



Stability switching and Hopf bifurcation in a multiple-delayed neural network with distributed delay

Israel Ncube

Department of Mathematics, Memorial University of Newfoundland, Corner Brook, NL, A2H 5G4, Canada



ARTICLE INFO

Article history:

Received 20 September 2012

Available online 13 May 2013

Submitted by W.L. Wendland

Keywords:

Neural network

Delay differential equations

Symmetry

Discrete delay

Distributed delay

Characteristic equation

Absolute stability

Hopf bifurcation

ABSTRACT

We consider a network of three identical neurons incorporating distributed and discrete signal transmission delays. The model for such a network is a system of coupled nonlinear delay differential equations. It is established that two cases of a single Hopf bifurcation may occur at the trivial equilibrium of the system, as a consequence of the \mathbb{D}_3 symmetry of the network. These single Hopf bifurcations are the simple and the double root. The present paper looks at the simple root case, and addresses the issue of absolute stability of the trivial equilibrium and stability switching, leading up to calculation of the critical delay and formulation of a Hopf bifurcation theorem.

© 2013 Elsevier Inc. All rights reserved.

1. Introduction

In this article, we study a system of delay differential equations given by

$$x'_i(t) = -x_i(t) + \alpha \int_{-\infty}^t k(t-s)f(x_i(s))ds + \beta[f(x_{i-1}(t-\tau_n)) + f(x_{i+1}(t-\tau_n))], \quad (1)$$

where $i \bmod 3$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ is a sufficiently smooth sigmoid amplification function, normalised such that $f(0) = 0$ and $f'(0) = 1$. A special case to be used throughout this paper is

$$f(x) = \tanh(x), \quad x \in \mathbb{R}.$$

A discrete delay analogue of (1) has been the subject of many previous studies—for example [3,9], and references listed therein. The system models a network of three identical neurons, connected in such a way that there is signal transmission delay in self-connections and nearest-neighbour connections. In the present study, we have introduced distributed signal transmission delay in the self-connection of the network and maintained the usual discrete signal transmission delay in the nearest-neighbour connection. To the best of our knowledge, this type of architecture of the three-cell network is new. Whereas the cited previous studies investigated the Hopf bifurcation of equilibria and its criticality, the present study focuses on the asymptotic stability of the trivial equilibrium of (1). The goal is to determine conditions on the parameters of the equation guaranteeing delay-independent stability and, when such conditions are violated, showing which choices lead to destabilisation through Hopf bifurcation. The parameters α and β measure, respectively, the normalised synaptic strength of self-connection and nearest-neighbour connection. The parameter τ_n denotes the signal transmission delay of the nearest-neighbour connections, whereas $k(s)$ is the delay kernel of the self-connection. This delay kernel is a non-negative bounded

E-mail address: incube@grenfell.mun.ca.

function defined on $[0, \infty)$. Assume that the presence of distributed delay does not affect the equilibrium values, so we normalise the kernel such that

$$\int_0^\infty k(s)ds = 1.$$

The average time delay is defined as [7]

$$\int_0^\infty sk(s)ds < \infty.$$

Conventionally, the weak kernel $k(s) = re^{-rs}$ and the strong kernel $k(s) = r^2se^{-rs}$, $r > 0$, are frequently used. The average time delays for the weak and strong kernels are $T = \frac{1}{r}$ and $T = \frac{2}{r}$, respectively [7]. In the system (1), let

$$w_i(t) = \int_{-\infty}^t k(t-s)f(x_i(s))ds.$$

We obtain the system

$$\begin{aligned} x'_i(t) &= -x_i(t) + \alpha w_i(t) + \beta[f(x_{i-1}(t - \tau_n)) + f(x_{i+1}(t - \tau_n))], \\ w'_i(t) &= r[f(x_i(t)) - w_i(t)], \end{aligned} \quad (2)$$

where $i \bmod 3$, and where we have used the weak kernel to obtain the second equation. Let $w_0(t) = x_3(t)$, $w_1(t) = x_4(t)$, and $w_2(t) = x_5(t)$. Then the system (2) becomes

$$\begin{aligned} x'_0(t) &= -x_0(t) + \alpha x_3(t) + \beta[f(x_2(t - \tau_n)) + f(x_1(t - \tau_n))], \\ x'_1(t) &= -x_1(t) + \alpha x_4(t) + \beta[f(x_0(t - \tau_n)) + f(x_2(t - \tau_n))], \\ x'_2(t) &= -x_2(t) + \alpha x_5(t) + \beta[f(x_1(t - \tau_n)) + f(x_0(t - \tau_n))], \\ x'_3(t) &= r[f(x_0(t)) - x_3(t)], \\ x'_4(t) &= r[f(x_1(t)) - x_4(t)], \\ x'_5(t) &= r[f(x_2(t)) - x_5(t)]. \end{aligned} \quad (3)$$

The linearisation of (3) about the trivial equilibrium of (1) is

$$\begin{aligned} u'_0(t) &= -u_0(t) + \alpha u_3(t) + \beta[u_2(t - \tau_n) + u_1(t - \tau_n)], \\ u'_1(t) &= -u_1(t) + \alpha u_4(t) + \beta[u_0(t - \tau_n) + u_2(t - \tau_n)], \\ u'_2(t) &= -u_2(t) + \alpha u_5(t) + \beta[u_1(t - \tau_n) + u_0(t - \tau_n)], \\ u'_3(t) &= r[u_0(t) - u_3(t)], \\ u'_4(t) &= r[u_1(t) - u_4(t)], \\ u'_5(t) &= r[u_2(t) - u_5(t)]. \end{aligned} \quad (4)$$

2. Stability and bifurcation of the trivial equilibrium

Consider solutions of (4) of the form

$$x_j(t) = c_j e^{\lambda t}, \quad j \bmod 6,$$

where $\lambda \in \mathbb{C}$ and $c_j \in \mathbb{R}$. Nontrivial solutions of (4) exist if and only if

$$S(\lambda) = \Delta_1(\lambda)\Delta_2^2(\lambda) = 0, \quad (5)$$

where

$$\begin{aligned} \Delta_1(\lambda) &= (r + \lambda)(\lambda + 1) - r\alpha - 2\beta(r + \lambda)e^{-\lambda\tau_n}, \\ \Delta_2(\lambda) &= (r + \lambda)(\lambda + 1) - r\alpha + \beta(r + \lambda)e^{-\lambda\tau_n}. \end{aligned} \quad (6)$$

This gives the characteristic equation of (4), which is needed in the study of steady state bifurcations of the trivial solution of (3). This factorisation of the characteristic equation is due to the \mathbb{D}_3 symmetry of the network (and of system (1)). The characteristic equation (5) has a simple zero when $r(1 - \alpha - 2\beta) = 0$ and a double zero when $r(1 - \alpha + \beta) = 0$. The simple zero is a potential steady state bifurcation point while the double zero is a potential equivariant steady state bifurcation point [2].

In the absence of delay, Eq. (5) reduces to

$$\lambda^6 + c_5\lambda^5 + c_4\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0, \quad (7)$$

where

$$\begin{aligned} c_5 &= 3(1+r), \\ c_4 &= -3r\alpha + 3r^2 + 9r + 3 - 3\beta^2, \\ c_3 &= -3\beta^2 + r^3 - 6r\alpha - 6r^2\alpha + 9r - 2\beta^3 + 9r^2 - 9r\beta^2 + 1, \\ c_2 &= 3r^3 - 12r^2\alpha - 6\beta^3r - 3r\alpha - 9r\beta^2 - 9\beta^2r^2 + 3r + 9r^2 + 3r\alpha\beta^2 - 3r^3\alpha + 3r^2\alpha^2, \\ c_1 &= 6r^2\alpha\beta^2 + 3r^2 - 6\beta^3r^2 - 6r^3\alpha - 6r^2\alpha + 3r^3\alpha^2 + 3r^2\alpha^2 + 3r^3 - 3\beta^2r^3 - 9\beta^2r^2, \\ c_0 &= 3r^3\alpha\beta^2 - 3r^3\alpha + r^3 - 3\beta^2r^3 - r^3\alpha^3 - 2\beta^3r^3 + 3r^3\alpha^2. \end{aligned}$$

Using the well-known Routh–Hurwitz criterion, we conclude that (7) is stable if and only if all of the following inequalities are satisfied:

$$\begin{aligned} c_5 &> 0, \\ c_5c_4 - c_3 &> 0, \\ c_5(c_4c_3 - c_5c_2) - c_3^2 + c_1c_5 &> 0, \\ c_5c_4c_3c_2 - c_5c_4^2c_1 + c_5^2c_4c_0 - c_5^2c_2^2 + 2c_5c_2c_1 - c_5c_3c_0 - c_3^2c_2 + c_3c_4c_1 - c_1^2 &> 0, \\ c_5c_4c_3c_2c_1 - c_5c_4c_3^2c_0 - c_5c_4^2c_1^2 + 2c_4c_5^2c_1c_0 - c_5^2c_2^2c_1 + c_5^2c_2c_3c_0 \\ &\quad - c_5^3c_0^2 + 2c_5c_1^2c_2 - 3c_3c_5c_1c_0 - c_3^2c_2c_1 + c_3^3c_0 + c_3c_4c_1^2 - c_1^3 > 0, \\ [c_5c_4c_3c_2c_1 - c_5c_4c_3^2c_0 - c_5c_4^2c_1^2 + 2c_4c_5^2c_1c_0 - c_5^2c_2^2c_1 + c_5^2c_2c_3c_0 \\ &\quad - c_5^3c_0^2 + 2c_5c_1^2c_2 - 3c_3c_5c_1c_0 - c_3^2c_2c_1 + c_3^3c_0 + c_3c_4c_1^2 - c_1^3]c_0 > 0. \end{aligned} \quad (8)$$

Consequently, all the roots of (7) have negative real parts if and only if all of the inequalities in (8) are satisfied.

Proposition 1. *In the absence of delay, the trivial equilibrium of (3) is asymptotically stable if the inequalities in (8) are satisfied.*

Let $\lambda = \nu + i\omega$, $\nu, \omega \in \mathbb{R}$ in the two factors of (5) and separate into real and imaginary parts to obtain $\Delta_j(\lambda) = R_j(\nu, \omega) + iI_j(\nu, \omega)$, where

$$\begin{aligned} R_1(\nu, \omega) &= (r + \nu)(1 + \nu) - \omega^2 - r\alpha - 2\beta e^{-\nu\tau_n}[(r + \nu)\cos(\omega\tau_n) + \omega\sin(\omega\tau_n)], \\ I_1(\nu, \omega) &= \omega + \nu\omega + \omega(r + \nu) - 2\beta e^{-\nu\tau_n}[\omega\cos(\omega\tau_n) - (r + \nu)\sin(\omega\tau_n)], \end{aligned} \quad (9)$$

and

$$\begin{aligned} R_2(\nu, \omega) &= (r + \nu)(1 + \nu) - \omega^2 - r\alpha - \beta e^{-\nu\tau_n}[(r + \nu)\cos(\omega\tau_n) + \omega\sin(\omega\tau_n)], \\ I_2(\nu, \omega) &= \omega(\nu + 1) + \omega(r + \nu) + \beta e^{-\nu\tau_n}[\omega\cos(\omega\tau_n) - (r + \nu)\sin(\omega\tau_n)]. \end{aligned} \quad (10)$$

We begin our investigation by focusing on the factors $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ of (5), separately. First, we establish conditions under which these factors have purely imaginary roots. To this end, let $\nu = 0$ in (9) and (10). This leads to the following simplifying relations:

$$\begin{aligned} 2\beta r \cos(\omega\tau_n) + 2\beta\omega \sin(\omega\tau_n) &= r - \omega^2 - r\alpha, \\ 2\beta r \sin(\omega\tau_n) - 2\beta\omega \cos(\omega\tau_n) &= -(\omega + \omega r), \end{aligned} \quad (11)$$

and

$$\begin{aligned} \beta r \cos(\omega\tau_n) + \beta\omega \sin(\omega\tau_n) &= r - \omega^2 - r\alpha, \\ \beta r \sin(\omega\tau_n) - \beta\omega \cos(\omega\tau_n) &= \omega + \omega r, \end{aligned} \quad (12)$$

respectively. Solving for $\sin(\omega\tau_n)$ and $\cos(\omega\tau_n)$ in (11) and (12) yields

$$\begin{aligned} \sin(\omega\tau_n) &= \frac{-\omega(r^2 + \omega^2 + r\alpha)}{2\beta(r^2 + \omega^2)}, \\ \cos(\omega\tau_n) &= \frac{r^2 + \omega^2 - \alpha r^2}{2\beta(r^2 + \omega^2)}, \end{aligned} \quad (13)$$

and

$$\begin{aligned}\sin(\omega\tau_n) &= \frac{\omega(2r + r^2 - \omega - r\alpha)}{\beta(r^2 + \omega^2)}, \\ \cos(\omega\tau_n) &= \frac{r^2 - r\omega - r^2\alpha - \omega^2 - \omega^2r}{\beta(r^2 + \omega^2)},\end{aligned}\quad (14)$$

respectively. From this point on, we will concentrate our analysis on the simple root case, i.e. $\Delta_1(\lambda) = 0$. The repeated root case, i.e. $\Delta_2^2(\lambda) = 0$, will be the subject of a future article. Recalling that $\sin^2(\omega\tau_n) + \cos^2(\omega\tau_n) = 1$, squaring both sides of the equations in (13), adding, and rearranging gives the hexic polynomial

$$\begin{aligned}\phi(\omega) &:= \omega^6 + (2r\alpha + 1 + 2r^2 - 4\beta^2)\omega^4 + (r^4 - 2r^2\alpha + r^2\alpha^2 - 8\beta^2r^2 + 2r^2 + 2r^3\alpha)\omega^2 \\ &\quad - 2r^4\alpha + r^4 + r^4\alpha^2 - 4\beta^2r^4 = 0.\end{aligned}\quad (15)$$

Similarly, Eq. (12) yields the quartic polynomial

$$\gamma(\omega) := a_4\omega^4 + a_3\omega^3 + a_2\omega^2 + a_1\omega + a_0 = 0, \quad (16)$$

where

$$\begin{aligned}a_4 &= -\beta^2 + 1 + 2r^2 + 2r, \\ a_3 &= 2r^3\alpha + 2r^2 + 2r - 4r^3 - 2r^4, \\ a_2 &= -r^2 + 4r^4 + 2r^3\alpha + 4r^5 - 2\beta^2r^2 + r^4\alpha^2 - 2r^5\alpha + 2r^2\alpha - 2r^3 + r^6 - 4r^4\alpha, \\ a_1 &= 2r^3\alpha - 2r^3, \\ a_0 &= -2r^4\alpha - \beta r^4 + r^4 + r^4\alpha^2.\end{aligned}\quad (17)$$

Claim 1. If $-2r^4\alpha + r^4 + r^4\alpha^2 - 4\beta^2r^4 < 0$, then Eq. (15) has at least one positive root.

Proof. By hypothesis, $\phi(0) = -2r^4\alpha + r^4 + r^4\alpha^2 - 4\beta^2r^4 < 0$, and $\lim_{z \rightarrow \infty} \phi(z) = +\infty$. Hence, there exists an $\omega_0^* \in (0, \infty)$ such that $\phi(\omega_0^*) = 0$. This completes the proof. \square

Recall that we set $\lambda = \nu + i\omega$ in the two factors of (5). We then decomposed both factors into real and imaginary parts. Further, we set the real part $\nu = 0$, and thus obtained (15), for example. Consequently, Claim 1 implies that the characteristic equation (5) has a simple pair of purely imaginary roots, $\pm i\omega_0^*$, $\omega_0^* \in \mathbb{R}$. That is, $\Delta_1(\pm i\omega_0^*) = 0$.

In Eq. (15), we make the change of variable $z := \omega^2$. This gives the polynomial

$$\phi(z) := \psi_3z^3 + \psi_2z^2 + \psi_1z + \psi_0 = 0, \quad (18)$$

where

$$\begin{cases} \psi_3 = 1, \\ \psi_2 = 2r\alpha + 1 + 2r^2 - 4\beta^2, \\ \psi_1 = r^4 - 2r^2\alpha + r^2\alpha^2 - 8\beta^2r^2 + 2r^2 + 2r^3\alpha, \\ \psi_0 = -2r^4\alpha + r^4 + r^4\alpha^2 - 4\beta^2r^4. \end{cases}\quad (19)$$

Claim 2. If $\psi_0 \geq 0$ and $\psi_1 > 0$, then the Eq. (18) has no positive real roots.

Proof. We note that

$$\phi'(z) = 3z^2 + 2\psi_2z + \psi_1. \quad (20)$$

Now set

$$3z^2 + 2\psi_2z + \psi_1 = 0. \quad (21)$$

The roots of (21) are given by

$$z_{\pm} = \frac{-\psi_2 \pm \sqrt{\psi_2^2 - 3\psi_1}}{3}. \quad (22)$$

If $\psi_1 > 0$, then $\sqrt{\psi_2^2 - 3\psi_1} < \psi_2$. Consequently, both z_+ and z_- are negative. This means that Eq. (21) has no positive roots. Since $\phi(0) = \psi_0 \geq 0$, it follows that Eq. (18) has no positive roots. This completes the proof. \square

Since $z := \omega^2$, Claim 2 implies that there is no $\omega \in \mathbb{R}$ such that $i\omega$ is an eigenvalue of $\Delta_1(\lambda) = 0$. We recall that $\operatorname{Re}(\lambda) = \nu$. Thus, the real parts of all the eigenvalues of $\Delta_1(\lambda) = 0$ are negative for all delays $\tau_n \geq 0$.

We now investigate Eq. (16), which is a quartic polynomial in ω . We shall employ rudiments of Ferrari's method [1, 6] to analyse this polynomial. This method reduces the study of delay-induced bifurcation to the problem of determining whether a related polynomial equation has simple positive real roots [4,5,8]. Forde et al. [5] employed the technique of Sturm sequences to study quadratic and cubic related polynomials. In this work, we will extend their results to study a related quartic polynomial. The analysis of a quartic polynomial using the approach of [5] is onerous, whereas Ferrari's method as enunciated in [1,6] seems to be relatively tractable.

Claim 3. If $a_4 > 0$ and $a_0 < 0$, Eq. (16) has at least one positive real root.

Proof. The proof is straightforward. \square

Now suppose that $a_4 > 0$ and $a_0 > 0$. Since the polynomial (16) is of even degree, we are not guaranteed a negative root. The only way to have a simple positive real root in this case is to have two positive real roots. Similarly, the only way to have a simple negative real root is to have two negative real roots. That is, all of the four roots are real. To see this, let $a_4 \neq 0$ in (16) and define the coefficients $\bar{a}_3 = \frac{a_3}{a_4}$, $\bar{a}_2 = \frac{a_2}{a_4}$, $\bar{a}_1 = \frac{a_1}{a_4}$, and $\bar{a}_0 = \frac{a_0}{a_4}$. This transforms polynomial (16) into

$$\bar{\gamma}(\omega) = \omega^4 + \bar{a}_3\omega^3 + \bar{a}_2\omega^2 + \bar{a}_1\omega + \bar{a}_0 = 0. \quad (23)$$

Now, define the following parameters [1,6]:

$$\begin{aligned} I &= \bar{a}_2^2 + 12\bar{a}_0 - 3\bar{a}_3\bar{a}_1, \\ J &= 72\bar{a}_2\bar{a}_0 + 9\bar{a}_3\bar{a}_2\bar{a}_1 - 2\bar{a}_2^3 - 27\bar{a}_1^2 - 27\bar{a}_0\bar{a}_3^2, \\ \Delta &= 4I^3 - J^2, \\ H &= 8\bar{a}_2 - 3\bar{a}_3^2, \\ F &= 16\bar{a}_2^2 + 3\bar{a}_3^4 - 16\bar{a}_2\bar{a}_3^2 - 64\bar{a}_0 + 16\bar{a}_3\bar{a}_1. \end{aligned} \quad (24)$$

Thus, the quartic polynomial (16) has four simple real roots if and only if [6]

$$\Delta > 0, \quad H < 0, \quad F > 0. \quad (25)$$

The above conditions guarantee that a nondegenerate bifurcation occurs in the case $a_4 > 0$, $a_0 > 0$. We have now established conditions to guarantee the existence of four simple real roots of (16). We need to finally derive conditions to guarantee that one of these roots is positive. This happens if at least one of the critical points of (23) is positive. The derivative function is

$$\bar{\gamma}_1 = \omega^3 + c_1\omega^2 + c_2\omega + c_3, \quad (26)$$

where $c_1 = \frac{3}{4}\bar{a}_3$, $c_2 = \frac{1}{2}\bar{a}_2$, and $c_3 = \frac{1}{4}\bar{a}_1$. Clearly, Eq. (26) has a positive root if and only if $c_3 = \frac{1}{4}\bar{a}_1 < 0$. Since $\bar{a}_1 = \frac{a_1}{a_4}$ and $a_4 > 0$, it follows that $c_3 < 0$ if and only if $a_1 < 0$. Therefore we conclude that one of the critical points of (23) is positive if and only if $a_1 < 0$.

Let $\lambda(\tau_n) = \eta(\tau_n) + i\omega(\tau_n)$ be the root of (5) such that $\eta(\tau_n^0) = 0$ and $\omega(\tau_n^0) = \omega_0$. From Eq. (11), we obtain

$$\tau_n^j = \frac{1}{\omega_0} \cos^{-1} \left[\frac{r^2 + \omega_0^2 - \alpha r^2}{2\beta(\omega_0^2 + r^2)} \right] + \frac{2j\pi}{\omega_0}, \quad j = 0, 1, 2, \dots \quad (27)$$

Now let $\lambda = \lambda(\tau_n)$ in (5), and differentiate with respect to τ_n . For the simple root case, we have that $\Delta_1(i\omega) = 0$. This realisation leads to

$$\operatorname{Re} \left(\frac{d\lambda}{d\tau_n} \Big|_{\lambda=i\omega} \right) = \frac{L_1(D_1K_1 - D_2K_2) + L_2(D_1K_2 + D_2K_1)}{(D_1K_1 - D_2K_2)^2 + (D_1K_2 + D_2K_1)^2}, \quad (28)$$

where

$$\begin{aligned} D_1 &= (r - \omega^2 - r\alpha + \beta[r \cos(\omega\tau_n) + \omega \sin(\omega\tau_n)])^2 - (\omega(1+r) + \beta[\omega \cos(\omega\tau_n) - r \sin(\omega\tau_n)])^2, \\ D_2 &= 2(r^2 - \omega^2 - r\alpha + \beta[r \cos(\omega\tau_n) + \omega \sin(\omega\tau_n)])(\omega(1+r) + \beta[\omega \cos(\omega\tau_n) - r \sin(\omega\tau_n)]), \\ K_1 &= 1 + r - 2\beta \cos(\omega\tau_n) + 2\beta\tau_n[r \cos(\omega\tau_n) + \omega \sin(\omega\tau_n)], \\ K_2 &= 2\omega + 2\beta \sin(\omega\tau_n) + 2\beta\tau_n[\omega \cos(\omega\tau_n) - r \sin(\omega\tau_n)], \\ L_1 &= -2\beta [-(D_1\omega^2 + D_2\omega r) \cos(\omega\tau_n) + (D_1\omega r - D_2\omega^2) \sin(\omega\tau_n)], \\ L_2 &= (D_1\omega^2 + D_2\omega r) \sin(\omega\tau_n) + (D_1\omega r - D_2\omega^2) \cos(\omega\tau_n). \end{aligned} \quad (29)$$

Hence, the usual transversality condition is satisfied if and only if

$$\operatorname{Re}(\lambda'|_{\lambda=i\omega}) \neq 0,$$

which is equivalent to

$$L_1(D_1K_1 - D_2K_2) + L_2(D_1K_2 + D_2K_1) \neq 0. \quad (30)$$

By continuity, $\operatorname{Re}[\lambda(\tau_n)]$ becomes positive when $\tau_n > \tau_n^0$ and the trivial equilibrium of (3) becomes unstable. A simple root Hopf bifurcation occurs when τ_n passes through the critical value τ_n^0 , where

$$\tau_n^0 = \frac{1}{\omega_0} \cos^{-1} \left[\frac{r^2 + \omega_0^2 - r^2\alpha}{2\beta(r^2 + \omega_0^2)} \right].$$

We summarise the foregoing discussion in the form of the following proposition. In particular, the proposition brings together the results of Proposition 1, Claims 1, 2, and the transversality condition (30).

Proposition 2. Suppose that:

- (i) Conditions (8) and (30) hold, and that Eq. (16) has no positive real roots (at least for $0 < \tau_n \leq \tilde{\tau}_n$ with $\tilde{\tau}_n > \tau_n^0$).
If either
- (ii) $\psi_0 < 0$
or
- (iii) $\psi_0 \geq 0$, $\psi_1 < 0$, and $2(3\psi_1 - \psi_2^2)\sqrt{\psi_2^2 - 3\psi_1} \leq 9\psi_2\psi_1 - 27\psi_0 - 2\psi_2^3$

is satisfied, then the trivial equilibrium of (3) with characteristic equation (5) is asymptotically stable when $\tau_n < \tau_n^0$ and unstable when $\tau_n^0 < \tau_n < \min\{\tilde{\tau}_n, \tau_n^1\}$, where

$$\tau_n^0 = \frac{1}{\omega_0} \cos^{-1} \left[\frac{r^2 + \omega_0^2 - r^2\alpha}{2\beta(r^2 + \omega_0^2)} \right],$$

and

$$\tau_n^1 = \frac{1}{\omega_0} \cos^{-1} \left[\frac{r^2 + \omega_0^2 - r^2\alpha}{2\beta(r^2 + \omega_0^2)} \right] + \frac{2\pi}{\omega_0}.$$

When $\tau_n = \tau_n^0$, a simple root Hopf bifurcation occurs; that is, a family of periodic solutions bifurcates from the trivial equilibrium of (3) as τ_n passes through the critical value τ_n^0 .

Acknowledgments

I would like to express my sincere gratitude to the anonymous reviewer whose meticulous attention to detail led to great improvement of the presentation of this article. In addition, I would like to thank the reviewer for pointing out an error in my earlier version of Claim 2.

References

- [1] R.S. Ball, Note on the algebraical solution of biquadratic equations, Quart. J. Pure Appl. Math. 7 (6–9) (1866) 358–369.
- [2] S. Bungay, S.A. Campbell, Patterns of oscillation in a ring of identical cells with delayed coupling, Internat. J. Bifur. Chaos 17 (9) (2007) 3109–3125.
- [3] S.A. Campbell, I. Ncube, J. Wu, Multistability and stable asynchronous periodic oscillations in a multiple-delayed neural system, Physica D 214 (2006) 101–119.
- [4] J. Forde, Delay differential equation models in mathematical biology, Ph.D. Thesis, The University of Michigan, 2005.
- [5] J. Forde, P. Nelson, Applications of Sturm sequences to bifurcation analysis of delay differential equation models, J. Math. Anal. Appl. 300 (2004) 273–284.
- [6] E.I. Jury, M. Mansour, Positivity and nonnegativity conditions of a quartic equation and related problems, IEEE Trans. Automat. Control AC-26 (2) (1981) 444–451.
- [7] N. MacDonald, Time Lags in Biological Models, Springer-Verlag, Berlin, 1970.
- [8] M. Marden, Geometry of Polynomials, in: Mathematical Surveys, vol. 3, American Mathematical Society, Providence, Rhode Island, 1966.
- [9] I. Ncube, S.A. Campbell, J. Wu, Change in criticality of synchronous Hopf bifurcation in a multiple-delayed neural system, Fields Inst. Commun. 36 (2003) 179–193.