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On the Toeplitzness of the adjoint of composition operators[☆]

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ABSTRACT

Building on techniques developed by Cowen (1988) [3] and Nazarov–Shapiro (2007) [10], it is shown that the adjoint of a composition operator, induced by a unit disk-automorphism, is not strongly asymptotically Toeplitz. This result answers Nazarov–Shapiro's question in Nazarov and Shapiro (2007) [10].

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1. Introduction

In the early 60s, Brown and Halmos [2] characterized the classical Toeplitz operators on the Hardy space H^2 of the unit disk with a simple operator equation:

The operator $T \in \mathcal{B}(H^2)$ is a Toeplitz operator if and only if $T_z^* T T_z = T$, where T_z is the unilateral forward shift.

From the matricial point of view, this fact also reveals an interesting characterization of (classical) Toeplitz operators, on H^2: T is a (classical) Toeplitz operator when its matrix, with respect to the monomial basis of H^2, has constant diagonals. Indeed, the point here, as noted by Barría and Halmos [1], is that the matrix of composing TT_z is obtained from that of T by erasing the first column, while the matrix of composing T_z^*T is obtained from that of T by erasing the first row. Hence, the matrix of $T_z^*TT_z$ is obtained from that of T by moving one step down the main diagonal, and so leaves the matrix unchanged if and only if each diagonal is constant.

Twenty years later, Barría and Halmos [1] introduced a (natural) asymptotic generalization of that operator-theoretic characterization. According to them, an operator $T \in \mathcal{B}(H^2)$ is *(strongly) asymptotically Toeplitz* if the *Toeplitz sequence of* T, given by,

$$(\mathcal{T}_n(T))_{n=0}^{\infty} := \left(T_z^{*n} T T_z^n\right)_{n=0}^{\infty}$$

converges in the strong operator topology. In 1989, A. Feintuch [7] extended their definition considering other usual topologies on $\mathcal{B}(H^2)$. We thus have three flavors of asymptotic Toeplitzness: uniform, strong and weak. More precisely, an operator $T \in \mathcal{B}(H^2)$ is called *uniformly asymptotically Toeplitz*, *strongly asymptotically Toeplitz*, and *weakly asymptotically Toeplitz*, if its Toeplitz sequence is convergent in the uniform operator topology, the strong operator topology, and the weak operator topology, respectively. For each of them the operator-limit of $(\mathcal{T}_n(T))_{n=0}^{\infty}$ is a (classical) Toeplitz operator whose symbol is called the *asymptotic symbol* of T.

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It is worth mentioning that the class of uniformly asymptotically Toeplitz operators forms a (uniformly closed) subspace of all bounded operators on H^2, and it contains both Toeplitz and compact operators. Hence, any compact perturbation of a Toeplitz operator belongs to this class of operators. But, surprisingly, Feintuch proved that these are the only uniformly asymptotically Toeplitz operators [7, Theorem 4.1]:

Theorem (*Feintuch's Characterization of Uniform Asymptotic Toeplitzness*). *A bounded operator on H^2 is uniformly asymptotically Toeplitz if and only if it is a compact perturbation of a Toeplitz operator.*

Hence, if the difference of a bounded operator, on H^2, from any Toeplitz operator is not a compact operator, then it does not respect uniform asymptotical Toeplitzness. And this is one of the major tools we use to prove Theorems 4.8 and 4.10.

Recently, Nazarov and Shapiro [10], studied the Toeplitz sequence of composition operators, on H^2, in the weak, strong, and uniform operator topology, and showed that the study of such phenomena led to surprising results and interesting open problems. Among other things, they also established a weakened variant of the weak asymptotic Toeplitzness: the 'arithmetic means' of the Toeplitz sequence of a composition operator, namely,

$$\frac{1}{N+1}\sum_{n=0}^{N}\mathscr{T}_n(C_\varphi), \quad (N = 0, 1, 2, \ldots),$$

and proved that for every composition operator, except the identity, these means converge in the weak operator topology to zero [10, Theorem 2.2]. Since, among the three flavors of Toeplitzness, only strongly asymptotic Toeplitzness fails to respect adjoints [1, Example 12], they also studied the behavior of the Toeplitz sequence of the adjoint of composition operators, and proved, under each of these hypotheses:

(i) $\varphi(0) = 0$, or
(ii) $|\varphi| < 1$ a.e. on $\partial\mathbb{U}$,

on H^2, C_φ^* is strongly asymptotically Toeplitz [10, Proposition 4.1 and Theorem 4.2]. At the end of their paper [10], they stated that "We do not know any non-rotational examples of composition operators whose adjoints are not strongly asymptotically Toeplitz. Perhaps they are all!"; But, in this paper, we provide a class of composition operators whose adjoints are not strongly asymptotically Toeplitz:

Theorem. *The adjoint of composition operators, induced by non-trivial $\mathbb{U}$-automorphisms, are not strongly asymptotically Toeplitz.*

The work we describe here has its roots in [1], but, is mainly inspired by Nazarov and Shapiro [10]. Here is a brief outline of what follows. In Section 2, we set up the notation and introduce the main concepts required for what follows. Section 3 provides us with more tools and techniques to prove our result on the asymptotic Toeplitzness of adjoint of $\mathbb{U}$-automorphic composition operators.

2. Prerequisites

This introductory section is dedicated to setting up the notation and introducing the main concepts along with a collection of some fundamental facts required for what is to follow.

2.1. Notations

- The symbol $\mathbb{U}$ denotes the open unit disk of the complex plane, and $\partial\mathbb{U}$ the unit circle.
- The symbol φ always denotes a holomorphic self-mapping of $\mathbb{U}$.
- $\mathrm{Hol}(\mathbb{U})$ stands for the space of all functions holomorphic on $\mathbb{U}$.
- $\mathcal{B}(\mathcal{H})$ is the space of all bounded linear operators on some Hilbert space $\mathcal{H}$.
- the usual Lebesgue space L^2, as always, is the space of (equivalence classes of) measurable functions on $\partial\mathbb{U}$ which are square-integrable with respect to the normalized arc-length measure m ($m(\partial\mathbb{U}) = 1$).
- L^∞ denotes the (Banach) space of essentially bounded measurable functions on $\partial\mathbb{U}$, equipped with the essential supremum norm, defined as
$$\|f\|_{\mathrm{ess}} := \inf\{C \geq 0 \mid |f(e^{\imath\theta})| \leq C \text{ for almost every } e^{\imath\theta}\}.$$
- We write H^∞ for the space of bounded holomorphic functions on $\mathbb{U}$, and denote its natural norm by $\|\cdot\|_\infty$, i.e.,
$$\|f\|_\infty := \sup_{z\in\mathbb{U}}|f(z)|, \quad (f \in H^\infty).$$
- For $f \in \mathrm{Hol}(\mathbb{U})$, we adopt the notation $\hat{f}(n)$ for the n-th coefficient in the power series expansion of f about the origin.

2.2. The space H^2

The *Hardy space* H^2 is the collection of all $f \in \mathrm{Hol}(\mathbb{U})$ for which

$$\|f\|_{H^2}^2 := \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty.$$

The above formula defines a norm that turns H^2 into a Hilbert space whose inner product is given by

$$\langle f, g\rangle_{H^2} := \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)} \quad (f, g \in H^2).$$

There is also a "boundary version" setting for H^2 in which H^2 is isometrically isomorphic to a (closed) subspace of $L^2 = L^2(\partial\mathbb{U}, m)$ consisting of (boundary) functions whose Fourier coefficients of negative index all vanish. These boundary functions turn out to be just the radial limits of each H^2-function, i.e.,

$$f^*(e^{\imath\theta}) = \lim_{r\uparrow 1} f(re^{\imath\theta}),$$

which is known to exist for (m-) almost every point $e^{\imath\theta} \in \partial\mathbb{U}$. We will write $f(e^{\imath\theta})$ instead of $f^*(e^{\imath\theta})$, for each $e^{\imath\theta} \in \partial\mathbb{U}$ at which this radial limit exists, relying on the context to determine what we mean by the symbol f. With this identification the norm and inner product in H^2 can be computed on $\partial\mathbb{U}$ as

$$\|f\|_{H^2}^2 = \int_0^{2\pi} |f(e^{\imath\theta})|^2 \frac{d\theta}{2\pi} \quad \text{and} \quad \langle f, g\rangle_{H^2} = \int_0^{2\pi} f(e^{\imath\theta})\overline{g(e^{\imath\theta})}\frac{d\theta}{2\pi} \quad (f, g \in H^2).$$

These materials can be found in detail in [6] or [11].

2.3. Composition operators

Composing functions, in $\mathrm{Hol}(\mathbb{U})$, with a holomorphic self-mapping φ of $\mathbb{U}$, define a linear transformation C_φ, called a *composition operator* and φ as its *symbol*:

$$C_\varphi f := f \circ \varphi, \quad (\forall f \in \mathrm{Hol}(\mathbb{U})).$$

This transformation is even continuous if that space is given its natural topology of uniform convergence on compact sets. On the Hardy space H^2, if φ fixes the origin, Littlewood's Subordination Principle [9], assures us that C_φ is a contraction. And a consequence of it, asserts that every composition operator restricts to a bounded operator on the Hardy space H^2 [12, pp. 13–15].

2.4. Toeplitz operators

Any essentially bounded function ϕ on $\partial\mathbb{U}$ induces, in a natural way, two bounded operators: one on L^2 and one on H^2, as follows:

- the *Multiplication operator* M_ϕ is just multiplication by ϕ: $M_\phi f = \phi f$, for each $f \in L^2$;
- the *Toeplitz operator* T_ϕ is defined, in terms of the orthogonal projection P from L^2 onto H^2, as the compression of M_ϕ to H^2: $T_\phi f = PM_\phi f$, for each $f \in H^2$.

If ϕ is the boundary function of an H^∞-function, also denoted by ϕ, then M_ϕ takes H^2 into itself, so T_ϕ is the restriction of M_ϕ to H^2. In this case, T_ϕ can be identified with the operator of pointwise multiplication by the holomorphic function ϕ, acting on H^2, now viewed as a space of functions holomorphic on $\mathbb{U}$.

The best known such operator is the one induced by the coordinate function $\phi(z) = z$, and is denoted by T_z, and called the *unilateral forward shift* on H^2 because it shifts the Taylor series coefficients of H^2-functions one unit to the right, placing a zero in the empty initial position:

$$(T_z f)(z) = zf(z), \quad (z \in \mathbb{U}, f \in H^2).$$

A routine adjoint computation shows that $T_\phi^* = T_{\bar{\phi}}$, for $\phi \in L^\infty$. Again the best known example is T_z^*, the (Hilbert-space) adjoint of the unilateral forward shift, and is called the *unilateral backward shift*. It is easy to check the following representation for the unilateral backward shift on H^2:

$$\left(T_z^* f\right)(z) = \frac{f(z) - f(0)}{z}, \quad (f \in H^2, z \in \mathbb{U}).$$

Thus for each $f \in H^2$, we have

$$f(z) = \hat{f}(0) + zT_z^* f(z), \quad (z \in \mathbb{U}).$$

2.5. Reproducing kernel for H^2

H^2-functions can blow out to infinity near $\partial\mathbb{U}$, but not too fast, for example, $\ln \frac{1}{1-z}$. This property can be obtained from the boundedness of pointwise evaluation of H^2-functions [12, p. 10]; precisely, for each $\alpha \in \mathbb{U}$

$$|f(\alpha)| \leq \frac{1}{\sqrt{1-|\alpha|^2}}\|f\|_{H^2}, \quad (f \in H^2).$$

Hence, by the Riesz Representation theorem, we know that there is a unique function (*kernel function*) $K_\alpha \in H^2$, with

$$\|K_\alpha\|^2_{H^2} = \frac{1}{1-|\alpha|^2},$$

such that the *reproducing kernel property* holds in H^2; Indeed, the "reproducing kernel" terminology comes from the fact that for each $\alpha \in \mathbb{U}$ and $f \in H^2$, the function K_α "reproduces the value of f at α" in the following sense:

$$f(\alpha) = \langle f, K_\alpha\rangle_{H^2}, \quad (\alpha \in \mathbb{U}).$$

Since the functions $\{z^k \mid k = 0, 1, 2, \ldots\}$ are an orthonormal system of functions in H^2, one can show that

$$K_\alpha(z) = \frac{1}{1-\bar{\alpha}z}, \quad (z \in \mathbb{U}).$$

The behavior of the reproducing kernels near $\partial\mathbb{U}$ will play a pivotal role in the proof of Theorems 4.8 and 4.10. Indeed, the *normalized reproducing kernels* $k_\alpha := K_\alpha/\|K_\alpha\|_{H^2}$ converge weakly to zero as $|\alpha| \to 1^-$.

2.6. The adjoint of a composition operator induced by a linear fractional transformation

One of the most fundamental questions related to composition operators is how to obtain a reasonable representation for their adjoints. By definition the adjoint of a composition operator C_φ, is the operator C^*_φ given by the equation

$$\langle C^*_\varphi f, g\rangle_{H^2} = \langle f, C_\varphi g\rangle_{H^2}, \quad (f, g \in H^2),$$

from which we derive the fact that such adjoints permute reproducing kernels, i.e., for any holomorphic self-mapping φ of $\mathbb{U}$, and any point $\alpha \in \mathbb{U}$, we have [12, p. 43]

$$C^*_\varphi K_\alpha = K_{\varphi(\alpha)}, \tag{1}$$

i.e., the set of reproducing kernel functions is invariant under the adjoint of any composition operator. This property also characterizes the composition operators on any functional Banach space [4, Theorem 1.4].

Although reproducing kernels span a dense subspace of H^2, the equation above cannot be regarded as a formula for C^*_φ.

In 1988, using an algebraic manipulation based on the fact in (1), Cowen [3, Theorem 2] established the first major and general result on the adjoint problem:

Theorem 2.1 (*Cowen's Adjoint Formula*). *Let*

$$\varphi(z) = \frac{az+b}{cz+d} \tag{2}$$

be a nonconstant $(ad-bc \neq 0)$ *linear fractional self-mapping of* $\mathbb{U}$*, i.e.,* $(\varphi(\mathbb{U}) \subset \mathbb{U})$*. The adjoint* C^*_φ *can be written as*

$$T_g C_{\sigma_\varphi} T^*_h, \tag{3}$$

where

$$g(z) := \frac{1}{-\bar{b}z+\bar{d}}, \qquad \sigma_\varphi(z) := \frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}, \quad \text{and} \quad h(z) := cz+d,$$

and T_g *and* T_h *denote the Toeplitz operators.*

Cowen's adjoint formula (3) involves three functions constructed from the coefficients of the given linear fractional map:

(i) g and h are obviously in H^∞. Thus, both of them induce Toeplitz operators on H^2 which are just pointwise multiplications by g and h, respectively; moreover, they are invertible on H^2.
(ii) The linear fractional map σ_φ, associated to φ, is sometimes referred to as the "Krein adjoint" of φ [5]. When φ is a self-mapping of $\mathbb{U}$, σ_φ will be a $\mathbb{U}$-self-mapping [3], and so induces a composition operator on H^2, C_{σ_φ}.

Remark 2.2. Notice that the Cowen's adjoint formula also holds for constant maps. Indeed, if φ is a constant function, it easily turns out that $\sigma_\varphi(z) = 0$. In this case, C_φ and C_{σ_φ} are just point-evaluation functionals on H^2.

For our purpose here, we apply the nice formulation, obtained in [8], of the Cowen's adjoint formula, i.e., for $f \in H^2$ and $z \in \mathbb{U}$

$$C_\varphi^* f(z) = \left[T_{\frac{1}{-\bar{b}z+\bar{d}}} C_{\sigma_\varphi} T^*_{cz+d}(f)\right](z)$$
$$= \frac{(\overline{ad} - \overline{bc})z}{(\bar{a}z - \bar{c})(-\bar{b}z + \bar{d})} f(\sigma_\varphi(z)) + \frac{\bar{c}}{\bar{c} - \bar{a}z} f(0) \tag{4}$$

to give a concrete variant of the adjoint of a composition operator induced by a $\mathbb{U}$-automorphism:

for $\alpha \in \mathbb{U}$, the $\mathbb{U}$-automorphism φ_α, interchanging the point $\alpha \in \mathbb{U}$ and the origin, i.e.,

$$\varphi_\alpha(z) := \frac{\alpha - z}{1 - \bar{\alpha}z}, \quad (z \in \mathbb{U}), \tag{5}$$

induces a composition operator whose adjoint, acting on the monomial bases of H^2, has the following form:

$$\left(C^*_{\varphi_\alpha} z^n\right)(z) = \begin{cases} K_\alpha(z) & \text{if } n = 0 \\ \dfrac{z}{n}\left(\varphi_\alpha^n(z)\right)' & \text{if } n = 1, 2, \ldots \end{cases} \tag{6}$$

where K_α, as usual, is the reproducing kernel function. Indeed, applying Cowen's adjoint formula (3) for each $z \in \mathbb{U}$, we have

$$\left[C^*_{\varphi_\alpha} z^n\right](z) = K_\alpha(z)\left[(I - \alpha T_z^*)z^n\right](\varphi_\alpha(z)).$$

Now, if $n = 0$, $(I - \alpha T_z^*)1 = 1$, i.e., $\left[C^*_{\varphi_\alpha} 1\right](z) = K_\alpha(z)$. And, for $n = 1, 2, 3, \ldots$, using (4) along with comparing the coefficients of two linear fractional maps (2) and (5), we have

$$\left[C^*_{\varphi_\alpha} z^n\right](z) = \frac{|\alpha|^2 - 1}{(\alpha - z)(1 - \overline{\alpha}z)} z\varphi_\alpha^n(z)$$
$$= z\varphi'_\alpha(z)\varphi_\alpha^{n-1}(z)$$
$$= \frac{z}{n}\left(\varphi_\alpha^n\right)'(z).$$

3. Asymptotic Toeplitzness of the adjoint of $\mathbb{U}$-automorphic composition operators

We already have almost all the ingredients to prove our first result. But before that, we need to state the following two facts:

Using a clever application of Littlewood's Subordination Theorem [9, Theorem 215, p.168] Nazarov and Shapiro showed, in the proof of [10, Theorem 3.3], that

Lemma 3.1 (*Nazarov–Shapiro*). *For $\alpha \in \mathbb{U} \setminus \{0\}$, there is a positive constant C, which depends on α but not on n, such that $\sum_{k=n}^\infty \left|\widehat{\varphi_\alpha^n}(k)\right|^2 > C$, for $n = 0, 1, 2, \ldots$.*

And the next lemma assures us that the only possible asymptotic symbols for the adjoint of a composition operator are the constants 1 (for the identity operator) and 0:

Lemma 3.2. *Let φ be neither a rotation nor the identity map. If C_φ^* is strongly asymptotically Toeplitz, then its asymptotic symbol should be zero.*

Proof. Since C_φ^* is strongly asymptotically Toeplitz, by definition, there exists a $T \in \mathcal{B}(H^2)$ such that

$$\lim_{n\to\infty} \left\|\mathcal{T}_n(C_\varphi^*)f - Tf\right\|_{H^2} = 0, \quad \forall f \in H^2,$$

on the other hand, this shows

$$\lim_{n\to\infty} \left\langle \mathcal{T}_n(C_\varphi^*)f, g\right\rangle_{H^2} = \langle Tf, g\rangle_{H^2}, \quad \forall f, g \in H^2,$$

which implies

$$\lim_{N\to\infty} \left\langle \frac{1}{N+1}\sum_{n=0}^N \mathcal{T}_n(C_\varphi^*)f, g\right\rangle_{H^2} = \langle Tf, g\rangle_{H^2}, \quad \forall f, g \in H^2,$$

or, equivalently,

$$\lim_{N\to\infty} \left\langle \frac{1}{N+1}\sum_{n=0}^N \mathcal{T}_n(C_\varphi)f, g\right\rangle_{H^2} = \langle T^*f, g\rangle_{H^2}, \quad \forall f, g \in H^2.$$

But, [10, Theorem 2.2] asserts that T^* is a zero operator; which finishes the proof. □

Having necessary ingredients, we are ready to state and prove our main result:

Theorem 3.3. *For $\alpha \in \mathbb{U} \setminus \{0\}$, $C^*_{\varphi_\alpha}$ is not strongly asymptotically Toeplitz.*

Proof. If $C^*_{\varphi_\alpha}$ is strongly asymptotically Toeplitz, then, by Lemma 3.2, the asymptotic symbol of $C^*_{\varphi_\alpha}$ should be zero. Thus, it will suffice to show that the norms of the vectors $\mathscr{T}_n(C^*_{\varphi_\alpha})(1)$ are bounded away from zero.

Using (6), for $n = 1, 2, \ldots$, we obtain

$$\begin{aligned}\mathscr{T}_n(C^*_{\varphi_\alpha})(1) &= T_z^{*n} C^*_{\varphi_\alpha} T_z^n 1 = \frac{1}{n} T_z^{*n-1}\left(\varphi_\alpha^n(z)\right)' = \frac{1}{n} T_z^{*n-1}\left(\sum_{k=1}^{\infty} k\widehat{\varphi_\alpha^n}(k) z^{k-1}\right)\\ &= \frac{1}{n}\sum_{k=n}^{\infty} k\widehat{\varphi_\alpha^n}(k) z^{k-n}.\end{aligned}$$

Thus,

$$\left\|\mathscr{T}_n(C^*_{\varphi_\alpha})(1)\right\|^2 = \sum_{k=n}^{\infty}\frac{k^2}{n^2}|\widehat{\varphi_\alpha^n}(k)|^2 \geq \sum_{k=n}^{\infty}|\widehat{\varphi_\alpha^n}(k)|^2 \left(= \left\|\mathscr{T}_n(C_{\varphi_\alpha})(1)\right\|^2\right). \tag{7}$$

But Lemma 3.1 confirmed us that the last sum in (7) is bounded away from zero, which asserts that

$$\inf_n \left\|\mathscr{T}_n(C^*_{\varphi_\alpha})(1)\right\| > 0,$$

and contradicts our assumption and the proof is completed. $\square$

Remark 3.4. It is worth mentioning that Theorem 3.3 provides a class of bounded operators, on H^2, which are weakly asymptotically Toeplitz [10, Corollary 2.4], but not strongly asymptotically Toeplitz. Also, it gives a class of bounded operators, on H^2, neither themselves nor their adjoints [10, Theorem 3.3] are strongly asymptotically Toeplitz.

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References

[1] J. Barría, P.R. Halmos, Asymptotic Toeplitz operators, Trans. Amer. Math. Soc. 273 (1982) 621–630.
[2] A. Brown, P.R. Halmos, Algebraic properties of Toeplitz operators, J. Reine Angew. Math. 213 (1963/1964) 89–102.
[3] C.C. Cowen, Linear fractional composition operators on H^2, Integral Equations Operator Theory 11 (1988) 151–160.
[4] C.C. Cowen, B. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, 1995.
[5] C.C. Cowen, B. MacCluer, Linear fractional maps of the ball and their composition operators, Acta Sci. Math. (Szeged) 66 (2000) 351–376.
[6] P.L. Duren, Theory of H^p Spaces, Academic Press, 1970.
[7] A. Feintuch, On asymptotic Toeplitz and Hankel operators, in: Operator Theory, Advances and Applications, Vol. 41, Birkhaüser, Basel, 1989, pp. 241–254.
[8] C. Hammond, J. Moorehouse, M.E. Robbins, Adjoints of composition operators with rational symbol, J. Math. Anal. Appl. 341 (1) (2008) 626–639.
[9] J.E. Littlewood, On inequalities in the theory of functions, Proc. Lond. Math. Soc. 23 (1925) 481–519.
[10] F. Nazarov, J.H. Shapiro, On the Toeplitzness of composition operators, Complex Var. Elliptic Equ. 52 (2007) 193–210.
[11] W. Rudin, Real and Complex Analysis, third ed., McGraw-Hill, 1987.
[12] J.H. Shapiro, Composition Operators and Classical Function Theory, Springer-Verlag, 1993.