct calculations. □

4. A conjecture and a counterexample

Using the notation of the preceding section we have that the metric adjusted skew information associated to the functions

$$f_\alpha(x) := \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \quad \alpha \in (0, 1/2],$$

is exactly the WYD information $I_\rho^\alpha(A) := \frac{1}{2}\mathrm{Tr}(i[\rho^\alpha, A] \cdot i[\rho^{1-\alpha}, A])$. Therefore inequality (1.2) can be written as

$$U_\rho^{f_\alpha}(A) \cdot U_\rho^{f_\alpha}(B) \geqslant f_\alpha(0)\big|\mathrm{Tr}\big(\rho[A, B]\big)\big|^2, \quad \alpha \in (0, 1/2].$$

It is therefore natural to investigate the status of the following general inequality.

Conjecture 4.1. *For $f \in \mathcal{F}_{op}^r$, $A, B \in M_{n,sa}$, it holds*

$$U_\rho^f(A) \cdot U_\rho^f(B) \geqslant f(0)\big|\mathrm{Tr}\big(\rho[A, B]\big)\big|^2.$$

Actually the above conjecture is false. To find a counterexample it is enough to consider the 2×2 case. Recall that the Pauli matrices are the following

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Any 2×2 density matrix in the Stokes parameterization is written as

$$\rho = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} = \frac{1}{2}(I + x\sigma_1 + y\sigma_2 + z\sigma_3),$$

where $(x, y, z) \in \mathbb{R}^3$, and $x^2 + y^2 + z^2 \leqslant 1$. Let $r := \sqrt{x^2 + y^2 + z^2} \in [0, 1]$. The eigenvalues of ρ are $\lambda_1 = \frac{1-r}{2}$ and $\lambda_2 = \frac{1+r}{2}$.

Proposition 5.1 in [4] says that

$$I_\rho^f(A) = \left[1 - m_{\tilde{f}}(1-r, 1+r)\right] \cdot |a_{12}|^2.$$

A similar straightforward calculation gives

$$J_\rho^f(A) = 2\left[\lambda_1|a_{11}|^2 + \lambda_2|a_{22}|^2\right] + \left[1 + m_{\tilde{f}}(1-r, 1+r)\right]|a_{12}|^2.$$

Consider the case

$$\rho = \begin{pmatrix} \frac{1-r}{2} & 0 \\ 0 & \frac{1+r}{2} \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad f(x) = \frac{1+x}{2}.$$

Then

$$f(0) = \frac{1}{2}, \qquad \tilde{f}(x) = \frac{2x}{1+x}, \qquad m_{\tilde{f}}(x,y) = \frac{2xy}{x+y},$$
$$I_\rho^f(A) = r^2, \qquad I_\rho^f(B) = r^2,$$
$$J_\rho^f(A) = 2 - r^2, \qquad J_\rho^f(B) = 2 - r^2,$$
$$\operatorname{Tr}\bigl(\rho[A,B]\bigr) = -2ir.$$

Therefore one has for any $r \in (0,1)$,

$$f(0)^2 \cdot \left|\operatorname{Tr}\bigl(\rho[A,B]\bigr)\right|^4 = 4r^4 > r^4\left(2-r^2\right)^2 = I_\rho^f(A) \cdot J_\rho^f(A) \cdot I_\rho^f(B) \cdot J_\rho^f(B) = \left(U_\rho^f(A) \cdot U_\rho^f(B)\right)^2,$$

and this disproves the conjecture.

5. The main result

Now we show that a weakened form of inequality 1.3 holds in general (namely we prove inequality (1.4)).

Theorem 5.1. *For $f \in \mathcal{F}_{op}^r$, $A, B \in M_{n,sa}$, it holds*

$$U_\rho^f(A) \cdot U_\rho^f(B) \geqslant f(0)^2\left|\operatorname{Tr}\bigl(\rho[A,B]\bigr)\right|^2.$$

Proof. Let $\{e_1, \dots, e_n\}$ be a basis of eigenvectors of ρ, corresponding to the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$. Introduce $a_{jk} := \langle e_j, A_0 e_k\rangle$, $b_{jk} := \langle e_j, B_0 e_k\rangle$, so that

$$I_\rho^f(A) = \frac{1}{2}\sum_{j,k}(\lambda_j + \lambda_k)a_{jk}a_{kj} - \sum_{j,k} m_{\tilde{f}}(\lambda_j, \lambda_k)a_{jk}a_{kj},$$
$$J_\rho^f(A) = \frac{1}{2}\sum_{j,k}(\lambda_j + \lambda_k)a_{jk}a_{kj} + \sum_{j,k} m_{\tilde{f}}(\lambda_j, \lambda_k)a_{jk}a_{kj},$$
$$U_\rho^f(A)^2 = \frac{1}{4}\left(\sum_{j,k}(\lambda_j + \lambda_k)|a_{jk}|^2\right)^2 - \left(\sum_{j,k} m_{\tilde{f}}(\lambda_j, \lambda_k)|a_{j,k}|^2\right)^2.$$

Moreover one can prove (see, for example, Theorem 9.2 in [3])

$$\operatorname{Tr}\bigl(\rho[A_0, B_0]\bigr) = \sum_{j,k}(\lambda_j - \lambda_k)a_{jk}b_{kj},$$

from which it follows

$$\begin{aligned}
f(0)^2\left|\operatorname{Tr}\bigl(\rho[A,B]\bigr)\right|^2 &\leqslant \left(\sum_{ij} f(0)|\lambda_i - \lambda_j| \cdot |a_{ij}| \cdot |b_{ij}|\right)^2 \\
&\leqslant \left(\sum_{ij}\left[\frac{1}{4}(\lambda_i + \lambda_j)^2 - m_{\tilde{f}}(\lambda_i, \lambda_j)^2\right]^{\frac{1}{2}} \cdot |a_{ij}| \cdot |b_{ij}|\right)^2 \\
&\leqslant \left(\sum_{ij}\left[\frac{1}{2}(\lambda_i + \lambda_j) - m_{\tilde{f}}(\lambda_i, \lambda_j)\right] \cdot |a_{ij}|^2\right) \cdot \left(\sum_{ij}\left[\frac{1}{2}(\lambda_i + \lambda_j) + m_{\tilde{f}}(\lambda_i, \lambda_j)\right] \cdot |b_{ij}|^2\right) \\
&= I_\rho^f(A) \cdot J_\rho^f(B),
\end{aligned}$$

where the second inequality is a consequence of Corollary 2.8 while the third one derives from Cauchy–Schwarz inequality. Therefore, successively we get

$$I_\rho^f(A) \cdot J_\rho^f(A) \cdot I_\rho^f(B) \cdot J_\rho^f(B) \geqslant f(0)^4 \left|\mathrm{Tr}\big(\rho[A,B]\big)\right|^4,$$
$$U_\rho^f(A)^2 \cdot U_\rho^f(B)^2 \geqslant f(0)^4 \left|\mathrm{Tr}\big(\rho[A,B]\big)\right|^4,$$
$$U_\rho^f(A) \cdot U_\rho^f(B) \geqslant f(0)^2 \left|\mathrm{Tr}\big(\rho[A,B]\big)\right|^2. \qquad \square$$

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