Harnack inequalities for stochastic equations driven by Lévy noise ☆

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A B S T R A C T

By using coupling argument and regularization approximations of the underlying sub-ordinator, dimension-free Harnack inequalities are established for a class of stochastic equations driven by a Lévy noise containing a subordinate Brownian motion. The Harnack inequalities are new even for linear equations driven by Lévy noise, and the gradient estimate implied by our log-Harnack inequality considerably generalizes some recent results on gradient estimates and coupling properties derived for Lévy processes or linear equations driven by Lévy noise. The main results are also extended to semilinear stochastic equations in Hilbert spaces.

1. Introduction

Due to their broad range of applications in heat kernel estimates, functional inequalities, transportation-cost inequalities and properties of invariant measures, the dimension-free Harnack inequality with powers introduced in [14] and the log-Harnack inequality introduced in [9] have been intensively investigated for stochastic (partial) differential equations driven by Gaussian noises, see [18,19] and references therein. However, due to technical difficulty on construction of couplings for jump processes, the study for stochastic equations driven by purely jump Lévy noise is very limited. The only known results on this type of Harnack inequalities are presented in [19,22] for linear stochastic differential equations (i.e. O–U processes) driven by purely jump Lévy processes, where [19] uses coupling through the Mecke formula and [22] adopts known heat kernel bounds of the α-stable processes. Recently, using regularization approximations of the time-change, Zhang established in [24] the Bismut formula for stochastic differential equations with Lipschitz continuous drifts driven by the α-stable process. In this paper, we will make use of Zhang's argument together with a coupling method to derive Harnack inequalities for stochastic equations driven by Lévy noise, which provide explicit heat kernel estimates (see Remark 2.2 below).

Let $W := (W_t)_{t \geq 0}$, $S := (S(t))_{t \geq 0}$ and $V := (V_t)_{t \geq 0}$ be independent stochastic processes, where W is the Brownian motion on $\mathbb{R}^d$ with $W_0 = 0$; V is a locally bounded measurable process on $\mathbb{R}^d$ with $V_0 = 0$; and S is the subordinator induced by a Bernstein function B, i.e., S is a one-dimensional non-negative increasing Lévy process with $S(0) = 0$ and $\mathbb{E}e^{-rS(t)} = e^{-tB(r)}$, $t, r \geq 0$. Then $(W_{S(t)})_{t \geq 0}$ is a Lévy process with symbol $\psi := B(|\cdot|^2)$.

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We consider the following stochastic equation on $\mathbb{R}^d$:

$$X_t = X_0 + \int_0^t b_s(X_s)\,\mathrm{d}s + \int_0^t \sigma_s\,\mathrm{d}W_{S(s)} + V_t, \quad t \geqslant 0, \tag{1.1}$$

where $\sigma : [0,\infty) \to \mathbb{R}^d \otimes \mathbb{R}^d$ is measurable and locally bounded, and $b : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is measurable, locally bounded and continuous in the second variable.

We shall need the following conditions on σ and b:

(H1) σ_t^{-1} exists and is locally bounded, i.e. there exists an increasing function λ on $[0,\infty)$ such that $\|\sigma_t^{-1}\| \leqslant \lambda_t$ for $t \geqslant 0$.

(H2) There exists a locally bounded measurable function K on $[0,\infty)$ such that

$$\langle b_t(x) - b_t(y), x - y \rangle \leqslant K_t |x - y|^2, \quad x, y \in \mathbb{R}^d,\ t \geqslant 0. \tag{1.2}$$

It is easy to see that **(H2)** implies the existence, uniqueness and non-explosion of the solution, see e.g. the proof of [13, Theorem 177]. Now, for any $x \in \mathbb{R}^d$, let $(X_t(x))_{t \geqslant 0}$ be the unique solution to (1.1) for $X_0 = x$. We aim to establish Harnack inequalities for the associated Markov operator P_t on $\mathscr{B}_b(\mathbb{R}^d)$:

$$P_t f(x) := \mathbb{E} f\big(X_t(x)\big), \quad t \geqslant 0,\ x \in \mathbb{R}^d,\ f \in \mathscr{B}_b(\mathbb{R}^d).$$

Comparing with the O–U type equations studied in [19,22], our equation is with a more general time-dependent drift. Moreover, to compare the Lévy term in (1.1) with those in [19,22] under a lower bound condition of the Lévy measure, we may replace $W_{S(t)}$ in (1.1) by a Lévy process L_t with Lévy measure $\nu(\mathrm{d}x) \geqslant c|x|^{-d} B(|x|^{-2})\,\mathrm{d}x$ for some constant $c > 0$. In fact, in this case we may split L_t into two independent Lévy parts, where one of them has Lévy measure $c|x|^{-d} B(|x|^{-2})\,\mathrm{d}x$ and is thus a subordinate Brownian motion (cf. [16,3]), and the integral of σ w.r.t. the other can be combined with the term V_t.

In Section 2 we state our main results, which are then proved in Section 3 by using regularization approximations of $S(t)$ and the coupling by change of measure. Finally, the main results are extended in Section 4 to semilinear SPDEs by using finite-dimensional approximations.

2. Main results

Theorem 2.1. *Assume* **(H1)** *and* **(H2)**, *and let* $K(t) = \int_0^s K_u\,\mathrm{d}u$ *for* $t \geqslant 0$.

(1) *For any* $T > 0$ *and strictly positive* $f \in \mathscr{B}_b(\mathbb{R}^d)$,

$$P_T \log f(y) \leqslant \log P_T f(x) + \frac{|x - y|^2}{2} \inf_{t \in (0,T]} \mathbb{E}\left\{ \frac{\lambda_t^2}{\int_0^t \mathrm{e}^{-2K(s)}\,\mathrm{d}S(s)} \right\}, \quad x, y \in \mathbb{R}^d.$$

(2) *For any* $T > 0$ *and* $f \in \mathscr{B}_b(\mathbb{R}^d)$,

$$|\nabla P_T f|(x)^2 \leqslant \big\{ P_T f^2(x) - \big(P_T f(x)\big)^2 \big\} \inf_{t \in [0,T]} \mathbb{E}\left\{ \frac{\lambda_t^2}{\int_0^t \mathrm{e}^{-2K(s)}\,\mathrm{d}S(s)} \right\}, \quad x \in \mathbb{R}^d,$$

where $|\nabla P_T f|(x)$ *is the local Lipschitz constant of* $P_T f$ *at point* x, *i.e.*

$$|\nabla P_T f|(x) = \limsup_{y \to x} \frac{|P_T f(y) - P_T f(x)|}{|y - x|}.$$

(3) *For any* $T > 0$, $p > 1$ *and positive* $f \in \mathscr{B}_b(\mathbb{R}^d)$,

$$\big(P_T f(y)\big)^p \leqslant \big(P_T f^p(x)\big) \left(\mathbb{E} \inf_{t \in (0,T]} \exp\left[\frac{p\lambda_t^2 |x - y|^2}{2(p-1)^2 \int_0^t \mathrm{e}^{-2K(s)}\,\mathrm{d}S(s)} \right] \right)^{p-1}, \quad x, y \in \mathbb{R}^d.$$

Remark 2.1. (1) When $S(t) = t$ and $V_t = 0$, Eq. (1.1) reduces to the SDE driven by Brownian motion. In this case the assertions in Theorem 2.1 coincide with the corresponding ones derived in the diffusion setting, e.g. when $\sigma_t = \sqrt{2}I$ and $b_t = b$, assertions (1), (2) and (3) reduce respectively to (1.3), (iii) and (1.2) in [15] for $M = \mathbb{R}^d$. These inequalities are sharp as they are equivalent to the underlying curvature condition, see [15, Theorem 1.1].

(2) For general subordinator S, Theorem 2.1 improves [8, Theorem 1.1] and [6, Theorems 1.2 and 1.3] for generalized Mehler semigroups. Theorem 2.1(1) and (3) are new even for linear stochastic equations driven by Lévy processes, for which the Harnack inequality has been investigated in [19] by using the Mecke formula. Since in [19] the density of the Lévy

measure was used, so that the derived inequalities cannot be extended to infinite-dimensions as we did in Section 4 for our present Harnack inequalities.

(3) The gradient inequality in Theorem 2.1(2) generalizes the main results in [16,3,10] for Lévy processes or linear equations driven by Lévy noise. When $K \leqslant 0$ and λ is bounded, it follows from Theorem 2.1(2) that

$$\left| P_T f(x) - P_T f(y) \right| \leqslant \|\lambda\|_\infty \|f\|_\infty |x - y| \sqrt{\mathbb{E} \frac{1}{S(T)}}, \quad f \in \mathscr{B}_b(\mathbb{R}^d), \ f \geqslant 0.$$

This implies the coupling property provided $\mathbb{E} \frac{1}{S(T)} \to 0$ as $T \to \infty$. Thus, the main results in [17,3,11,12] on the coupling property of Lévy processes or linear equations driven by Lévy noise are generalized, see also [23] for the recent study of the coupling property of Lévy processes with drift. If furthermore $K \leqslant -\theta$ for some constant $\theta > 0$, we have $|X_t(x) - X_t(y)| \leqslant e^{-\theta t}|x - y|$, which together with Theorem 2.1(2), implies the exponential convergence of ∇P_T: there exists a constant $C > 0$ such that (see the proof of [16, Theorem 1.1] for details)

$$|\nabla P_T f|(x)^2 \leqslant C e^{-2\theta T} \{P_T f^2(x) - (P_T f(x))^2\}, \quad x \in \mathbb{R}^d, \ T \geqslant 1, \ f \in \mathscr{B}_b(\mathbb{R}^d).$$

To illustrate Theorem 2.1, we consider the equation driven by stable like processes. In the following result $W_{S(t)}$ is the α-stable process when $B(r) = r^{\alpha/2}, \alpha \in (0, 2)$.

Corollary 2.2. *Assume that the assumptions of* Theorem 2.1 *hold and assume that there exist some constants* $\theta \in (0, 1)$ *and* $c, r_0 > 0$ *such that* $B(r) \geqslant cr^\theta$ *holds for any* $r \geqslant r_0$. *Then there exists a constant* $C > 0$ *such that*

(1) *For any* $T > 0$ *and strictly positive* $f \in \mathscr{B}_b(\mathbb{R}^d)$,

$$P_T \log f(y) \leqslant \log P_T f(x) + \frac{C|x - y|^2}{(T \wedge 1)^{\frac{1}{\theta}}}, \quad x, y \in \mathbb{R}^d;$$

(2) *For any* $T > 0$ *and* $f \in \mathscr{B}_b(\mathbb{R}^d)$,

$$|\nabla P_T f|(x)^2 \leqslant \{P_T f^2(x) - (P_T f(x))^2\} \frac{C}{(T \wedge 1)^{\frac{1}{\theta}}}, \quad x \in \mathbb{R}^d;$$

(3) *When* $\theta \in (\frac{1}{2}, 1)$, *any* $T > 0, p > 1, x, y \in \mathbb{R}^d$ *and positive* $f \in \mathscr{B}_b(\mathbb{R}^d)$,

$$(P_T f(y))^p \leqslant (P_T f^p(x)) \exp\left[\frac{Cp|x - y|^2}{(p - 1)(T \wedge 1)^{\frac{1}{\theta}}} + \frac{C(p|x - y|^2)^{\frac{\theta}{2\theta - 1}}}{[(p - 1)(T \wedge 1)]^{\frac{1}{2\theta - 1}}}\right].$$

Remark 2.2. Among some other applications of Harnack inequalities summarized in [18, §1.4] (see also [15,20]), to save space we only mention here heat kernel estimates implied by Corollary 2.2. Let $p_T(x, y)$ be the density of P_T with respect to the Lebesgue measure (the existence is well known as the equation is non-degenerate). By [20, Proposition 3.1(4)] for μ replacing by the Lebesgue measure (note that the proof works also for σ-finite quasi-invariant measures), Corollary 2.2(1) and (3) imply the entropy inequality

$$\int_{\mathbb{R}^d} p_T(x, z) \log \frac{p_T(x, z)}{p_T(y, z)} \, dz \leqslant \frac{C|x - y|^2}{(T \wedge 1)^{\frac{1}{\theta}}}, \quad x, y \in \mathbb{R}^d, \ T > 0, \tag{2.1}$$

and when $\theta \in (\frac{1}{2}, 1)$, for any $p > 1$ and $T > 0$,

$$\int_{\mathbb{R}^d} p_T(x, z) \left(\frac{p_T(x, z)}{p_T(y, z)}\right)^{\frac{1}{p}} dz \leqslant \exp\left[\frac{C|x - y|^2}{(p - 1)(T \wedge 1)^{\frac{1}{\theta}}} + \frac{Cp^{\frac{1-\theta}{2\theta-1}}|x - y|^{\frac{2\theta}{2\theta-1}}}{[(p - 1)(T \wedge 1)]^{\frac{1}{2\theta-1}}}\right], \quad x, y \in \mathbb{R}^d. \tag{2.2}$$

Moreover, it is obvious that Corollary 2.2(2) implies

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left|\nabla \log p_T(\cdot, y)(x)\right|^2 p_T(x, y) \, dy \leqslant \frac{C}{(T \wedge 1)^{\frac{1}{\theta}}}, \quad T > 0. \tag{2.3}$$

3. Proofs of Theorem 2.1 and Corollary 2.2

We first explain the main idea of the proof. As in [24] we consider the following regularization of S:

$$S_\varepsilon(t) := \frac{1}{\varepsilon} \int\limits_t^{t+\varepsilon} S(s)\,ds + \varepsilon t, \quad t \geqslant 0, \ \varepsilon > 0.$$

Then S_ε is strictly increasing, absolutely continuous and $S_\varepsilon \downarrow S$ as $\varepsilon \downarrow 0$. For each $\varepsilon > 0$, we consider the approximation equation

$$X_t^\varepsilon = X_0 + \int\limits_0^t b_s(X_s^\varepsilon)\,ds + \int\limits_0^t \sigma_s\,dW_{S_\varepsilon(s)} + V_t, \quad t \geqslant 0. \tag{3.1}$$

Since S_ε is absolutely continuous, this equation is indeed driven by the Brownian motion so that we are able to establish the Harnack inequalities for the associated operator P_t^ε by using coupling. Finally, by proving $P_t^\varepsilon \to P_t$ as $\varepsilon \to 0$, we derive the corresponding Harnack inequalities for P_t.

3.1. The case of absolutely continuous time-change

Let ℓ be an absolutely continuous and strictly increasing function on $[0, \infty)$ with $\ell(0) = 0$, and let $v : [0, \infty) \to \mathbb{R}^d$ be measurable and locally bounded with $v_0 = 0$. We consider the following equation

$$X_t^{\ell,v} = X_0 + \int\limits_0^t b_s(X_s^{\ell,v})\,ds + \int\limits_0^t \sigma_s\,dW_{\ell(s)} + v_t, \quad t \geqslant 0. \tag{3.2}$$

Under our general assumptions, this equation has a unique solution. Let

$$P_t^{\ell,v} f(x) = \mathbb{E}f(X_t^{\ell,v}(x)), \quad x \in \mathbb{R}^d, \ t \geqslant 0, \ f \in \mathscr{B}_b(\mathbb{R}^d),$$

where $X_t^{\ell,v}(x)$ is the solution to (3.2) for $X_0 = x$.

Now, for fixed $T > 0$ and $x, y \in \mathbb{R}^d$, we intend to construct a coupling to derive the Harnack inequalities of $P_T^{\ell,v}$. To this end, let $(Y_t)_{t \geqslant 0}$ solve the equation

$$Y_t = y + \int\limits_0^t b_s(Y_s)\,ds + \int\limits_0^t \sigma_s\,dW_{\ell(s)} + v_t + \int\limits_0^t \{\xi_s 1_{[0,\tau)}(s)\} \cdot \frac{X_s^{\ell,v} - Y_s}{|X_s^{\ell,v} - Y_s|}\,d\ell(s), \tag{3.3}$$

where

$$\xi_t := \frac{|x - y| \exp[-\int_0^t K_s\,ds]}{\int_0^T \exp[-2\int_0^t K_s\,ds]\,d\ell(t)}, \quad t \geqslant 0$$

and

$$\tau := \inf\{t \geqslant 0 : X_t^{\ell,v} = Y_t\}.$$

To construct a solution to (3.3), we consider the equation

$$\tilde{Y}_t = y + \int\limits_0^t b_s(\tilde{Y}_s)\,ds + \int\limits_0^t \sigma_s\,dW_{\ell(s)} + v_t + \int\limits_0^t \{\xi_s 1_{\{X_s^{\ell,v} \neq \tilde{Y}_s\}}\} \cdot \frac{X_s^{\ell,v} - \tilde{Y}_s}{|X_s^{\ell,v} - \tilde{Y}_s|}\,d\ell(s). \tag{3.4}$$

Since $(z, z') \mapsto \frac{z-z'}{|z-z'|}$ is locally Lipschitz continuous on the domain $\{(z, z') \in \mathbb{R}^d \times \mathbb{R}^d : z \neq z'\}$, the joint equation of (3.2) and (3.4) has a unique solution up to the coupling time

$$\tilde{\tau} := \inf\{t \geqslant 0 : X_t^{\ell,v} = \tilde{Y}_t\}.$$

Let

$$Y_t = \tilde{Y}_t 1_{[0,\tilde{\tau})}(t) + X_t^{\ell,v} 1_{[\tilde{\tau},\infty)}(t).$$

Then $(Y_t)_{t \geqslant 0}$ is a solution to (3.3) with $\tau = \tilde{\tau}$.

Lemma 3.1. *For the above constructed coupling* $(X_t^{\ell,v}, Y_t)_{t \geqslant 0}$, *there holds* $\tau \leqslant T$, *i.e.* $X_T^{\ell,v} = Y_T$.

Proof. By (3.2) and (3.3) we have

$$d\big(X_t^{\ell,v} - Y_t\big) = \big(b_t\big(X_t^{\ell,v}\big) - b_t(Y_t)\big)dt - \xi_t \cdot \frac{X_t^{\ell,v} - Y_t}{|X_t^{\ell,v} - Y_t|} d\ell(t), \quad t < \tau.$$

Then **(H2)** and the absolutely continuity of ℓ yield

$$d\big|X_t^{\ell,v} - Y_t\big|^2 \leqslant 2K_t\big|X_t^{\ell,v} - Y_t\big|^2 dt - 2\xi_t\big|X_t^{\ell,v} - Y_t\big| d\ell(t), \quad t < \tau.$$

Note that for two continuous semimartingales M_t and $\tilde{M}_t$, the inequality $dM_t \leqslant d\tilde{M}_t$ means that they have the same martingale part and $\tilde{M}_t - M_t$ is an increasing process. Thus,

$$d\big\{\big|X_t^{\ell,v} - Y_t\big| e^{-\int_0^t K_s ds}\big\} \leqslant -\xi_t e^{-\int_0^t K_s ds} d\ell(t), \quad t < \tau.$$

Therefore, if $\tau > T$ then

$$0 < \big|X_T^{\ell,v} - Y_T\big| e^{-\int_0^T K_s\, ds} \leqslant |x - y| - \int_0^T \xi_t e^{-\int_0^t K_s ds} d\ell(t) = 0,$$

which is a contradiction. $\quad\square$

To derive the Harnack inequality, we define

$$\tilde{W}_t := W_t + \int_0^{t \wedge \ell(\tau)} \frac{\xi_{\ell^{-1}(s)}}{|X_{\ell^{-1}(s)}^{\ell,v} - Y_{\ell^{-1}(s)}|} \sigma_{\ell^{-1}(s)}^{-1}\big(X_{\ell^{-1}(s)}^{\ell,v} - Y_{\ell^{-1}(s)}\big) ds, \quad t \geqslant 0.$$

By the Girsanov theorem, $(\tilde{W}_t)_{t \geqslant 0}$ is the d-dimensional Brownian motion under the probability $d\mathbb{Q} := R\, d\mathbb{P}$, where

$$R := \exp\Bigg[-\int_0^{\ell(\tau)} \langle \eta_t, dW_t \rangle - \frac{1}{2}\int_0^{\ell(\tau)} |\eta_t|^2 dt\Bigg],$$

$$\eta_t := \frac{\xi_{\ell^{-1}(t)}}{|X_{\ell^{-1}(t)}^{\ell,v} - Y_{\ell^{-1}(t)}|} \sigma_{\ell^{-1}(t)}^{-1}\big(X_{\ell^{-1}(t)}^{\ell,v} - Y_{\ell^{-1}(t)}\big), \quad t \in \big[0, \ell(\tau)\big).$$

Reformulating (3.3) by

$$Y_t = y + \int_0^t b_s(Y_s)\, ds + \int_0^t \sigma_s d\tilde{W}_{\ell(s)} + v_t, \quad t \geqslant 0,$$

we conclude from the definition of $P_t^{\ell,v}$ and Lemma 3.1 that

$$P_T^{\ell,v} f(y) = \mathbb{E}_{\mathbb{Q}} f(Y_T) = \mathbb{E}\big[Rf(Y_T)\big] = \mathbb{E}\big[Rf\big(X_T^{\ell,v}\big)\big]. \tag{3.5}$$

It is now more or less standard that this formula implies the following result.

Proposition 3.2. *For any strictly positive* $f \in \mathscr{B}_b(\mathbb{R}^d)$,

$$P_T^{\ell,v} \log f(y) \leqslant \log P_T^{\ell,v} f(x) + \inf_{t \in (0,T]} \frac{\lambda_t^2 |x - y|^2}{2 \int_0^t e^{-2K(s)} d\ell(s)}, \quad x, y \in \mathbb{R}^d, \tag{3.6}$$

and for any $p > 1$,

$$\big(P_T^{\ell,v} f\big)^p(y) \leqslant \big(P_T^{\ell,v} f^p(x)\big) \inf_{t \in (0,T]} \exp\Bigg[\frac{p\lambda_t^2 |x - y|^2}{2(p-1)\int_0^t e^{-2K(s)} d\ell(s)}\Bigg], \quad x, y \in \mathbb{R}^d. \tag{3.7}$$

Proof. (1) By (3.5) and the Young inequality that for any probability measure ν on $\mathbb{R}^d$, if $g_1, g_2 \geqslant 0$ with $\nu(g_1) = 1$, then

$$\nu(g_1 g_2) \leqslant \log \nu(e^{g_2}) + \nu(g_1 \log g_1),$$

we obtain

$$P_T^{\ell,\nu} \log f(y) = \mathbb{E}\big[R \log f(X_T^{\ell,\nu})\big] \leqslant \log P_T^{\ell,\nu} f(x) + \mathbb{E}[R \log R].$$

By the definitions of R, η_t, ξ_t and noting that $\tau \leqslant T$, we have

$$\mathbb{E}[R \log R] = \mathbb{E}_{\mathbb{Q}} \log R = \mathbb{E}_{\mathbb{Q}} \left\{ - \int_0^{\ell(\tau)} \langle \eta_t, \mathrm{d}\tilde{W}_t \rangle + \frac{1}{2} \int_0^{\ell(\tau)} |\eta_t|^2 \, \mathrm{d}t \right\}$$

$$= \frac{1}{2} \mathbb{E} \int_0^{\ell(\tau)} |\eta_t|^2 \, \mathrm{d}t \leqslant \frac{\lambda_T^2}{2} \int_0^{\ell(T)} \xi_{\ell^{-1}(t)}^2 \, \mathrm{d}t = \frac{\lambda_T^2}{2} \int_0^T \xi_t^2 \, \mathrm{d}\ell(t)$$

$$= \frac{\lambda_T^2 |x - y|^2}{2 \int_0^T \exp[-2 \int_0^t K_s \, \mathrm{d}s] \, \mathrm{d}\ell(t)}.$$

This implies that

$$P_T^{\ell,\nu} \log f(y) \leqslant \log P_T^{\ell,\nu} f(x) + \frac{\lambda_T^2 |x - y|^2}{2 \int_0^T e^{-2K(s)} \, \mathrm{d}\ell(s)}, \quad x, y \in \mathbb{R}^d.$$

Now, for any $t \in (0, T]$, let $P_{t,T}^{\ell,\nu} f(x) = \mathbb{E}(f(X_{t,T}^{\ell,\nu}(x)))$, where $(X_{t,T}^{\ell,\nu}(x))_{t \leqslant T}$ solves the equation

$$X_{t,T}^{\ell,\nu}(x) = x + \int_t^T b_s\big(X_{t,s}^{\ell,\nu}(x)\big) \, \mathrm{d}s + \int_t^T \sigma_s \, \mathrm{d}W_{\ell(s)} + v_T - v_t, \quad t \leqslant T.$$

By the Markov property we have $P_T^{\ell,\nu} = P_t^{\ell,\nu} P_{t,T}^{\ell,\nu}$. So, applying the above inequality to t and $P_{t,T}^{\ell,\nu} f$ in place of T and f respectively, and noting that by the Jensen inequality

$$P_t^{\ell,\nu} \log P_{t,T}^{\ell,\nu} f \geqslant P_t^{\ell,\nu} P_{t,T}^{\ell,\nu} \log f = P_T^{\ell,\nu} \log f,$$

we obtain (3.6).

(2) By (3.5) and the Hölder inequality, we obtain

$$\big(P_T^{\ell,\nu} f(y)\big)^p = \big(\mathbb{E}\big[R f\big(X_T^{\ell,\nu}\big)\big]\big)^p \leqslant \big(P_T^{\ell,\nu} f^p(x)\big)\big(\mathbb{E} R^{\frac{p}{p-1}}\big)^{p-1}.$$

Since

$$\mathbb{E} R^{\frac{p}{p-1}} = \mathbb{E}\left(\exp\left[-\frac{p}{p-1} \int_0^{\ell(\tau)} \langle \eta_t, \mathrm{d}W_t \rangle - \frac{p}{2(p-1)} \int_0^{\ell(\tau)} |\eta_t|^2 \, \mathrm{d}t \right] \right)$$

$$= \mathbb{E}\left(\exp\left[-\frac{p}{p-1} \int_0^{\ell(\tau)} \langle \eta_t, \mathrm{d}W_t \rangle - \frac{p^2}{2(p-1)^2} \int_0^{\ell(\tau)} |\eta_t|^2 \, \mathrm{d}t \right] \exp\left[\frac{p}{2(p-1)^2} \int_0^{\ell(\tau)} |\eta_t|^2 \, \mathrm{d}t \right] \right)$$

$$\leqslant \exp\left[\frac{p \lambda_T^2}{2(p-1)^2} \int_0^{\ell(T)} \xi_{\ell^{-1}(t)}^2 \, \mathrm{d}t \right] = \exp\left[\frac{p \lambda_T^2 |x - y|^2}{2(p-1)^2 \int_0^T e^{-2K(s)} \, \mathrm{d}\ell(s)} \right],$$

we obtain

$$\big(P_T^{\ell,\nu} f\big)^p(y) \leqslant \big(P_T^{\ell,\nu} f^p(x)\big) \exp\left[\frac{p \lambda_T^2 |x - y|^2}{2(p-1) \int_0^T e^{-2K(s)} \, \mathrm{d}\ell(s)} \right], \quad x, y \in \mathbb{R}^d.$$

From this we obtain (3.7) similarly as in the first part of the proof. $\square$

3.2. Proofs of Theorem 2.1 and Corollary 2.2

To prove Theorem 2.1 using Proposition 3.2, we need the following lemma to ensure that $X_t^{(n)} \to X_t$ as $n \to \infty$, where $X^{(n)}$ solves (3.1) for $\varepsilon = \frac{1}{n}$.

Lemma 3.3. *Assume that*

(i) σ *is piecewise constant: there exists a sequence* $\{t_n\}_{n \geqslant 0}$ *with* $t_0 = 0$ *and* $t_n \uparrow \infty$ *such that* $\sigma_. = \sum_{i=1}^\infty 1_{[t_{i-1}, t_i)} \sigma_{t_{i-1}}$;
(ii) b *is globally Lipschitzian: for any* $T > 0$ *there exists a constant* $C > 0$ *such that*

$$\big| b_t(x) - b_t(y) \big| \leqslant C|x - y|, \quad t \in [0, T], \ x, y \in \mathbb{R}^d.$$

Then $\lim_{n \to \infty} X_t^{(n)} = X_t$ *holds for all* $t > 0$.

Proof. Let $T > 0$ be fixed. By (i) and (ii), for any $t \in [0, T]$, we have

$$\big| X_t^{(n)} - X_t \big| \leqslant C \int_0^t \big| X_s^{(n)} - X_s \big| \, ds + 2 \sup_{t \in [0, T]} \|\sigma_t\| \left(\sum_{t_i < t} |W_{S_{1/n}(t_i)} - W_{S(t_i)}| + |W_{S_{1/n}(t)} - W_{S(t)}| \right). \tag{3.8}$$

Moreover, it is easy to see that (ii) and the local boundedness of b, σ, V imply $\sup_{n \geqslant 1} |X_t^{(n)}| < \infty$ and thus, $\phi_t := \limsup_{n \to \infty} |X_t^{(n)} - X_t| < \infty$ for any $t \in [0, T]$. Combining these with (3.8) and using the Fatou lemma and the fact that $S_{1/n} \downarrow S$ as $n \uparrow \infty$, we arrive at $\phi_t \leqslant C \int_0^t \phi_s \, ds$ for any $t \in [0, T]$. Therefore, $\phi_t = 0$ for all $t \in [0, T]$ and the proof is thus finished. $\square$

Proof of Theorem 2.1. According to [2, Proposition 2.3], (2) follows from (1). So, we only prove (1) and (3). To apply Lemma 3.3, we shall make approximations of b and σ. Since $C_b(\mathbb{R}^d)$ is dense in $L^1(P_T(x, \cdot) + P_T(y, \cdot))$, where $P_T(z, dz')$ is the transition probability for P_T, we only consider strictly positive $f \in C_b(R^d)$.

(a) We first assume that (i) and (ii) in Lemma 3.3 hold. By applying Proposition 3.2 to $P_T^{S_{1/n}, V}$ and noting that Lemma 3.3 implies

$$P_T f = \lim_{n \to \infty} \mathbb{E} P_T^{S_{1/n}, V} f, \quad f \in C_b(\mathbb{R}^d),$$

we obtain (3.6) and (3.7) for (S, V) in place of (ℓ, v). Then the log-Harnack inequality follows by taking expectations to (3.6), and the Harnack inequality with power follows by taking expectations to (3.7) and using the Hölder inequality:

$$\begin{aligned}
P_T f(y) = \mathbb{E} P_T^{S, V} f(y) &\leqslant \mathbb{E} \left\{ \big(P_T^{S, V} f^p(x) \big)^{\frac{1}{p}} \inf_{t \in [0, T]} \exp\left[\frac{\lambda_t^2 |x - y|^2}{2(p-1) \int_0^t e^{-2K(s)} \, dS(s)} \right] \right\} \\
&\leqslant \big(\mathbb{E} P_T^{S, V} f^p(x) \big)^{\frac{1}{p}} \left(\mathbb{E} \inf_{t \in [0, T]} \exp\left[\frac{p \lambda_t^2 |x - y|^2}{2(p-1)^2 \int_0^t e^{-2K(s)} \, dS(s)} \right] \right)^{\frac{p-1}{p}} \\
&= \big(P_T f^p(x) \big)^{\frac{1}{p}} \left(\mathbb{E} \inf_{t \in [0, T]} \exp\left[\frac{p \lambda_t^2 |x - y|^2}{2(p-1)^2 \int_0^t e^{-2K(s)} \, dS(s)} \right] \right)^{\frac{p-1}{p}}.
\end{aligned}$$

(b) Assume that (ii) in Lemma 3.3 holds. Since for a fixed sample of S, the class of piecewise constant functions on $[0, \infty)$ is dense in $L^2_{loc}(dS)$, we may find out a sequence of $\mathbb{R}^d \otimes \mathbb{R}^d$-valued functions $\{\sigma^{(n)}\}_{n \geqslant 1}$ satisfying (i) in Lemma 3.3 such that $\sigma^{(n)} \to \sigma$ in $L^2([0, T]; dS)$ and $\|(\sigma_t^{(n)})^{-1}\| \leqslant \lambda_T$ for $t \in [0, T]$. Let $\tilde{X}_t^{(n)}$ solve (1.1) for $\sigma^{(n)}$ in place of σ, and let $\tilde{P}_t^{(n)}$ be the associated Markov operator. According to (a), the assertions in Theorem 2.1 hold for $\tilde{P}_T^{(n)}$ in place of P_T. By (ii) we have

$$\big| X_t - \tilde{X}_t^{(n)} \big| \leqslant C \int_0^t \big| X_s - \tilde{X}_s^{(n)} \big| \, ds + \left| \int_0^t \big(\sigma_s^{(n)} - \sigma_s \big) \, dW_{S(s)} \right|, \quad t \in [0, T]. \tag{3.9}$$

Since $\sigma^{(n)} \to \sigma$ in $L^2([0, T]; dS)$, we have (see e.g. [13, Theorem 88(v) on p. 53])

$$\lim_{n \to \infty} \mathbb{E}^S \left| \int_0^t \big(\sigma_s^{(n)} - \sigma_s \big) \, dW_{S(s)} \right|^2 = \lim_{n \to \infty} \int_0^t \big\| \sigma_s^{(n)} - \sigma_s \big\|_{HS}^2 \, dS(s) = 0,$$

where $\mathbb{E}^S$ is the conditional expectation given S. Then, as in the proof of Lemma 3.3, by letting $n \to \infty$ in (3.9) we obtain $\lim_{n\to\infty} \mathbb{E}^S |\tilde{X}_T^{(n)} - X_T| = 0$, so that

$$P_T f = \mathbb{E} P_T^{S,V} f = \lim_{n\to\infty} \mathbb{E}\big(\mathbb{E}^{S,V} f\big(\tilde{X}_T^{(n)}\big)\big) = \lim_{n\to\infty} \tilde{P}_T^{(n)} f.$$

Therefore, Theorem 2.1 also holds for P_T.

(c) In general, let $\tilde{b}_t(x) = b_t(x) - K_t x$. Then (1.2) is equivalent to the dissipative property of $\tilde{b}$:

$$\langle \tilde{b}_t(x) - \tilde{b}_t(y), x - y \rangle \leqslant 0, \quad t \geqslant 0, \ x, y \in \mathbb{R}^d.$$

Let $(\tilde{b}^{(n)})_{n\geqslant 1}$ be the Yoshida approximation of $\tilde{b}$, i.e.

$$\tilde{b}_t^{(n)}(x) = n\left\{\left(I - \frac{1}{n}\tilde{b}_t\right)^{-1}(x) - x\right\}, \quad t \geqslant 0, \ x \in \mathbb{R}^d.$$

Then (see e.g. [4, Section 2]), $\tilde{b}^{(n)}$ is dissipative and globally Lipschitzian in the sense of (ii) of Lemma 3.3, $|\tilde{b}^{(n)}| \leqslant |\tilde{b}|$, and $\lim_{n\to\infty} \tilde{b}^{(n)} = \tilde{b}$. Let $b_t^{(n)}(x) = \tilde{b}_t^{(n)}(x) + K_t x$. Then, $b_t^{(n)}$ satisfies (ii), and (1.2) holds for $b_t^{(n)}$ in place of b_t.

Now, let $\bar{X}_t^{(n)}$ solve (1.1) for $b^{(n)}$ in place of b. Then, according to (b), the associated Markov operator $\bar{P}_t^{(n)}$ satisfies the claimed inequalities in Theorem 2.1. So, if

$$\lim_{n\to\infty} \bar{P}_T^{(n)} f = P_T f, \quad f \in C_b\big(\mathbb{R}^d\big), \tag{3.10}$$

then we complete the proof by applying Theorem 2.1 to $\bar{P}_T^{(n)}$ and letting $n \to \infty$. The proof of (3.10) is straightforward by the constructions of $\tilde{b}$ and $\tilde{b}^{(n)}$, from which we have

$$\begin{aligned}
\mathrm{d}\big|X_t - \bar{X}_t^{(n)}\big|^2 &= 2K_t\big|X_t - \bar{X}_t^{(n)}\big|^2 \mathrm{d}t + 2\big\langle \tilde{b}_t(X_t) - \tilde{b}_t^{(n)}\big(\bar{X}_t^{(n)}\big), X_t - \bar{X}_t^{(n)}\big\rangle \mathrm{d}t \\
&\leqslant 2K_t\big|X_t - \bar{X}_t^{(n)}\big|^2 \mathrm{d}t + 2\big\langle \tilde{b}_t(X_t) - \tilde{b}_t^{(n)}(X_t), X_t - \bar{X}_t^{(n)}\big\rangle \mathrm{d}t \\
&\leqslant (2K_t + 1)\big|X_t - \bar{X}_t^{(n)}\big|^2 \mathrm{d}t + \big|\tilde{b}_t(X_t) - \tilde{b}_t^{(n)}(X_t)\big|^2 \mathrm{d}t.
\end{aligned} \tag{3.11}$$

Let $\tau_m = \inf\{t \geqslant 0: \ |X_t| \geqslant m\}$ for $m \geqslant 1$. We obtain from (3.11) that

$$\big|X_{T\wedge\tau_m} - \bar{X}_{T\wedge\tau_m}^{(n)}\big|^2 \leqslant \mathrm{e}^{2\int_0^T (K_t+1)\,\mathrm{d}t} \int_0^{T\wedge\tau_m} \big|\tilde{b}_t(X_t) - \tilde{b}_t^{(n)}(X_t)\big|^2 \mathrm{d}t.$$

Since $\{|\tilde{b}_t(X_t) - \tilde{b}_t^{(n)}(X_t)| : t \leqslant T \wedge \tau_m, n \geqslant 1\}$ is bounded and $\lim_{n\to\infty} \tilde{b}_t^{(n)} = \tilde{b}_t$, this implies $\lim_{n\to\infty} |X_{T\wedge\tau_m} - \bar{X}_{T\wedge\tau_m}^{(n)}|^2 = 0$ for all $m \geqslant 1$. Combining this with $\tau_m \uparrow \infty$ as $m \uparrow \infty$, we conclude that $\lim_{n\to\infty} \bar{X}_T^{(n)} = X_T$ a.s. and thus prove (3.10). $\quad\square$

Proof of Corollary 2.2. By Theorem 2.1, it suffices to prove for $T \in (0, 1]$. There exists a constant $c_1 \geqslant 1$ such that for any $k \geqslant 1$,

$$\begin{aligned}
\mathbb{E}\frac{1}{S(t)^k} &= \frac{1}{\Gamma(k)}\int_0^\infty r^{k-1}\mathrm{e}^{-tB(r)}\,\mathrm{d}r \leqslant \frac{\exp[cr_0^\theta t]}{\Gamma(k)}\int_0^\infty r^{k-1}\exp\big[-cr^\theta t\big]\,\mathrm{d}r \\
&= \frac{\exp[cr_0^\theta t]}{\Gamma(k)c^{\frac{1}{\theta}}}\int_0^\infty r^{k-1}\mathrm{e}^{-tr^\theta}\,\mathrm{d}r \leqslant c_1\mathbb{E}\frac{1}{\tilde{S}(t)^k}, \quad t \in (0, 1],
\end{aligned} \tag{3.12}$$

where $\tilde{S}$ is the subordinator associated to the Bernstein function $r \mapsto r^\theta$. Therefore, (1) and (2) follow from Theorem 2.1(1)–(2) and [5, (2.2)] by noting that

$$\frac{1}{\int_0^t \mathrm{e}^{-2K(s)}\,\mathrm{d}S(s)} \leqslant \frac{c_2}{S(t)}, \quad t \in (0, 1] \tag{3.13}$$

holds for some constant $c_2 > 0$. To prove (3), we make use of the third display from below in the proof of [5, Theorem 1.1] for $\kappa = 1$, i.e.

$$\mathbb{E}\mathrm{e}^{\lambda/\tilde{S}(t)} \leqslant 1 + \left(\exp\left[\frac{c_3\lambda^{\frac{\theta}{2\theta-1}}}{t^{\frac{1}{2\theta-1}}}\right] - 1\right)^{\frac{2\theta-1}{\theta}} \leqslant \exp\left[\frac{c_4\lambda}{t^{\frac{1}{\theta}}} + \frac{c_4\lambda^{\frac{\theta}{2\theta-1}}}{t^{\frac{1}{2\theta-1}}}\right], \quad \lambda, t \geqslant 0$$

for some constants $c_3, c_4 > 0$. This along with (3.12) yields that

$$\mathbb{E}e^{\lambda/S(t)} \leqslant 1 + c_1 \mathbb{E}\big(e^{\lambda/\tilde{S}(t)} - 1\big) \leqslant \mathbb{E}e^{c_1\lambda/\tilde{S}(t)} \leqslant \exp\left[\frac{c_5\lambda}{t^{\frac{1}{\theta}}} + \frac{c_5\lambda^{\frac{\theta}{2\theta-1}}}{t^{\frac{1}{2\theta-1}}}\right]$$

for some $c_5 > 0$. Combining this with (3.13) we prove (3) from Theorem 2.1(3). □

4. Extension to semilinear SPDEs

Let $(H, \langle\cdot,\cdot\rangle, |\cdot|)$ be a separable Hilbert space, $V := (V_t)_{t\geqslant 0}$ be a locally bounded measurable stochastic process on H, $W = (W_t)_{t\geqslant 0}$ be a cylindrical Brownian motion on H, and $S = (S_t)_{t\geqslant 0}$ be a one-dimensional non-negative increasing Lévy process associated to a Bernstein function B as introduced in Section 1. Recall that W can be formally formulated as

$$W_t = \sum_{i=1}^{\infty} B_t^i e_i, \tag{4.1}$$

where $\{B^i\}_{i\geqslant 1}$ is a family of independent one-dimensional Brownian motions, and $\{e_i\}_{i\geqslant 1}$ is an orthonormal basis of H. Thus, for any orthonormal family $\{e_i\}_{i=1}^n$, the process $(\langle W, e_1\rangle, \ldots, \langle W, e_n\rangle)$ is a Brownian motion on $\mathbb{R}^n$. As in the finite-dimensional case, we assume that W, S and V are independent. Let $\mathscr{L}(H)$ be the set of all bounded linear operators on H.

Consider the following stochastic equation on H:

$$X_t = e^{At} X_0 + \int_0^t e^{A(t-s)} F_s(X_s)\,ds + \int_0^t e^{A(t-s)}\sigma_s\,dW_{S(s)} + V_t, \quad t \geqslant 0, \tag{4.2}$$

where $\sigma : [0,\infty) \to \mathscr{L}(H)$ is measurable and locally bounded, and A and F satisfy

(A1) $(A, \mathscr{D}(A))$ is a negatively definite self-adjoint operator on H such that

$$\int_0^t \|e^{sA}\|_{HS}^2\,ds < \infty, \quad t > 0; \tag{4.3}$$

(A2) $F : [0,\infty) \times H \to H$ is measurable, bounded on bounded sets and satisfies

$$\big|F_s(x) - F_s(y)\big| \leqslant K_s|x - y|, \quad x, y \in H,\ s \geqslant 0 \tag{4.4}$$

holds for some locally bounded measurable function K on $[0,\infty)$.

Note that if $V_t = \int_0^t e^{(t-s)A}\tilde{\sigma}_s\,dL_s$ holds for some $\tilde{\sigma} : [0,\infty) \to \mathscr{L}(H)$ and some noise L, (4.2) is known as the definition of mild solutions to the stochastic differential equation

$$dX_t = AX_t\,dt + F_t(X_t)\,dt + \sigma_t\,dW_{S(t)} + \tilde{\sigma}_t\,dL_t.$$

We first confirm the existence and uniqueness of the solution (4.2). By **(A1)**, the operator $-A$ has discrete spectrum with eigenvalues $0 \leqslant \rho_1 \leqslant \rho_2 \leqslant \cdots \leqslant \rho_n \uparrow \infty$. From now on, we let $\{e_i\}_{i\geqslant 1}$ be the corresponding eigenbasis, i.e. an orthonormal basis of H such that $Ae_i = -\rho_i e_i$, $i \geqslant 1$.

Proposition 4.1. *Assume* **(A1)**–**(A2)***. Then*

(1) *Let* $B^k := \langle W, e_k\rangle$ *as in* (4.1)*. Then*

$$Y_t := \int_0^t e^{(t-s)A}\sigma_s\,dW_{S(s)} = \sum_{k,j=1}^{\infty}\left(\int_0^t \langle e^{(t-s)A}\sigma_s e_k, e_j\rangle\,dB_{S(s)}^k\right)e_j, \quad t \geqslant 0,$$

gives rise to a stochastically continuous process on H such that $\mathbb{E}^S \int_0^T |Y_t|^2\,dt < \infty$ for $T > 0$, where $\mathbb{E}^S$ is the conditional expectation given S. In particular, Y has a measurable modification.

(2) *Fix a measurable modification of $Y := (Y_t)_{t\geqslant 0}$, and denote it again by*

$$Y_t = \int_0^t e^{(t-s)A}\sigma_s\,dW_{S(s)}, \quad t \geqslant 0.$$

For any $X_0 \in H$, Eq. (4.2) *has a unique solution.*

Proof. (1) We first prove

$$U_t := \sum_{k,j=1}^{\infty} \int_0^t \langle e^{A(t-s)} \sigma_s e_k, e_j \rangle^2 \, dS(s) < \infty \text{ a.s.}, \quad t \geqslant 0, \tag{4.5}$$

and $\int_0^T U_t \, dt < \infty$ for any $T > 0$. Note that for each $t \geqslant 0$, $U_t < \infty$ a.s. implies that Y_t is a well defined H-valued random variable with $\mathbb{E}^S |Y_t|^2 = U_t < \infty$ (see e.g. [13, Theorem 88(v) on p. 53]), and $\int_0^T U_t \, dt < \infty$ implies $\mathbb{E}^S \int_0^T |Y_t|^2 \, dt < \infty$.

It is easy to see that

$$U_t = \int_0^t \big\| e^{A(t-s)} \sigma_s \big\|_{HS}^2 \, dS(s) \leqslant \left(\sup_{s \in [0,t]} \|\sigma_s\|^2 \right) \int_0^t \big\| e^{A(t-s)} \big\|_{HS}^2 \, dS(s) =: C_t \gamma_t, \quad t \geqslant 0. \tag{4.6}$$

It then follows from (4.3) that

$$\int_0^T U_t \, dt \leqslant C_T \int_0^T \gamma_t \, dt = C_T \int_0^T dS(s) \int_s^T \big\| e^{(t-s)A} \big\|_{HS}^2 \, dt \leqslant C_T S(T) \int_0^T \big\| e^{tA} \big\|_{HS}^2 \, dt < \infty.$$

So, $Y_. \in L^2([0,T] \to H; dt)$ and $U_t < \infty$ a.s. for a.e.-$t \geqslant 0$. It remains to show that $U_t < \infty$ a.s. for all $t \geqslant 0$, so that Y_t is an H-valued random variable for each $t \geqslant 0$. For any $t > 0$, there exists $t' \in (0,t)$ such that $\gamma_{t'}, \gamma_{t-t'} < \infty$ a.s. Since $S = S(\cdot + t') - S(t')$ in law, $\int_{t'}^t \big\| e^{A(t-s)} \big\|_{HS}^2 \, dS(s) = \gamma_{t-t'} = \int_0^{t-t'} \big\| e^{A(t-t'-s)} \big\|_{HS}^2 \, dS(s)$ in law as well. Thus,

$$\gamma_t = \int_0^{t'} \big\| e^{A(t-s)} \big\|_{HS}^2 \, dS(s) + \int_{t'}^t \big\| e^{A(t-s)} \big\|_{HS}^2 \, dS(s) = \gamma_{t'} + \int_{t'}^t \big\| e^{A(t-s)} \big\|_{HS}^2 \, dS(s) < \infty, \quad \text{a.s.}$$

Therefore, by (4.6) we have $U_t < \infty$ a.s. for all $t \geqslant 0$.

To prove the stochastic continuity of Y, we note that for any $t \geqslant 0$ and $h > 0$, we have

$$|Y_{t+h} - Y_t| \leqslant \big| e^{hA} Y_t - Y_t \big| + |I(h)|,$$

and

$$I(h) := \int_t^{t+h} e^{(t+h-s)A} \sigma_s \, dW_{S(s)}.$$

Note that $C_t = \sup_{s \in [0,t]} \|\sigma_s\|^2 < \infty$. We have, for $h \in (0,1)$,

$$\mathbb{E}^S \big| I(h) \big|^2 \leqslant C_{t+1} \int_{t-h}^t \big\| e^{(t-s)A} \big\|_{HS}^2 \, dS(s+h)$$

which in law equals to $C_{t+1} \int_{t-h}^t \big\| e^{(t-s)A} \big\|_{HS}^2 \, dS(s)$. Since $U_t < \infty$ a.s. and $S(t) = S(t-)$ a.s. for fixed t, we conclude that $I(h) \to 0$ in probability as $h \to 0$. Therefore, for any $\varepsilon > 0$,

$$\limsup_{h \downarrow 0} \mathbb{P} \big(|Y_{t+h} - Y_t| \geqslant \varepsilon \big) \leqslant \limsup_{h \downarrow 0} \left\{ \mathbb{P} \left(\big| e^{hA} Y_t - Y_t \big| \geqslant \frac{\varepsilon}{2} \right) + \mathbb{P} \left(|I(h)| \geqslant \frac{\varepsilon}{2} \right) \right\} = 0.$$

Similarly, we can prove $\lim_{s \uparrow t} \mathbb{P}(|Y_t - Y_s| \geqslant \varepsilon) = 0$ for any $t, \varepsilon > 0$. Due to the stochastic continuity, the process Y has a measurable modification (see [1, Theorem 3]).

(2) Once a measurable modification of Y is fixed, as explained in Section 1, we let $\tilde{X}_t = X_t - Y_t - V_t$ and reformulate (4.2) as

$$\tilde{X}_t = e^{At} X_0 + \int_0^t e^{A(t-s)} F_s(\tilde{X}_s + Y_s + V_s) \, ds, \quad t \geqslant 0,$$

which has a unique solution due to **(A2)**. □

We note that in the proof of Proposition 4.1, for different measurable modifications of Y, the corresponding solutions derived for Eq. (4.2) are equivalent, i.e. they are modifications of each other as well. When $V = 0$ and σ_s is independent of s with $\sigma e_i = \beta_i e_i$ holding for some sequence $\{\beta_i\} \subset \mathbb{R}$, solutions to (4.2) have been investigated in [7].

By Proposition 4.1, we define

$$P_t f(x) = \mathbb{E} f(X_t(x)), \quad t \geqslant 0, \ x \in H, \ f \in \mathscr{B}_b(H),$$

where $X(x)$ is the solution to (4.2) for $X_0 = x$. We shall make use of finite-dimensional approximations to derive the Harnack inequalities from Theorem 2.1. Note that P_t is independent of modifications of Y, and is thus unique due to Proposition 4.1.

For $n \geqslant 1$, let $H_n = \text{span}\{e_1, \ldots, e_n\}$, and let π_n be the orthogonal projection from H onto H_n. Let

$$A^{(n)} = \pi_n A, \qquad F^{(n)} = \pi_n F, \qquad \sigma^{(n)} = \pi_n \sigma, \qquad W^{(n)} = \pi_n W, \qquad V^{(n)} = \pi_n V.$$

For any $n \geqslant 1$, consider the following equation on H_n:

$$X_t^{(n)} = \pi_n X_0 + \int_0^t \left\{ A^{(n)} X_s^{(n)} + F_s^{(n)}(X_s^{(n)}) \right\} ds + \int_0^t \sigma_s^{(n)} dW_{S(s)}^{(n)} + V_t^{(n)}, \quad t \geqslant 0. \tag{4.7}$$

Let $P_t^{(n)}$ be the associated Markov operator. It is easy to see that assertions in Theorem 2.1 hold for $P_T^{(n)}$. Letting $n \to \infty$, we conclude that assertions in Theorem 2.1, and hence in Corollary 2.2, hold for the present P_T.

Theorem 4.2. *Assume* **(A1)** *and* **(A2)**. *If* $\|(\sigma_t^{(n)})^{-1}\| \leqslant \lambda_t$ *for some increasing function* λ *on* $[0, \infty)$ *and large n, then assertions in* Theorem 2.1 *and* Corollary 2.2 *hold for H in place of* $\mathbb{R}^d$.

Proof. As explained above we only consider positive $f \in C_b(H)$. In this case, by the assertions for $P_T^{(n)}$ and the dominated convergence theorem, it suffices to prove that $\lim_{n \to \infty} X_T^{(n)} = X_T$ in law. This follows from

$$\lim_{n \to \infty} \mathbb{E}^{S,V} |X_t^{(n)} - X_t| = 0, \quad t > 0,$$

which can be easily verified as in the proof of [21, Theorem 2.1]. $\quad\square$

Similarly to the finite-dimensional situation, if P_T has a quasi-invariant measure μ then according to [20, Proposition 3.1], the assertions in Corollary 2.2 imply that P_T has a heat kernel $p_T(x, y)$ with respect to μ and estimates (2.1), (2.2) and (2.3) hold for μ and H replacing the Lebesgue measure and $\mathbb{R}^d$ respectively.

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