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The Szegő curve and Laguerre polynomials with large negative parameters

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ABSTRACT

We study the asymptotic zero distribution of the rescaled Laguerre polynomials, $L_n^{(\alpha_n)}(nz)$, with the parameter α_n varying in such a way that $\lim_{n\to\infty} \alpha_n/n = -1$. The connection with the so-called Szegő curve is shown.

1. Introduction

The definition and many properties of the Laguerre polynomials $L_n^{(\alpha)}$ can be found in Chapter V of Szegő's classic memoir [20]. Given explicitly by

$$L_n^{(\alpha)}(z) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-z)^k}{k!}, \tag{1}$$

or, equivalently, by the well-known Rodrigues formula

$$L_n^{(\alpha)}(z) = \frac{(-1)^n}{n!} z^{-\alpha} e^z \left(\frac{d}{dz}\right)^n \left[z^{n+\alpha} e^{-z}\right], \tag{2}$$

they are defined for arbitrary values of the parameter $\alpha \in \mathbb{C}$. In particular, (1) shows that each $L_n^{(\alpha)}$ depends analytically on α and no degree reduction occurs: $\deg L_n^{(\alpha)} = n$ for all $\alpha \in \mathbb{C}$.

For $\alpha > -1$ it is well known that $L_n^{(\alpha)}(x)$ are orthogonal on $[0, +\infty)$ with respect to the weight function $x^\alpha e^{-x}$; in particular, all their zeros are simple and belong to $[0, +\infty)$. In the general case, $\alpha \in \mathbb{C}$, $L_n^{(\alpha)}(z)$ may have complex zeros; the only multiple zero can appear at $z = 0$, which occurs if and only if $\alpha \in \{-1, -2, \ldots, -n\}$. In this case we have

$$L_n^{(-k)}(z) = (-z)^k \frac{(n-k)!}{n!} L_{n-k}^{(k)}(z), \tag{3}$$

which shows that $z = 0$ is a zero of multiplicity k for $L_n^{(-k)}(z)$.

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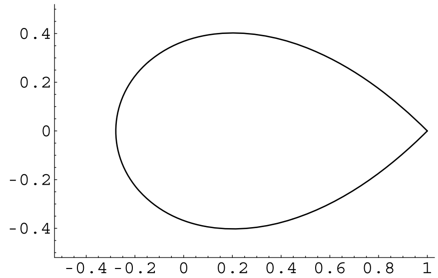

Fig. 1. The Szegő curve.

In a series of papers [7,8,12], asymptotics for rescaled Laguerre polynomials $L_n^{(\alpha_n)}(nz)$ were analyzed, under the assumption that $\lim_{n\to\infty} \alpha_n/n = A \in \mathbb{R}$. In [12] the authors obtained the weak zero asymptotics for the case where $A < -1$, by means of classical (logarithmic) potential theory. To this end, a key role was played by a full set of non-hermitian orthogonality relations satisfied by Laguerre polynomials in a class of open contours in $\mathbb{C}$. Unfortunately, this analysis could not be extended to the other cases, since essential for this approach is the connectedness of the complement to the support of the asymptotic distribution of zeros (see e.g. [5] and [18]). However, the authors formulated in [12] a conjecture for the case $-1 < A < 0$, which was confirmed in some cases and refused in others in [8], by means of the Riemann–Hilbert approach. This approach has been previously used by the same authors in [7] to obtain strong asymptotics in the case $A < -1$. A similar study for Jacobi polynomials with varying nonstandard parameters has been carried out in [9,11,13].

Jacobi or Laguerre polynomials with real parameters (and in general, depending on the degree n) appear naturally as polynomial solutions of hypergeometric differential equations, or in the expressions of the wave functions of many classical systems in quantum mechanics (see e.g. [2]).

In [12], the authors also formulated a conjecture for the case $A = -1$, but up to now this problem has remained open. Observe that, by (3), when $k = n$ we have

$$L_n^{(-n)}(z) = (-1)^n \frac{1}{n!} z^n.$$

There is another particular situation corresponding to the case $A = -1$ which is very well known in the literature: when $\alpha_n = -n - 1$, we have

$$L_n^{(-n-1)}(z) = (-1)^n \sum_{k=0}^{n} \frac{z^k}{k!},$$

and thus, in this case the Laguerre polynomials agree (up to a possible sign) with the partial sums of the exponential series. In a seminal paper, G. Szegő [19] showed that the zeros of the rescaled partial sums of the exponential series, $\sum_{k=0}^{n} \frac{(nz)^k}{k!} = (-1)^n L_n^{(-n-1)}(nz)$, approach the so-called Szegő curve:

$$\Gamma = \{ z \in \mathbb{C},\ |ze^{1-z}| = 1,\ |z| \leqslant 1 \}, \tag{4}$$

which is a closed curve around the origin passing through $z = 1$ and crossing once the negative real semiaxis $(-\infty, 0)$ (see Fig. 1). See also [15] for a detailed study of the Szegő curve and some related problems in approximation of functions. Recently, T. Kriecherbauer et al. [6] obtained uniform asymptotic expansions for the partial sums of the exponential series by means of the Riemann–Hilbert analysis. Also, in [3] the authors studied the asymptotics of orthogonal polynomials with respect to modified Laguerre weights of the type

$$z^{-n+\nu} e^{-Nz} (z-1)^{2b},$$

where $n, N \to \infty$ with $N/n \to 1$ and ν is a fixed number in $\mathbb{R} \setminus \mathbb{N}$.

In this paper, the weak zero asymptotics of rescaled Laguerre polynomials $L_n^{(\alpha_n)}(nz)$, with $\lim_{n\to\infty} \alpha_n/n = -1$, will be analyzed. We will prove that such rescaled Laguerre polynomials are asymptotically extremal on certain well-defined curves in the complex plane.

The outline of the paper is as follows. In Section 2, the main result about the weak zero asymptotics of the rescaled Laguerre polynomials is announced, and in Section 3, some basic facts on the theory of logarithmic potentials and asymptotically extremal polynomials are recalled. Finally, the proofs are given in Section 4.

2. Main result

Along with the Szegő curve (4), we introduce the family of level curves:

$$\Gamma_r = \{ z \in \mathbb{C},\ |ze^{1-z}| = e^{-r},\ |z| \leqslant 1 \}, \quad 0 \leqslant r < +\infty, \tag{5}$$

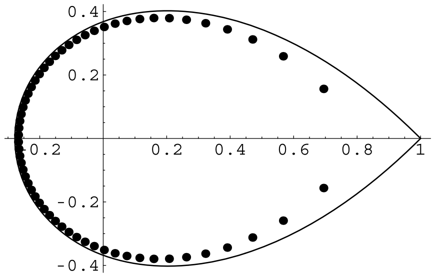

Fig. 2. The Szegő curve and the zeros of $L_{60}^{(-60.1)}(60z)$.

while for $r = \infty$ we take $\Gamma_\infty = \{0\}$. Observe that $\Gamma_0 = \Gamma$, the Szegő curve. We consider the usual counterclockwise orientation. All the level curves Γ_r $(0 \leqslant r < +\infty)$ are closed contours such that $\{0\} \subset \operatorname{Int}(\Gamma_r)$ and $\Gamma_{r'} \subset \operatorname{Int}(\Gamma_r)$, for $r' > r$. On the sequel, the interior of Γ_r will be denoted by G_r. Associated with this family of curves, consider for $0 \leqslant r < +\infty$ the family of measures:

$$d\mu_r(z) = \frac{1}{2\pi i}\frac{1-z}{z}\,dz, \quad z \in \Gamma_r, \tag{6}$$

and set $d\mu_\infty(z) = \delta_0$, the Dirac delta at $z = 0$.

Let us recall the definition of balayage (or sweeping out) of a measure (see e.g. [16]). Given an open set Ω with compact boundary $\partial\Omega$ and a positive measure σ with compact support in Ω, there exists a positive measure $\widehat{\sigma}$, supported in $\partial\Omega$, such that $\|\sigma\| = \|\widehat{\sigma}\|$ and

$$V^{\widehat{\sigma}}(z) - V^{\sigma}(z) = \text{const}, \quad \text{qu.e. } z \notin \Omega, \tag{7}$$

where

$$V^{\sigma}(z) = -\int \log|z-x|\,d\sigma(x), \tag{8}$$

const $= 0$ when Ω is a bounded set, and a property is said to be satisfied for "quasi-every" (qu.e.) z in a certain set, if it holds except for a possible subset of zero (logarithmic) capacity. Then, $\widehat{\sigma}$ is said to be the balayage of σ from Ω onto $\partial\Omega$.

Now, we have the following:

Lemma 2.1. *The a priori complex measure* (6) *is a unit positive measure in Γ_r, for $0 \leqslant r < +\infty$. Moreover, μ_r is the balayage of δ_0 from G_r onto Γ_r.*

Now, for each $n \in \mathbb{N}$, consider the "pathological" subset of negative integers $\mathbb{S}_n = \{-n, -(n-1), \dots, -2, -1\}$. Hereafter, suppose that $\alpha_n \notin \mathbb{S}_n$.

Finally, denote by $\operatorname{dist}(\alpha_n, \mathbb{S}_n) > 0$ the minimal distance between α_n and the set $\mathbb{S}_n$.

Theorem 2.1. *Let $\{L_n^{(\alpha_n)}(nz)\}_{n\in\mathbb{N}}$ be a sequence of rescaled Laguerre polynomials, such that $\lim_{n\to\infty}\frac{\alpha_n}{n} = -1$ and*

$$\lim_{n\to\infty}\bigl[\operatorname{dist}(\alpha_n, \mathbb{S}_n)\bigr]^{1/n} = e^{-r}, \tag{9}$$

for some $r \geqslant 0$. Then, the contracted zeros of Laguerre polynomials asymptotically follow the measure $d\mu_r$ on the curve Γ_r. For $r = +\infty$, the limit measure is $d\mu_\infty = \delta_0$.

Remark 2.1. The results above also hold for infinite subsequences $\{L_n^{(\alpha_n)}(nz)\}_{n\in\Lambda}$, $\Lambda \subset \mathbb{N}$.

Remark 2.2. Observe that the case $r = 0$ in Theorem 2.1 is generic, because it takes place when the parameters α_n do not approach, or, at least, do not approach exponentially fast, the set of integers $\mathbb{S}_n$ (see Fig. 2). On the other hand, when $r > 0$, the parameters approach the set of integers $\mathbb{S}_n$ exponentially fast, and the Szegő curve Γ is replaced by a level curve Γ_r which surrounds $z = 0$ and is strictly contained in the interior of Γ (see Fig. 3). Finally, when $r = \infty$, i.e., when the parameters approach the set $\mathbb{S}_n$ faster than exponentially, the limit measure reduces to a Dirac mass at $z = 0$.

Remark 2.3. The weak asymptotics in the case $A = -1$, characterized by the set of measures (6) and the corresponding set of closed curves (5), is the natural matching between the solutions in the cases $A < -1$ (see [7] and [12]) and $-1 < A < 0$

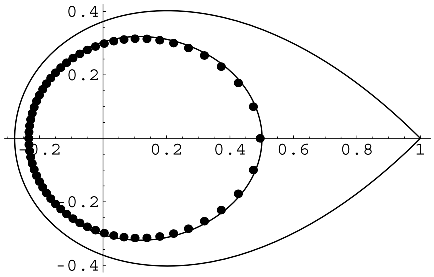

Fig. 3. Zeros of $L_{60}^{(-60+10^{-5})}(60z)$ and the curve Γ_r, for $r = \frac{1}{12}\ln 10$.

(see [8]). For these cases the following full set of non-hermitian orthogonality relations for the Laguerre polynomials with parameters $\alpha \in \mathbb{C}$ was used:

$$\int_{\Sigma} L_n^{(\alpha)}(z) z^k z^{\alpha} e^{-z}\, dz = 0, \quad k = 0, \ldots, n-1,$$

where Σ is any unbounded contour in $\mathbb{C} \setminus [0, \infty)$, connecting $+\infty + iy$ and $+\infty - iy$, for some $y > 0$, and the branch of z^{α} is taken with the cut along the positive real axis (see [7, Lemma 2.1]). In [12] this full set of orthogonality relations allowed the application of the seminal results by H. Stahl [18] and A.A. Gonchar and E.A. Rakhmanov [5] on the asymptotic behavior of complex orthogonal polynomials. Indeed, it was proved that zeros of the rescaled Laguerre polynomials accumulate on a closed contour C in $\mathbb{C} \setminus [0, \infty)$ which is "symmetric" (in the "Stahl's sense", see [17,18]) with respect to the external field $\varphi(z) = \frac{1}{2}(-A \log|z| + \operatorname{Re} z)$, and that they asymptotically follow the equilibrium distribution on C in presence of the external field φ. In the proof of the main result in this paper, it will be shown that the zeros of the rescaled Laguerre polynomials in the present case also asymptotically follow the equilibrium distribution of Γ_r in presence of the external field φ above (for $A = -1$), Γ_r being a symmetric contour with respect to this external field. That is, although the theorems by H. Stahl and A.A. Gonchar–E.A. Rakhmanov cannot be applied in this case since the complement to the support is disconnected, the conclusions still hold.

3. On asymptotically extremal polynomials

Throughout this section, some topics in potential theory which are needed for the proof of our main result will be recalled. For more details the reader can consult the monograph [16].

First, we introduce the notion of admissible weights.

Definition 3.1. Given a closed set $\Sigma \subset \mathbb{C}$, we say that a function $\omega : \Sigma \to [0, \infty)$ is an admissible weight on Σ if the following conditions are satisfied (see [16, Definition I.1.1]):

(a) ω is upper semi-continuous;
(b) the set $\{z \in \Sigma\colon\ \omega(z) > 0\}$ has positive (logarithmic) capacity;
(c) if Σ is unbounded, then $\lim_{|z|\to\infty,\, z\in\Sigma} |z|\omega(z) = 0$.

Given such an admissible weight ω in the closed set Σ, and setting $\varphi(z) = -\log \omega(z)$, we know (see e.g. [16, Ch. I]) that there exists a unique measure μ_{ω}, with compact support in Σ, for which the infimum of the weighted logarithmic energy

$$I_{\omega}(\mu) = -\int\int \log|z - x|\, d\mu(z)\, d\mu(x) + 2\int \varphi(x)\, d\mu(x)$$

is attained. Moreover, setting $F_{\omega} = I_{\omega}(\mu_{\omega}) - \int \varphi\, d\mu_{\omega}$, which is called the modified Robin constant, we have the following property, which uniquely characterizes the extremal measure μ_{ω}:

$$V^{\mu_{\omega}}(z) + \varphi(z) \begin{cases} = F_{\omega}, & \text{qu.e. } z \in \operatorname{supp} \mu_{\omega}, \\ \geqslant F_{\omega}, & \text{qu.e. } z \in \Sigma, \end{cases}$$

where for a measure σ, V^{σ} denotes its logarithmic potential defined by (8).

Now, let Σ be a closed set and ω be an admissible weight on Σ. Then, a sequence of monic polynomials $\{p_n\}_{n\in\mathbb{N}}$ is said to be asymptotically extremal with respect to the weight ω if (see [16]):

$$\lim_{n\to\infty} \left\|\omega^n p_n\right\|_{\Sigma}^{1/n} = \exp(-F_{\omega}), \tag{10}$$

where, as usual, $\|\cdot\|_K$ denotes the sup-norm in the set K. The study of weighted polynomials of the form $\omega(z)^n P_n(z)$ has applications to many problems in approximation theory (see e.g. the monographs [16] and [21]). It is well known that if for each $n \in \mathbb{N}$, T_n^{ω} is the n-th (weighted) Chebyshev polynomial with respect to the weight ω^n, that is, if it is the (unique) monic polynomial of degree n for which the infimum

$$t_n^{\omega} = \inf\{\|\omega^n P\|_{\Sigma},\ P(z) = z^n + \cdots\}$$

is attained, then the sequence $\{T_n^{\omega}\}$ satisfies the asymptotic behavior given in (10) (see [16, Ch. III]).

Under mild conditions on the weight ω, in [16, Ch. III] it is shown that the zeros of such sequences of polynomials asymptotically follow the equilibrium measure μ_{ω}, in the sense of the weak-* convergence. Indeed, we have the following result (see [16, Theorem III.4.1] or [14]):

Theorem 3.1. *Let ω be an admissible weight such that the support S_{ω} of the corresponding equilibrium measure μ_{ω} has zero Lebesgue planar measure. Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of monic polynomials of respective degrees $n = 1, 2, \ldots$ satisfying*

$$\lim_{n\to\infty} \|\omega^n p_n\|_{S_{\omega}}^{1/n} = \exp(-F_{\omega}), \tag{11}$$

where F_{ω} denotes the modified (by the external field $\varphi = -\ln\omega$) Robin constant. Then, the following statements are equivalent:

(a) *$\nu(p_n) \to \mu_{\omega}$ in the weak-* sense, where $\nu(p_n)$ denotes the unit zero counting measure associated with p_n, that is, $d\nu(p_n) = \frac{1}{n}\sum_{p_n(\zeta)=0}\delta_{\zeta}$.*

(b) *For each bounded component R of $\mathbb{C}\setminus S_{\omega}$ and each infinite sequence $N \subset \mathbb{N}$, there exist $z_0 \in R$ and $N_1 \subset N$ such that*

$$\lim_{n\to\infty,\, n\in N_1} |p_n(z_0)|^{1/n} = \exp(-V^{\mu_{\omega}}(z_0)). \tag{12}$$

Remark 3.1. In [4, Theorem 5], condition (11) is replaced by the weaker condition:

$$\limsup_{n\to\infty} \omega(z)|p_n(z)|^{1/n} \leqslant \exp(-F_{\omega}), \quad \text{qu.e. } z \in S_{\omega}. \tag{13}$$

Remark 3.2. It is clear that the balayage of a measure (see (7)) is a very particular case of equilibrium measure in an external field. Since Lemma 2.1 says that measure μ_r is the balayage of δ_0 from G_r onto its boundary Γ_r, it means that

$$V^{\mu_r}(z) = -\log|z|, \quad z \in \Gamma_r. \tag{14}$$

Taking into account the definition of Γ_r, (14) implies both

$$V^{\mu_r}(z) + \operatorname{Re} z = r + 1, \quad z \in \Gamma_r, \tag{15}$$

and

$$V^{\mu_r}(z) + \varphi(z) = \frac{r+1}{2}, \quad z \in \Gamma_r, \tag{16}$$

where

$$\varphi(z) = \frac{1}{2}(\log|z| + \operatorname{Re} z) \tag{17}$$

(see Remark 2.3 above).

For the proof of Theorem 2.1, taking into account Theorem 3.1, it will be proved that the rescaled Laguerre polynomials are asymptotically extremal with respect to the weight $\omega = e^{-\varphi}$ in the compact set given by the closed contour Γ_r (using (13)), along with the fact that they satisfy the local behavior (12).

4. Proofs

4.1. Proof of Lemma 2.1

By (5), the level curves Γ_r, for $0 \leqslant r < \infty$, have the representation

$$\Gamma_r = \left\{ z \in \mathbb{C} \;\middle|\; \operatorname{Re}\int_1^z \left(1 - \frac{1}{t}\right) dt = r \right\}. \tag{18}$$

Expression (18) shows that (6) is real-valued in Γ_r and does not change its sign. Moreover, by a straightforward application of the Cauchy theorem, we get

$$\mu_r(\Gamma_r) = \int_{\Gamma_r} d\mu_r(t) = 1.$$

Now, we will prove that μ_r is the balayage of δ_0 from $G_r = \operatorname{Int}(\Gamma_r)$ onto Γ_r.

To this end, consider the function $\phi(z) = ze^{1-z}$. It is easy to see that ϕ conformally maps G_r onto the disk $\mathbb{D}_r = \{w \in \mathbb{C}/|w| < r\}$, $0 \leqslant r < \infty$, in the w-plane (see [19] and [15]). Thus, from (6), we have

$$d\mu_r(z) = \frac{1}{2\pi i}\left(\frac{1}{z} - 1\right)dz = \frac{1}{2\pi i}\frac{\phi'(z)}{\phi(z)}\,dz = \frac{1}{2\pi i}\frac{dw}{w} = \frac{d\theta}{2\pi},$$

where $w = re^{i\theta} = \phi(z)$, and $z \in \Gamma_r$. Therefore, (6) is the preimage of the normalized arc-length measure on the circle $\mathbb{T}_r = \partial\mathbb{D}_r$ under the mapping $w = \phi(z)$, that is, the harmonic measure at $z = 0$ with respect to the domain G_r. But this fact implies that (6) is the balayage of δ_0 from G_r onto Γ_r (see [10, p. 222]).

4.2. Proof of Theorem 2.1

In Remark 3.2, it was shown that μ_r is also the equilibrium measure on Γ_r for the external field φ in (17).
Moreover, (16) shows that the corresponding modified Robin constant is given by

$$F_\omega = \frac{r+1}{2}. \tag{19}$$

On the other hand, the function $g(z) = V^{\mu_r}(z) + \operatorname{Re} z$ is harmonic in $\overline{G_r}$ and, by (15), $g(z) \equiv r+1$, $z \in \Gamma_r$. Then, $g(z) \equiv r+1$, $z \in \overline{G_r}$. In particular,

$$V^{\mu_r}(0) = r+1. \tag{20}$$

From (19), in order to prove (13) we need to show that

$$\limsup_{n\to\infty} \omega(z)\left|p_n(z)\right|^{1/n} \leqslant e^{-\frac{r+1}{2}}, \quad \text{qu.e. } z \in \Gamma_r,$$

for the monic polynomial $p_n(z) = \widehat{L}_n^{(\alpha_n)}(nz)$ and the weight $\omega(z) = e^{-\varphi(z)}$, which by (5) is equivalent to proving that

$$\limsup_{n\to\infty} e^{-\operatorname{Re} z}\left|p_n(z)\right|^{1/n} \leqslant e^{-(r+1)}, \quad \text{qu.e. } z \in \Gamma_r. \tag{21}$$

Now, since by (1) $L_n^{(\alpha_n)}(nz) = l_n^{\alpha_n} z^n + \cdots$, with

$$l_n^{\alpha_n} = (-1)^n \frac{n^n}{n!}, \tag{22}$$

we get that (21) is equivalent to

$$\limsup_{n\to\infty} e^{-\operatorname{Re} z}\left|L_n^{(\alpha_n)}(nz)\right|^{1/n} \leqslant e^{-r}, \quad \text{qu.e. } z \in \Gamma_r. \tag{23}$$

In addition, we must show that there exists a point $z_0 \in G_r$ for which (12) is attained. Thus, choosing $z_0 = 0$, and taking into account (20), it is enough to show that

$$\lim_{n\to\infty}\left|p_n(0)\right|^{1/n} = e^{-(r+1)},$$

which in view of (22) is equivalent to

$$\lim_{n\to\infty}\left|L_n^{(\alpha_n)}(0)\right|^{1/n} = e^{-r}. \tag{24}$$

Now, we are going to prove (24) and (23) under the conditions in Theorem 2.1.

4.2.1. Proof of (24)

By (1), it follows that

$$L_n^{(\alpha_n)}(0) = \binom{n+\alpha_n}{n}.$$

Let $h_n \in \{1, 2, \ldots, n\}$ be such that

$$\operatorname{dist}(\alpha_n, \mathbb{S}_n) = |\alpha_n + h_n|.$$

Thus, by (9) we have

$$\lim_{n\to\infty} |\alpha_n + h_n|^{1/n} = e^{-r},$$

and, therefore, to prove (24) it should be satisfied

$$\lim_{n\to\infty}\left(\frac{|(n+\alpha_n)(n+\alpha_n-1)\cdots(1+\alpha_n)|}{n!|\alpha_n+h_n|}\right)^{\frac{1}{n}} = 1. \tag{25}$$

We have that

$$\left|(n+\alpha_n)(n+\alpha_n-1)\cdots(1+\alpha_n)\right| = |\alpha_n+h_n| \prod_{k=1}^{n-h_n} |\alpha_n+h_n+k| \prod_{k=1}^{h_n-1} |\alpha_n+h_n-k|.$$

First assume that $\alpha_n \geqslant -n-\frac{1}{2}$. Since $\frac{2k-1}{2} \leqslant |\alpha_n+h_n \pm k| \leqslant \frac{2k+1}{2}$, for any integer $k \geqslant 1$, it follows that

$$\begin{aligned}\prod_{k=1}^{h_n-1} \frac{2k-1}{2} \prod_{k=1}^{n-h_n} \frac{2k-1}{2} &\leqslant \frac{|(n+\alpha_n)(n+\alpha_n-1)\cdots(1+\alpha_n)|}{|\alpha_n+h_n|} \\ &\leqslant \prod_{k=1}^{h_n-1} \frac{2k+1}{2} \prod_{k=1}^{n-h_n} \frac{2k+1}{2}.\end{aligned}$$

Set $a_l = \prod_{k=1}^{l} \frac{2k+1}{2} = \frac{(2l+2)!}{2^{2l+1}(l+1)!}$, $l \geqslant 1$. Then, for $1 \leqslant h_n \leqslant n-1$, $a_{-1}=2$, $a_0=1$,

$$\begin{aligned}\frac{1}{2^2} a_{h_n-2} a_{n-h_n-1} &\leqslant \frac{|(n+\alpha_n)(n+\alpha_n-1)\cdots(1+\alpha_n)|}{|\alpha_n+h_n|} \\ &\leqslant a_{h_n-1} a_{n-h_n}.\end{aligned} \tag{26}$$

On the other hand, if $\alpha_n < -n-\frac{1}{2}$ (and thus, $h_n = n$),

$$\begin{aligned}\left|(n+\alpha_n-1)(n+\alpha_n-2)\cdots(1+\alpha_n)\right| &\leqslant \prod_{k=1}^{n-1}(-\alpha_n-k) \\ &= \frac{\Gamma(-\alpha_n)}{\Gamma(-(\alpha_n+n-1))} = \frac{\Gamma(-\alpha_n)}{\Gamma(\delta_n+1)},\end{aligned}$$

which yields

$$a_{n-1} \leqslant \left|(n+\alpha_n-1)\cdots(1+\alpha_n)\right| \leqslant \frac{\Gamma(-\alpha_n)}{\Gamma(\delta_n+1)}. \tag{27}$$

Take into account that if $\{b_n\}$ is a sequence such that $\lim_{n\to\infty} \frac{b_n}{n} = 1$, then by Stirling formula we get $\lim_{n\to\infty} \frac{\sqrt[n]{\Gamma(b_n)}}{n} = 1$. Therefore, we have

$$\lim_{n\to\infty} \left(\frac{a_{h_n-1-s} a_{n-h_n-s}}{n!}\right)^{\frac{1}{n}} = 1, \quad s = 0, 1,$$

and

$$\lim_{n\to\infty} \left(\frac{\Gamma(-\alpha_n)}{n!\,\Gamma(\delta_n+1)}\right)^{\frac{1}{n}} = 1.$$

Finally, by (26)–(27), (25) follows.

4.2.2. Proof of (23)

Let us denote

$$k_n = \min\bigl([-\alpha_n], n\bigr), \quad \alpha_n = -k_n-\delta_n,\ \delta_n > 0, \tag{28}$$

where $[\cdot]$ denotes the integer part of a real number. It is clear that $-k_n \in \mathbb{S}_n$ and if $k_n < n$, then $0 < \delta_n < 1$.

Moreover,

$$\operatorname{dist}(\alpha_n, \mathbb{S}_n) = \begin{cases} \delta_n, & \text{if } \alpha_n < -n, \\ \min(\delta_n, 1-\delta_n), & \text{if } \alpha_n > -n. \end{cases}$$

In order to prove (23), the following integral representation will be used (see [1, formula (6.2.22)]):

$$e^{-x} L_n^{(\alpha)}(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_x^\infty (t-x)^{\beta-\alpha-1} e^{-t} L_n^{(\beta)}(t)\,dt, \tag{29}$$

where $\beta > \alpha$ and the path of integration is any simple smooth path connecting $x \in \mathbb{C}$ with $+\infty$. Thus, setting $\beta = -k_n$ and $\alpha = \alpha_n$ in (29) and taking into account (28), we have

$$e^{-x} L_n^{(\alpha_n)}(x) = \frac{1}{\Gamma(\delta_n)} \int_x^\infty (t-x)^{\delta_n - 1} e^{-t} L_n^{(-k_n)}(t)\, dt,$$

which after some calculations becomes

$$e^{-nx} L_n^{(\alpha_n)}(nx) = \frac{n^{\delta_n}}{\Gamma(\delta_n)} \int_x^\infty (t-x)^{\delta_n - 1} e^{-nt} L_n^{(-k_n)}(nt)\, dt. \tag{30}$$

Now, since $k_n \in \{1, \dots, n\}$, making use of (3), (30) may be written in the form:

$$e^{-nx} L_n^{(\alpha_n)}(nx) = (-1)^{k_n} \frac{n^{\delta_n + k_n}(n-k_n)!}{n!\,\Gamma(\delta_n)} \int_x^\infty (t-x)^{\delta_n - 1} t^{k_n} e^{-nt} L_{n-k_n}^{(k_n)}(nt)\, dt. \tag{31}$$

On the other hand, by the Rodrigues formula (2), (31) yields

$$\begin{aligned} e^{-nx} L_n^{(\alpha_n)}(nx) &= (-1)^n e^{-n} \frac{n^{\delta_n + k_n}}{n!\,\Gamma(\delta_n)} \int_x^\infty (t-x)^{\delta_n - 1} \bigl[\phi(t)^n\bigr]^{(n-k_n)}\, dt \\ &= K_n F_n(x), \end{aligned}$$

where, as above, $\phi(t) = t e^{1-t}$ and

$$F_n(x) = \int_x^\infty (t-x)^{\delta_n - 1} \bigl[\phi(t)^n\bigr]^{(n-k_n)}\, dt.$$

Let $x_0 = x_0(r)$ denote the unique point where the curve Γ_r meets the positive real semiaxis. By the freedom of the choice of the path of integration, we select a path that consists of two arcs: the first goes from x to x_0 along the curve Γ_r (by the shortest way), and the corresponding integral will be denoted by $G_n(x)$; the second goes from x_0 to ∞ along the positive real semiaxis, and we will denote this integral by $H_n(x)$. Thus, $F_n(x) = G_n(x) + H_n(x)$.

We will estimate $G_n(x)$ for $x \in \Gamma_r \setminus \{x_0\}$.

Suppose first that $k_n = n$, and hence,

$$G_n(x) = \int_x^{x_0} (t-x)^{\delta_n - 1} \phi(t)^n\, dt.$$

For this integral, consider the natural arc-length parametrization: $t = t(s)$, so that $t(0) = x$ and $t(s_0) = x_0$, for some positive real number s_0. Recall that $|\phi(t)| = e^{-r}$ on Γ_r. Since the path of integration is a smooth rectifiable Jordan arc (even when $r = 0$, in which case the path is entirely contained in the upper or lower half of $\Gamma_0 = \Gamma$), we have

$$\begin{aligned} \bigl|G_n(x)\bigr| &= \left| \int_0^{s_0} \bigl(t(s) - t(0)\bigr)^{\delta_n - 1} \bigl(\phi\bigl(t(s)\bigr)\bigr)^n t'(s)\, ds \right| \\ &\leqslant \|\phi\|_{\Gamma_r}^n \int_0^{s_0} \bigl|t(s) - t(0)\bigr|^{\delta_n - 1} \bigl|t'(s)\bigr|\, ds \\ &\leqslant e^{-rn} \int_0^{s_0} \bigl|t(s) - t(0)\bigr|^{\delta_n - 1} \bigl|t'(s)\bigr|\, ds. \end{aligned} \tag{32}$$

Now, let c and C be two positive constants such that $c \leqslant |t'(s)| \leqslant C$, $s \in [0, s_0]$, and set

$$A_n = \begin{cases} C^{\delta_n}, & \text{if } \delta_n \geqslant 1, \\ C c^{\delta_n - 1}, & \text{if } 0 < \delta_n < 1. \end{cases}$$

Then, by the classical mean value theorem, (32) implies

$$\bigl|G_n(x)\bigr| \leqslant A_n e^{-rn} \frac{s_0^{\delta_n}}{\delta_n}, \tag{33}$$

where $\lim_{n\to\infty} A_n^{1/n} = 1$. On the other hand, when $k_n < n$, it follows

$$G_n(x) = \int_x^{x_0} (t-x)^{\delta_n - 1}\big[\big(\phi(t)\big)^n\big]^{(n-k_n)}\,dt.$$

Proceeding analogously as above, we can show that

$$\big|G_n(x)\big| \leqslant A_n \big\|\big[\phi^n\big]^{(n-k_n)}\big\|_{\Gamma_r} \frac{s_0^{\delta_n}}{\delta_n}.$$

Applying the Cauchy integral formula in an arbitrarily small circle around t, for t in the segment of curve Γ_r connecting x to x_0, we obtain the estimate

$$\begin{aligned}\big|\big[\phi(t)^n\big]^{(n-k_n)}\big| &\leqslant (n-k_n)!\epsilon^{-n+k_n} e^{\epsilon n}\big(\big|\phi(t)\big| + \epsilon e^2\big)^n \\ &= (n-k_n)!\epsilon^{-n+k_n} e^{\epsilon n}\big(e^{-r} + \epsilon e^2\big)^n,\end{aligned}$$

for $\epsilon > 0$ arbitrarily small. Hence,

$$\big|G_n(x)\big| \leqslant A_n (n-k_n)!\epsilon^{-n+k_n} e^{n\epsilon}\big(e^{-r} + \epsilon e^2\big)^n \frac{s_0^{\delta_n}}{\delta_n}, \tag{34}$$

for $\epsilon > 0$. Since $\lim_{n\to\infty} \frac{k_n}{n} = 1$, we have

$$\lim_{n\to\infty}\big((n-k_n)!\epsilon^{-n+k_n} e^{n\epsilon}\big(e^{-r} + \epsilon e^2\big)^n\big)^{1/n} = e^{\epsilon}\big(e^{-r} + \epsilon e^2\big), \tag{35}$$

for $\varepsilon > 0$. Taking the limit $\epsilon \to 0^+$ in (35) shows that (34) implies (33). Since $\lim_{n\to\infty} \frac{\alpha_n}{n} = -1$, we have

$$\lim_{n\to\infty}\left(|K_n| \frac{s_0^{\delta_n}}{\delta_n}\right)^{1/n} = \lim_{n\to\infty} \frac{n^{\delta_n/n}}{\Gamma(1+\delta_n)^{1/n}} = 1, \tag{36}$$

where Stirling formula has been used when δ_n is unbounded (recall that $\delta_n = o(n)$). Therefore, by (33)–(36), it follows that

$$\limsup_{n\to\infty}\big(\big|K_n G_n(x)\big|\big)^{1/n} \leqslant e^{-r}, \quad x \in \Gamma_r \setminus \{x_0\}, \tag{37}$$

after taking limits when $\varepsilon \to 0^+$, if necessary. Note that in this part of the proof (9) has not been used.

Now, we proceed with $H_n(x)$. As above, suppose first that $k_n = n$, and thus,

$$H_n(x) = \int_{x_0}^{\infty} (t-x)^{\delta_n - 1}\phi(t)^n\,dt,$$

where now the path of integration is taken along the positive real semiaxis. Then, we have

$$H_n(x) = e^n \int_{x_0}^{\infty}\left(1 - \frac{x}{t}\right)^{\delta_n - 1} t^{n+\delta_n-1} e^{-(n-1)t} e^{-t}\,dt. \tag{38}$$

Let M and N be two positive constants such that $M \leqslant |1 - \frac{x}{t}| \leqslant N$, $t \in [x_0, \infty)$, and set

$$B_n = \begin{cases} N^{\delta_n - 1}, & \text{if } \delta_n \geqslant 1, \\ M^{\delta_n - 1}, & \text{if } 0 < \delta_n < 1. \end{cases}$$

Then,

$$\big|H_n(x)\big| \leqslant e^n B_n \|h\|_{[0,+\infty)} \int_{x_0}^{\infty} e^{-t}\,dt \leqslant e^n B_n \|h\|_{[0,+\infty)},$$

where $\lim_{n\to\infty} B_n^{1/n} = 1$ and $h(t) = t^{n+\delta_n-1} e^{-(n-1)t}$. It is easy to see that

$$\|h\|_{[0,+\infty)} = h\left(\frac{n+\delta_n-1}{n-1}\right) = \left(\frac{n+\delta_n-1}{n-1}\right)^{n-\delta_n-1} e^{-(n+\delta_n-1)}.$$

Hence,

$$\big|H_n(x)\big| \leqslant B_n \left(\frac{n+\delta_n-1}{n-1}\right)^{n-\delta_n-1} e^{-(\delta_n-1)}.$$

Therefore,

$$\begin{aligned}\left|K_n H_n(x)\right| &\leqslant \frac{e^{-n}n^{n+\delta_n}}{n!\Gamma(1+\delta_n)} B_n \left(\frac{n+\delta_n-1}{n-1}\right)^{n-\delta_n-1} e^{-(\delta_n-1)}\delta_n \\ &\leqslant C_n\delta_n = C_n \operatorname{dist}(\alpha_n, \mathbb{S}_n),\end{aligned} \tag{39}$$

where $\lim_{n\to\infty} C_n^{1/n} = 1$.

On the other hand, when $k_n < n$, we have

$$H_n(x) = \int_{x_0}^{\infty} (t-x)^{\delta_n-1}\left[\phi(t)^n\right]^{(n-k_n)} dt,$$

and integrating by parts, it yields

$$H_n(x) = (x_0-x)^{\delta_n-1}\left[\phi(t)^n\right]_{t=x_0}^{(n-k_n-1)} + (1-\delta_n)\int_{x_0}^{\infty} (t-x)^{\delta_n-2}\left[\phi(t)^n\right]^{(n-k_n-1)} dt. \tag{40}$$

Now, applying again the Cauchy integral formula for $t \in [x_0, \infty) \subset \mathbb{R}^+$, we obtain

$$\left|\left[\left(\phi(t)\right)^n\right]^{(l)}\right| \leqslant l!\epsilon^{-l}e^{2\epsilon n}\phi(t+\varepsilon)^n, \tag{41}$$

for arbitrarily small $\epsilon > 0$.

Then, taking into account (40)–(41) and setting

$$D_n = (n-k_n-1)!\epsilon^{-n+k_n+1}e^{2\epsilon n}|x_0-x|^{\delta_n-1},$$

we have

$$\begin{aligned}\left|H_n(x)\right| &\leqslant D_n\left(\phi(x_0+\varepsilon)^n + (1-\delta_n)\int_{x_0}^{\infty}|t-x|^{-1}\phi(t+\varepsilon)^n\,dt\right) \\ &\leqslant D_n\left(\phi(x_0+\varepsilon)^n + (1-\delta_n)\int_{x_0}^{\infty}\left|1-\frac{x-\varepsilon}{t}\right|^{-1}t^{-1}\phi(t)^n\,dt\right).\end{aligned}$$

Finally, we can bound the integral above as in (38), which yields

$$\left|H_n(x)\right| \leqslant D_n\left(\phi(x_0+\varepsilon)^n + (1-\delta_n)\widetilde{M}^{-1}e^{-1}\right),$$

where we denote by $\widetilde{M}$ the lower bound of the function $|1-\frac{x-\varepsilon}{t}|$, $t \in [x_0, \infty)$.

Therefore,

$$\begin{aligned}\left|K_n H_n(x)\right| &\leqslant \frac{e^{-n}n^{k_n+\delta_n}}{n!\Gamma(1+\delta_n)} D_n\left(\phi(x_0+\varepsilon)^n\delta_n + \delta_n(1-\delta_n)\widetilde{M}^{-1}e^{-1}\right) \\ &\leqslant R_n\delta_n\phi(x_0+\varepsilon)^n + S_n\delta_n(1-\delta_n),\end{aligned} \tag{42}$$

where $\lim_{n\to\infty} R_n^{1/n} = \lim_{n\to\infty} S_n^{1/n} = 1$. Taking into account (9), we have

$$\lim_{n\to\infty}\left[\operatorname{dist}(\alpha_n, \mathbb{S}_n)\right]^{1/n} = \lim_{n\to\infty}\left[\delta_n(1-\delta_n)\right]^{1/n} = e^{-r}. \tag{43}$$

Now, from (39), (42) and (43), it follows that

$$\limsup_{n\to\infty}\left(\left|K_n H_n(x)\right|\right)^{1/n} \leqslant e^{-r}, \quad x \in \Gamma_r \setminus \{x_0\}, \tag{44}$$

after taking limits when $\varepsilon \to 0^+$, if necessary. Thus, from (37) and (44), it follows that

$$\limsup_{n\to\infty}\left(\left|K_n F_n(x)\right|\right)^{1/n} \leqslant e^{-r}, \quad x \in \Gamma_r \setminus \{x_0\}.$$

It only remains to consider the limit case $r = \infty$, which occurs when

$$\lim_{n\to\infty}\left[\operatorname{dist}(\alpha_n, \mathbb{S}_n)\right]^{1/n} = 0.$$

Having in mind the method above, it is not hard to see that in this case, we have

$$\limsup_{n\to\infty} e^{-\operatorname{Re} x}\left|L_n^{(\alpha_n)}(nx)\right|^{1/n} \leqslant e^{-s}, \quad x \in \Gamma_s \setminus \{x_0(s)\}, \tag{45}$$

for any $s > 0$. Applying [4, Theorem 5] to (45) we can show that $\operatorname{supp}\mu_\infty \subset \overline{G_s}$, for any $s > 0$. Since $\bigcap_{s>0}\overline{G_s} = \{0\}$, the conclusion easily follows.

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