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The Generalized Riemann Hypothesis and the discriminant of number fields

Pei-Chu Hu [a,1], Zhuan Ye [b,*]

[a] *Department of Mathematics, Shandong University, Jinan 250100, Shandong, PR China*
[b] *Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115, USA*

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ABSTRACT

Under the Generalized Riemann Hypothesis for the Dedekind zeta-function ζ_κ, we obtain a formula for the discriminant $D_{\kappa/\mathbb{Q}}$ of the algebraic number field κ in terms of an integral of ζ_κ on the critical line.

1. Introduction and main result

Let κ be a number field of degree $n = r_1 + 2r_2$, where r_1, r_2 are the number of real, complex places respectively. The Dedekind zeta-function of the number field κ is defined by the series

$$\zeta_\kappa(s) = \sum_{\mathfrak{a}} \frac{1}{\mathcal{N}(\mathfrak{a})^s},$$

where $\mathfrak{a}$ varies over the non-zero integral ideals of κ, and $\mathcal{N}(\mathfrak{a})$ denotes the absolute norm of $\mathfrak{a}$. Denote by $D_{\kappa/\mathbb{Q}}$ the discriminant of κ.

The Dedekind function $\zeta_\kappa(s)$ admits a holomorphic continuation with the exclusion of a simple pole at $s = 1$, and satisfies the following functional equation

$$\zeta_\kappa(1-s) = A(s)\zeta_\kappa(s), \tag{1}$$

where

$$A(s) = |D_{\kappa/\mathbb{Q}}|^{s-\frac{1}{2}} \left(\cos\frac{\pi s}{2}\right)^{r_1+r_2} \left(\sin\frac{\pi s}{2}\right)^{r_2} 2^{(1-s)n} \pi^{-sn} \Gamma^n(s).$$

A straightforward computation gives $A(1/2) = 1$. The *Generalized Riemann Hypothesis* (GRH) for κ is the conjecture that all the zeros of the zeta-function $\zeta_\kappa(s)$ that lie within the *critical strip* $0 < \mathrm{Re}(s) < 1$ actually lie on the *critical line* $\mathrm{Re}(s) = 1/2$.

* Corresponding author.
E-mail addresses: pchu@sdu.edu.cn (P.-C. Hu), ye@math.niu.edu (Z. Ye).

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Further, let

$$\zeta_\kappa(s) = \alpha_\kappa\left(s - \frac{1}{2}\right)^\mu + \beta_\kappa\left(s - \frac{1}{2}\right)^{\mu+1} + \cdots,$$

which is the Taylor expansion of $\zeta_\kappa(s)$ at $s = 1/2$. It is clear from (1) that μ is a non-negative even integer, and $\alpha_\kappa \neq 0$ and β_κ are real numbers. Recently, we [3] proved the following identity

$$|D_{\kappa/\mathbb{Q}}|^{1/n} = \left(8\pi e^{\gamma+\pi/2}\right)^{r_1/n}\left(8\pi e^\gamma\right)^{2r_2/n} e^{-2\beta_\kappa/(n\alpha_\kappa)}, \tag{2}$$

where γ is the Euler constant. Thus, the computation of $|D_{\kappa/\mathbb{Q}}|$ is equivalent to the computation of $\beta_\kappa/\alpha_\kappa$.

It is well known [4] that

$$|D_{\kappa/\mathbb{Q}}|^{1/n} \geqslant \left(4\pi e^{\gamma+1}\right)^{r_1/n}\left(4\pi e^\gamma\right)^{2r_2/n} - O\left(n^{-2/3}\right), \tag{3}$$

and further, if the Generalized Riemann Hypothesis is assumed, a much stronger inequality states

$$|D_{\kappa/\mathbb{Q}}|^{1/n} \geqslant \left(8\pi e^{\gamma+\pi/2}\right)^{r_1/n}\left(8\pi e^\gamma\right)^{2r_2/n} - O\left(\log^{-2} n\right). \tag{4}$$

We refer the reader to the survey paper [4] for the history of these bounds. Many mathematicians such as A. Odlyzko, G. Poitou, J.-P. Serre and H. Stark, have all contributed to this theory. Moreover, explicit forms of these inequalities have proven quite useful in several types of application in algebraic number theory, as described in [4] for example.

After a little computation with inequalities (2), (3) and (4), we obtain

$$\frac{\beta_\kappa}{\alpha_\kappa} \leqslant O(n)$$

and, if the Generalized Riemann Hypothesis is assumed,

$$\frac{\beta_\kappa}{\alpha_\kappa} \leqslant O\left(\frac{n}{\log^2 n}\right),$$

as $n \to \infty$.

In general, to obtain the estimates (3) and (4), it is common (e.g. see [4]) to use the well-known formula

$$\begin{aligned}\log|D_{\kappa/\mathbb{Q}}| = {} & \frac{\pi r_1}{2} + \{\gamma + \log(8\pi)\}n - n\int_0^\infty \frac{1 - F(x)}{2\sinh(x/2)}\,dx - r_1\int_0^\infty \frac{1 - F(x)}{2\cosh(x/2)}\,dx \\ & - 4\int_0^\infty F(x)\cosh\frac{x}{2}\,dx + \sum_\rho{}' \Phi(\rho) + 2\sum_{\mathfrak{p}}\sum_{m=1}^\infty \frac{\log\mathcal{N}(\mathfrak{p})}{\mathcal{N}(\mathfrak{p})^{m/2}} F\big(m\log\mathcal{N}(\mathfrak{p})\big),\end{aligned} \tag{5}$$

where $\mathfrak{p}$ runs over prime ideals of κ, ρ runs over the zeros of $\zeta_\kappa(s)$ in the critical strip, $\sum_\rho'$ means that the ρ and $\bar{\rho}$ terms are to be taken together, and

$$\Phi(s) = \int_{-\infty}^\infty F(x)e^{(s-1/2)x}\,dx$$

in which $F: \mathbb{R} \to \mathbb{R}$ is a differentiable function with $F(-x) = F(x)$, $F(0) = 1$, and such that for some constants $c, \varepsilon > 0$,

$$\left|F(x)\right|, \left|F'(x)\right| \leqslant ce^{-(1/2+\varepsilon)|x|}$$

as $x \to \infty$.

Note that the identity (2) is equivalent to the identity

$$\log|D_{\kappa/\mathbb{Q}}| = \{\gamma + \log(8\pi)\}n + \frac{\pi}{2}r_1 - 2\frac{\beta_\kappa}{\alpha_\kappa}. \tag{6}$$

In this short paper, we obtain a new identity and change the term $\frac{\beta_\kappa}{\alpha_\kappa}$ in (2) to a simple integral form as follows:

Theorem 1.1. *The Generalized Riemann Hypothesis for $\zeta_\kappa(s)$ is true if and only if*

$$\log|D_{\kappa/\mathbb{Q}}| = \{\gamma + \log(8\pi)\}n + \frac{\pi}{2}r_1 - 8 - \frac{2}{\pi}\int_0^\infty t^{-2}\log\frac{|\zeta_\kappa(\frac{1}{2}+it)|^2}{|\alpha_\kappa|^2 t^{2\mu}}\,dt. \tag{7}$$

2. Preliminaries

Similar to the case of Riemann ζ-function, we use the following fact [6] to give an estimate of $\zeta_\kappa(s)$ in the critical strip.

Lemma 2.1. *If $f(s)$ is holomorphic, and for $M > 1$,*

$$\left|\frac{f(s)}{f(s_0)}\right| < e^M$$

in the disk $|s - s_0| \leqslant r$, then for $|s - s_0| \leqslant r/4$,

$$\left|\frac{f'(s)}{f(s)} - \sum \frac{1}{s-\rho}\right| < \frac{AM}{r}$$

where ρ runs through the zeros of $f(s)$ in $|s - s_0| \leqslant r/2$.

Using the Phragmén–Lindelöf principle, one can obtain upper bounds for the order of growth of $\zeta_\kappa(s)$ inside the critical strip [5], Theorem 6.8.

Lemma 2.2. *Uniformly in σ, as $|t| \to \infty$,*

$$|t|^{(1/2-\sigma)n}\left|\zeta_\kappa(1-\sigma+it)\right| \ll \left|\zeta_\kappa(\sigma+it)\right| \ll |t|^{(1/2-\sigma)n}\left|\zeta_\kappa(1-\sigma+it)\right|.$$

In particular,

$$\limsup_{t\to\pm\infty} \frac{\log|\zeta_\kappa(\sigma+\mathrm{i}t)|}{\log|t|} \leqslant \begin{cases} 0, & \text{if } \sigma > 1; \\ n(1-\sigma)/2, & \text{if } 0 \leqslant \sigma \leqslant 1; \\ (\frac{1}{2}-\sigma)n, & \text{if } \sigma < 0. \end{cases} \tag{8}$$

By using Lemma 2.1 and Lemma 2.2, we can prove the following lemma, which modifies a lemma in [6], p. 27.

Lemma 2.3. *Let T be any positive real number. If $\rho = \xi + i\eta$ runs through zeros of $\zeta_\kappa(s)$, then*

$$\frac{\zeta_\kappa'(s)}{\zeta_\kappa(s)} - \sum_{|T-\eta|<1} \frac{1}{s-\rho} = O(\log T) \tag{9}$$

uniformly for $-1 \leqslant \sigma \leqslant 2$, where $s = \sigma + it$. Moreover, for

$$-1 \leqslant \sigma = \mathrm{Re}(s) \leqslant 2, \qquad T - \frac{1}{2} \leqslant t = \mathrm{Im}(s) \leqslant T + \frac{1}{2},$$

we have

$$\begin{aligned} \log\left|\zeta_\kappa(s)\right| &= \sum_{|T-\eta|<1} \log|s-\rho| + O(\log T) \\ &\geqslant \sum_{|T-\eta|<1} \log|t-\eta| + O(\log T). \end{aligned} \tag{10}$$

Proof. Take $s_0 = \frac{3}{2} + iT$, $r = 10$ in Lemma 2.1. Note that when $\sigma > 1$,

$$\frac{1}{|\zeta_\kappa(s)|} = \left|\prod_{\mathfrak{p}}\left(1 - \frac{1}{\mathcal{N}(\mathfrak{p})^s}\right)\right| \leqslant \prod_{\mathfrak{p}}\left(1 + \frac{1}{\mathcal{N}(\mathfrak{p})^\sigma}\right) < \zeta_\kappa(\sigma),$$

which implies from (8) that

$$\left|\frac{\zeta_\kappa(s)}{\zeta_\kappa(s_0)}\right| \leqslant e^{C\log T},$$

where C is a positive real number. It follows from Lemma 2.1 that

$$\frac{\zeta_\kappa'(s)}{\zeta_\kappa(s)} - \sum_{|\rho-s_0|<5} \frac{1}{s-\rho} = O(\log T)$$

for $|s - s_0| \leqslant \frac{5}{2}$, and in particular for $-1 \leqslant \sigma \leqslant 2$.

Let $N(T)$ count the number of zeros ρ of $\zeta_\kappa(s)$ satisfying $\mathrm{Re}(\rho) > 0$, $|\mathrm{Im}(\rho)| \leqslant T$. It is known (e.g. Theorem 7.7 in [5]) that

$$N(T) = \frac{n}{\pi} T \log \frac{T}{e} + \frac{T}{\pi} \log \frac{|D_{\kappa/\mathbb{Q}}|}{(2\pi)^n} + O(\log T), \tag{11}$$

which easily yields

$$N(T+1) - N(T) = O(\log T).$$

Therefore, the number of terms in one of the above sums but not in the other is $O(\log T)$, and each such term is $O(1)$. So,

$$\frac{\zeta_\kappa'(s)}{\zeta_\kappa(s)} = \sum_{|T-\eta|<1} \frac{1}{s-\rho} + O(\log T), \tag{12}$$

since the number of terms included in this sum, but not in the above sums or vice versa, is $O(\log T)$, and each term is $O(1)$. Thus (10) is proved.

Further, for

$$-1 \leqslant \sigma = \mathrm{Re}(s) \leqslant 2, \qquad T - \frac{1}{2} \leqslant t = \mathrm{Im}(s) \leqslant T + \frac{1}{2},$$

integrating (9) from s to $2 + iT$ (here we suppose that T is not equal to the ordinate of any zero of ζ_κ), we obtain

$$\log \zeta_\kappa(2+iT) - \log \zeta_\kappa(s) = \sum_{|T-\eta|<1} \big\{\log(2+iT-\rho) - \log(s-\rho)\big\} + O(\log T)|s-2-iT|. \tag{13}$$

Since $\log \zeta_\kappa(2+iT)$ and $\log(2+iT-\rho)$ are bounded, $|s-(2+iT)| \leqslant 5$, $|s-\rho| \geqslant |t-\eta|$, and there are $O(\log T)$ terms in the sum, we conclude from (13) that (10) holds. $\square$

3. Proof of Theorem 1.1

We will use the method in [2] to prove Theorem 1.1, which is a straightforward consequence of (6) and the following two theorems. Let $R > 0$ and set

$$\mathbb{C}(0; R) = \big\{z \in \mathbb{C} \,\big|\, |z| < R\big\}, \qquad \mathbb{C}[0; R] = \big\{z \in \mathbb{C} \,\big|\, |z| \leqslant R\big\}.$$

Lemma 3.1 *(Carleman's formula). (See [2].) Let $f(z)$ be meromorphic in $\mathbb{C}[0; R] \cap \{\mathrm{Re}(z) \geqslant 0\}$ with $f(0) = 1$, and suppose that it has the zeros $r_1 e^{i\theta_1}, r_2 e^{i\theta_2}, \ldots, r_m e^{i\theta_m}$ and the poles $s_1 e^{i\varphi_1}, s_2 e^{i\varphi_2}, \ldots, s_n e^{i\varphi_n}$ inside $\mathbb{C}(0; R) \cap \{\mathrm{Re}(z) > 0\}$. Then*

$$\sum_{\mu=1}^{m} \left(\frac{1}{r_\mu} - \frac{r_\mu}{R^2}\right) \cos\theta_\mu - \sum_{\nu=1}^{n} \left(\frac{1}{s_\nu} - \frac{s_\nu}{R^2}\right) \cos\varphi_\nu = \mathcal{C}_f(R) - \frac{1}{2}\mathrm{Re}\big(f'(0)\big),$$

where

$$\mathcal{C}_f(R) = \frac{1}{\pi R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\big|f\big(\mathrm{R}e^{i\theta}\big)\big| \cos\theta \, d\theta + \frac{1}{2\pi} \int_0^R \left(\frac{1}{y^2} - \frac{1}{R^2}\right) \log\big|f(iy) f(-iy)\big| \, dy. \tag{14}$$

Theorem 3.2. *The Generalized Riemann Hypothesis is true for $\zeta_\kappa(s)$ if and only if*

$$\lim_{R\to\infty} \mathcal{C}_f(R) = \frac{\beta_\kappa}{2\alpha_\kappa} - 2,$$

where

$$f(s) = \frac{\zeta_\kappa(s+\frac{1}{2})}{\alpha_\kappa s^\mu}. \tag{15}$$

Proof. Let

$$z_\mu - \frac{1}{2} = r_\mu e^{i\theta_\mu} \quad \left(r_\mu > 0,\ 0 < \theta_\mu < \frac{\pi}{2}\right), \qquad \bar{z}_\mu - \frac{1}{2}$$

be the zeros of f in the half-plane $\mathrm{Re}(s) > 0$, but on the critical line. It is clear that $s = \frac{1}{2}$ is the unique pole of f in $\mathrm{Re}(s) > 0$. Hence Lemma 3.1 yields

$$2 \sum_{r_\mu < R} \left(\frac{1}{r_\mu} - \frac{r_\mu}{R^2}\right) \cos\theta_\mu - \left(2 - \frac{1}{2R^2}\right) = \mathcal{C}_f(R) - \frac{1}{2}\mathrm{Re}\bigl(f'(0)\bigr). \tag{16}$$

Since f is of order 1, then the convergence exponent of zeros for f is at most 1. Hence the series

$$\sum_\mu \frac{1}{r_\mu^{1+\varepsilon}}$$

is convergent for any $\varepsilon > 0$, and so

$$0 \leqslant \sum_\mu \frac{\cos\theta_\mu}{r_\mu} = \sum_\mu \frac{r_\mu \cos\theta_\mu}{r_\mu^2} \leqslant \frac{1}{2} \sum_\mu \frac{1}{r_\mu^2} < \infty.$$

Further, the formula (11) implies

$$0 \leqslant \sum_{r_\mu < R} \frac{r_\mu \cos\theta_\mu}{R^2} \leqslant \frac{N(R)}{2R^2} \to 0 \quad (R \to \infty).$$

Thus we obtain from (16) and $\mathrm{Re}(f'(0)) = \beta_\kappa / \alpha_\kappa$ that

$$\lim_{R\to\infty} \mathcal{C}_f(R) = 2 \sum_\mu \frac{\cos\theta_\mu}{r_\mu} + \frac{\beta_\kappa}{2\alpha_\kappa} - 2.$$

It is clear that the Generalized Riemann Hypothesis is true if and only if the zeros z_μ do not exist, that is,

$$\sum_\mu \frac{\cos\theta_\mu}{r_\mu} = 0,$$

which implies

$$\lim_{R\to\infty} \mathcal{C}_f(R) = \frac{\beta_\kappa}{2\alpha_\kappa} - 2. \qquad \square$$

Theorem 3.3. *If f is defined by* (15)*, we have*

$$\lim_{R\to\infty} \mathcal{C}_f(R) = \frac{1}{2\pi} \int_0^\infty \log\bigl|f(it)\bigr|^2 \frac{dt}{t^2}.$$

Proof. It is known [1] that

$$\int_0^R \left|\zeta_\kappa\left(\frac{1}{2} + it\right)\right|^2 dt = O\bigl(R^{n/2} (\log R)^n\bigr),$$

where n the degree of the number field κ. By the concavity of the logarithmic function, we obtain

$$\frac{1}{R^2} \int_0^R \log\bigl|f(it)\bigr|^2 dt \leqslant \frac{1}{R} \log\left\{\frac{1}{R} \int_0^R \bigl|f(it)\bigr|^2 dt\right\} = O\left(\frac{n \log R}{R}\right). \tag{17}$$

On the other hand, it is easily seen from a graph that the integral

$$\int_{T-\frac{1}{2}}^{T+\frac{1}{2}} \log|t - \eta|\, dt,$$

considered as a function of η, takes the minimum value $-\log 2-1$ when $\eta=T$. Since there are $O(\log T)$ terms in the following sum, it follows that

$$\int_{T-\frac{1}{2}}^{T+\frac{1}{2}} \sum_{|T-\eta|<1} \log|t-\eta|\,dt > -A\log T,$$

where A is a positive constant. In the sequel, A denotes an absolute positive constant and its value may be different in each appearance. Thus we have from (10) that

$$\int_{T-\frac{1}{2}}^{T+\frac{1}{2}} \log\bigl|\zeta_\kappa(\sigma+it)\bigr|\,dt > -A\log T.$$

Hence

$$\int_{\frac{1}{2}}^{[R]-\frac{1}{2}} \log\bigl|\zeta_\kappa(\sigma+it)\bigr|\,dt = \sum_{k=1}^{[R]-1}\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \log\bigl|\zeta_\kappa(\sigma+it)\bigr|\,dt > -A\log\bigl(\bigl([R]-1\bigr)!\bigr).$$

Similarly, we can show

$$\int_{[R]-\frac{1}{2}}^{R} \sum_{|[R]-\eta|<1} \log|t-\eta|\,dt > -A\log[R],$$

when $R \leqslant [R]+\frac{1}{2}$, and

$$\int_{[R]+\frac{1}{2}}^{R} \sum_{|[R]+1-\eta|<1} \log|t-\eta|\,dt > -A\log\bigl([R]+1\bigr),$$

when $R > [R]+\frac{1}{2}$. Thun we have, for any large R,

$$\int_{[R]-\frac{1}{2}}^{R} \log\bigl|\zeta_\kappa(\sigma+it)\bigr|\,dt > -A\log[R].$$

Therefore

$$\int_{\frac{1}{2}}^{R} \log\bigl|\zeta_\kappa(\sigma+it)\bigr|\,dt > -A\log\bigl([R]!\bigr) = -AR\log R,$$

since *Stirling's formula* yields

$$\log\bigl([R]!\bigr) = \left([R]+\frac{1}{2}\right)\log[R]-[R]+O(1).$$

It follows that

$$\frac{1}{R^2}\int_0^R \log\bigl|f(it)\bigr|^2\,dt > -A\frac{\log R}{R}. \tag{18}$$

Finally, combining (17) and (18) gives

$$\frac{1}{R^2}\int_0^R \log\bigl|f(it)\bigr|^2\,dt = O\left(\frac{\log R}{R}\right),$$

which implies

$$\lim_{R\to\infty} \frac{1}{R^2} \int_0^R \log\left|f(it)\right|^2 dt = 0. \tag{19}$$

In order to estimate the first integral in (14), we set

$$\delta = \arcsin \frac{1}{R}.$$

Then we can take R sufficient large such that

$$R - R\cos\delta \leqslant \frac{1}{2}.$$

For $-\delta + \pi/2 \leqslant \theta \leqslant \pi/2$, we have

$$-1 \leqslant \sigma = \mathrm{Re}\left(\frac{1}{2} + Re^{i\theta}\right) \leqslant 2, \qquad R - \frac{1}{2} \leqslant \mathrm{Im}\left(\frac{1}{2} + Re^{i\theta}\right) \leqslant R + \frac{1}{2}.$$

Thus, the inequality (10) implies

$$\log\left|\zeta_\kappa\left(\frac{1}{2} + Re^{i\theta}\right)\right| > \sum_{|R-\eta|<1} \log|R\sin\theta - \eta| + O(\log R),$$

where $\rho = \alpha + i\eta$ runs though the zeros of $\zeta_\kappa(s)$. Note that

$$\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} \sum_{|R-\eta|<1} \log|R\sin\theta - \eta| \cos\theta\, d\theta = \frac{1}{R} \int_{R\cos\delta}^{R} \sum_{|R-\eta|<1} \log|t-\eta|\, dt > -\frac{A\log R}{R}.$$

Then

$$\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} \log\left|\zeta_\kappa\left(\frac{1}{2} + Re^{i\theta}\right)\right| \cos\theta\, d\theta > -A\log R.$$

The estimates (8) yield immediately

$$\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} \log\left|\zeta_\kappa\left(\frac{1}{2} + Re^{i\theta}\right)\right| \cos\theta\, d\theta < O(\log R).$$

Thus we obtain

$$\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}} \log\left|\zeta_\kappa\left(\frac{1}{2} + Re^{i\theta}\right)\right| \cos\theta\, d\theta = O(\log R).$$

If $\sigma > 1$, then

$$\left|\zeta_\kappa(s)\right| \leqslant \zeta_\kappa(\sigma) \quad \text{and} \quad \frac{1}{|\zeta_\kappa(s)|} \leqslant \zeta_\kappa(\sigma)$$

for all values of t. Therefore

$$\int_0^{\frac{\pi}{2}-\delta} \log\left|\zeta_\kappa\left(\frac{1}{2} + Re^{i\theta}\right)\right| \cos\theta\, d\theta = O(1).$$

Since $\log|\zeta_\kappa(\frac{1}{2} + Re^{i\theta})|$ is an even function of θ, these estimates give

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left|\zeta_{\kappa}\left(\frac{1}{2}+Re^{i\theta}\right)\right|\cos\theta\,d\theta = O(\log R). \tag{20}$$

Therefore Theorem 3.3 follows from (14), (19) and (20). □

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