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A simple solution of some composition conjectures for Abel equations

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ABSTRACT

Trigonometric Abel differential equations appear in the study of the number of limit cycles and the center-focus problem for certain families of planar polynomial systems. The composition centers are a class of centers for trigonometric Abel equations which have been widely studied during last years. We characterize this type of centers as the ones given by couples of trigonometric polynomials for which all the generalized moments vanish. They also coincide with the strongly and the highly persistent centers. Our result gives a simple and self-contained proof of the so called *Composition Conjecture for trigonometric* Abel differential equations. We also prove a similar version of this result for Abel equations with polynomial coefficients.

1. Introduction and main results

The study of Abel differential equations of the form

$$\dot{r} = \frac{dr}{d\theta} = A(\theta)\, r^3 + B(\theta)\, r^2 + C(\theta) r,$$

provides a useful tool for knowing either the number of limit cycles of certain planar polynomial differential equations or for studying the center-focus problem for them; see for instance [1–5]. These equations are also interesting in applications; see [6,7].

In this paper we consider Abel differential equations of the form

$$\dot{r} = A(\theta)\, r^3 + B(\theta)\, r^2, \tag{1}$$

defined on the cylinder $(r, \theta) \in \mathbb{R} \times \mathbb{R}/(2\pi\mathbb{Z})$, with A and B being trigonometric polynomials. We focus on the *center-focus* problem, and in particular, on obtaining conditions for A and B to ensure that all the solutions $r = r(\theta, r_0)$, with initial condition $r(0, r_0) = r_0$ and $|r_0|$ small enough, are 2π-periodic. Shortly, if this property holds, we will say that the Abel equation has a *center*. This question is relevant in the context of planar polynomial equations with homogeneous non-linearities, because the center-focus problem for them can be reduced to it; see [8,4].

In this work we give simple proofs of some *composition conjectures*. These are conjectures about the relation between a special type of centers, the ones satisfying the *composition condition*, the cancellation of some *moments* computed from A and B (see [9,10]) and the persistence under certain perturbations of the centers. To be more precise we introduce some definitions.

When there exist $\mathcal{C}^1$-functions A_1, B_1 and u, with u being 2π-periodic, such that

$$\widetilde{A}(\theta) := \int_0^\theta A(\psi)\, d\psi = A_1(u(\theta)) \quad \text{and} \quad \widetilde{B}(\theta) := \int_0^\theta B(\psi)\, d\psi = B_1(u(\theta)), \tag{2}$$

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it is said that the corresponding Abel equation satisfies the *composition condition*. This condition was introduced in [2] and ensures that the Abel equation has a center. In this situation we will say that the Abel equation has a *CC-center*.

This condition plays a similar role for Abel equations that being Hamiltonian, or reversible with respect to one line, for planar vector fields with homogeneous non-linearities. This is because it can be seen that if one of these systems has one of these types of center, the corresponding Abel equation, constructed from the Cherkas transformation, has a CC-center. Centers for (1) which are not CC-centers are given for instance in [11,9,12].

Another interesting family of centers is the class of *persistent centers*. Recall that it is said that Eq. (1) has a persistent center if the family of equations

$$\dot{r} = \varepsilon\, A(\theta)\, r^3 + B(\theta)\, r^2, \tag{3}$$

has a center for all ε small enough; see [9] and the references therein. This definition is equivalent to say that

$$\dot{r} = \alpha\, A(\theta)\, r^3 + \delta\, B(\theta)\, r^2, \tag{4}$$

has a center for all $\alpha, \delta \in \mathbb{R}$; see [12]. It is known that persistent centers satisfy the following *moment conditions*

$$\int_0^{2\pi} \widetilde{B}^p(\theta)\, A(\theta)\, d\theta = 0 \tag{5}$$

and

$$\int_0^{2\pi} \widetilde{A}^p(\theta) B(\theta)\ d\theta = 0, \tag{6}$$

for all natural numbers $p \in \mathbb{N} \cup \{0\}$; see [9,12].

Many authors have considered composition problems when A and B are polynomials instead of being trigonometric polynomials; see [10,13–19]. In Section 4 we give the precise definitions of center, CC-center, persistent center and moment conditions in this situation. In contrast to the trigonometric case, in the polynomial case all known centers are CC-centers. Whether all the centers for polynomial Abel equations are CC-centers is an interesting open question, which we will not treat here.

The relation between the above three concepts: CC-centers, persistent centers and moment conditions is of current interest. For instance it is clear that CC-centers are persistent centers and the corresponding A and B satisfy the moment conditions (5) and (6). In particular, to know whether conditions (5), either when A and B are trigonometric polynomials or when A and B are polynomials, imply that the corresponding Abel equation (1) has a CC-center has been known as the *Composition Conjecture*. In the polynomial case it has been shown to be false in [20]. In the trigonometric case, even assuming that (5) and (6) hold, it also turns out to be false; see [12]. The trigonometric counterexample given in that paper is

$$\dot{r} = \big(a\cos(2\theta) + b\sin(2\theta) + c\,\sin(6\theta)\big)\, r^3 + \frac{1}{32}\cos(3\theta)\, r^2. \tag{7}$$

For $a(a^2 - 3b^2) \neq 0$ it has a center which is not a CC-center but the moment conditions (5) and (6) for the corresponding functions A and B are satisfied. It was constructed from the class of integrable Lotka–Volterra quadratic systems in the plane.

Therefore, to characterize CC-centers, more restrictive conditions that the moment conditions (5) and (6) have to be given. Following [12] we will say that (1) has a *strongly persistent center* if

$$\frac{dr}{d\theta} = \big(\alpha\, A(\theta) + \beta\, B(\theta)\big)\, r^3 + \big(\gamma\, A(\theta) + \delta\, B(\theta)\big)\, r^2, \tag{8}$$

has a center for all $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. We introduce a new concept here. When, either

$$\frac{dr}{d\theta} = A(\theta)\, r^3 + \big(\gamma\, A(\theta) + \delta\, B(\theta)\big)\, r^2 \tag{9}$$

has a center for all $\gamma, \delta \in \mathbb{R}$, or

$$\frac{dr}{d\theta} = \big(\alpha\, A(\theta) + \beta\, B(\theta)\big)\, r^3 + B(\theta)\, r^2, \tag{10}$$

has a center for all $\alpha, \beta \in \mathbb{R}$ we will say that (1) has a *highly persistent center*. Although, in principle, strongly persistent centers are a subclass of the highly persistent ones, we will prove that both classes coincide. Notice also that as a consequence of this fact it is equivalent to say that Eq. (9) has a center for all the values of the parameters that to impose the same property for Eq. (10).

Given A and B, their associated *generalized moment conditions (GMC)* are

$$\int_0^{2\pi} \widetilde{A}^p(\theta)\, \widetilde{B}^q(\theta)\, A(\theta)\, d\theta = 0 \tag{11}$$

and

$$\int_0^{2\pi} \widetilde{A}^p(\theta)\,\widetilde{B}^q(\theta)\,B(\theta)\,d\theta = 0, \tag{12}$$

for all $p, q \in \mathbb{N} \cup \{0\}$. It is easy to see that for the Abel equation (7),

$$\int_0^{2\pi} \widetilde{A}^3(\theta)\,\widetilde{B}(\theta)\,B(\theta)\,d\theta \neq 0 \quad \text{and} \quad \int_0^{2\pi} \widetilde{A}^2(\theta)\,\widetilde{B}^2(\theta)\,A(\theta)\,d\theta \neq 0.$$

Therefore the corresponding A and B do not satisfy neither the GMC (11) nor (12). As far as we know, conditions (11)–(12) already appear in [21], studying analytic functions in a neighborhood of S^1, and in [22,19], in the context of Abel equations and the center problem.

Our main result is as follows.

Theorem 1. *Consider the Abel equation (1). The following statements are equivalent:*

(i) *The equation has a strongly persistent center.*
(ii) *The equation has a highly persistent center.*
(iii) *For the corresponding A and B the GMC (11) are satisfied and $\widetilde{B}(2\pi) = 0$.*
(iv) *For the corresponding A and B the GMC (12) are satisfied and $\widetilde{A}(2\pi) = 0$.*
(v) *For the corresponding A and B the GMC (11)–(12) are satisfied.*
(vi) *The equation has a CC-center.*

The most difficult part of the proof is to show that (v) implies (vi). Although this implication is already proved in a more general context in [23], in our paper we present a different, simple and self-contained proof. It relies on the characterization of all the subfields, containing some trigonometric polynomial, of the field of quotients of real trigonometrical polynomials; see Theorem 5. We learned of this characterization from [24]. Although the proof of [24] and our proof of this result are both based on Lüroth's Theorem, they are slightly different.

Using the results of [25,26] it can also be seen that all the classes appearing in the theorem are also equivalent to the class of so called *universal centers*, introduced in these papers.

Notice that Theorem 1 can be interpreted as the solution of the Composition Conjecture in the trigonometric setting, because it characterizes the CC-centers in terms of the cancellation of certain moments associated to A and B. Moreover, to the best of our knowledge, Theorem 1 is the first result that shows the equivalence of both concepts with more dynamic ones, the strongly and the highly persistence of the centers of the Abel equation.

The above questions have also been widely studied when the functions A and B, instead of being trigonometric polynomials are usual polynomials; see [10,13–19]. The equivalent version of Theorem 1 also holds in this context; see Theorem 9. In Section 4 we prove it using similar tools that for demonstrating Theorem 1. A proof that, in the polynomial case, the GMC imply that the Abel equation has a CC center, has already appeared in [23,27].

2. Preliminary results

From now on, $\mathbb{R}[x]$ will denote the ring of polynomials with real coefficients and $\mathbb{R}(x)$ its quotient field. Also we will denote by $\mathbb{R}[\theta]$ the ring of trigonometric polynomials with real coefficients and by $\mathbb{R}(\theta)$ its quotient field. It is well known that $\mathbb{R}(\theta)$ is isomorphic to $\mathbb{R}(x)$ by means of the map $\Phi : \mathbb{R}(\theta) \longrightarrow \mathbb{R}(x)$ defined by

$$\Phi(\sin\theta) = \frac{2x}{1+x^2} \quad \text{and} \quad \Phi(\cos\theta) = \frac{1-x^2}{1+x^2}. \tag{13}$$

In particular, this morphism satisfies that

$$\Phi\left(\tan(\theta/2)\right) = \Phi\left(\frac{\sin\theta}{1+\cos\theta}\right) = x.$$

The next lemma characterizes the image by Φ of the set of trigonometric polynomials.

Lemma 2. *It holds that*

$$\Phi(\mathbb{R}[\theta]) = \left\{ \frac{p(x)}{(1+x^2)^n} : p(x) \in \mathbb{R}[x] \text{ and } \deg(p(x)) \leq 2n \right\} =: \mathbb{T}(x).$$

Proof. From the definition of Φ it follows that $\Phi(\mathbb{R}[\theta]) \subset \mathbb{T}(x)$. To prove the converse inclusion it suffices to show that $\frac{x^i}{(1+x^2)^n} \in \Phi(\mathbb{R}[\theta])$ for all $i \leq 2n$ and all $n \in \mathbb{N}$. We will prove this fact by induction on n. For $n = 0$ the statement follows because $1 = \Phi(1) \in \Phi(\mathbb{R}[\theta])$. Assume that the statement holds for n and we prove it for $n+1$. Set $i \leq 2(n+1)$. If in addition $i \leq 2n$ then $\frac{x^i}{(1+x^2)^{n+1}} = \frac{x^i}{(1+x^2)^n} \cdot \frac{1}{1+x^2}$ which belongs to $\mathbb{T}(x)$ by the induction hypothesis. If $i \in \{2n+1, 2n+2\}$ then $\frac{x^i}{(1+x^2)^{n+1}} = \frac{x^{i-2}}{(1+x^2)^n} \cdot \frac{x^2}{1+x^2}$ that also belongs to $\mathbb{T}(x)$, again by the induction hypothesis. $\square$

Given $r, s \in \mathbb{R}(x)$ (respectively $r, s \in \mathbb{R}(\theta)$) we will say that they are *equivalent*, and we write $r \sim s$, if there exists a Möbius transformation γ such that $\gamma(r) = s$. Recall that a Möbius transformation γ is a rational map given by $\gamma(z) = \frac{az+b}{cz+d}$ for some fixed $a, b, c, d \in \mathbb{R}$ such that $ad - bc \neq 0$.

For $\xi \in \mathbb{R}(x)$ (respectively $\xi \in \mathbb{R}(\theta)$) we denote by $\mathbb{R}(\xi)$ the minimum field containing $\mathbb{R}$ and ξ. It is well known that $\mathbb{R}(r) = \mathbb{R}(s)$ if and only if $r \sim s$; see [28].

To state next result we need to introduce some definitions. For $\alpha \in \mathbb{R}$, let

$$\Delta, R_\alpha : \mathbb{R}[x] \times \mathbb{R}[x] \longrightarrow \mathbb{R}[x] \times \mathbb{R}[x]$$

be the maps defined by

$$\Delta(P, Q) = (P + xQ, Q - xP)$$

and

$$R_\alpha(P, Q) = (P \cos\alpha + Q \sin\alpha, -P \sin\alpha + Q \cos\alpha).$$

Easy computations show that both maps commute, that is $\Delta \circ R_\alpha = R_\alpha \circ \Delta$.

Proposition 3. *Consider the equation*

$$P^2 + Q^2 = (1 + x^2)^n \tag{14}$$

with $P, Q \in \mathbb{R}[x]$. The following assertions holds:

(a) *If (P, Q) satisfies Eq. (14) with $n = k$ then $\Delta(P, Q)$ and $\Delta(P, -Q)$ satisfy Eq. (14) with $n = k + 1$.*
(b) *If (P, Q) satisfies Eq. (14) and $\gcd(P, Q) = 1$ then either $\gcd(\Delta(P, Q)) = 1$ and $\gcd(\Delta(P, -Q)) \neq 1$ or vice versa.*
(c) *For any $n \geq 1$ Eq. (14) has a solution with $\gcd(P, Q) = 1$.*
(d) *If (P_1, Q_1) and (P_2, Q_2) are solutions of (14) with $\gcd(P_1, Q_1) = \gcd(P_2, Q_2) = 1$ then $P_1/Q_1 \sim P_2/Q_2$.*

Proof. To prove (a) assume that $P^2 + Q^2 = (1 + x^2)^k$. Then

$$(P + xQ)^2 + (Q - xP)^2 = P^2 + Q^2 + x^2Q^2 + x^2P^2 = (1 + x^2)^{k+1}.$$

To see (b) assume that (P, Q) satisfies Eq. (14), $\gcd(P, Q) = 1$ and $\gcd(\Delta(P, Q)) \neq 1$. From (a), $(P + xQ)^2 + (Q - xP)^2 = (1 + x^2)^{n+1}$. Hence it follows that the only common irreducible factor of $P + xQ$ and $Q - xP$ is $1 + x^2$ and the same situation holds for $P + x(-Q)$ and $-Q - xP$. Then if $\gcd(\Delta(P, -Q)) \neq 1$ we will obtain that $1 + x^2$ is a common factor of $P + xQ$ and $P + x(-Q)$. However this implies that $1 + x^2$ is a common factor of P and Q contradicting that $\gcd(P, Q) = 1$. On the other hand, since

$$(P + xQ)(P - xQ) = P^2 - x^2Q^2 = P^2 + Q^2 - (1 + x^2)Q^2 = (1 + x^2)((1 + x^2)^{n-1} - Q^2),$$

it follows that either $P + xQ$ or $P - xQ$ is a multiple of $1 + x^2$. In the first case we will have that $\gcd(\Delta(P, Q)) \neq 1$ and, in the second one, we will get that $\gcd(\Delta(P, -Q)) \neq 1$.

Now we prove (c) inductively. For $n = 1$ we have that $P = ax + b$, $Q = cx + d$ with $a^2 + c^2 = 1$, $b^2 + d^2 = 1$ and $ab + cd = 0$. Clearly all the solutions (P, Q) verify that $\gcd(P, Q) = 1$ and $\frac{P}{Q} \sim \frac{x}{1}$. Now assume the result holds for $n = k$ and we show it for $n = k + 1$. Let (P, Q) satisfy Eq. (14) with $n = k > 1$ and $\gcd(P, Q) = 1$. Then from (a) and (b) the result follows.

To see (d) we take a pair (P, Q) satisfying Eq. (14) with $n = k > 1$ and $\gcd(P, Q) = 1$ and we look for a pair satisfying (14) with $n = k - 1$.

As we have noticed either $P + xQ$ or $P - xQ$ is a multiple of $(1 + x^2)$. Assume for example that $P - xQ = (1 + x^2)R$ with $R \in \mathbb{R}[x]$. Then we get that

$$Q + xP = Q + x((1 + x^2)R + xQ) = (1 + x^2)(xR + Q).$$

Thus we will have that also $Q + xP$ is a multiple of $(1 + x^2)$. In this case we can consider $\Upsilon(P, Q) = (\frac{P - xQ}{1 + x^2}, \frac{Q + xP}{1 + x^2}) \in \mathbb{R}[x] \times \mathbb{R}[x]$. In the other case we can consider $\Upsilon(P, -Q) = (\frac{P + xQ}{1 + x^2}, \frac{-Q + xP}{1 + x^2})$. Note that in both situations we have that $\Delta(\Upsilon(P, Q)) = (P, Q)$. Also an easy computation shows that $\Upsilon(P, Q)$ satisfies (14) with $n = k - 1$. Moreover, if $\gcd(\Upsilon(P, Q)) \neq 1$ since $(P, Q) = \Delta(\Upsilon(P, Q))$ we obtain that $\gcd(P, Q) \neq 1$ which gives a contradiction.

Easy computations show that if (P, Q) and (R, S) satisfy Eq. (14) and $P/Q \sim R/S$, then necessarily either $(R, S) = R_\alpha(P, Q)$ or $(R, S) = R_\alpha(P, -Q)$ for some $\alpha \in [0, 2\pi]$.

Assume that for $n = k + 1$ Eq. (14) has two solutions (P_1, Q_1) and (P_2, Q_2) with $\gcd(P_1, Q_1) = \gcd(P_2, Q_2) = 1$.

Assume also, without loss of generality, that $P_1 - xQ_1$ and $P_2 - xQ_2$ are multiple of $(1 + x^2)$. Then we will have that $(\widehat{P}_1, \widehat{Q}_1) := \Upsilon(P_1, Q_1)$ and $(\widehat{P}_2, \widehat{Q}_2) := \Upsilon(P_2, Q_2)$ are solutions of Eq. (14) with $n = k$ and $\gcd(\Upsilon(P_1, Q_1)) = \gcd(\Upsilon(P_2, Q_2)) = 1$. Thus from the induction hypothesis we will have that $\widehat{P}_2/\widehat{Q}_2 \sim \widehat{P}_1/\widehat{Q}_1$. From the previous observation we will have that either $(\widehat{P}_2, \widehat{Q}_2) = R_\alpha(\widehat{P}_1, \widehat{Q}_1)$ or $(\widehat{P}_2, \widehat{Q}_2) = R_\alpha(\widehat{P}_1, -\widehat{Q}_1)$. In the first case we obtain

$$(P_2, Q_2) = \Delta(\widehat{P}_2, \widehat{Q}_2) = \Delta(R_\alpha(\widehat{P}_1, \widehat{Q}_1)) = R_\alpha(\Delta(\widehat{P}_1, \widehat{Q}_1)) = R_\alpha(P_1, Q_1)$$

and hence $P_1/Q_1 \sim P_2/Q_2$. In the second case we will have

$$(P_2, Q_2) = \Delta(\widehat{P}_2, \widehat{Q}_2) = \Delta(R_\alpha(\widehat{P}_1, -\widehat{Q}_1)) = R_\alpha(\Delta(\widehat{P}_1, -\widehat{Q}_1)).$$

Since $\Delta(\widehat{P}_1, \widehat{Q}_1) = (P_1, Q_1)$ and $\gcd(P_1, Q_1) = 1$ it follows from (b) that $\gcd(\Delta(\widehat{P}_1, -\widehat{Q}_1)) \neq 1$ and hence the same occurs for $(P_2, Q_2) = R_\alpha(\Delta(\widehat{P}_1, -\widehat{Q}_1))$. This contradicts the fact that $\gcd(P_2, Q_2) = 1$ and shows that this second possibility does not occur. This ends the proof of the proposition. □

Lemma 4. *Let P_n, Q_n be such that* $\gcd(P_n, Q_n) = 1$, $P_n^2(0) + Q_n^2(0) = 1$ *and* $\Phi(\tan(\frac{n\theta}{2})) = \frac{P_n}{Q_n}$. *Then* $P_n^2 + Q_n^2 = (1 + x^2)^n$.

Proof. We prove the lemma by induction. For $n = 1$ we have $\Phi(\tan(\frac{\theta}{2})) = x$. So $P_1 = x$ and $Q_1 = 1$. Thus we have $P_1^2 + Q_1^2 = 1 + x^2$.

Now assume that the lemma holds for $n = k$ and we prove it for $n = k + 1$. First of all note that

$$\tan\left(\frac{(k+1)\theta}{2}\right) = \frac{\tan(\frac{k\theta}{2}) + \tan(\frac{\theta}{2})}{1 - \tan(\frac{k\theta}{2})\tan(\frac{\theta}{2})}$$

and hence

$$\Phi\left(\tan\left(\frac{(k+1)\theta}{2}\right)\right) = \frac{\frac{P_k}{Q_k} + x}{1 - \frac{P_k}{Q_k}x} = \frac{P_k + Q_k x}{Q_k - P_k x} = \frac{P_{k+1}}{Q_{k+1}}.$$

With the notation introduced in the previous lemma we have that $(P_{k+1}, Q_{k+1}) = \Delta(P_k, Q_k)$ and then from Proposition 3(a) we get that $P_{k+1}^2 + Q_{k+1}^2 = (1 + x^2)^{k+1}$. Therefore to prove the result it remains to show that $\gcd(P_{k+1}, Q_{k+1}) = 1$.

If $\gcd(P_{k+1}, Q_{k+1}) \neq 1$ from Proposition 3(b) we will have that $\gcd(\Delta(P_k, -Q_k)) = 1$. If we write $\Delta(P_k, -Q_k) = (\widehat{P}_{k+1}, \widehat{Q}_{k+1})$ we have

$$\frac{\widehat{P}_{k+1}}{\widehat{Q}_{k+1}} = \frac{P_k - Q_k x}{-Q_k - P_k x} = -\frac{\frac{P_k}{Q_k} - x}{1 + \frac{P_k}{Q_k}x} = -\Phi\left(\tan\left(\frac{(k-1)\theta}{2}\right)\right)$$

and by the induction hypothesis we obtain that $\widehat{P}_{k+1}^2 + \widehat{Q}_{k+1}^2 = (1+x^2)^{k-1}$ in contradiction with the fact that $\widehat{P}_{k+1}^2 + \widehat{Q}_{k+1}^2 = (1 + x^2)^{k+1}$ because $(\widehat{P}_{k+1}, \widehat{Q}_{k+1}) = \Delta(P_k, -Q_k)$. Therefore $\gcd(\widehat{P}_{k+1}, \widehat{Q}_{k+1}) \neq 1$ and hence $\gcd(P_{k+1}, Q_{k+1}) = 1$. This ends the proof of the lemma. □

Next result is also proved in [24]. We include here a proof slightly different.

Theorem 5. *Let $\mathbb{K}$ be a subfield of $\mathbb{R}(\theta)$ containing a non-constant trigonometric polynomial. Then either $\mathbb{K} = \mathbb{R}(\tan(\frac{n\theta}{2}))$ for some $n \in \mathbb{N}$ or $\mathbb{K} = \mathbb{R}(p)$ for some trigonometric polynomial p.*

Proof. By Lüroth‘s Theorem it holds that $\mathbb{K} = \mathbb{R}(\xi)$ for some quotient of trigonometric polynomials ξ; see [28]. Set $\Phi(\xi) = \frac{p}{q}$, with $p, q \in \mathbb{R}[x]$ and $\gcd(p, q) = 1$, where Φ is defined in (13). By Lemma 2, the hypothesis that $\mathbb{K}$ contains some trigonometric polynomial is translated into the fact that $\mathbb{R}(\frac{p}{q})$ contains some element of the form $\frac{M}{(1+x^2)^n}$, with M a polynomial of degree at most $2n$. Changing p/q, if necessary, by a Möbius transformation we can assume that $\deg(p) > \deg(q)$. Let $R, S \in \mathbb{R}[x]$ be such that $\gcd(R, S) = 1$ and

$$\frac{R(\frac{p}{q})}{S(\frac{p}{q})} = \frac{M}{(1 + x^2)^n}.$$

Note that since $\deg(p) > \deg(q)$ necessarily $\deg(S) \geq 1$. Thus we obtain

$$\frac{q^s \widehat{R}(p, q)}{q^r \widehat{S}(p, q)} = \frac{M}{(1 + x^2)^n}, \tag{15}$$

where $\widehat{R}, \widehat{S}$ denotes the homogenization of R and S and r, s are the degrees of R and S respectively. We claim that $\gcd(\widehat{S}(p, q), q^s\widehat{R}(p, q)) = 1$. To see this it suffices to show that $\widehat{S}(p, q)$ does not share roots (real or complex) with q^s or $\widehat{R}(p, q)$.

Let $z \in \mathbb{C}$ be a root of $\widehat{S}(p, q)$ and suppose first that z is also a root of q. If $S = \sum_{i=0}^{s} a_i x^i$ with $a_s \neq 0$ then

$$\widehat{S}(p, q) = \sum_{i=0}^{s} a_i p^i q^{s-i} \quad \text{and} \quad \widehat{S}(p, q)(z) = a_s\, p^s(z) = 0.$$

Since $a_s \neq 0$, it holds that $p(z) = 0$ which contradicts that $\gcd(p, q) = 1$. So $q(z) \neq 0$.

Suppose now that z is also a root $\widehat{R}(p,q)$. Since $q(z)\neq 0$ we will obtain that $R(\frac{p(z)}{q(z)}) = S(\frac{p(z)}{q(z)}) = 0$ which contradicts that $\gcd(R,S)=1$.

Thus from (15) we obtain that $\widehat{S}(p,q) = (1+x^2)^k$ for some $k\geq 0$. Since $\widehat{S}$ is a homogeneous polynomial it decomposes in a product of real irreducible homogeneous polynomials of degrees 1 or 2. So we will have $\prod_{i=1}^{l}\widehat{S}_i(p,q) = (1+x^2)^k$. Clearly this implies that for each i, $\widehat{S}_i(p,q) = (1+x^2)^{k_i}$ for some $k_i\geq 0$. If there is some linear $\widehat{S}_i$ we have that there exists $0\neq a\in\mathbb{R}$ and $b\in\mathbb{R}$ such that $ap+bq = (1+x^2)^{k_i}$. Set $c,d\in\mathbb{R}$ such that $ad-bc\neq 0$. We will have that

$$\frac{p}{q}\sim\frac{c\frac{p}{q}+d}{a\frac{p}{q}+b} = \frac{cp+dq}{ap+bq} = \frac{cp+dq}{(1+x^2)^{k_i}}.$$

Since $\deg(cp+dq)\leq\deg(p)=\deg(ap+bq)=2k_i$ we get that $\mathbb{R}(\frac{p}{q})$ admits a generator of the form $\frac{N}{(1+x^2)^{k_i}}$ with $\deg(N)\leq 2k_i$. From Lemma 2 we get that $\mathbb{K}=\mathbb{R}(\xi)$ admits a polynomial generator. So the result follows in this case.

Now suppose that all $\widehat{S}_i$ are quadratic. Then, for each i, $\widehat{S}_i(p,q) = (1+x^2)^{k_i}$ with $\widehat{S}_i$ irreducible. Thus

$$(ap+bq)^2 + c^2q^2 = (1+x^2)^{k_i}$$

for some non-zero real numbers a and c. Therefore, considering

$$\frac{\overline{p}}{\overline{q}} = \frac{ap+bq}{cq},$$

we have that $\frac{\overline{p}}{\overline{q}}\sim\frac{p}{q}$ and $\overline{p}^{\,2}+\overline{q}^{\,2} = (1+x^2)^{k_i}$. Finally, from Proposition 3(c) and Lemma 4 we obtain that

$$\frac{\overline{p}}{\overline{q}}\sim\Phi\left(\tan\left(\frac{k_i\theta}{2}\right)\right),$$

and thus that $\xi\sim\tan(\frac{k_i\theta}{2})$, as we wanted to prove. $\square$

Finally we state two results for trigonometric Abel equations.

Proposition 6. *([12]) The following four conditions are equivalent:*

(i) *The differential equation $\dot r = A(\theta)r^3+\epsilon B(\theta)r^2$ has a center for all ϵ small enough.*
(ii) *The differential equation $\dot r = \epsilon\, A(\theta)r^3 + B(\theta)r^2$ has a center for all ϵ small enough.*
(iii) *The differential equation $\dot r = \alpha\, A(\theta)r^3 + \delta\, B(\theta)r^2$ has a center for all $\alpha,\delta\in\mathbb{R}$.*

Moreover, if these conditions are satisfied then the following moment conditions hold:

$$\int_0^{2\pi}\widetilde{B}^p(\theta)\,A(\theta)\,d\theta = \int_0^{2\pi}\widetilde{A}^p(\theta)B(\theta)\;d\theta = 0,$$

for all $p\in\mathbb{N}\cup\{0\}$.

Recall that the centers appearing in the above proposition are the so called persistent centers.

The next lemma relates the GMC (11) and (12).

Lemma 7. *Given two continuous functions A and B in $[0,2\pi]$ then:*

(i) *If A and B satisfy the GMC (12) and $\widetilde{A}(2\pi)=0$ then they also satisfy the GMC (11).*
(ii) *If A and B satisfy the GMC (11) and $\widetilde{B}(2\pi)=0$ then they also satisfy the GMC (12).*

Proof. It is clear that it suffices to prove item (i). By integration by parts,

$$\begin{aligned}\int_0^{2\pi}\widetilde{A}^p(\theta)\,\widetilde{B}^q(\theta)\,A(\theta)\,d\theta &= \frac{1}{p+1}\widetilde{A}^{p+1}(\theta)\widetilde{B}^q(\theta)\bigg|_0^{2\pi} - \frac{q}{p+1}\int_0^{2\pi}\widetilde{A}^{p+1}(\theta)\widetilde{B}^{q-1}(\theta)B(\theta)\,d\theta\\ &= -\frac{q}{p+1}\int_0^{2\pi}\widetilde{A}^{p+1}(\theta)\widetilde{B}^{q-1}(\theta)B(\theta)\,d\theta = 0.\quad\square\end{aligned}$$

3. Proof of Theorem 1

(iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v). These equivalences are a consequence of Lemma 7.
(i) $\Rightarrow$ (ii). This implication is clear.
(ii) $\Rightarrow$ (v). Assume for instance that we are in the case when condition (9) holds, i.e.

$$\frac{dr}{d\theta} = A(\theta)\, r^3 + \big(\gamma\, A(\theta) + \delta\, B(\theta)\big)\, r^2, \tag{16}$$

has a center for all $\gamma, \delta \in \mathbb{R}$. Then, for any fixed γ and δ, the above equation has a persistent center. In particular, from Proposition 6, we have

$$\int_0^{2\pi} \big(\gamma\, \widetilde{A}(\theta) + \delta\, \widetilde{B}(\theta)\big)^k A(\theta)\, d\theta = 0, \quad k \geq 0.$$

Then

$$F(\gamma, \delta) := \int_0^{2\pi} \big(\gamma\, \widetilde{A}(\theta) + \delta\, \widetilde{B}(\theta)\big)^k A(\theta)\, d\theta = \sum_{i=0}^{k} \gamma^i\, \delta^{k-i} \binom{k}{i} \int_0^{2\pi} \widetilde{A}^i(\theta)\, \widetilde{B}^{k-i}(\theta)\, A(\theta)\, d\theta = 0.$$

Since $F(\gamma, \delta)$ is a polynomial in γ and δ we obtain that all its coefficients are zero. Therefore we have proved that for all $k \in \mathbb{N},\ 0 \leq i \leq k$,

$$\int_0^{2\pi} \widetilde{A}^i(\theta)\, \widetilde{B}^{k-i}(\theta)\, A(\theta)\, d\theta = 0.$$

Moreover $\widetilde{B}(2\pi) = 0$ because the Abel equation (16) has a center. Then we have proved that (ii) $\Rightarrow$ (iii) and therefore that (v) holds.

The case where condition (10) holds can be treated similarly.

(v) $\Rightarrow$ (vi). Assume that all the generalized moments vanish and consider the field $\mathbb{K} := \mathbb{R}(\widetilde{A}(\theta), \widetilde{B}(\theta))$. Notice that since $\int_0^{2\pi} A(\psi)\, d\psi = \int_0^{2\pi} B(\psi)\, d\psi = 0$, the functions $\widetilde{A}$ and $\widetilde{B}$ are trigonometric polynomials. Therefore we can apply Theorem 5 and $\mathbb{K} = \mathbb{R}(\xi)$, with ξ either a trigonometric polynomial or $\xi = \tan(\frac{n\theta}{2})$ for some $n > 0$. Now we will see that the second possibility does not occur. Assume that

$$\frac{P(\widetilde{A}(\theta), \widetilde{B}(\theta))}{Q(\widetilde{A}(\theta), \widetilde{B}(\theta))} = \tan\left(\frac{n\theta}{2}\right),$$

for some $P, Q \in \mathbb{R}[x, y]$. Derivating with respect to θ we get

$$\frac{(QP_x - PQ_x)(\widetilde{A}(\theta), \widetilde{B}(\theta))A(\theta) + (QP_y - PQ_y)(\widetilde{A}(\theta), \widetilde{B}(\theta))B(\theta)}{Q^2(\widetilde{A}(\theta), \widetilde{B}(\theta))} = \frac{n}{2}\left(1 + \tan^2\left(\frac{n\theta}{2}\right)\right).$$

So

$$(QP_x - PQ_x)(\widetilde{A}(\theta), \widetilde{B}(\theta))A(\theta) + (QP_y - PQ_y)(\widetilde{A}(\theta), \widetilde{B}(\theta))B(\theta) = \frac{n}{2}(P^2 + Q^2)(\widetilde{A}(\theta), \widetilde{B}(\theta)).$$

Note that the integral in the interval $[0, 2\pi]$ on the left side of this equality is zero because it is the sum of a finite number of generalized moments, but the right side of the equality is a positive continuous function. This gives the desired contradiction.

So we conclude that $\mathbb{R}(\widetilde{A}(\theta), \widetilde{B}(\theta))$ is generated by a trigonometric polynomial p. Then $\widetilde{A}(\theta) = \frac{R_1}{S_1}(p(\theta))$ and $\widetilde{B}(\theta) = \frac{R_2}{S_2}(p(\theta))$ with $\frac{R_i}{S_i} \in \mathbb{R}(x)$ and $\gcd(R_i, S_i) = 1$ for $i = 1, 2$. We are going to prove that we can choose $S_1 = S_2 = 1$. We prove this fact for S_1. From Lemma 2 we have that

$$\frac{R_1}{S_1}\left(\frac{M}{(1 + x^2)^i}\right) = \frac{N}{(1 + x^2)^j},$$

with $M, N \in \mathbb{R}[x],\ \ \gcd(M, (1 + x^2)) = \gcd(N, (1 + x^2)) = 1,\ \ \deg(M) \leq 2i$ and $\deg(N) \leq 2j$. Adding, if necessary, a constant to $p(\theta)$ we can assume that $\deg(M) < 2i$. Now assume, in order to get a contradiction, that $\deg S_1 \geq 1$. Thus we obtain

$$\frac{(1 + x^2)^{is}\widehat{R}(M, (1 + x^2)^i)}{(1 + x^2)^{ir}\widehat{S}(M, (1 + x^2)^i)} = \frac{N}{(1 + x^2)^j},$$

where $\widehat{R}$ and $\widehat{S}$ denote the homogenization of R_1 and S_1 and r and s are the corresponding degrees of R_1 and S_1. Arguing as in the proof of Theorem 5 we obtain that $\widehat{S}(M, (1 + x^2)^i) = (1 + x^2)^k$ for some $k \leq j$. Since $\widehat{S}(M, (1 + x^2)^i) = a_s M^s + (1 + x^2)^i L$

with $L \in K[x]$, $a_s \neq 0$ and $\gcd(M, (1+x^2)) = 1$ we obtain that $k = 0$ and $\widehat{S}(M, (1+x^2)^i) = 1$. If we decompose the homogeneous polynomial $\widehat{S}$ in its real irreducible components we will obtain that for each one of them, say T,

$$T(M, (1+x^2)^i) \in \mathbb{R}.$$

If $\deg(T) = 2$ this last property does not hold because it is impossible that

$$(aM + b(1+x^2)^i)^2 + c^2(1+x^2)^{2i} \in \mathbb{R},$$

with a, b, c real numbers and $a \neq 0$ and $c \neq 0$. If $\deg(T) = 1$ we obtain $aM + b(1+x^2)^i \in \mathbb{R}$ for some $a, b \in \mathbb{R}$. Since $\deg M < 2i$ the only possibility is $b = 0$ and $M \in \mathbb{R}$. Then the only possible irreducible factor of T is x. Hence $S_1 = x^s$. However since $\gcd(R_1, S_1) = 1$, this implies that $R_1(0) \neq 0$ and $\deg \widehat{R}(M, (1+x^2)^i) = 2ir$. Since

$$\frac{(1+x^2)^{is}\widehat{R}(M, (1+x^2)^i)}{(1+x^2)^{ir}} = \frac{N}{(1+x^2)^j}$$

and $\deg(N) \leq 2j$ we get $s = 0$ and $S_1 = 1$. So, $\widetilde{A} = R_1(p)$. Similarly $\widetilde{B} = R_2(p)$ and the result follows.
(vi) $\Rightarrow$ (i). This implication is trivial because if Eq. (1) has a CC-center the same holds for Eq. (8).

Remark 8. Notice that if an Abel equation has a CC-center then there exist infinitely many functions A_1, B_1 and u satisfying (2), because if A_1, A_2, u satisfy the CC-condition, all the triplets $A_1 \circ h, A_2 \circ h, h^{-1} \circ u$, with h being a diffeomorphism satisfy (2) as well. As a consequence of the proof of Theorem 1 we will see that the trigonometric CC-centers always admit functions A_1, B_1 and u with A_1, B_1 polynomials and u a trigonometric polynomial.

4. Polynomial Abel equations

Similarly to the trigonometric case, for each two real numbers $a < b$ we can consider the problem of giving necessary and sufficient conditions for the two real polynomials $A(t)$ and $B(t)$ to ensure that the solutions of the equation

$$\frac{dx}{dt} = A(t)\,x^3 + B(t)\,x^2, \tag{17}$$

satisfy $x(a) = x(b)$, for all initial conditions close enough to the solution $x = 0$. This question is considered in several papers; see for instance [10,13–19]. The notions of center, CC-center, persistent center, strongly or highly persistent center, moment conditions and generalized moment conditions are similar to the ones presented for trigonometric Abel equations. For instance the GMC read as

$$\int_a^b \widetilde{A}^p(t)\,\widetilde{B}^q(t)\,A(t)\,dt = 0 \tag{18}$$

and

$$\int_a^b \widetilde{A}^p(t)\,\widetilde{B}^q(t)\,B(t)\,dt = 0, \tag{19}$$

for all $p, q \in \mathbb{N} \cup \{0\}$ and the condition of having a CC-center like

$$\widetilde{A}(t) := \int_a^t A(s)\,ds = A_1(u(t)) \quad \text{and} \quad \widetilde{B}(t) := \int_a^t B(s)\,ds = B_1(u(t)), \tag{20}$$

for some $\mathcal{C}^1$-functions A_1, B_1 and u, where u is such that $u(a) = u(b)$. The following result solves the Composition Conjecture in this setting.

Theorem 9. *Consider the polynomial Abel equation (17). The following statements are equivalent:*

(i) *The equation has a strongly persistent center.*
(ii) *The equation has a highly persistent center.*
(iii) *For the corresponding A and B the GMC (18) are satisfied and $\widetilde{B}(b) = 0$.*
(iv) *For the corresponding A and B the GMC (19) are satisfied and $\widetilde{A}(b) = 0$.*
(v) *For the corresponding A and B the GMC (18)–(19) are satisfied.*
(vi) *The equation has a CC-center.*

Remark 10. When a polynomial Abel equation (17) has a CC-center it follows from the proof of Theorem 9 that it is possible to choose A_1, B_1 and u in (20) being polynomials.

Our proof of Theorem 9 is based on the following result, which is quite similar to Theorem 5.

Theorem 11. *Let $\mathbb{K}$ be a subfield of $\mathbb{R}(x)$ containing a non-constant polynomial. Then $\mathbb{K} = \mathbb{R}(p)$ for some polynomial p. Moreover, if a polynomial $t \in \mathbb{K}$ then $t = R(p)$ for some polynomial R.*

Proof. By Lüroth's Theorem there exists a rational function $p/q \in \mathbb{R}(x)$, with $\gcd(p, q) = 1$, such that $\mathbb{K} = \mathbb{R}(\frac{p}{q})$. By using a Möbius transformation, if necessary, we can also assume that $\deg p > \deg q$. By hypothesis there exists $t \in \mathbb{R}[x] \cap \mathbb{K}$. Let $R, S \in \mathbb{R}[x]$ be such that $\frac{R}{S}(\frac{p}{q}) = t$ and $\gcd(R, S) = 1$. Equivalently,

$$\frac{q^s\widehat{R}(p, q)}{q^r\widehat{S}(p, q)} = t,$$

where $\widehat{R}$ and $\widehat{S}$ denote the homogenization of R and S and r, s denote the degrees of R and S, respectively. By using similar arguments as that in the proof of Theorem 5 we obtain that $\gcd(\widehat{R}(p, q), \widehat{S}(p, q)) = \gcd(q, \widehat{S}(p, q)) = 1$. Hence $\widehat{S}(p, q) = 1$. Since $\deg p > \deg q$ it follows that $\deg \widehat{S}(p, q) = s \deg p$ and hence $s = 0$ and S is constant. So we can assume $S = 1$. Therefore $\frac{\widehat{R}(p,q)}{q^r} = t$. Since $\gcd(\widehat{R}(p, q), q) = 1$ we get that q is also a constant polynomial. Thus, $t = R(p)$ and the result follows. $\square$

Proof of Theorem 9. The proofs of all the implications, except that (v) $\Rightarrow$ (vi), are similar to the corresponding ones in the trigonometric case and we omit them.

Next we show that (v) $\Rightarrow$ (vi). By Theorem 11, since $\widetilde{A}, \widetilde{B}$ are polynomials, we have that $\mathbb{R}(\widetilde{A}, \widetilde{B}) = \mathbb{R}(p)$ with $p \in \mathbb{R}[x]$. To prove the implication it suffices to show that $p(a) = p(b)$. We know that

$$p = \frac{P(\widetilde{A}, \widetilde{B})}{Q(\widetilde{A}, \widetilde{B})},$$

for some $P, Q \in \mathbb{R}[x, y]$. Derivating this expression we obtain

$$p' = \frac{(QP_x - PQ_x)(\widetilde{A}, \widetilde{B})A + (QP_y - PQ_y)(\widetilde{A}, \widetilde{B})B}{Q^2(\widetilde{A}, \widetilde{B})}.$$

Since $\widetilde{A}$ and $\widetilde{B}$ are polynomial functions of p we have that

$$Q^2(\widetilde{A}, \widetilde{B}) = Q^2(A_1(p), A_2(p)) =: M(p) := N'(p),$$

for some polynomials A_1, A_2 and M, and $N \in \mathbb{R}(x)$ such that $N' = M$. Thus

$$N'(p)p' = Q^2(\widetilde{A}, \widetilde{B})p' = (QP_x - PQ_x)(\widetilde{A}, \widetilde{B})A + (QP_y - PQ_y)(\widetilde{A}, \widetilde{B})B.$$

Integrating both sides of this equality in $[a, b]$ and using that the generalized moments vanish we obtain that $N(p(b)) - N(p(a)) = 0$. Since $N'(p) = Q^2(\widetilde{A}, \widetilde{B}) \geq 0$ we have that $N'(x) \geq 0$ for all x in the interval with extremes $p(a)$ and $p(b)$. Therefore N is increasing on this interval and $p(b) = p(a)$, as we wanted to prove. $\square$

Final remarks and open questions

We have given a simple proof of the Composition Conjecture for Abel equations in the polynomial and trigonometric polynomial settings. Both results can be easily extended for general equations of the form

$$\dot{r} = \sum_{k\geq 2} A_k(\theta)r^k,$$

having either a finite or an infinite sum, with the natural generalizations of the concepts appearing in this paper. We have only focused on the case of Abel equations because it already contains the main difficulties.

From our point of view, there are two problems that still deserve to be studied in this context. The first one is to know if all the persistent centers are also CC-centers.

The second one appears only in the polynomial case. It turns out that there is no known example that satisfies both moment conditions (5) and (6), and is not a CC-center. Recall that the example given in [20], with A and B constructed by using some Chebyshev polynomials, is not a CC-center but only the moments (5) vanish. The problem is to know whether such an example exists. The results of [29] seem to be a good starting point to investigate this question.

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