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Pairs of nontrivial solutions for resonant Neumann problems

Leszek Gasiński [a,*], Nikolaos S. Papageorgiou [b]

[a] Jagiellonian University, Faculty of Mathematics and Computer Science, ul. Łojasiewicza 6, 30-348 Kraków, Poland

[b] National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece

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ABSTRACT

We study a semilinear Neumann problem which is resonant at $\pm\infty$ with respect to any eigenvalue different from the first and the second eigenvalue of $-\Delta_N$ (the negative Neumann Laplacian). Using a combination of variational methods with Morse theoretic techniques, we show that the problem has at least two nontrivial smooth solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following semilinear Neumann problem:

$$\begin{cases} -\Delta u(z) = f\big(z, u(z)\big) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$ and $f(z,\zeta)$ is a measurable function which is C^1 in the ζ-variable. It is well-known that the existence and multiplicity of solutions for problem (1.1) relies heavily on the interaction of the limit

$$\lim_{|\zeta|\to+\infty} \frac{f(z,\zeta)}{\zeta}$$

with the spectrum of the negative Neumann Laplacian, hereafter denoted by $-\Delta_N$. The difficult and interesting case is when the above limit belongs in the spectrum of $-\Delta_N$. This is called a "resonant problem". Such problems were investigated primarily in the context of Dirichlet equations. We mention the works of Hirano and Nishimura [1], Landesman et al. [2], Liang–Su [3], Li–Willem [4], de Paiva [5], Su–Tang [6], and Zou [7]. The Neumann case has not been studied so extensively. In this direction, we mention the works of Gasiński–Papageorgiou [8], Li [9], Li–Li [10], Qian [11] and Tang–Wu [12]. The hypotheses in these works are in general different from ours and in many respects more restrictive, since they impose symmetry conditions on $f(z,\cdot)$ and have a left hand side differential operator of the form

$$-\Delta u + au,$$

* Corresponding author.
E-mail addresses: Leszek.Gasinski@ii.uj.edu.pl (L. Gasiński), npapg@math.ntua.gr (N.S. Papageorgiou).

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with $a > 0$ (see [9–11]) or use an anticoercivity condition on the primitive

$$F(z, \zeta) = \int_0^\zeta f(z, s)\, ds$$

(hence the energy functional of the problem is coercive) and assume resonance only at zero (see [12]) or finally they require that $f(z, \cdot)$ exhibits an oscillatory behaviour near zero (see [8]). Consequently their conclusions are different. Some other types of Neumann boundary value problems can be found in recent papers [13–16].

Here, we combine variational methods based on the critical point theory with Morse theory (critical groups) and prove the existence of two nontrivial smooth solutions, when resonance occurs. More precisely, we assume that $f(z, \cdot)$ is resonant at $\pm\infty$ with respect to any eigenvalue of $-\Delta_N$ different from the first and the second eigenvalue of $-\Delta_N$ (see (2.2)), while at zero we have nonuniform nonresonance with respect to a lower spectral interval.

In the next section, for easy reference, we review the main mathematical tools that we will use in this paper.

2. Mathematical background

Let X be a Banach space and let X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that φ satisfies the *Cerami condition*, if every sequence $\{x_n\}_{n \geqslant 1} \subseteq X$, such that $\{\varphi(x_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$\big(1 + \|x_n\|\big)\varphi'(x_n) \longrightarrow 0 \quad \text{in } X^*,$$

admits a strongly convergent subsequence. This compactness-type condition is in general weaker than the usual Palais–Smale condition. Nevertheless, this weaker condition suffices to have a deformation theorem and from it derives the minimax theory of the critical values of $\varphi \in C^1(X)$. In this minimax theory, the topological notion of linking sets plays a crucial role.

Definition 2.1. Let Y be a Hausdorff topological space, C_0, C and D three nonempty subsets of Y, such that $C_0 \subseteq C$. We say that the pair $\{C, C_0\}$ is linking with D in Y, if

(a) $C_0 \cap D = \emptyset$;
(b) for every $\gamma \in C(C, Y)$, such that $\gamma|_{C_0} = \mathrm{id}|_{C_0}$, we have that $\gamma(C) \cap D \neq \emptyset$.

Using this notion, we can prove the following general minimax principle concerning the critical values of a C^1-functional (see e.g., [17]).

Theorem 2.2. *If X is a Banach space, C_0, C and D are three nonempty, closed subsets of X, the pair $\{C, C_0\}$ is lining with D in X, $\varphi \in C^1(X)$, φ satisfies the Cerami condition,*

$$\sup_{C_0} \varphi < \inf_D \varphi$$

and

$$c = \inf_{\gamma\in\Gamma} \sup_{u\in C} \varphi\big(\gamma(u)\big),$$

where

$$\Gamma = \big\{\gamma \in C(C, X) : \gamma|_{C_0} = \mathrm{id}|_{C_0}\big\},$$

then $c \geqslant \inf_D \varphi$ and c is a critical value of φ.

Remark 2.3. With suitable choices of the linking sets, we obtain as corollaries of Theorem 2.2, well known results, such as the mountain pass theorem, the saddle point theorem and the generalized mountain pass theorem (see e.g., [17]).

Next, we recall a few basic facts about the spectrum of $-\Delta_N$. So, let $m \in L^\infty(\Omega)$, $m \geqslant 0$, $m \neq 0$. We consider the following weighted linear eigenvalue problem:

$$\begin{cases} -\Delta u(z) = \widehat{\lambda} m(z) u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an eigenvalue of $-\Delta_N$ with weight m (denoted by $(-\Delta_N, m)$), if problem (2.1) admits a nontrivial solution $\widehat{u} \in H^1(\Omega)$. Evidently, a necessary condition for $\widehat{\lambda}$ to be an eigenvalue, is that $\widehat{\lambda} \geqslant 0$. In fact $\widehat{\lambda}_0 = \widehat{\lambda}_0(m) = 0$ is an eigenvalue of (2.1) with corresponding eigenspace $\mathbb{R}$ (the space of constant functions). Using the spectral theorem for compact self-adjoint operators, we can show that $(-\Delta_N, m)$ has a sequence of distinct eigenvalues $\{\widehat{\lambda}_k(m)\}_{k\geqslant 0}$, such that

$$\widehat{\lambda}_0(m) = 0 \quad \text{and} \quad \widehat{\lambda}_k(m) \longrightarrow +\infty \quad \text{as } k \to +\infty. \tag{2.2}$$

If $m \equiv 1$, then we write $\widehat{\lambda}_k(1) = \widehat{\lambda}_k$ for all $k \geqslant 0$.

In what follows, for every $k \geqslant 0$, by $E(\widehat{\lambda}_k(m))$, we denote the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_k(m)$. Standard regularity theory (see e.g., [17]), implies that

$$E\big(\widehat{\lambda}_k(m)\big) \subseteq C^1(\overline{\Omega}) \quad \forall k \geqslant 0.$$

In addition, these eigenspaces have the so-called *unique continuation property*, namely, if $u \in E\big(\widehat{\lambda}_k(m)\big)$ vanishes on a set of positive Lebesgue measures, then $u \equiv 0$ (see [18,19]). We set

$$\overline{H}_l = \bigoplus_{k=0}^{l} E\big(\widehat{\lambda}_k(m)\big) \quad \text{and} \quad \widehat{H}_l = \overline{H}_l^{\perp} = \overline{\bigoplus_{k\geqslant l+1} E\big(\widehat{\lambda}_k(m)\big)}.$$

Evidently, $\overline{H}_l$ is finite dimensional. Using these spaces we have the following variational characterizations of the eigenvalues $\big\{\widehat{\lambda}_k(m)\big\}_{k\geqslant 0}$:

$$0 = \widehat{\lambda}_0(m) = \inf\left\{\frac{\|\nabla u\|_2^2}{\int_\Omega mu^2 dz} : u \in H_0^1(\Omega),\ u \neq 0\right\} \tag{2.3}$$

and for every $l \geqslant 1$, we have

$$\begin{aligned}\widehat{\lambda}_l(m) &= \inf\left\{\frac{\|\nabla u\|_2^2}{\int_\Omega mu^2 dz} : u \in \widehat{H}_{l-1},\ u \neq 0\right\}\\ &= \sup\left\{\frac{\|\nabla u\|_2^2}{\int_\Omega mu^2 dz} : u \in \overline{H}_l,\ u \neq 0\right\}.\end{aligned} \tag{2.4}$$

In (2.3), the infimum is actually attained on $E\big(\widehat{\lambda}_0(m)\big) = \mathbb{R}$. Similarly in (2.4), both the infimum and the supremum are realized on $E\big(\widehat{\lambda}_l(m)\big)$.

As an easy consequence of the unique continuation property and (2.3) and (2.4), we have the following two results concerning the eigenvalues $\big\{\widehat{\lambda}_k(m)\big\}_{k\geqslant 0}$ and the component spaces $\overline{H}_l$ and $\widehat{H}_l$ (see [8]).

Proposition 2.4. *If $m, m' \in L^\infty(\Omega)_+ \setminus \{0\}$, $m(z) \leqslant m'(z)$ for almost all $z \in \Omega$ and $m \neq m'$, then*

$$\widehat{\lambda}_l(m') < \widehat{\lambda}_l(m) \quad \forall l \geqslant 1.$$

Proposition 2.5. (a) *If $l \geqslant -1$ is an integer and $\eta \in L^\infty(\Omega)$, $\eta(z) \leqslant \widehat{\lambda}_{l+1}$ for almost all $z \in \Omega$, $\eta \neq \widehat{\lambda}_{l+1}$, then there exists $\xi_0 > 0$, such that*

$$\|\nabla u\|_2^2 - \int_\Omega \eta u^2\, dz \geqslant \xi_0 \|u\|^2 \quad \forall u \in \widehat{H}_l.$$

(b) *If $l \geqslant 0$ is an integer and $\eta \in L^\infty(\Omega)$, $\eta(z) \geqslant \widehat{\lambda}_l$ for almost all $z \in \Omega$, $\eta \neq \widehat{\lambda}_l$, then there exists $\xi_1 > 0$, such that*

$$\|\nabla u\|_2^2 - \int_\Omega \eta u^2\, dz \leqslant -\xi_1 \|u\|^2 \quad \forall u \in \overline{H}_l.$$

We mention that only the principal eigenvalue $\widehat{\lambda}_0(m) = 0$ has eigenfunctions of constant sign. All the other eigenvalues $\big\{\widehat{\lambda}_k(m)\big\}_{k\geqslant 1}$ have nodal (i.e., sign changing) eigenfunctions.

Next, let us recall a few basic facts from Morse theory. So, as before X is a Banach space and we consider $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the following sets:

$$\begin{aligned}\varphi^c &= \big\{x \in X : \varphi(x) \leqslant c\big\},\\ K_\varphi &= \big\{x \in X : \varphi'(x) = 0\big\},\\ K_\varphi^c &= \big\{x \in K_\varphi : \varphi(x) = c\big\}.\end{aligned}$$

Let (Y_1, Y_2) be a topological pair, such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geqslant 0$, by $H_k(Y_1, Y_2)$ we denote the k-th relative singular homology group with integer coefficients for the pair (Y_1, Y_2). The critical groups of φ at an isolated $u \in K_\varphi^c$ are defined by

$$C_k(\varphi, u) = H_k\big(\varphi^c \cap U,\ \varphi^c \cap U \setminus \{u\}\big) \quad \forall k \geqslant 0,$$

where U is a neighbourhood of u, such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology, implies that this definition is independent of the particular choice of the neighbourhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the Cerami condition. Assume that $-\infty < \inf \varphi(K_\varphi)$ and let

$$c < \inf \varphi(K_\varphi).$$

Then, the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \forall k \geqslant 0.$$

The second deformation theorem (see e.g., [17, p. 628]), implies that this definition is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$. Note that, if $C_k(\varphi, \infty) \neq 0$, then we can find $u \in K_\varphi$, such that $C_k(\varphi, u) \neq 0$.

Now, let $X = H$ be a Hilbert space, $\varphi \in C^2(H)$ and $u \in K_\varphi$. The Morse index $\mu(u)$ of u is defined to be the supremum of the dimensions of the vector subspaces of H on which $\varphi''(u)$ is negative definite. The nullity $\nu(u)$ of u is defined to be the dimension of $\ker \varphi''(u)$. We say that $u \in K_\varphi$ is nondegenerate, if $\varphi''(u)$ is invertible. Suppose that $u \in K_\varphi$ is nondegenerate with Morse index $\mu(u) = m$. Then

$$C_k(\varphi, u) = \delta_{k,m}\mathbb{Z} \quad \forall k \geqslant 0, \tag{2.5}$$

where $\delta_{k,m}$ is the Kronecker symbol, defined by

$$\delta_{k,m} = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k \neq m. \end{cases}$$

As a consequence of the so-called "shifting theorem", we have the following proposition (see e.g., [20]).

Proposition 2.6. *If H is a Hilbert space, $\varphi \in C^2(H)$ and $u \in K_\varphi$ has finite Morse index m and nullity n, then one of the following holds:*

(a) $C_k(\varphi, u) = 0$ *for all $k \leqslant m$ and $k \geqslant m + n$; or*
(b) $C_k(\varphi, u) = \delta_{k,m}\mathbb{Z}$ *for all $k \geqslant 0$; or*
(c) $C_k(\varphi, u) = \delta_{k,m+n}\mathbb{Z}$ *for all $k \geqslant 0$.*

The next proposition due to Liang–Su [3] is useful in computing the critical groups at infinity.

Proposition 2.7. *If H is a Hilbert space, $\{h_t\}_{t\in[0,1]} \subseteq C^1(H)$, the maps $u \longmapsto (h_t)'(u)$ and $u \longmapsto \partial_t h_t(u)$ are both locally Lipschitz, h_0 and h_1 satisfy the Cerami condition and there exist $a \in \mathbb{R}$ and $\delta > 0$, such that*

$$h_t(u) \leqslant a, \Longrightarrow \big(1 + \|u\|\big)\big\|(h_t)'(u)\big\|_* \geqslant \delta \quad \forall t \in [0, 1],$$

then

$$C_k(h_0, \infty) = C_k(h_1, \infty) \quad \forall k \geqslant 0.$$

Now suppose that X is a Banach space.

Definition 2.8. Suppose that $X = Y \oplus V$. We say that $\varphi \in C^1(X)$ has a local linking at 0, if there exists $\varrho > 0$, such that

$$\begin{cases} \varphi(y) \leqslant 0 & \text{for all } y \in Y,\ \|y\| \leqslant \varrho, \\ \varphi(v) \geqslant 0 & \text{for all } v \in V,\ \|v\| \leqslant \varrho. \end{cases}$$

From Su [21], we have the following result.

Proposition 2.9. *If H is a Hilbert space, $\varphi \in C^2(H)$ and φ has a local linking at 0 with respect to $H = Y \oplus V$ and $k = \dim Y < +\infty$, then if $k = m_0 = \mu(0)$, we have*

$$C_k(\varphi, 0) = \delta_{k,m_0}\mathbb{Z} \quad \forall k \geqslant 0$$

and if $k = d_0 = \mu(0) + \nu(0)$, we have

$$C_k(\varphi, 0) = \delta_{k,d_0}\mathbb{Z} \quad \forall k \geqslant 0.$$

3. The two solutions theorem

In this section we show that under resonance conditions at $\pm\infty$, problem (1.1) has at least two nontrivial smooth solutions. The hypotheses on the reaction $f(z, \zeta)$ are the following:

H: $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function, such that for almost all $z \in \Omega$, we have $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

(i) there exist $a \in L^\infty(\Omega)_+$, $c > 0$ and $r \in [2, q)$, where q is the Sobolev critical exponent, defined by

$$q = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geqslant 2, \\ +\infty & \text{if } N = 1, 2, \end{cases}$$

such that

$$\left|f'_\zeta(z,\zeta)\right| \leqslant a(z) + c|\zeta|^{r-2} \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \in \mathbb{R};$$

(ii) there exists an integer $m \geqslant 3$, such that

$$\lim_{\zeta\to\pm\infty} \frac{f(z,\zeta)}{\zeta} = \widehat{\lambda}_m \quad \text{uniformly for almost all } z \in \Omega;$$

(iii) if

$$F(z,\zeta) = \int_0^\zeta f(z,s)\,ds,$$

then there exist $\mu \in (1,2]$ and $\beta_0 > 0$, such that

$$\liminf_{\zeta\to\pm\infty} \frac{f(z,\zeta)\zeta - 2F(z,\zeta)}{|\zeta|^\mu} \geqslant \beta_0 > 0 \quad \text{uniformly for almost all } z \in \Omega;$$

(iv) there exists integer $k \geqslant 0$, such that $k < m-2$ and

$$\frac{f(z,\zeta)}{\zeta} \leqslant f'_\zeta(z,\zeta) \quad \text{and} \quad \frac{f(z,\zeta)}{\zeta} < \widehat{\lambda}_m \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \neq 0$$

and

$$f'_\zeta(z,0) \in \left[\widehat{\lambda}_k,\ \widehat{\lambda}_{k+1}\right] \quad \text{for almost all } z \in \Omega,$$

with $f'_\zeta(\cdot,0) \neq \widehat{\lambda}_k$ and $f'_\zeta(\cdot,0) \neq \widehat{\lambda}_{k+1}$.

Remark 3.1. The condition

$$\frac{f(z,\zeta)}{\zeta} \leqslant f'_\zeta(z,\zeta) \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \neq 0,$$

implies that for almost all $z \in \Omega$, we have that the map

$$\zeta \longmapsto \frac{f(z,\zeta)}{\zeta} \quad \begin{cases} \text{is nondecreasing on } (0,+\infty), \\ \text{is nonincreasing on } (-\infty,0). \end{cases}$$

Indeed, note that

$$\left(\frac{f(z,\zeta)}{\zeta}\right)' = \frac{f'_\zeta(z,\zeta)\zeta - f(z,\zeta)}{\zeta^2} \quad \begin{cases} \geqslant 0 & \text{if } \zeta > 0, \\ \leqslant 0 & \text{if } \zeta < 0. \end{cases}$$

Therefore, from hypothesis H(iv), we have

$$\widehat{\lambda}_k \leqslant \frac{f(z,\zeta)}{\zeta} \leqslant f'_\zeta(z,\zeta) \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \neq 0.$$

Evidently, if for almost all $z \in \Omega$, the map $\zeta \longmapsto f(z,\zeta)$ is convex in $(0,+\infty)$ and concave in $(-\infty,0)$, then we have

$$\frac{f(z,\zeta)}{\zeta} \leqslant f'_\zeta(z,\zeta) \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \neq 0.$$

This convexity–concavity hypothesis was used by de Paiva [5]. In Bartsch et al. [22], the authors have

$$f(z,\zeta) = f(\zeta),$$

with $f \in C^1(\mathbb{R})$ and assume that

$$\frac{f(\zeta)}{\zeta} < f'_\zeta(\zeta) \quad \forall \zeta \neq 0.$$

Let $\varphi\colon H^1(\Omega) \longrightarrow \mathbb{R}$ be the energy functional for problem (1.1), defined by

$$\varphi(u) = \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F\big(z,u(z)\big)\,dz \quad \forall u \in H^1(\Omega).$$

Evidently $\varphi \in C^2\big(H^1(\Omega)\big)$ and we have

$$\varphi'(u) = A(u) - N_f(u), \tag{3.1}$$

where $A \in \mathcal{L}\big(H^1(\Omega),\ H^1(\Omega)^*\big)$ is defined by

$$\langle A(u),\ y\rangle = \int_\Omega (\nabla u,\ \nabla y)_{\mathbb{R}^N}\, dz \quad \forall u, y \in H^1(\Omega)$$

and

$$N_f(u)(\cdot) = f\big(\cdot, u(\cdot)\big) \quad \forall u \in H^1(\Omega).$$

Proposition 3.2. *If hypotheses* H *hold, then* φ *satisfies the Cerami condition.*

Proof. Let $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ be a sequence, such that

$$\big|\varphi(u_n)\big| \leqslant M_1 \quad \forall n \geqslant 1, \tag{3.2}$$

for some $M_1 > 0$ and

$$\big(1 + \|u_n\|\big)\varphi'(u_n) \longrightarrow 0 \quad \text{in } H^1(\Omega)^*. \tag{3.3}$$

From (3.3), we have

$$\big|\langle \varphi'(u_n),\ h\rangle\big| \leqslant \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \forall h \in H^1(\Omega),$$

with $\varepsilon_n \to 0^+$, so

$$\left|\langle A(u_n),\ h\rangle - \int_\Omega f(z, u_n) h\, dz\right| \leqslant \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \forall n \geqslant 1 \tag{3.4}$$

(see (3.1)). In (3.4) we choose $h = u_n \in H^1(\Omega)$. Then

$$-\|\nabla u_n\|_2^2 + \int_\Omega f(z, u_n) u_n\, dz \leqslant \varepsilon_n \quad \forall n \geqslant 1. \tag{3.5}$$

On the other hand, from (3.2), we have

$$\|\nabla u_n\|_2^2 - \int_\Omega 2F(z, u_n)\, dz \leqslant 2M_1 \quad \forall n \geqslant 1. \tag{3.6}$$

Adding (3.5) and (3.6), we obtain

$$\int_\Omega \big(f(z, u_n) u_n - 2F(z, u_n)\big)\, dz \leqslant M_2 \quad \forall n \geqslant 1,$$

for some $M_2 > 0$. Suppose that $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is unbounded. By passing to a subsequence if necessary, we may assume that $\|u_n\| \longrightarrow +\infty$. Then

$$\limsup_{n\to+\infty} \frac{1}{\|u_n\|^p} \int_\Omega \big(f(z, u_n) u_n - 2F(z, u_n)\big)\, dz \leqslant 0. \tag{3.7}$$

Let

$$y_n = \frac{u_n}{\|u_n\|} \quad \forall n \geqslant 1.$$

Then $\|y_n\| = 1$ for all $n \geqslant 1$ and so, passing to a subsequence if necessary, we may assume that

$$y_n \longrightarrow y \quad \text{weakly in } H^1(\Omega), \tag{3.8}$$

$$y_n \longrightarrow y \quad \text{in } L^2(\Omega). \tag{3.9}$$

From (3.4), we have

$$\left|\langle A(y_n),\ h\rangle - \int_\Omega \frac{f(z, u_n)}{\|u_n\|} h\, dz\right| \leqslant \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \forall h \in H^1(\Omega),\ n \geqslant 1. \tag{3.10}$$

Let $h = y_n - y \in H^1(\Omega)$. Note that the sequence $\big\{\frac{f(\cdot, u_n(\cdot))}{\|u_n\|}\big\}_{n\geqslant 1} \subseteq L^2(\Omega)$ is bounded (see hypotheses H(i) and (ii)). So, if we pass to the limit as $n \to +\infty$ in (3.10), then

$$\lim_{n\to+\infty} \langle A(y),\ y_n - y\rangle = 0,$$

so

$$\|\nabla y_n\|_2 \longrightarrow \|\nabla y\|_2$$

(since $A(y_n) \longrightarrow A(y)$ weakly in $H^1(\Omega)^*$). From (3.8), we also have that

$$\nabla y_n \longrightarrow \nabla y \quad \text{weakly in } L^2(\Omega;\mathbb{R}^N).$$

So, by virtue of the Kadec–Klee property of Hilbert spaces, we have

$$\nabla y_n \longrightarrow \nabla y \quad \text{in } L^2(\Omega;\mathbb{R}^N),$$

so

$$y_n \longrightarrow y \quad \text{in } H^1(\Omega) \tag{3.11}$$

(see (3.8)), hence $\|y\| = 1$.

Since the sequence $\left\{\frac{N_f(u_n)}{\|u_n\|}\right\}_{n\geqslant 1} \subseteq L^2(\Omega)$ is bounded and using hypothesis H(ii), we have (at least for a subsequence), that

$$\frac{N_f(u_n)}{\|u_n\|} \longrightarrow \widehat{\lambda}_m y \quad \text{weakly in } L^2(\Omega).$$

Therefore, passing to the limit as $n \to +\infty$ in (3.10), we obtain

$$\langle A(y),\ h\rangle = \int_\Omega \widehat{\lambda}_m yh\,dz \quad \forall h \in H^1(\Omega),$$

so

$$A(y) = \widehat{\lambda}_m y,$$

thus

$$\begin{cases} -\Delta y(z) = \widehat{\lambda}_m y(z) & \text{a.e. in } \Omega, \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

and so

$$y \in E(\widehat{\lambda}_m) \setminus \{0\}$$

(see (3.11)).

By virtue of the unique continuation property, we have that $y(z) \neq 0$ for almost all $z \in \Omega$. Then

$$|u_n(z)| \longrightarrow +\infty \quad \text{for almost all } z \in \Omega$$

and so by virtue of hypothesis H(iii) and Fatou's lemma, we have

$$\liminf_{n\to+\infty} \frac{1}{\|u_n\|^\mu} \int_\Omega \big(f(z,u_n)u_n - 2F(z,u_n)\big)\,dz \geqslant \beta_1 > 0. \tag{3.12}$$

Comparing (3.7) and (3.12), we reach a contradiction. This proves that the sequence $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is bounded. So, we may assume that

$$u_n \longrightarrow u \quad \text{weakly in } H^1(\Omega),$$
$$u_n \longrightarrow u \quad \text{in } L^2(\Omega).$$

Hence, if in (3.4) we choose $h = u_n - u \in H^1(\Omega)$ and pass to the limit as $n \to +\infty$, then

$$\lim_{n\to+\infty} \langle A(y_n),\ y_n - y\rangle = 0,$$

so

$$u_n \longrightarrow u \quad \text{in } H^1(\Omega)$$

(as before, using the Kadec–Klee property of Hilbert spaces).

This proves that φ satisfies the Cerami condition. $\square$

In the next proposition, we determine the behaviour of φ near zero. So, let

$$\overline{H}_k = \bigoplus_{i=0}^{k} E(\widehat{\lambda}_i) \quad \text{and} \quad \widehat{H}_k = \overline{H}_k^\perp = \overline{\bigoplus_{i\geq k+1} E(\lambda)_i}.$$

Proposition 3.3. *If hypotheses* H *hold, then* φ *has a local linking at the origin with respect to the orthogonal direct sum decomposition*

$$H^1(\Omega) = \overline{H}_k \oplus \widehat{H}_k.$$

Proof. By virtue of hypothesis H(iv), for a given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$, such that

$$\frac{f(z,\zeta)}{\zeta} \geqslant f'_\zeta(z,0) - \varepsilon \quad \text{for almost all } z \in \Omega, \text{ all } 0 < |\zeta| \leqslant \delta,$$

so

$$F(z,\zeta) \geqslant \frac{1}{2}\big(f'_\zeta(z,0) - \varepsilon\big)\zeta^2 \quad \text{for almost all } z \in \Omega, \text{ all } |\zeta| \leqslant \delta. \tag{3.13}$$

Let $u \in \overline{H}_k$. Since $\overline{H}_k$ is finite dimensional all norms are equivalent. So, we can find $\varrho_0 > 0$, such that

$$\|u\| \leqslant \varrho_0 \Longrightarrow \big|u(z)\big| \leqslant \delta \quad \text{for almost all } z \in \overline{\Omega}. \tag{3.14}$$

Therefore, if $u \in \overline{H}_k$ with $\|u\| \leqslant \varrho_0$, then

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(z,u)\,dz \\ &\leqslant \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_\Omega f'_\zeta(z,0)u^2\,dz + \frac{\varepsilon}{2}\|u\|^2 \\ &\leqslant \frac{\varepsilon - \xi_1}{2}\|u\|^2 \end{aligned}$$

(see (3.13) and (3.14) and Proposition 2.5(b)).

Choosing $\varepsilon \in (0, \xi_1)$, we infer that

$$\varphi(u) \leqslant 0 \quad \forall u \in \overline{H}_k, \ \|u\| \leqslant \varrho_0.$$

On the other hand, hypotheses H(i) and (iv) imply that for a given $\varepsilon > 0$, we can find $c_\varepsilon > 0$, such that

$$F(z,\zeta) \leqslant \frac{1}{2}\big(f'_\zeta(z,0) + \varepsilon\big)\zeta^2 + c_\varepsilon|\zeta|^{r_0} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}, \tag{3.15}$$

with $2 < r_0$. Therefore, if $u \in \widehat{H}_k$, then

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(z,u)\,dz \\ &\geqslant \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\int_\Omega f'_\zeta(z,0)u^2\,dz - \frac{\varepsilon}{2}\|u\|^2 - c_1\|u\|^{r_0} \\ &\geqslant \frac{\xi_0 - \varepsilon}{2}\|u\|^2 - c_1\|u\|^{r_0}, \end{aligned}$$

for some $c_1 > 0$ (see (3.15) and Proposition 2.5(a)).

Choosing $\varepsilon \in (0, \xi_0)$, we have

$$\varphi(u) \geqslant c_2\|u\|^2 - c_1\|u\|^{r_0} \quad \forall u \in \widehat{H}_k \tag{3.16}$$

for some $c_2 > 0$.

Since $r_0 > 2$, from (3.16), we see that, if we choose $\varrho_1 > 0$ small, then

$$\varphi(u) \geqslant 0 \quad \forall u \in \widehat{H}_k, \ \|u\| \leqslant \varrho_1.$$

Therefore, for $\varrho = \min\{\varrho_0, \varrho_1\}$, we have

$$\varphi(u) \begin{cases} \leqslant 0 & \text{if } u \in \overline{H}_k, \ \|u\| \leqslant \varrho, \\ \geqslant 0 & \text{if } u \in \widehat{H}_k, \ \|u\| \leqslant \varrho. \end{cases}$$

This proves that φ has a local linking at the origin with respect to the orthogonal direct sum decomposition $H^1(\Omega) = \overline{H}_k \oplus \widehat{H}_k$ (see Definition 2.8). $\square$

Invoking Proposition 2.9, we have the following.

Proposition 3.4. *If hypotheses* H *hold, then*

$$C_l(\varphi, 0) = \delta_{l,d_k}\mathbb{Z} \quad \forall l \geqslant 0,$$

where

$$d_k = \dim \bigoplus_{i=0}^{k} E(\widehat{\lambda}_i).$$

Next, we compute the critical groups of φ at infinity.

Proposition 3.5. *If hypotheses* H *hold, then*

$$C_l(\varphi, \infty) = \delta_{l,d_{m-1}}\mathbb{Z} \quad \forall l \geqslant 0,$$

where

$$d_{m-1} = \dim \bigoplus_{i=0}^{m-1} E(\widehat{\lambda}_i).$$

Proof. Let $\eta \in (\lambda_{m-1}, \lambda_m)$ and consider the C^2-functional $\psi : H^1(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\psi(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\eta}{2}\|u\|_2^2 \quad \forall u \in H_0^1(\Omega).$$

We consider the homotopy h: $[0, 1] \times H^1(\Omega) \longrightarrow \mathbb{R}$, defined by

$$h_t(u) = (1-t)\varphi(u) + t\psi(u) \quad \forall (t, u) \in [0, 1] \times H^1(\Omega).$$

Claim 1. There exist $a \in \mathbb{R}$ and $\delta > 0$, such that

$$h_t(u) \leqslant a \Longrightarrow \big(1 + \|u\|\big)\big\|(h_t)'(u)\big\|_* \geqslant \delta \quad \forall t \in [0, 1].$$

We argue by contradiction. So, suppose that the Claim is not true. Since $(t, u) \longmapsto h_t(u)$ is bounded (i.e., maps bounded sets to bounded sets), we can find two sequences $\{t_n\}_{n\geqslant 1} \subseteq [0, 1]$ and $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$, such that

$$t_n \longrightarrow t, \qquad \|u_n\| \longrightarrow +\infty, \qquad h_{t_n}(u_n) \longrightarrow -\infty \tag{3.17}$$

and

$$(h_{t_n})'(u_n) \longrightarrow 0 \quad \text{in } H^1(\Omega)^*. \tag{3.18}$$

From the convergence (3.18), we have

$$\left|\big\langle A(u_n),\ h\big\rangle - (1-t_n)\int_\Omega f(z, u_n)h\,dz - t_n\eta\int_\Omega u_n h\,dz\right| \leqslant \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \quad \forall h \in H^1(\Omega), \tag{3.19}$$

with $\varepsilon_n \to 0^+$.

Let

$$y_n = \frac{u_n}{\|u_n\|} \quad \forall n \geqslant 1.$$

Then $\|y_n\| = 1$ for all $n \geqslant 1$. So, by passing to a suitable subsequence if necessary, we may assume that

$$y_n \longrightarrow y \quad \text{weakly in } H^1(\Omega), \tag{3.20}$$

$$y_n \longrightarrow y \quad \text{in } L^2(\Omega). \tag{3.21}$$

From (3.19), we have

$$\left|\big\langle A(y_n),\ h\big\rangle - (1-t_n)\int_\Omega \frac{f(z, u_n)}{\|u_n\|}h\,dz - t_n\eta\int_\Omega y_n h\,dz\right| \leqslant \frac{\varepsilon_n\|h\|}{(1 + \|u_n\|)\|u_n\|} \quad \forall n \geqslant 1. \tag{3.22}$$

Clearly the sequence $\left\{\frac{N_f(u_n)}{\|u_n\|}\right\}_{n\geqslant 1} \subseteq L^2(\Omega)$ is bounded (see hypotheses H(i) and (ii)). So, if in (3.22) we choose $h = y_n - y$, pass to the limit as $n \to +\infty$ and use (3.20), then

$$\lim_{n\to+\infty}\big\langle A(y_n),\ y_n - y\big\rangle = 0,$$

so

$$y_n \longrightarrow y \quad \text{in } H^1(\Omega) \tag{3.23}$$

(as before, via the Kadec–Klee property), hence $\|y\| = 1$. Hypothesis H(ii) implies that at least for a subsequence, we have

$$\frac{N_f(u_n)}{\|u_n\|} \longrightarrow \widehat{\lambda}_m u \quad \text{weakly in } L^2(\Omega). \tag{3.24}$$

Therefore, if in (3.22) we pass to the limit as $n \to +\infty$ and use (3.23) and (3.24), then

$$\langle A(y),\ h\rangle = \eta_t \int_\Omega yh\,dz \quad \forall h \in H^1(\Omega),$$

with $\eta_t = (1-t)\widehat{\lambda}_m + t\eta$, so

$$A(y) = \eta_t y$$

and thus

$$\begin{cases} -\Delta y(z) = \eta_t y(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.25}$$

If $t \neq 0$, then $\eta_t \in (\widehat{\lambda}_{m-1}, \widehat{\lambda}_m)$ and so from (3.25) we infer that $y = 0$, which contradicts (3.23).

If $t = 0$, then $\eta_t = \widehat{\lambda}_m$ and so from (3.23) and (3.25), we have $y \in E(\widehat{\lambda}_m) \setminus \{0\}$. Hence by the unique continuation property, we have

$$y(z) \neq 0 \quad \text{for almost all } z \in \Omega$$

and so

$$\big|u_n(z)\big| \longrightarrow +\infty \quad \text{for almost all } z \in \Omega.$$

Hypothesis H(iii) and Fatou's lemma, imply that

$$\liminf_{n\to+\infty} \frac{1}{\|u_n\|^\mu} \int_\Omega \big(f(z,u_n)u_n - 2F(z,u_n)\big)\,dz \geqslant \beta_2 > 0. \tag{3.26}$$

On the other hand, from the third convergence in (3.17), we can find an integer $n_0 \geqslant 1$, such that

$$h_{t_n}(u_n) \leqslant 0 \quad \forall n \geqslant n_0,$$

so

$$\|\nabla u_n\|_2^2 - (1-t_n)\int_\Omega 2F(z,u_n)\,dz - t_n\eta\|u_n\|_2^2 \leqslant 0 \quad \forall n \geqslant n_0. \tag{3.27}$$

In (3.19) we choose $h = u_n$ and obtain

$$-\|\nabla u_n\|_2^2 + (1-t_n)\int_\Omega f(z,u_n)u_n\,dz + t_n\eta\|u_n\|_2^2 \leqslant \varepsilon_n \quad \forall n \geqslant n_0. \tag{3.28}$$

Adding (3.27) and (3.28), we have

$$(1-t_n)\int_\Omega \big(f(z,u_n)u_n - 2F(z,u_n)\big)\,dz \leqslant M_2 \quad \forall n \geqslant n_0,$$

for some $M_2 > 0$.

Since $t = 0$ and $t_n \longrightarrow t = 0$, we may assume that $1 - t_n > 0$ for all $n \geqslant n_0$. Then

$$\frac{1}{\|u_n\|^\mu}\int_\Omega \big(f(z,u_n)u_n - 2F(z,u_n)\big)\,dz \leqslant \frac{M_2}{(1-t_n)\|u_n\|^\mu} \quad \forall n \geqslant n_0,$$

so

$$\limsup_{n\to+\infty} \frac{1}{\|u_n\|} \int_\Omega \big(f(z,u_n)u_n - 2F(z,u_n)\big)\,dz \leqslant 0. \tag{3.29}$$

Comparing (3.26) and (3.29), we reach a contradiction. This proves the Claim.

From Proposition 3.2, we know that $h_0 = \varphi$ satisfies the Cerami condition. Also, since $\eta \in (\widehat{\lambda}_{m-1}, \widehat{\lambda}_m)$, $h_1 = \psi$ satisfies the Cerami condition. Moreover, it is clear that the maps $u \longmapsto (h_t)'(u)$ and $u \longmapsto \partial_t h_t(u)$ are both locally Lipschitz. These facts and the Claim, permit the use of Proposition 2.7 and we have

$$C_l(\varphi, \infty) = C_l(\psi, \infty) \quad \forall l \geqslant 0. \tag{3.30}$$

Since $\eta \in (\widehat{\lambda}_{m-1}, \widehat{\lambda}_m)$, we have that $K_\psi = \{0\}$, hence

$$C_l(\psi, \infty) = C_l(\psi, 0) \quad \forall l \geqslant 0. \tag{3.31}$$

Clearly $u = 0$ is a nondegenerate critical point of ψ with Morse index

$$d_{m-1} = \dim \bigoplus_{i=0}^{m-1} E(\widehat{\lambda}_i).$$

Therefore

$$C_l(\psi, 0) = \delta_{l, d_{m-1}} \mathbb{Z} \quad \forall l \geqslant 0$$

(see (2.5)), so

$$C_l(\varphi, \infty) = \delta_{l, d_{m-1}} \mathbb{Z} \quad \forall l \geqslant 0$$

(see (3.30) and (3.31)). $\square$

This proposition leads to the first nontrivial smooth solution of (1.1).

Proposition 3.6. *If hypotheses* H *hold, then problem* (1.1) *has a solution* $u_0 \in C^1(\overline{\Omega})$, $u_0 \neq 0$, *such that*

$$C_l(\varphi, u_0) = \delta_{l, d_{m-1}} \mathbb{Z} \quad \forall l \geqslant 0.$$

Proof. From Proposition 3.5, we have

$$C_l(\varphi, \infty) = \delta_{l, d_{m-1}} \mathbb{Z} \quad \forall l \geqslant 0.$$

Hence, there is $u_0 \in K_\varphi$, such that

$$C_{d_{m-1}}(\varphi, u_0) \neq 0.$$

Since $d_k < d_{m-1}$ (recall that $k < m-2$), from Proposition 3.4 it follows that $u_0 \neq 0$. Since $u_0 \in K_\varphi$, we have that u_0 is a nontrivial solution of (1.1) and by standard regularity theory, we have $u_0 \in C^1(\overline{\Omega})$. $\square$

With the next two propositions, we produce a second nontrivial smooth solution of (1.1).

Proposition 3.7. *If hypotheses* H *hold, then problem* (1.1) *has a solution* $\widehat{u} \in C^1(\overline{\Omega})$, $\widehat{u} \neq 0$, *such that*

$$C_{d_{k+1}}(\varphi, \widehat{u}) \neq 0.$$

Proof. Recall that

$$\overline{H}_k = \bigoplus_{i=0}^{k} E\big(\widehat{\lambda}_i\big) \quad \text{and} \quad \widehat{H}_k = \overline{H}_k^{\perp} = \overline{\bigoplus_{i \geqslant k+1} E\big(\widehat{\lambda}_i\big)}.$$

We have

$$H^1(\Omega) = \overline{H}_k \oplus \widehat{H}_k.$$

From (3.16), we see that we can find $\varrho \in (0, 1)$ small and $\xi_2 > 0$, such that

$$\varphi(u) \geqslant \xi_2 > 0 \quad \forall u \in \widehat{H}_k \cap \partial B_\varrho, \tag{3.32}$$

where $\partial B_\varrho = \big\{u \in H^1(\Omega) : \ \|u\| = \varrho\big\}$.

By virtue of hypotheses H(i) and (ii), for a given $\varepsilon > 0$, we can find $M_\varepsilon > 0$, such that

$$F(z, \zeta) \geqslant \frac{1}{2}\big(\widehat{\lambda}_m - \varepsilon\big)\zeta^2 - M_\varepsilon \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}.$$

Let $e \in E(\widehat{\lambda}_{k+1})$ be such that $\|e\| = 1$ and let $u = \overline{u} + \vartheta e$, where $\overline{u} \in \overline{H}_k$ and $\vartheta \in \mathbb{R}$. Then exploiting the orthogonality of $\overline{H}_k$ and $E(\widehat{\lambda}_{k+1})$, we have

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(z,u)\,dz \\ &\leqslant \frac{1}{2}\|\nabla \overline{u}\|_2^2 + \frac{\vartheta^2}{2}\|\nabla e\|_2^2 - \frac{\widehat{\lambda}_m}{2}\|\overline{u}\|_2^2 - \frac{\widehat{\lambda}_m\vartheta^2}{2}\|e\|_2^2 + \frac{\varepsilon}{2}\|\overline{u}\|^2 + \frac{\vartheta^2\varepsilon}{2}\|e\|^2 \\ &\leqslant \frac{\varepsilon-\xi_1}{2}\|\overline{u}\|^2 + \frac{\vartheta^2}{2}\big(\widehat{\lambda}_k - \widehat{\lambda}_m + \varepsilon\big)\|e\|_2^2,\end{aligned}$$

so

$$\varphi|_{\overline{H}_k\oplus\mathbb{R}e} \leqslant 0 \tag{3.33}$$

(by choosing $\varepsilon \in \big(0, \widehat{\lambda}_m - \widehat{\lambda}_k\big)$, since $k < m$).

Let

$$\begin{aligned} C &= \big\{u = \overline{u} + \vartheta e :\ \overline{u} \in \overline{H}_k,\ \vartheta \geqslant 0,\ \|u\| \leqslant R\big\}, \\ C_0 &= \big\{u = \overline{u} + \vartheta e :\ \overline{u} \in \overline{H}_k,\ \big(\vartheta \geqslant 0,\ \|u\| = R\big) \text{ or } \big(\vartheta = 0,\ \|u\| \leqslant \mathbb{R}\big)\big\},\end{aligned}$$

with $R > \varrho$. Also, let

$$D = \widehat{H}_k \cap \partial B_\varrho.$$

Then from Papageorgiou–Kyritsi [23, p. 278], we know that the pair $\{C, C_0\}$ is linking with D in $H^1(\Omega)$ (see Definition 2.1). Because of (3.32) and (3.33) and since φ satisfies the Cerami condition (see Proposition 3.2), we can apply Theorem 2.2 and find $\widehat{u} \in K_\varphi$, such that

$$\varphi(0) = 0 < \xi_2 \leqslant \varphi(\widehat{u}),$$

so

$$\widehat{u} \neq 0 \quad \text{and} \quad \widehat{u} \in C^1(\overline{\Omega}) \text{solves (1.1)}.$$

Moreover, from Chang [24, p. 84], we have

$$C_{d_{k+1}}(\varphi, \widehat{u}) \neq 0, \quad \text{with } d_{k+1} = \dim \bigoplus_{i=0}^{k+1} E(\widehat{\lambda}_i). \quad \Box \tag{3.34}$$

We need to show that $\widehat{u} \neq u_0$. This is done with the next proposition.

Proposition 3.8. *If hypotheses* H *hold and* $\widehat{u} \in C^1(\overline{\Omega})$ *is the solution of problem* (1.1) *obtained in* Proposition 3.7, *then*

$$C_{d_{m-1}}(\varphi, \widehat{u}) = 0.$$

Proof. Let $h \in L^\infty(\Omega)_+$ be defined by

$$h(z) = \begin{cases} \dfrac{f(z, \widehat{u}(z))}{\widehat{u}(z)} & \text{if } \widehat{u}(z) \neq 0, \\ f'_\zeta(z, 0) & \text{if } \widehat{u}(z) = 0. \end{cases}$$

We consider the following linear weighted eigenvalue problem:

$$\begin{cases} -\Delta u(z) = \lambda h(z) u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.35}$$

Since $\widehat{u} \in C^1(\overline{\Omega})$ is a nontrivial solution of (1.1) (see Proposition 3.7), we see that $\lambda = 1$ is an eigenvalue of (3.35).

Note that

$$h(z) \geqslant \widehat{\lambda}_k \quad \text{for almost all } z \in \Omega.$$

If

$$h(z) = \widehat{\lambda}_k \quad \text{for almost all } z \in \Omega,$$

then $\widehat{u} = \xi \widehat{u}_k$, where $\xi \in \mathbb{R} \setminus \{0\}$ and $\widehat{u}_k \in E(\widehat{\lambda}_k) \setminus \{0\}$. To fix things, we assume that $\widehat{\xi} > 0$ (the analysis is similar if $\widehat{\xi} < 0$). Let

$$\Omega_+^k = \{\widehat{u}_k > 0\}, \qquad \Omega_-^k = \{\widehat{u}_k < 0\}.$$

If by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$, then by the unique continuation property, we have

$$|\Omega|_N = |\Omega_+^k|_N + |\Omega_-^k|_N.$$

Recall that hypothesis H(iv) implies that for almost all $z \in \Omega$, we have

$$\text{the map } \zeta \longmapsto \frac{f(z,\zeta)}{\zeta} \text{ is nondecreasing on } (0, +\infty)$$

and

$$\text{the map } \zeta \longmapsto \frac{f(z,\zeta)}{\zeta} \text{ is nonincreasing on } (-\infty, 0).$$

Therefore, if $\xi \in (0, \widehat{\xi}]$, then

$$\frac{f(z, \xi \widehat{u}_k(z))}{\xi \widehat{u}_k(z)} \leqslant \frac{f(z, \widehat{\xi} \widehat{u}_k(z))}{\widehat{\xi} \widehat{u}_k(z)} \quad \text{for almost all } z \in \Omega_+^k,$$

so

$$\frac{f(z, \xi \widehat{u}_k(z))}{\xi \widehat{u}_k(z)} = \widehat{\lambda}_k \quad \text{for almost all } z \in \Omega_+^k.$$

Similarly, we have

$$\frac{f(z, \xi \widehat{u}_k(z))}{\xi \widehat{u}_k(z)} \leqslant \frac{f(z, \widehat{\xi} \widehat{u}_k(z))}{\widehat{\xi} \widehat{u}_k(z)} \quad \text{for almost all } z \in \Omega_-^k,$$

so

$$\frac{f(z, \xi \widehat{u}_k(z))}{\xi \widehat{u}_k(z)} = \widehat{\lambda}_k \quad \text{for almost all } z \in \Omega_-^k.$$

It follows that

$$\frac{f(z, \xi \widehat{u}_k(z))}{\xi \widehat{u}_k(z)} = \widehat{\lambda}_k \quad \text{for almost all } z \in \Omega$$

and thus

$$f\big(z, \xi \widehat{u}_k(z)\big) = \widehat{\lambda} \xi \widehat{u}_k(z) \quad \text{for almost all } z \in \Omega.$$

We have

$$-\Delta\big(\xi \widehat{u}_k(z)\big) = \widehat{\lambda}_k \xi \widehat{u}_k(z) = f\big(z, \xi \widehat{u}_k(z)\big) \quad \text{for almost all } z \in \Omega,$$

so $\{\xi \widehat{u}_k\}_{\xi \in (0, \widehat{\xi}]} \subseteq C^1(\overline{\Omega})$ are all distinct smooth solutions of (1.1) and so we are done.

Therefore, we may assume that $h \neq \widehat{\lambda}_k$. Since $h(z) \geqslant \widehat{\lambda}_k$ for almost all $z \in \Omega$, from Proposition 2.4, we have

$$\widehat{\lambda}_i(h) < \widehat{\lambda}_i(\widehat{\lambda}_k) \leqslant 1 \quad \forall i \leqslant k.$$

Since $\lambda = 1$ is an eigenvalue of (3.35), we infer that

$$\widehat{\lambda}_{k+1}(h) \leqslant 1. \tag{3.36}$$

By virtue of hypothesis H(iv), we have

$$f'_\zeta\big(z, \widehat{u}(z)\big) \geqslant h(z) \quad \text{for almost all } z \in \Omega.$$

If $f'_\zeta\big(\cdot, \widehat{u}(\cdot)\big) \neq h(\cdot)$, then from Proposition 2.4 and (3.36), we have

$$\widehat{\lambda}_{k+1}\Big(f'_\zeta\big(\cdot, \widehat{u}(\cdot)\big)\Big) < 1,$$

so

$$\mu(\widehat{u}) \geqslant d_{k+1}. \tag{3.37}$$

On the other hand, from (3.34), we have

$$C_{d_{k+1}}(\varphi, \widehat{u}) \neq 0,$$

so

$$\mu(\widehat{u}) \leqslant d_{k+1} \tag{3.38}$$

(see Proposition 2.6). From (3.37) and (3.38), it follows that

$$\mu(\widehat{u}) = d_{k+1},$$

so

$$C_l(\varphi, \widehat{u}) = \delta_{l,d_{k+1}}\mathbb{Z} \quad \forall l \geqslant 0$$

(see Proposition 2.6) and thus

$$C_{d_{m-1}}(\varphi, \widehat{u}) = 0$$

(since $k+1 < m-1$).

Next, suppose that $f'_\zeta\big(\cdot, \widehat{u}(\cdot)\big) = h(\cdot)$. By virtue of hypothesis H(iv) and Proposition 2.4, we have

$$\widehat{\lambda}_m\big(f'_\zeta\big(\cdot, \widehat{u}(\cdot)\big)\big) > \widehat{\lambda}_m(\widehat{\lambda}_m) = 1,$$

so

$$\mu(\widehat{u}) + \nu(\widehat{u}) \leqslant m-1.$$

If $\mu(\widehat{u}) = k+1$, then from Proposition 2.6(b), we have

$$C_l(\varphi, \widehat{u}) = \delta_{l,d_{k+1}}\mathbb{Z} \quad \forall l \geqslant 0$$

(see (3.34)), so

$$C_{d_{m-1}}(\varphi, \widehat{u}) = 0.$$

If $\mu(\widehat{u}) < k+1$, then from Proposition 2.6(a), we have

$$C_{d_{m-1}}(\varphi, \widehat{u}) = 0$$

(see (3.34)). □

From Propositions 3.6 and 3.8, it follows that $\widehat{u} \neq u_0$ and so $\widehat{u} \in C^1(\overline{\Omega})$ is the second nontrivial smooth solution of (1.1). Therefore, we can state the following multiplicity theorem for problem (1.1).

Theorem 3.9. *If hypotheses* H *hold, then problem* (1.1) *has at least two distinct nontrivial smooth solutions*

$$u_0,\ \widehat{u} \in C^1(\overline{\Omega}).$$

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