



A note on generalized q -difference equations for q -beta and Andrews–Askey integral



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ABSTRACT

Two q -difference equations with solutions expressed by q -exponential operator identities are investigated. As applications, two extensions of Ramanujan's formulas for q -beta integral are given, two generalizations of Andrews–Askey integral are obtained. In addition, generating functions for generalized Al-Salam–Carlitz polynomials are deduced. At last, a generalized transformation identity is gained.

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1. Introduction

The q -polynomials constitute a very important and interesting set of special functions and more specifically of orthogonal polynomials. Their generating functions appear in several branches of the natural sciences [4,20,25,30], e.g., continued fractions, Eulerian series, theta functions, elliptic functions, quantum groups and algebras, discrete mathematics (combinatorics, graph theory), coding theory, etc.

In this paper, we follow the notations and terminology in [20] and suppose that $0 < q < 1$. The q -series and its compact factorials are defined respectively by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k) \quad (1.1)$$

and $(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$, where m is a positive integer and n is a non-negative integer or ∞ .

The following two q -difference operators are defined by [17,20]

$$D_a \{f(a)\} = \frac{f(a) - f(aq)}{a}, \quad \theta_a \{f(a)\} = \frac{f(aq^{-1}) - f(a)}{q^{-1}a}. \quad (1.2)$$

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Chen and Liu [16,17] employed the technique of Parameter Augmentation by constructing the following two q -exponential operators

$$\mathbb{T}(bD_a) = \sum_{n=0}^{\infty} \frac{(bD_a)^n}{(q; q)_n}, \quad \mathbb{E}(b\theta_a) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (b\theta_a)^n}{(q; q)_n}. \quad (1.3)$$

Later, authors [15,18] researched the following general q -exponential operators

$$\mathbb{T}(a, bD_c) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} (bD_c)^n, \quad \mathbb{E}(a, -b\theta_c) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} (-b\theta_c)^n. \quad (1.4)$$

Fang [19], and Zhang and Yang [31] considered the following generalized q -exponential operators

$$\mathbb{T} \left[\begin{matrix} q^{-N}, w \\ v \end{matrix}; q, bD_c \right] = \sum_{n=0}^N \frac{(q^{-N}, w; q)_n}{(q, v; q)_n} (bD_c)^n, \quad \mathbb{E} \left[\begin{matrix} q^{-N}, w \\ v \end{matrix}; q, -b\theta_c \right] = \sum_{n=0}^N \frac{(q^{-N}, w; q)_n}{(q, v; q)_n} (-b\theta_c)^n. \quad (1.5)$$

The method of q -exponential operator is a rich and powerful tool for q -series, especially it makes many famous results easily fall into this framework. For more information, please refer to [10,15–19,21,31].

Liu [22,23] obtained several results involving Bailey's ${}_6\psi_6$, q -Mehler formulas for Rogers–Szegő polynomials and q -integral of Sears' transformation by the following q -difference equations, see also [24].

Proposition 1. (See [22, Theorems 1 and 2].) Let $f(a, b)$ be a two-variable analytic function in a neighborhood of $(a, b) = (0, 0) \in \mathbb{C}^2$.

(I.1) If $f(a, b)$ satisfies the difference equation

$$bf(aq, b) - af(a, bq) = (b - a)f(a, b), \quad (1.6)$$

then we have

$$f(a, b) = \mathbb{T}(bD_a)\{f(a, 0)\}. \quad (1.7)$$

(I.2) If $f(a, b)$ satisfies the difference equation

$$af(aq, b) - bf(a, bq) = (a - b)f(aq, bq), \quad (1.8)$$

then we have

$$f(a, b) = \mathbb{E}(b\theta_a)\{f(a, 0)\}. \quad (1.9)$$

Lu [24] constructed the following generalized equations and studied some generating functions.

Proposition 2. (See [24, Proposition 1.2].) Let $f(a, b, c)$ be a three-variable analytic function in a neighborhood of $(a, b, c) = (0, 0, 0) \in \mathbb{C}^3$.

(II.1) If $f(a, b, c)$ satisfies the difference equation

$$(c - b)f(a, b, c) = abf(a, bq, cq) - bf(a, b, cq) + (c - ab)f(a, bq, c), \quad (1.10)$$

then we have

$$f(a, b, c) = \mathbb{T}(a, bD_c)\{f(a, 0, c)\}. \quad (1.11)$$

In this paper, we give the following two q -difference equations for the generalized q -exponential operators.

Theorem 3. Let $f(a, b, c, d, e)$ be a five-variable analytic function in a neighborhood of $(a, b, c, d, e) = (0, 0, 0, 0, 0) \in \mathbb{C}^5$.

(III.1) If $f(w, r, v, b, c)$ satisfies the q -difference equation

$$\begin{aligned} & b[f(w, r, v, b, c) - (1 + q^{-1}v)f(w, r, v, b, qc) + q^{-1}vf(w, r, v, b, q^2c)] \\ &= c\{[f(w, r, v, b, c) - f(w, r, v, qb, c)] - (w + r)[f(w, r, v, b, qc) - f(w, r, v, qb, qc)] \\ &+ wr[f(w, r, v, b, q^2c) - f(w, r, v, qb, q^2c)]\}, \end{aligned} \quad (1.12)$$

then we have

$$f(w, r, v, b, c) = \mathbb{T} \left[\begin{matrix} w, r \\ v \end{matrix}; q, cD_b \right] f(w, r, v, b, 0). \quad (1.13)$$

(III.2) If $f(w, r, v, b, c)$ satisfies the q -difference equation

$$\begin{aligned} & q^{-1}b[f(w, r, v, b, c) - (1 + q^{-1}v)f(w, r, v, b, qc) + q^{-1}vf(w, r, v, b, q^2c)] \\ &= c\{[f(w, r, v, b, c) - f(w, r, v, q^{-1}b, c)] - (w + r)[f(w, r, v, b, qc) - f(w, r, v, q^{-1}b, qc)] \\ &+ wr[f(w, r, v, b, q^2c) - f(w, r, v, q^{-1}b, q^2c)]\}, \end{aligned} \quad (1.14)$$

then we have

$$f(w, r, v, b, c) = \mathbb{E} \left[\begin{matrix} w, r \\ v \end{matrix}; q, -c\theta_b \right] f(w, r, v, b, 0). \quad (1.15)$$

Corollary 4. Let $f(a, b, c)$ be a three-variable analytic function in a neighborhood of $(a, b, c) = (0, 0, 0) \in \mathbb{C}^3$.

(II.2) If $f(a, b, c)$ satisfies the difference equation

$$(q^{-1}c - b)f(a, b, c) = abf(a, qb, q^{-1}c) - bf(a, b, q^{-1}c) + (q^{-1}c - ab)f(a, qb, c), \quad (1.16)$$

then we have

$$f(a, b, c) = \mathbb{E}(a, -b\theta_c)\{f(a, 0, c)\}. \quad (1.17)$$

Remark 5. Let $w = v = 0$ and $(r, b, c) = (a, c, b)$ in Theorem 3, Eqs. (1.12) and (1.14) reduce to (1.10) and (1.16) respectively. Set $r = 1$ in (1.12) and (1.14), $f(w, 1, v, b, c)$ is equal to $f(w, r, v, b, 0)$ by (1.13) and (1.15) respectively.

Theorem 3 shows that if an analytic function $f(w, r, v, b, c)$ satisfies q -difference equation (1.12) or (1.14), we can obtain it from q -exponential operators. The method is simple and direct. Two cases of the theorem are similar, so the proof of the second case is omitted.

Proof of Theorem 3. From the theory of several complex variables [29], we assume that

$$f(w, r, v, b, c) = \sum_{n=0}^{\infty} A_n(w, r, v, b)c^n. \quad (1.18)$$

Let the above definition in (1.12) yield

$$\begin{aligned} & b \sum_{n=0}^{\infty} (1 - q^n)(1 - vq^{n-1})A_n(w, r, v, b)c^n \\ &= \sum_{n=1}^{\infty} \{(1 - wq^{n-1})(1 - rq^{n-1})[A_{n-1}(w, r, v, b) - A_{n-1}(w, r, v, qb)]\}c^n. \end{aligned} \quad (1.19)$$

Equating coefficients of c^n , we have

$$A_n(w, r, v, b) = \frac{(1 - wq^{n-1})(1 - rq^{n-1})}{(1 - q^n)(1 - vq^{n-1})} D_b A_{n-1}(w, r, v, b). \quad (1.20)$$

Repeating the process, we have

$$A_n(w, r, v, b) = \frac{(w, r; q)_n}{(q, v; q)_n} D_b^n A_0(w, r, v, b). \quad (1.21)$$

Letting $c = 0$ in (1.18), we obtain $A_0(w, r, v, b) = f(w, r, v, b, 0)$ and gain (1.13) immediately by definition (1.18). Similarly, assuming that

$$f(w, r, v, b, c) = \sum_{n=0}^{\infty} B_n(w, r, v, b)c^n \quad (1.22)$$

and substituting (1.22) into (1.14) yields

$$B_n(w, r, v, b) = \frac{(1 - wq^{n-1})(1 - rq^{n-1})}{(q^n - 1)(1 - vq^{n-1})} \theta_b B_{n-1}(w, r, v, b), \quad (1.23)$$

which becomes (1.15) by (1.22) after simplification. The proof is complete. \square

The rest of the paper will be organized as follows. In Section 2, two generalizations of Ramanujan's formulas for q -beta integrals are treated by the method of q -difference equation. In Section 3, two expansions of Andrews–Askey integrals are deduced by the method of q -difference equation. In Section 4, two generating functions for generalized Al-Salam–Carlitz polynomials are obtained by the method of q -difference equation. In Section 5, a generalized transformation identity from q -Chu–Vandermonde sums is gained by the method of q -difference equation.

2. Generalizations of Ramanujan's formulas for q -beta integrals

The following two q -beta integrals due to Ramanujan are found by Andrews [3] in 1976 and evaluated by Askey [6, Eqs. (2) and (3)] in 1982. These two integrals are connected with the q -beta function and are analogues of the Euler and Cauchy integrals, see details in [3,6,8,26].

Proposition 6. (See [6, Eqs. (2) and (3)].) For $0 < q = \exp(-2k^2) < 1$ and $m \in \mathbb{R}$, supposing that $|abq| < 1$, we have

$$\int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q)_{\infty}} dx = \sqrt{\pi} e^{m^2} \frac{(-aqe^{2mki}, -bqe^{-2mki}; q)_{\infty}}{(abq; q)_{\infty}}. \quad (2.1)$$

Supposing that $\max\{|aq^{1/2}e^{2mk}|, |bq^{1/2}e^{-2mk}|\} < 1$, we have

$$\int_{-\infty}^{\infty} e^{-x^2+2mx} (-aqe^{2kx}, -bqe^{-2kx}; q)_{\infty} dx = \sqrt{\pi} e^{m^2} \frac{(abq; q)_{\infty}}{(aq^{1/2}e^{2mk}, bq^{1/2}e^{-2mk}; q)_{\infty}}. \quad (2.2)$$

Remark 7. Derivations of Ramanujan's integrals (2.1) and (2.2) for real values of the parameter m have been deduced by R. Askey [6]. Later on it has become clear that these integrals are in fact valid for arbitrary complex values of the parameter m and they are thus instances of the standard Fourier transform with the exponential kernel by N.M. Atakishiyev and P. Feinsilver [8].

The Ramanujan q -beta integrals (2.1) and (2.2) are important in q -series. Hardy remarked [28, p. xxv]: “There is always more in one of Ramanujan's formulas than meets the eye, as anyone who gets to work to verify those which look the easiest will soon discover. In some the interest lies very deep, in others comparatively near the surface; but there is not one which is not curious and entertaining”. For more information, please refer to [28].

There are many clever methods to deduce (2.1) and (2.2). For example, Askey [6] evaluated these equations by the technique of integral. Pastro [26] proved them again by the method of recurrence. For more information about Ramanujan's q -beta integrals and their generalizations, please refer to [3,6,8,21,27,28].

In this section, we give the generalizations of (2.1) and (2.2) by the method of q -difference equation.

Theorem 8. For $m \in \mathbb{R}$, $N \in \mathbb{N}$ and $r = q^{-N}$, if $|abq| < 1$, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q)_{\infty}} {}_3\phi_1 \left[\begin{matrix} r, w, -aqe^{2mki} \\ qrw/v \end{matrix}; q, -\frac{q^{1/2}e^{2ik(x-m)}}{v} \right] dx \\ &= \sqrt{\pi} e^{m^2} \frac{(-aqe^{2mki}, -bqe^{-2mki}, qw/v, qr/v; q)_{\infty}}{(abq, qwr/v, q/v; q)_{\infty}} {}_2\phi_1 \left[\begin{matrix} r, w \\ v \end{matrix}; q, -\frac{qb}{e^{2mki}} \right]. \end{aligned} \quad (2.3)$$

Theorem 9. For $m \in \mathbb{R}$, $N \in \mathbb{N}$ and $r = q^{-N}$, if $\max\{|aq^{1/2}e^{2mk}|, |bq^{1/2}e^{-2mk}|\} < 1$, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{-x^2+2mx} (-aqe^{2kx}, -bqe^{-2kx}; q)_{\infty} {}_3\phi_1 \left[\begin{matrix} r, w, q^{1/2}e^{-2mk}/a \\ qrw/v \end{matrix}; q, -aqe^{2kx} \right] dx \\ &= \sqrt{\pi} e^{m^2} \frac{(v/(wr), v, abq; q)_{\infty}}{(aq^{1/2}e^{2mk}, bq^{1/2}e^{-2mk}, v/w, v/r; q)_{\infty}} {}_2\phi_1 \left[\begin{matrix} r, w \\ v \end{matrix}; q, \frac{bvq^{1/2}e^{-2mk}}{wr} \right]. \end{aligned} \quad (2.4)$$

Remark 10. Let $r = 1$ in Theorems 8 and 9, Eqs. (2.3) and (2.4) reduce to (2.1) and (2.2) respectively.

To prove the above results, the following lemmas are necessary.

Lemma 11. (See [31, Eqs. (2.9) and (2.10)].) For $N \in \mathbb{N}$ and $r = q^{-N}$, supposing that $|at| < 1$, we have

$$\mathbb{T} \left[\begin{matrix} w, r \\ v \end{matrix}; q, cD_a \right] \left\{ \frac{1}{(at; q)_\infty} \right\} = \frac{1}{(at; q)_\infty} {}_2\phi_1 \left[\begin{matrix} w, r \\ v \end{matrix}; q, ct \right], \quad (2.5)$$

$$\mathbb{E} \left[\begin{matrix} w, r \\ v \end{matrix}; q, -c\theta_a \right] \{ (at; q)_\infty \} = (at; q)_\infty {}_2\phi_1 \left[\begin{matrix} w, r \\ v \end{matrix}; q, ct \right]. \quad (2.6)$$

Lemma 12. (See [31, Eqs. (2.11) and (2.14)].) For $N \in \mathbb{N}$ and $r = q^{-N}$, supposing that $\max\{|as|, |at|, |qw/v|, |qr/v|\} < 1$, we have

$$\mathbb{T} \left[\begin{matrix} w, r \\ v \end{matrix}; q, \frac{qD_a}{t} \right] \left\{ \frac{1}{(as, at; q)_\infty} \right\} = \frac{(qwr/v, q/v; q)_\infty}{(as, at, qw/v, qr/v; q)_\infty} {}_3\phi_1 \left[\begin{matrix} r, w, at \\ qr w/v \end{matrix}; q, \frac{qs}{vt} \right]. \quad (2.7)$$

Supposing that $\max\{|v/(wr)|, |v|\} < 1$, we have

$$\mathbb{E} \left[\begin{matrix} w, r \\ v \end{matrix}; q, -\frac{v\theta_a}{wrt} \right] \{ (as, at; q)_\infty \} = \frac{(as, at, v/w, v/r; q)_\infty}{(v/(wr), v; q)_\infty} {}_3\phi_1 \left[\begin{matrix} r, w, q/(at) \\ qr w/v \end{matrix}; q, as \right]. \quad (2.8)$$

Proof of Theorems 8 and 9. Eq. (2.3) can be written equivalently as

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(bq^{1/2}e^{-2ikx}; q)_\infty} \left\{ \frac{(qwr/v, q/v; q)_\infty}{(aq^{1/2}e^{2ikx}, -aqe^{2mki}, qw/v, qr/v; q)_\infty} {}_3\phi_1 \left[\begin{matrix} r, w, -aqe^{2mki} \\ qr w/v \end{matrix}; q, -\frac{q^{1/2}e^{2ik(x-m)}}{v} \right] \right\} dx \\ &= \sqrt{\pi} e^{m^2} (-bqe^{-2mki}; q)_\infty \left\{ \frac{1}{(abq; q)_\infty} {}_2\phi_1 \left[\begin{matrix} r, w \\ v \end{matrix}; q, -\frac{qb}{e^{2mki}} \right] \right\}. \end{aligned} \quad (2.9)$$

For $s = -e^{-2mki}$, if we use $F(w, r, v, a, s)$ to denote the right-hand side of (2.9) and let $f(w, r, v, a, s)$ to denote

$$\frac{1}{(abq; q)_\infty} {}_2\phi_1 \left[\begin{matrix} r, w \\ v \end{matrix}; q, -\frac{qb}{e^{2mki}} \right],$$

noting the fact that

$$\begin{aligned} & \mathbb{T} \left[\begin{matrix} w, r \\ v \end{matrix}; q, -\frac{D_a}{e^{2mki}} \right] \left\{ \frac{1}{(aq^{1/2}e^{2ikx}, -aqe^{2mki}; q)_\infty} \right\} \\ &= \frac{(qwr/v, q/v; q)_\infty}{(aq^{1/2}e^{2ikx}, -aqe^{2mki}, qw/v, qr/v; q)_\infty} {}_3\phi_1 \left[\begin{matrix} r, w, -aqe^{2mki} \\ qr w/v \end{matrix}; q, -\frac{q^{1/2}e^{2ik(x-m)}}{v} \right] \end{aligned} \quad (2.10)$$

and

$$\mathbb{T} \left[\begin{matrix} w, r \\ v \end{matrix}; q, -\frac{D_a}{e^{2mki}} \right] \left\{ \frac{1}{(abq; q)_\infty} \right\} = \frac{1}{(abq; q)_\infty} {}_2\phi_1 \left[\begin{matrix} r, w \\ v \end{matrix}; q, -\frac{qb}{e^{2mki}} \right], \quad (2.11)$$

it's easy to verify that $f(w, r, v, a, s)$ satisfies (1.12), so does $F(w, r, v, a, s)$. By (1.13), we have

$$\begin{aligned} F(w, r, v, a, s) &= \mathbb{T} \left[\begin{matrix} w, r \\ v \end{matrix}; q, -\frac{D_a}{e^{2mki}} \right] \{ F(w, r, v, a, 0) \} \\ &= \mathbb{T} \left[\begin{matrix} w, r \\ v \end{matrix}; q, -\frac{D_a}{e^{2mki}} \right] \{ F(w, 1, v, a, s) \} \quad \text{by Remark 5} \\ &= \mathbb{T} \left[\begin{matrix} w, r \\ v \end{matrix}; q, -\frac{D_a}{e^{2mki}} \right] \left\{ \sqrt{\pi} e^{m^2} \frac{(-bqe^{-2mki}; q)_\infty}{(abq; q)_\infty} \right\} \\ &= \mathbb{T} \left[\begin{matrix} w, r \\ v \end{matrix}; q, -\frac{D_a}{e^{2mki}} \right] \left\{ \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, -aqe^{2mki}, bq^{1/2}e^{-2ikx}; q)_\infty} dx \right\} \quad \text{by (2.1)} \\ &= \int_{-\infty}^{+\infty} \frac{e^{-x^2+2mx}}{(bq^{1/2}e^{-2ikx}; q)_\infty} \left\{ \mathbb{T} \left[\begin{matrix} w, r \\ v \end{matrix}; q, -\frac{D_a}{e^{2mki}} \right] \left\{ \frac{1}{(aq^{1/2}e^{2ikx}, -aqe^{2mki}; q)_\infty} \right\} \right\} dx, \end{aligned}$$

which is the left-hand side of (2.9) after using (2.10). By the same method, we have (2.4). The proof is complete. \square

3. Generalizations of Andrews–Askey integrals

The following famous formula is the Andrews–Askey integral [5], which can be derived from Ramanujan's ${}_1\psi_1$ summation.

Proposition 13. (See [5, Eq. (2.1)].) For $\max\{|ac|, |ad|, |bc|, |bd|\} < 1$, we have

$$\int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty} d_q t = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty}. \quad (3.1)$$

The Andrews–Askey integral is an important formula in q -series [16]. In this section, we give the following two generalizations of Andrews–Askey integral by the method of q -difference equation.

Theorem 14. For $N \in \mathbb{N}$ and $r = q^{-N}$, supposing that $\max\{|ac|, |ad|, |bc|, |bd|, |qwr/v|, |q/v|\} < 1$, we have

$$\begin{aligned} & \int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty} {}_4\phi_2 \left[\begin{matrix} r, w, c/t, abcd \\ ac, qrw/v \end{matrix}; q, \frac{qt}{vbcd} \right] d_q t \\ &= \frac{d(1-q)(q, dq/c, c/d, abcd, qw/v, qr/v; q)_\infty}{(ac, ad, bc, bd, qwr/v, q/v; q)_\infty} {}_2\phi_1 \left[\begin{matrix} w, r \\ v \end{matrix}; q, \frac{q}{bc} \right]. \end{aligned} \quad (3.2)$$

Theorem 15. For $N \in \mathbb{N}$ and $r = q^{-N}$, supposing that $\max\{|ac|, |ad|, |bc|, |bd|, |v/w|, |v/r|\} < 1$, we have

$$\begin{aligned} & \int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty} {}_4\phi_2 \left[\begin{matrix} r, w, c/t, q/(ad) \\ q/(at), qrw/v \end{matrix}; q, q \right] d_q t \\ &= \frac{d(1-q)(q, dq/c, c/d, abcd, v/(wr), v; q)_\infty}{(ac, ad, bc, bd, v/w, v/r; q)_\infty} {}_2\phi_1 \left[\begin{matrix} w, r \\ v \end{matrix}; q, \frac{vbc}{wr} \right]. \end{aligned} \quad (3.3)$$

Remark 16. Set $r = 1$ in Theorems 14 and 15, Eqs. (3.2) and (3.3) reduce to (3.1) respectively.

Before the proof of Theorems 14 and 15, the following lemma is necessary.

Lemma 17. (See [31, Eq. (2.11)].) For $N \in \mathbb{N}$ and $r = q^{-N}$, supposing that $\max\{|as|, |at|, |qw/v|, |qr/v|\} < 1$, we have

$$\mathbb{T} \left[\begin{matrix} w, r \\ v \end{matrix}; q, \frac{qD_a}{t} \right] \left\{ \frac{(au; q)_\infty}{(as, at; q)_\infty} \right\} = \frac{(au, qwr/v, q/v; q)_\infty}{(as, at, qw/v, qr/v; q)_\infty} {}_4\phi_2 \left[\begin{matrix} r, w, u/s, at \\ au, qrw/v \end{matrix}; q, \frac{qs}{vt} \right]. \quad (3.4)$$

Supposing that $\max\{|au|, |v/(wr)|, |v|\} < 1$, we have

$$\mathbb{E} \left[\begin{matrix} w, r \\ v \end{matrix}; q, -\frac{v\theta_a}{wrt} \right] \left\{ \frac{(as, at; q)_\infty}{(au; q)_\infty} \right\} = \frac{(as, at, v/w, v/r; q)_\infty}{(au, v/(wr), v; q)_\infty} {}_4\phi_2 \left[\begin{matrix} r, w, s/u, q/(at) \\ q/(au), qrw/v \end{matrix}; q, q \right]. \quad (3.5)$$

Proof of Theorems 14 and 15. Eq. (3.2) can be rewritten as

$$\begin{aligned} & \int_c^d \frac{(qt/c, qt/d; q)_\infty}{(bt; q)_\infty} \frac{(ac, qwr/v, q/v; q)_\infty}{(at, abcd, qw/v, qr/v; q)_\infty} {}_4\phi_2 \left[\begin{matrix} r, w, c/t, abcd \\ ac, qrw/v \end{matrix}; q, \frac{qt}{vbcd} \right] d_q t \\ &= \frac{d(1-q)(q, dq/c, c/d; q)_\infty}{(bc, bd; q)_\infty} \frac{1}{(ad; q)_\infty} {}_2\phi_1 \left[\begin{matrix} w, r \\ v \end{matrix}; q, \frac{q}{bc} \right]. \end{aligned} \quad (3.6)$$

By the fact that

$$\mathbb{T} \left[\begin{matrix} w, r \\ v \end{matrix}; q, \frac{qD_a}{bcd} \right] \left\{ \frac{(ac; q)_\infty}{(at, abcd; q)_\infty} \right\} = \frac{(ac, qwr/v, q/v; q)_\infty}{(at, abcd, qw/v, qr/v; q)_\infty} {}_4\phi_2 \left[\begin{matrix} r, w, c/t, abcd \\ ac, qrw/v \end{matrix}; q, \frac{qt}{vbcd} \right] \quad (3.7)$$

and

$$\mathbb{T} \left[\begin{matrix} w, r \\ v \end{matrix}; q, \frac{qD_a}{bcd} \right] \left\{ \frac{1}{(ad; q)_\infty} \right\} = \frac{1}{(ad; q)_\infty} {}_2\phi_1 \left[\begin{matrix} w, r \\ v \end{matrix}; q, \frac{q}{bc} \right], \quad (3.8)$$

if we denote the right-hand side of (3.6) by $H(w, r, v, a, s)$ and $\frac{1}{(ad; q)_\infty} {}_2\phi_1\left[\begin{smallmatrix} w, r \\ v \end{smallmatrix}; q, \frac{q}{bc}\right]$ by $h(w, r, v, a, s)$ respectively, it's easy to verify that $h(w, r, v, a, s)$ and $H(w, r, v, a, s)$ satisfy (1.12) with $s = q/(bcd)$, so we have

$$\begin{aligned} H(w, r, v, a, s) &= \mathbb{T}\left[\begin{smallmatrix} w, r \\ v \end{smallmatrix}; q, \frac{qDa}{bcd}\right] \{H(w, r, v, a, 0)\} \\ &= \mathbb{T}\left[\begin{smallmatrix} w, r \\ v \end{smallmatrix}; q, \frac{qDa}{bcd}\right] \{H(w, 1, v, a, s)\} \\ &= \mathbb{T}\left[\begin{smallmatrix} w, r \\ v \end{smallmatrix}; q, \frac{qDa}{bcd}\right] \left\{ \frac{d(1-q)(q, dq/c, c/d; q)_\infty}{(bc, bd, ad; q)_\infty} \right\} \quad \text{by (3.1)} \\ &= \mathbb{T}\left[\begin{smallmatrix} w, r \\ v \end{smallmatrix}; q, \frac{qDa}{bcd}\right] \left\{ \int_c^d \frac{(qt/c, qt/d; q)_\infty}{(bt; q)_\infty} \frac{(ac; q)_\infty}{(at, abcd; q)_\infty} dqt \right\}, \end{aligned}$$

which is the right-hand side of (3.6) after using (3.7). Similarly, we have

$$\mathbb{E}\left[\begin{smallmatrix} w, r \\ v \end{smallmatrix}; q, -\frac{v\theta_a}{wrd}\right] \left\{ \frac{(ac, ad; q)_\infty}{(at; q)_\infty} \right\} = \frac{(ac, ad, v/w, v/r; q)_\infty}{(at, v/(wr), v; q)_\infty} {}_4\phi_2\left[\begin{smallmatrix} r, w, c/t, q/(ad) \\ q/(at), qr w/v \end{smallmatrix}; q, q \right] \quad (3.9)$$

and

$$\mathbb{E}\left[\begin{smallmatrix} w, r \\ v \end{smallmatrix}; q, -\frac{v\theta_a}{wrd}\right] \{(abcd; q)_\infty\} = (abcd; q)_\infty {}_2\phi_1\left[\begin{smallmatrix} w, r \\ v \end{smallmatrix}; q, \frac{vbc}{wr}\right]. \quad (3.10)$$

By the same method, we have (3.3). The proof is complete. \square

4. Generating functions for generalized Al-Salam–Carlitz polynomials

The Rogers–Szegő polynomials [1]

$$h_n(x|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} x^k \quad \text{and} \quad g_n(x|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} x^k = h_n(x|q^{-1}) \quad (4.1)$$

are closely related to the continuous q -Hermite polynomials, whose corresponding bilinear generating functions are [13, Eqs. (3.9) and (3.13)]

$$\sum_{n=0}^{\infty} h_n(x|q) h_n(y|q) \frac{t^n}{(q; q)_n} = \frac{(xyt^2; q)_\infty}{(t, xt, yt, xyt; q)_\infty}, \quad \max\{|t|, |xt|, |yt|, |xyt|\} < 1, \quad (4.2)$$

$$\sum_{n=0}^{\infty} g_n(x|q) g_n(y|q) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} = \frac{(t, xt, yt, xyt; q)_\infty}{(xyt^2/q; q)_\infty}, \quad |xyt^2/q| < 1. \quad (4.3)$$

The Al-Salam–Carlitz polynomials [14, p. 92]

$$\phi_n^{(a)}(x|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (a; q)_k x^k \quad \text{and} \quad \psi_n^{(a)}(x|q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k+1}{2} - kn} (-ax)^k (1/a; q)_k \quad (4.4)$$

are generalized Rogers–Szegő polynomials, whose generating functions are [9, Eqs. (1.14) and (1.15)]

$$\sum_{n=0}^{\infty} \phi_n^{(a)}(x|q) \frac{t^n}{(q; q)_n} = \frac{(axz; q)_\infty}{(z, xz; q)_\infty}, \quad \max\{|z|, |xz|\} < 1, \quad (4.5)$$

$$\sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \psi_n^{(a)}(x|q) \frac{z^n}{(q; q)_n} = \frac{(z, xz; q)_\infty}{(axz; q)_\infty}, \quad |axz| < 1. \quad (4.6)$$

The Al-Salam–Carlitz polynomials are q -orthogonal polynomials whose applications and generalizations arise in many applications such as the q -harmonic oscillator, theta functions, quantum groups and coding theory. For more details the reader is referred to [1, 2, 7, 9–12].

In this section, we define the following generalized Al-Salam–Carlitz polynomials

$$\Phi_n^{(a, b; c)}(x, y|q) \triangleq \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a, b; q)_k}{(c; q)_k} x^k y^{n-k}$$

and

$$\Psi_n^{(a,b;c)}(x, y|q) \triangleq \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a, b; q)_k}{(c; q)_k} (-1)^k q^{\binom{k+1}{2} - nk} x^k y^{n-k}, \quad (4.7)$$

and give the following two generating functions by the method of q -difference equation.

Theorem 18. For $M, N \in \mathbb{N}$ and $a = q^{-M}$, we have

$$\sum_{n=0}^{\infty} \Phi_n^{(a,b;c)}(x, y|q) h_n(d|q) \frac{t^n}{(q; q)_n} = \frac{1}{(yt, ydt; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(a, b; q)_n (xdt)^n}{(q, c; q)_n} {}_2\phi_0 \left[\begin{matrix} q^{-n}, ydt \\ - \end{matrix}; q, \frac{q^n}{d} \right], \quad (4.8)$$

where $\max\{|yt|, |ydt|\} < 1$. For $a = q^{-N}$, we have

$$\sum_{n=0}^{\infty} \Psi_n^{(a,b;c)}(x, y|q) g_n(d|q) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} = (yt, ydt; q)_{\infty} \sum_{n=0}^{\infty} \frac{(a, b; q)_n (dxt)^n}{(q, c; q)_n} {}_2\phi_1 \left[\begin{matrix} q^{-n}, q/(ydt) \\ 0 \end{matrix}; q, yt \right]. \quad (4.9)$$

Remark 19. Set $(a, b, c, y) = (0, 0, 0, 1)$ in Theorem 18, Eqs. (4.8) and (4.9) reduce to (4.2) and (4.3) respectively. Let $d = 0$ in Theorem 18, Eqs. (4.8) and (4.9) reduce to

$$\sum_{n=0}^{\infty} \Phi_n^{(a,b;c)}(x, y|q) \frac{t^n}{(q; q)_n} = \frac{1}{(yt; q)_{\infty}} {}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix}; q, xt \right], \quad (4.10)$$

$$\sum_{n=0}^{\infty} \Psi_n^{(a,b;c)}(x, y|q) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q; q)_n} = (yt; q)_{\infty} {}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix}; q, xt \right] \quad (4.11)$$

respectively.

To prove the above main results, the following lemmas are necessary.

Lemma 20 (Leibniz formula). For $n \in \mathbb{N}$, we have

$$\theta_a^n \{f(a)g(a)\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \theta_a^k \{f(a)\} \theta_a^{n-k} \{g(aq^{-k})\}, \quad (4.12)$$

$$D_a^n \{f(a)g(a)\} = \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_a^k \{f(a)\} D_a^{n-k} \{g(aq^k)\}. \quad (4.13)$$

Lemma 21. For $M \in \mathbb{N}$ and $a = q^{-M}$, supposing that $\max\{|yt|, |ydt|\} < 1$, we have

$$\mathbb{T} \left[\begin{matrix} a, b \\ c \end{matrix}; q, xD_y \right] \left\{ \frac{1}{(yt, dyt; q)_{\infty}} \right\} = \frac{1}{(yt, ydt; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(a, b; q)_n (xdt)^n}{(q, c; q)_n} {}_2\phi_0 \left[\begin{matrix} q^{-n}, ydt \\ - \end{matrix}; q, \frac{q^n}{d} \right], \quad (4.14)$$

$$\mathbb{E} \left[\begin{matrix} a, b \\ c \end{matrix}; q, -x\theta_y \right] \{(yt, dyt; q)_{\infty}\} = (yt, ydt; q)_{\infty} \sum_{n=0}^{\infty} \frac{(a, b; q)_n (dxt)^n}{(q, c; q)_n} {}_2\phi_1 \left[\begin{matrix} q^{-n}, q/(ydt) \\ 0 \end{matrix}; q, yt \right]. \quad (4.15)$$

Proof. By the definitions of q -difference operators (1.2) and q -exponential operators (1.5), the left-hand side of (4.14) becomes

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{(a, b; q)_n x^n}{(q, c; q)_n} D_y^n \left\{ \frac{1}{(yt, dyt; q)_{\infty}} \right\} \\ &= \sum_{n=0}^{\infty} \frac{(a, b; q)_n x^n}{(q, c; q)_n} \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_y^k \left\{ \frac{1}{(yt; q)_{\infty}} \right\} D_y^{n-k} \left\{ \frac{1}{(ydtq^k; q)_{\infty}} \right\} \\ &= \sum_{n=0}^{\infty} \frac{(a, b; q)_n x^n}{(q, c; q)_n} \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} \frac{t^k}{(yt; q)_{\infty}} \frac{(dtq^k)^{n-k}}{(ydtq^k; q)_{\infty}}, \end{aligned}$$

which is the right-hand side of (4.14) after simplification. Similarly, the left-hand side of (4.15) is equal to

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{(a, b; q)_n (-x)^n}{(q, c; q)_n} \theta_y^n \{ (yt, ydt; q)_{\infty} \} \\ &= \sum_{n=0}^{\infty} \frac{(a, b; q)_n (-x)^n}{(q, c; q)_n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \theta_y^k \{ (yt; q)_{\infty} \} \theta_y^{n-k} \{ (ydtq^{-k}; q)_{\infty} \} \\ &= \sum_{n=0}^{\infty} \frac{(a, b; q)_n (-x)^n}{(q, c; q)_n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-t)^k (yt; q)_{\infty} (-dtq^{-k})^{n-k} (ydtq^{-k}; q)_{\infty}, \end{aligned}$$

which equals the right-hand side of (4.15). The proof is complete. \square

Proof of Theorem 18. The left-hand side of (4.8) and (4.9) are equal to

$$\mathbb{T} \left[\begin{matrix} a, b \\ c \end{matrix}; q, xD_y \right] \left\{ \sum_{n=0}^{\infty} h_n(d|q) \frac{(yt)^n}{(q; q)_n} \right\} = \mathbb{T} \left[\begin{matrix} a, b \\ c \end{matrix}; q, xD_y \right] \left\{ \frac{1}{(yt, dyt; q)_{\infty}} \right\} \quad (4.16)$$

and

$$\mathbb{E} \left[\begin{matrix} a, b \\ c \end{matrix}; q, -x\theta_y \right] \left\{ \sum_{n=0}^{\infty} g_n(d|q) \frac{(-1)^n q^{\binom{n}{2}} (yt)^n}{(q; q)_n} \right\} = \mathbb{E} \left[\begin{matrix} a, b \\ c \end{matrix}; q, -x\theta_y \right] \{ (yt, dyt; q)_{\infty} \} \quad (4.17)$$

respectively. By Theorem 3 and Lemma 21, we have the proof of Theorem 18. The proof is complete. \square

Remark 22. Based on the way of q -difference equation, we can deduce Carlitz type generating functions for generalized Al-Salam–Carlitz polynomials in [14].

5. A generalized transformation identity from q -Chu–Vandermonde sums

The following q -Chu–Vandermonde formula reads as [20, Eq. (II.6)]

$${}_2\phi_1 \left[\begin{matrix} q^{-n}, d \\ e \end{matrix}; q, q \right] = \frac{(e/d; q)_n}{(e; q)_n} d^n. \quad (5.1)$$

In this section, we obtain the following generalized transformation identity from q -Chu–Vandermonde sums by the way of q -difference equation.

Theorem 23. For $n, M \in \mathbb{N}$, if $a = q^{-M}$ (or $b = q^{-M}$), we have

$$\begin{aligned} &\sum_{k=0}^n \frac{(q^{-n}, d; q)_k q^k}{(q, ey; q)_k} \sum_{j=0}^{\infty} \frac{(a, b; q)_j (exq^k)^j}{(q, c; q)_j} {}_2\phi_1 \left[\begin{matrix} q^{-j}, q^{1-k}/(ye) \\ 0 \end{matrix}; q, \frac{eyq^n}{d} \right] \\ &= \frac{(ey/d; q)_n d^n}{(ey; q)_n} \sum_{j=0}^{\infty} \frac{(a, b; q)_j (exq^n)^j}{(q, c; q)_j} {}_2\phi_1 \left[\begin{matrix} q^{-j}, q^{1-n}/(ye) \\ 0 \end{matrix}; q, \frac{ey}{d} \right]. \end{aligned} \quad (5.2)$$

Remark 24. Set $x = 0$ in Theorem 23, Eq. (5.2) reduces to (5.1) immediately.

Proof of Theorem 23. On the one hand, we can rewrite Eq. (5.1) as

$$\sum_{k=0}^n \frac{(q^{-n}, d; q)_k q^k}{(q; q)_k} (eyq^k, eyq^n/d; q)_{\infty} = (eyq^n, ey/d; q)_{\infty} d^n. \quad (5.3)$$

On the other hand, Eq. (5.2) can be written as

$$\begin{aligned} &\sum_{k=0}^n \frac{(q^{-n}, d; q)_k q^k}{(q; q)_k} \cdot (eyq^n/d, eyq^k; q)_{\infty} \sum_{j=0}^{\infty} \frac{(a, b; q)_j (exq^k)^j}{(q, c; q)_j} {}_2\phi_1 \left[\begin{matrix} q^{-j}, q^{1-k}/(ye) \\ 0 \end{matrix}; q, \frac{eyq^n}{d} \right] \\ &= d^n \cdot (eyq^n, ey/d; q)_{\infty} \sum_{j=0}^{\infty} \frac{(a, b; q)_j (exq^n)^j}{(q, c; q)_j} {}_2\phi_1 \left[\begin{matrix} q^{-j}, q^{1-n}/(ye) \\ 0 \end{matrix}; q, \frac{ey}{d} \right]. \end{aligned} \quad (5.4)$$

Using (4.15), we have

$$\begin{aligned} \mathbb{E} \left[\begin{matrix} a, b \\ c \end{matrix}; q, -x\theta_y \right] \{ (eyq^k, eyq^n/d; q)_\infty \} \\ = (eyq^n/d, eyq^k; q)_\infty \sum_{j=0}^{\infty} \frac{(a, b; q)_j (exq^k)^j}{(q, c; q)_j} {}_2\phi_1 \left[\begin{matrix} q^{-j}, q^{1-k}/(ye) \\ 0 \end{matrix}; q, \frac{eyq^n}{d} \right], \end{aligned} \quad (5.5)$$

$$\mathbb{E} \left[\begin{matrix} a, b \\ c \end{matrix}; q, -x\theta_y \right] \{ (eyq^n, ey/d; q)_\infty \} = (eyq^n, ey/d; q)_\infty \sum_{j=0}^{\infty} \frac{(a, b; q)_j (exq^n)^j}{(q, c; q)_j} {}_2\phi_1 \left[\begin{matrix} q^{-j}, q^{1-n}/(ye) \\ 0 \end{matrix}; q, \frac{ey}{d} \right]. \quad (5.6)$$

If we denote the right-hand side of (5.4) by $J(a, b, c, y, x)$, we can verify that $J(a, b, c, y, x)$ satisfies q -difference equation (1.14) by (5.6), so we have

$$\begin{aligned} J(a, b, c, y, x) &= \mathbb{E} \left[\begin{matrix} a, b \\ c \end{matrix}; q, -x\theta_y \right] \{ J(a, b, c, y, 0) \} \\ &= \mathbb{E} \left[\begin{matrix} a, b \\ c \end{matrix}; q, -x\theta_y \right] \{ d^n (eyq^n, ey/d; q)_\infty \} \\ &= \mathbb{E} \left[\begin{matrix} a, b \\ c \end{matrix}; q, -x\theta_y \right] \left\{ \sum_{k=0}^n \frac{(q^{-n}, d; q)_k q^k}{(q; q)_k} (eyq^k, eyq^n/d; q)_\infty \right\} \quad \text{by (5.3),} \end{aligned}$$

which is the left-hand side of (5.4) after using (5.5). The proof is complete. \square

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