J. Math. Anal. Appl. 420 (2014) 1744–1761

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Scaling laws and the rate of convergence in thin magnetic films

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A R T I C L E I N F O

Article history:
Received 4 February 2014
Available online 1 July 2014
Submitted by A. Daniilidis

Keywords:
Thin magnetic films
Magnetic wires
Magnetization reversal
Domain wall

A B S T R A C T

We study static 180 degree domain walls in thin infinite magnetic films. We establish the scaling of the minimal energy by Γ-convergence and the energy minimizer profile, which turns out to be the so-called *transverse wall* as predicted in earlier numerical and experimental work. Surprisingly, the minimal energy decays faster than the area of the film cross section at an infinitesimal cross section diameter. We establish a rate of convergence of the rescaled energies as well.

Contents

1. Introduction

1.1. Micromagnetics

In the theory of micromagnetics the energy of micromagnetics of a ferromagnetic body $\Omega \in \mathbb{R}^3$ is given by

$$E(m) = A_{ex}\int_{\Omega} |\nabla m|^2 + K_d\int_{\mathbb{R}^3} |\nabla u|^2 + Q\int_{\Omega} \varphi(m) - 2\int_{\Omega} H_{ext}\cdot m,$$

http://dx.doi.org/10.1016/j.jmaa.2014.06.056
0022-247X/

where $m: \Omega \to \mathbb{S}^2$ with $m = 0$ in $\mathbb{R}^3 \setminus \Omega$ is a unit vector field representing the magnetization vector, A_{ex}, K_d, Q are material parameters, H_{ext} is the externally applied magnetic field, φ is the anisotropy energy density and u is the induced field potential, obtained from Maxwell's equations of magnetostatics,

$$\begin{cases} \operatorname{curl} H_{ind} = 0 & \text{in } \mathbb{R}^3, \\ \operatorname{div}(H_{ind} + m) = 0 & \text{in } \mathbb{R}^3, \end{cases}$$

where $H_{ind} = \nabla u$. Namely, u is a weak solution of

$$\Delta u = \operatorname{div} m \quad \text{in } \mathbb{R}^3,$$

i.e., ∇u is the Helmoltz projection of m onto the L^2 closure of the gradient fields in $L^2(\mathbb{R}^3)$. The energy density φ is a non-negative function called the anisotropy energy density. It is typically a polynomial, with the symmetry properties inherited from those of the underlying crystalline lattice. The zeroes of φ form the set of preferred directions of magnetization (easy axes), e.g., [6]. According to the theory of micromagnetics, stable magnetization patterns are described by the minimizers (global and local) of the micromagnetic energy functional, e.g., [14,6–8]. This is a non-convex and nonlocal minimization problem due to the non-convex constraint $|m| = 1$ in Ω. This theory is used for the analysis and design of magnetic devices. It explains observations on many length scales, and it also explains the magnetic hysteresis, through the multiplicity of local minima, e.g., [6].

1.2. Motivation

In recent years the study of thin structures in micromagnetics, in particular thin films and wires, has been of great interest, see [1,2,5,9,16–21] for nanowires and [4,7,8,10,15,17]. It was suggested in [1] that magnetic nanowires can be used as storage devices. It is known that the magnetization pattern reversal time is closely related to the writing and reading speed of such a device, thus it has been suggested to study the magnetization reversal and switching processes. In [9] the magnetizetion reversal process has been studied numerically in cobalt nanowires by the Landau–Lishitz–Gilbert equation. In thin wires the transverse mode has been observed: the magnetization in almost constant on each cross section forming a domain wall that propagates along the wire, while in relatively thick wires the vortex wall has been observed: the magnetization is approximately tangential to the boundary and forms a vortex which propagates along the wire. In [13] similar study has been done for thin nickel wires and the same results have been observed. When a homogenous external field is applied in the axial direction of the wire facing the homogenous magnetization direction, then at a critical strength the reversal of the magnetization typically starts at one end of the wire creating a domain wall, which moves along the wire. The domain wall separates the reversed and the not yet reversed parts of the wire. In [3] Cantero-Alvarez and Otto considered the problem of finding the scaling of critical field in terms of the thin film cross section and material parameters. The authors found four different scalingS and corresponding four different regimes. In Fig. 1 one can see the transverse and the vortex wall longitudinal and cross section pictures for wires with a rectangular cross section.

A distinctive crossover has been observed between the two different modes, which is expected to occur at a critical diameter of the wire. It has been suggested that the magnetization switching process can be understood by analyzing the micromagnetics energy minimization problem for different diameters of the cross section. In [16] K. Kühn studied 180 degree static domain walls in magnetic wires with circular cross sections. Kühn proved that indeed, the transverse mode must occur in thin magnetic wires as was predicted by experimental and numerical analysis before in [9] and in [13], while in thick wires a vortex wall has the optimal energy scaling. Some of the results proven by K. Kühn for thin wires has been later generalized in [12] to any wires with a bounded, Lipschitz and rotationally symmetric cross sections, see also [11]. Slastikov

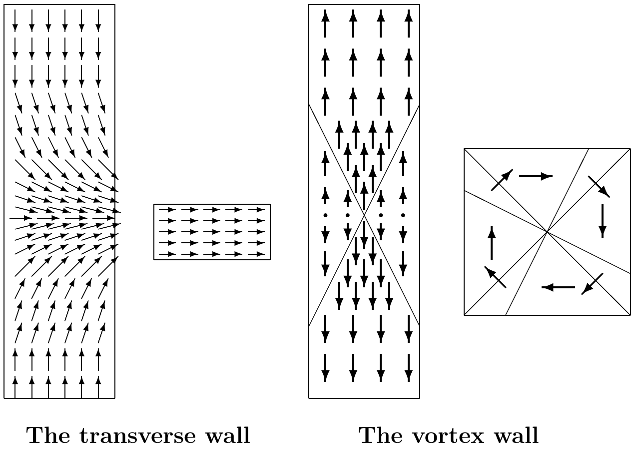

Fig. 1.

and Sonnenberg proved the energy Γ-convergence result in [20] for any C^1 cross sections in finite curved wires. It is shown in [16,12,20] that the minimal energy scales like d^2, where d is the diameter of the wire, provided the wire cross section has comparable dimensions. It turns out that if the dimensions of the cross section are not comparable, then the minimal energies decay faster than d^2 and a logarithmic term occurs. In this paper we study the minimal energy scaling in infinite thin films, as both sides of the cross section go to zero, but one faster that the other. The minimal appears to scale like $d^2(\ln l - \ln d)$, where $0 < d < l$ are the dimensions of the cross section. The paper is organized as follows: In Section 2 we make some notations and formulate the main results. In Section 3 we prove that for small cross section diameters the magnetostatic energy can ba approximated by a quadratic form in the second and the third components of the magnetization m. In Section 4 we prove when the diameter goes to zero the energy minimization problems Γ-converge to a one-dimensional problem. In Section 5 we prove a rate of convergence on the minimal energies as the diameter of the film goes to zero. Finally, in Appendix A we prove two auxiliary lemmas.

2. The main results

Denote $\Omega(l,d) = \mathbb{R} \times R(l,d)$, where $R(l,d) = [-l,l] \times [-d,d]$ and throughout this work it will be assumed that $0 < d \leq l$. Denote the aspect ratio $c = \frac{d}{l}$. Consider the energy of micromagnetics without an external field and anisotropy energy, i.e., the energy of an isotropic ferromagnet with the absence of an external field:

$$E(m) = A_{ex} \int\limits_{\Omega} |\nabla m|^2 + K_d \int\limits_{\mathbb{R}} |\nabla u|^2.$$

By scaling of all coordinates one can reach the situation when $A_{ex} = K_d$, so we will henceforth assume that $A_{ex} = K_d = 1$. Denote

$$A(\Omega) = \big\{ m \colon \Omega \to \mathbb{S}^2 \ : \ m \in H^1_{loc}(\Omega),\ E(m) < \infty \big\},$$

and also let us introduce 180 degree domain walls

$$\tilde{A}(\Omega) = \big\{ m \colon \Omega \to \mathbb{S}^2 \ : \ m - \bar{e} \in H^1(\Omega) \big\},$$

where

$$\bar{e}(x,y,z) = \begin{cases} (-1,0,0) & \text{if } x < -1 \\ (x,0,0) & \text{if } -1 \le x \le 1 \\ (1,0,0) & \text{if } 1 < x \end{cases}$$

Roughly speaking, we are considering the set of all magnetizations that satisfy $\lim_{x\to\pm\infty} m(x,y,z) = \pm\vec{e}_x$ for all y and z. The target of this paper is the study of the minimal energy scaling and the minimizers in minimization problem

$$\inf_{m\in\tilde{A}(\Omega(l,d))} E(m) \tag{2.1}$$

when $l \to 0$ and $c \to 0$. It turns out that after rescaling the energy by a suitable factor, the new rescaled energies Γ-converge to a one dimensional energy, e.g., [16,12,20]. Consider sequences of domain–magnetization–energy triples $(\Omega(l_n,d_n), m^n, E(m^n))$ such that $d_n, l_n \to 0$ and $c_n = \frac{d_n}{l_n} \to 0$ as n goes to infinity. Denote for simplicity $\Omega_n = \Omega(l_n,d_n)$, $A_n = A(\Omega(l_n,d_n))$, $\tilde{A}_n = \tilde{A}(\Omega(l_n,d_n))$. Set $\lambda_n = \frac{1}{c_n|\ln c_n|}$, $\mu_n = \frac{l_n d_n}{\lambda_n}$ and rescale the magnetization m as follows: $\acute{m}(x,y,z) = m(\lambda_n x, l_n y, d_n z)$. Note that, in contrast to [16,20,11], we rescale m in the x direction as well. Denote now $\acute{E}(\acute{m}^n) = \frac{E(m^n)}{\mu_n}$ and consider the rescaled minimization problems

$$\inf_{m\in\tilde{A}_n} \acute{E}(\acute{m}) \tag{2.2}$$

instead of the original problem

$$\inf_{m\in\tilde{A}_n} E(m).$$

The rescaled energy functional will have the form:

$$\acute{E}\big(\acute{m}^n\big) = \frac{1}{\mu_n}\int\limits_{\Omega(1,1)} \left(\big|\partial_x \acute{m}^n(\xi)\big|^2 + \frac{\lambda_n^2}{l_n^2}\big|\partial_y \acute{m}^n(\xi)\big|^2 + \frac{\lambda_n^2}{d_n^2}\big|\partial_z \acute{m}^n(\xi)\big|^2\right) \mathrm{d}\xi + \frac{E_{mag}(m^n)}{\mu_n}.$$

The limit (reduced) energy functional E_0 turns out to be

$$E_0(m) = \begin{cases} 4\int_{\mathbb{R}} |\partial_x m|^2\,\mathrm{d}x + \frac{4}{\pi}\int_{\mathbb{R}} |m_2|^2\,\mathrm{d}x, & \text{if } m_3 \equiv 0 \\ +\infty, & \text{otherwise} \end{cases}$$

and the admissible set A_0 for the reduced variational problem is

$$A_0 = \big\{m: \mathbb{R} \to \mathbb{S}^2 \;\big|\; m(\pm\infty) = \pm 1\big\}.$$

The reduced or limit variational problem is to minimize the reduced energy functional E_0 over the admissible set A_0, i.e.,

$$\inf_{m\in A_0} E_0(m). \tag{2.3}$$

Define furthermore

$$A_0^3 = \{m \in A_0 \mid m_3 \equiv 0\}.$$

The equality $\min_{m\in A_0} E_0(m) = \min_{m\in A_0^3} E_0(m)$ suggests considering the minimization problem $\min_{m\in A_0^3} E_0(m)$ instead of $\min_{m\in A_0} E_0(m)$. Next, define the notion of convergence of the magnetizations like in [16,12].

Definition 2.1. The sequence $\{m^n\} \subset A(\Omega)$ is said to converge to $m^0 \in A(\Omega)$ as n goes to infinity if:

(i) $\nabla m^n \rightharpoonup \nabla m^0$ weakly in $L^2(\Omega)$,
(ii) $m^n \to m^0$ strongly in $L^2_{loc}(\Omega)$.

The following result is then the main contribution of the work:

Theorem 2.2 *(Γ-convergence). The reduced variational problem is the Γ-limit of the full variational problem with respect to the convergence stated in Definition 2.1. This amounts to the following three statements:*

- ***Lower semicontinuity.*** *If a sequence of rescaled magnetizations $\{\acute{m}^n\}$ with $m^n \in A_n$ converges to some $m^0 \in A(\Omega)$ in the sense of Definition 2.1 then*

$$E_0\big(m^0\big) \le \liminf_{n\to\infty} \acute{E}\big(\acute{m}^n\big)$$

- ***Recovery sequence.*** *For every $m^0 \in A_0$ and every sequence of pairs $\{(l_n, d_n)\}$ with $l_n, d_n \to 0$, $c_n \to 0$, there exists a sequence $\{m^n\}$ with $m^n \in \tilde{A}_n$ such that*

$$\acute{m}^n \to m^0 \quad \text{in the sense of Definition 2.1}$$
$$E_0\big(m^0\big) = \lim_{n\to\infty} \acute{E}\big(\acute{m}^n\big)$$

- ***Compactness.*** *Let $\{(l_n, d_n)\}$ be such that $l_n, d_n \to 0$ and $c_n \to 0$. Assume $m^n \in \tilde{A}_n$ and $\acute{E}(\acute{m}^n) \le C$ for all $n \in \mathbb{N}$. Then there exists a subsequence of $\{m^n\}$ (not relabeled) such that after a translation in the x direction the sequence $\acute{m}^n$ converges to some $m^0 \in A_0^3$ in the sense of Definition 2.1.*

From the above Γ-convergence result, we get the convergence of the minimal energies as a property of Γ-convergence:

Corollary 2.3. *Due to the above theorem, we have*

$$\lim_{n\to\infty} \min_{m^n \in \tilde{A}_n} \acute{E}\big(\acute{m}^n\big) = \min_{m\in A_0} E(m). \tag{2.4}$$

As will be seen later $\min_{m\in A_0} E(m) = \frac{16}{\sqrt{\pi}}$. The next theorem establishes a rate of convergence for (2.4), which is again contribution of our work:

Theorem 2.4 *(Rate of convergence). For sufficiently small d and c the following bound holds:*

$$\Big|\min_{m\in\tilde{A}} \acute{E}(\acute{m}) - \min_{m\in A_0} E_0(m)\Big| \le \frac{200}{\sqrt{|\ln c|}} + 20Cl,$$

where C is the Poincaré constant for rectangles.

3. An approximation of the magnetostatic energy

Recall that the map u is a weak solution of $\Delta u = \operatorname{div} m$ if and only if

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla\varphi = \int_{\Omega} m \cdot \nabla\varphi, \quad \text{for all } \varphi \in C_0^\infty\big(\mathbb{R}^3\big). \tag{3.1}$$

The left hand side of the above equality can be written as a sum volume and surface contributions as:

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi = -\int_{\Omega} \operatorname{div} m \cdot \varphi + \int_{\partial\Omega} m \cdot n\varphi, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^3), \tag{3.2}$$

where n is the outward unit normal to $\partial\Omega$. Denoting

$$u_v(\xi) = -\int_{\Omega} \Gamma(\xi - \xi_1)(\operatorname{div} m)(\xi)\,\mathrm{d}\xi \quad \text{and} \quad u_s(\xi) = \int_{\partial\Omega} \Gamma(\xi - \xi_1)(m \cdot n)(\xi)\,\mathrm{d}\xi,$$

where $\Gamma(\xi) = \frac{1}{4\pi|\xi|}$ is the Green function in $\mathbb{R}^3$, we obtain

$$\int_{\mathbb{R}^3} \nabla u_v \cdot \nabla \varphi = \int_{\Omega} \operatorname{div} m \cdot \varphi, \qquad \int_{\mathbb{R}^3} \nabla u_s \cdot \nabla \varphi = \int_{\partial\Omega} m \cdot n \cdot \varphi, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^3). \tag{3.3}$$

Denote furthermore

$$E_v = \int_{\mathbb{R}^3} |\nabla u_v|^2, \qquad E_s = \int_{\mathbb{R}^3} |\nabla u_s|^2, \qquad E_{vs} = \int_{\mathbb{R}^3} \nabla u_v \cdot \nabla u_s.$$

Following Kohn and Slastikov as in [15], define the average of the magnetization vector over the cross section:

$$\bar{m}(x, y, z) = \frac{1}{4ld} \int_{R(l,d)} m\,\mathrm{d}y\,\mathrm{d}z, \quad (x, y, z) \in \Omega.$$

Like m, we extend $\bar{m}$ as 0 outside Ω. In this section we prove upper and lower bounds on the magnetostatic energy for thin films. We start with the E_s part of the energy. If the parametrization

$$\begin{cases} y = y(t), & t \in [0, 2] \\ z = z(t), & t \in [0, 2] \end{cases}$$

of $\partial R(l, d)$ is chosen by symmetry, so that $y(t + 1) = -y(t)$, $z(t + 1) = -z(t)$, then Theorem 3.3.5 of [12] delivers a formula for $E_s(m)$ in Fourier space for $m = m(x)$, namely:

Theorem 3.1. *For every $m = m(x) \in A(\Omega)$ there holds:*

$$\begin{aligned} E_s(m) = \frac{1}{4\pi^2} \int_{\mathbb{R}^3} \frac{1}{|k|^2} \Big\{ |a|^2 \big|\hat{m}_2(k_1)\big|^2 + |b|^2 \big|\hat{m}_3(k_1)\big|^2 \\ + \bar{a}b\big(\hat{m}_2(k_1)\overline{\hat{m}_3(k_1)} + \overline{\hat{m}_2(k_1)}\hat{m}_3(k_1)\big) \Big\}\,\mathrm{d}k, \end{aligned}$$

where

$$a(k_2, k_3, \omega) = -2i \int_0^1 z'(t) \sin\big(k_2 y(t) + k_3 z(t)\big)\,\mathrm{d}t,$$

$$b(k_2, k_3, \omega) = 2i \int_0^1 y'(t) \sin\big(k_2 y(t) + k_3 z(t)\big)\,\mathrm{d}t.$$

Observe, that when the cross section is the rectangle $R(l, d)$ then the formula for E_s can be easily simplified in further steps, namely, for any $m = m(x) \in A(\Omega)$, we have the following representation formula:

$$E_s(m)=\frac{4}{\pi^2}\int\limits_{\mathbb{R}^3}\frac{\sin^2(ly)\sin^2(dz)}{|\xi|^2}\left(\frac{|\widehat{m}_2(x)|^2}{z^2}+\frac{|\widehat{m}_3(x)|^2}{y^2}\right)\mathrm{d}\xi.$$

Set now for convenience

$$I(l,d,x)=\int\limits_{\mathbb{R}^2}\frac{\sin^2(ly)\sin^2(dz)}{y^2|\xi|^2}\,\mathrm{d}y\,\mathrm{d}z,$$

then

$$E_s(m)=\frac{4}{\pi^2}\int\limits_{\mathbb{R}}\left(I(l,d,x)\left|\widehat{m}_3(x)\right|^2+I(d,l,x)\left|\widehat{m}_2(x)\right|^2\right)\mathrm{d}x.$$

The following functions will play an important role in this work. Denote for any $c>0$,

$$a_c=\frac{c}{2}\int\limits_0^\infty\frac{\sin^2 t}{t^2}\cdot\frac{1-e^{-\frac{2t}{c}}}{t}\,\mathrm{d}t,\qquad b_c=a_{\frac{1}{c}}. \tag{3.4}$$

Lemma 3.2. *For any $0<d\le l$, we have*

(i) $I(d,l,x)\le 2\pi lda_c$ *and* $I(l,d,x)\le 2\pi ldb_c$, *for all* $x\in\mathbb{R}$,
(ii) $I(d,l,x)\le \pi ldc(3-\ln c)$, *for all* $x\in\mathbb{R}$,
(iii) $I(d,l,x)\ge \pi ldc|\ln c|(1-\frac{5}{\sqrt{|\ln c|}})$, *for all* $x\in[-\frac{1}{l},\frac{1}{l}]$.

Proof. We will use the following two identities, that are well known and can be found in most advanced calculus and complex analysis textbooks:

$$\int\limits_0^\infty\frac{\sin^2 t}{t^2}\,\mathrm{d}t=\frac{\pi}{2},\qquad \int\limits_0^\infty\frac{\sin^2(pt)}{t^2+q^2}\,\mathrm{d}t=\frac{\pi}{4q}\left(1-e^{-2pq}\right),\quad p,q>0. \tag{3.5}$$

For any $x\neq 0$, we have by making a change of variables $y\to|x|y$, $z\to|x|z$ and putting $a=l|x|, b=d|x|$,

$$\begin{aligned}I(l,d,x)&=4\int\limits_0^\infty\int\limits_0^\infty\frac{\sin^2(ly)\sin^2(dz)}{y^2|\xi|^2}\,\mathrm{d}y\,\mathrm{d}z\\&=\frac{4}{x^2}\int\limits_0^\infty\int\limits_0^\infty\frac{\sin^2(ay)\sin^2(bz)}{y^2(1+y^2+z^2)}\,\mathrm{d}y\,\mathrm{d}z.\end{aligned}$$

Utilizing now the second identity of (3.5) and making a change of variables $y=\frac{t}{a}$, we obtain

$$\begin{aligned}I(l,d,x)&=\frac{\pi}{x^2}\int\limits_0^\infty\frac{\sin^2(ay)}{y^2}\cdot\frac{1-e^{-2b\sqrt{y^2+1}}}{\sqrt{y^2+1}}\,\mathrm{d}y\\&=\frac{2\pi ab}{x^2}\int\limits_0^\infty\frac{\sin^2 t}{t^2}\cdot\frac{1-e^{-\frac{2b}{a}\sqrt{t^2+a^2}}}{\frac{2b}{a}\sqrt{t^2+a^2}}\,\mathrm{d}t\\&=2\pi ld\int\limits_0^\infty\frac{\sin^2 t}{t^2}\cdot\frac{1-e^{-\frac{2d}{l}\sqrt{t^2+l^2x^2}}}{\frac{2d}{l}\sqrt{t^2+l^2x^2}}\,\mathrm{d}t.\end{aligned}$$

By the inequality

$$\frac{2l}{d}\sqrt{t^2+d^2x^2} \geq \frac{2l}{d}t = \frac{2t}{c}$$

and the fact that the function $\frac{1-e^{-t}}{t}$ decreases over $(0,+\infty)$, we get

$$I(d,l,x) \leq 2\pi ld \int\limits_0^{+\infty} \frac{\sin^2 t}{t^2} \cdot \frac{1-e^{-\frac{2t}{c}}}{\frac{2t}{c}}\,\mathrm{d}t = 2\pi lda_c. \tag{3.6}$$

Similarly, we have $I(l,d,x) \leq 2\pi ldb_c$.

For (ii) we have that $I(d,l,x) \leq I_1 + I_2 + I_3$, where

$$I_1 = \pi ldc \int\limits_0^{c} \frac{\sin^2 t}{t^2} \cdot \frac{1-e^{-\frac{2t}{c}}}{t}\,\mathrm{d}t,$$

$$I_2 = \pi ldc \int\limits_c^{1} \frac{\sin^2 t}{t^2} \cdot \frac{1-e^{-\frac{2t}{c}}}{t}\,\mathrm{d}t,$$

$$I_3 = \pi ldc \int\limits_1^{+\infty} \frac{\sin^2 t}{t^2} \cdot \frac{1-e^{-\frac{2t}{c}}}{t}\,\mathrm{d}t.$$

It is clear that

$$I_1 = 2\pi ld \int\limits_0^{c} \frac{\sin^2 t}{t^2} \cdot \frac{1-e^{-\frac{2t}{c}}}{\frac{2t}{c}}\,\mathrm{d}t \leq 2\pi ld \int\limits_0^{c} \mathrm{d}t = 2\pi ldc,$$

$$I_2 \leq \pi ldc \int\limits_c^{1} \frac{1}{t}\,\mathrm{d}t = -\pi ldc \ln c,$$

$$I_3 \leq \pi ldc \int\limits_1^{+\infty} \frac{\sin^2 t}{t^2}\,\mathrm{d}t \leq \pi ldc \int\limits_1^{+\infty} \frac{1}{t^2}\,\mathrm{d}t = \pi ldc.$$

Therefore, we obtain $I(d,l,x) \leq \pi ldc(3-\ln c)$ and (ii) is proved.

To get a lower bound on $I(d,l,x)$, note that the main contribution to the integral comes from the interval $[c,1]$. The idea is replacing in the previous argument $[c,1]$ by $[c^{1-\epsilon}, c^{\epsilon}]$ where ϵ is a small positive number yet to be chosen. Assume $\epsilon < \frac{1}{3}$ and $x \in [-\frac{1}{l}, \frac{1}{l}]$. For any $t \in [c^{1-\epsilon}, c^{\epsilon}]$, we have

$$\frac{2l}{d}\sqrt{t^2+x^2d^2} \geq \frac{2t}{c} \geq 2c^{-\epsilon},$$

and

$$\sqrt{t^2+x^2d^2} \leq t + |x|d \leq t + \frac{d}{l} = t + c,$$

hence

$$I(d,l,x) \geq \pi ldc \int\limits_{c^{1-\epsilon}}^{c^{\epsilon}} \frac{\sin^2 t}{t^2} \cdot \frac{1-e^{-2c^{-\epsilon}}}{t+c}\,\mathrm{d}t. \tag{3.7}$$

If we choose now $\epsilon = \frac{1}{\sqrt{|\ln c|}}$ then $c^\epsilon \to 0$, thus we get

$$1 - e^{-2c^{-\epsilon}} > 1 - \frac{1}{2c^{-\epsilon}} = 1 - \frac{c^\epsilon}{2},$$
$$\frac{\sin^2 t}{t^2} \geq \frac{(t - \frac{t^3}{6})^2}{t^2} \geq 1 - t^2, \quad t \in \big[0, c^\epsilon\big].$$

Thus we obtain by (3.7),

$$\begin{aligned} I(d,l,x) &\geq \pi l d c\big(1 - c^{2\epsilon}\big)\left(1 - \frac{c^\epsilon}{2}\right) \int\limits_{c^{1-\epsilon}}^{c^\epsilon} \frac{1}{t+c}\,\mathrm{d}t \\ &\geq \pi l d c\big(1 - 2c^\epsilon\big)\big(\ln\big(c + c^\epsilon\big) - \ln\big(c + c^{1-\epsilon}\big)\big). \end{aligned}$$

It is clear that

$$\begin{aligned} \ln\big(c + c^{1-\epsilon}\big) &= \ln c + \ln\big(1 + c^{-\epsilon}\big) \\ &\leq \ln c + \ln\big(2c^{-\epsilon}\big) \\ &\leq (1 - 2\epsilon)\ln c, \end{aligned}$$

and

$$\ln\big(c + c^\epsilon\big) \geq \ln c^\epsilon = \epsilon \ln c,$$
$$1 - 2c^\epsilon = 1 - 2e^{\epsilon \ln c} > 1 - 2e^{-\frac{1}{\epsilon}} > 1 - 2\epsilon.$$

Concluding, we obtain

$$\begin{aligned} I(d,l,x) &\geq \pi\big(1 - 2c^\epsilon\big)(1 - 3\epsilon) l d c |\ln c| \\ &\geq \pi l d c |\ln c|(1 - 5\epsilon) \\ &= \pi l d c |\ln c|\left(1 - \frac{5}{\sqrt{|\ln c|}}\right). \qquad \square \end{aligned}$$

Corollary 3.3. *We have that*

$$\lim_{c \to 0} \frac{a_c}{c|\ln c|} = \frac{1}{2}.$$

Proof. The proof follows from (ii) and (iii) parts of the above lemma. $\square$

It is straightforward to see that due to the symmetry of the cross section $R(l,d)$ one has $E_{vs}(m) = 0$ for all $m = m(x) \in A(\Omega)$. We estimate now the volume contribution E_v to E_{mag}.

Lemma 3.4. *For any $0 < d \leq l$ and $m = m(x) \in A$ the following bound holds:*

$$E_v(m) \leq M_m\left(l^2 d^2 + l d^2\left(1 + \ln\frac{l}{d}\right)\right), \tag{3.8}$$

where M_m is a constant depending on the magnetization m.

Proof. By density argument (3.3) holds for $\varphi = u_v$ thus,

$$E_v(m) = \int_{\mathbb{R}^3} |\nabla u_v|^2 = -\int_{\Omega} \operatorname{div} m \cdot u_v = \int_{\Omega}\int_{\Omega} \Gamma(\xi - \xi_1) \operatorname{div} m(\xi) \operatorname{div} m(\xi_1)\, \mathrm{d}\xi\, \mathrm{d}\xi_1.$$

For any $m = m(x) \in A$, we have $\operatorname{div} m = \partial_x m_1(x)$, thus

$$E_v(m) = \frac{1}{4\pi} \int_{\Omega}\int_{\Omega} \frac{\partial_x m_1(x)\partial_x m_1(x_1)}{|\xi - \xi_1|}\, \mathrm{d}\xi\, \mathrm{d}\xi_1$$

where $\xi = (x, y, z)$ and $\xi_1 = (x_1, y_1, z_1)$. We have by integration by parts

$$\begin{aligned}\int_{\mathbb{R}} \frac{\partial_x m_x(x)}{|\xi - \xi_1|}\, \mathrm{d}x &= \int_{-\infty}^{0} \frac{\mathrm{d}m^*(x)}{|\xi - \xi_1|} + \int_{0}^{+\infty} \frac{\mathrm{d}m^*(x)}{|\xi - \xi_1|} \\ &= \frac{2}{\sqrt{x_1^2 + (y - y_1)^2 + (z - z_1)^2}} - \int_{\mathbb{R}} \frac{(x - x_1)m^*(x)}{|\xi - \xi_1|^3}\, \mathrm{d}x,\end{aligned}$$

where

$$m^*(x) = \begin{cases} m_1(x) + 1 & \text{if } x \le 0 \\ m_1(x) - 1 & \text{if } x > 0. \end{cases}$$

Then it has been shown in [12] that $m^*(x) \in L^2(\mathbb{R})$. We can estimate $E_v(m) \le I_1 + I_2$, where

$$\begin{aligned} I_1 &= \frac{1}{2\pi} \int_{R(l,d)} \int_{\Omega} \frac{|\partial_x m_1(x_1)|}{\sqrt{x_1^2 + (y - y_1)^2 + (z - z_1)^2}}\, \mathrm{d}\xi_1\, \mathrm{d}y\, \mathrm{d}z, \\ I_2 &= \int_{\Omega}\int_{\Omega} \frac{|\partial_x m_1(x_1) m^*(x)|}{|\xi - \xi_1|^2}\, \mathrm{d}\xi\, \mathrm{d}\xi_1. \end{aligned}$$

We have furthermore

$$\begin{aligned} &\int_{\mathbb{R}} \frac{|\partial_x m_1(x_1)|}{\sqrt{x_1^2 + (y - y_1)^2 + (z - z_1)^2}}\, \mathrm{d}x_1 \\ &\le \frac{1}{2} \int_{\mathbb{R}} \left(\left|\partial_x m_1(x_1)\right|^2 + \frac{1}{x_1^2 + (y - y_1)^2 + (z - z_1)^2} \right) \mathrm{d}x_1 \\ &= \frac{1}{2} \|\partial_x m_1\|_{L^2(\mathbb{R})}^2 + \frac{\pi}{2\sqrt{(y - y_1)^2 + (z - z_1)^2}}. \end{aligned}$$

Recall now Lemma A2 from [11], which asserts that for any point $(y_1, z_1) \in \mathbb{R}^2$ one has

$$\int_{R(l,d)} \frac{1}{\sqrt{(y - y_1)^2 + (z - z_1)^2}}\, \mathrm{d}y\, \mathrm{d}z \le 10d\left(1 + \ln \frac{d}{l}\right). \tag{3.9}$$

Thus we obtain for I_1,

$$I_1 \le \frac{4}{\pi}\|\partial_x m_1\|^2_{L^2(\mathbb{R})} l^2 d^2 + \frac{1}{4}\int\limits_{R(l,d)}\int\limits_{R(l,d)} \frac{1}{\sqrt{(y-y_1)^2+(z-z_1)^2}}\,\mathrm{d}y_1\,\mathrm{d}z_1\,\mathrm{d}y\,\mathrm{d}z$$
$$\le \frac{4}{\pi}\|\partial_x m_1\|^2_{L^2(\mathbb{R})} l^2 d^2 + 10ld^2\left(1+\ln\frac{l}{d}\right).$$

By making a change of variables $\xi_2 = \xi_1 - \xi$ and utilizing again (3.9), we can estimate,

$$I_2 = \int\limits_{\Omega}\int\limits_{\mathbb{R}\times[-l-y,l-y]\times[-d-z,d-z]} \frac{|m^\star(x)|\cdot|\partial_x m_1(x_2+x)|}{|\xi_2|^2}\,\mathrm{d}\xi_2\,\mathrm{d}\xi$$
$$\le \frac{1}{2}\int\limits_{R(l,d)}\int\limits_{\mathbb{R}\times[-l-y,l-y]\times[-d-z,d-z]}\int\limits_{\mathbb{R}} \frac{|m^*(x)|^2+|\partial_x m_1(x_2+x)|^2}{|\xi_2|^2}\,\mathrm{d}x\,\mathrm{d}\xi_2\,\mathrm{d}y\,\mathrm{d}z$$
$$= 2ld\left(\|m^*\|^2_{L^2(\mathbb{R})} + \|\partial_x m_1\|^2_{L^2(\mathbb{R})}\right)\int\limits_{\mathbb{R}\times[-l-y,l-y]\times[-d-z,d-z]} \frac{\mathrm{d}\xi_2}{|\xi_2|^2}$$
$$= 2\pi ld\left(\|m^*\|^2_{L^2(\mathbb{R})} + \|\partial_x m_1\|^2_{L^2(\mathbb{R})}\right)\int\limits_{R(l,d)} \frac{1}{\sqrt{(y_1-y)^2+(z_1-z)^2}}\,\mathrm{d}y_1\,\mathrm{d}z_1$$
$$\le 20\pi ld^2\left(1+\ln\frac{l}{d}\right)\left(\|m^*\|^2_{L^2(\mathbb{R})} + \|\partial_x m_1\|^2_{L^2(\mathbb{R})}\right).$$

The summary of the estimates on I_1 and I_2 completes the proof. $\square$

4. The convergence of the energies

Consider a sequence of domain–magnetization–energy triples $\{(\Omega_n, m^n, E(m^n))\}$ where $\Omega_n = \mathbb{R}\times R(l_n, d_n)$, $m^n \in \tilde{A}_n = \tilde{A}(\Omega_n)$ and $l_n, c_n \to 0$. Lemma 3.2 suggests that for sufficiently big n one can formally write for any $m = m(x)$,

$$E_s\left(m^n\right) \approx \frac{8}{\pi} l_n d_n a_{c_n}\int\limits_{\mathbb{R}}\left|m_2^n(x)\right|^2\mathrm{d}x + \frac{8}{\pi} l_n d_n b_{c_n}\int\limits_{\mathbb{R}}\left|m_3^n(x)\right|^2\mathrm{d}x$$

Next, Lemma A.2 asserts that a_{c_n} scales like $c_n \ln c_n$ and $b_{c_n} \to \frac{\pi}{2}$. Furthermore, by Lemma 3.4, for a fixed $m^n = m^n(x)$ the summand $E_v(m^n)$ decays at least like $l_n d_n^2 \ln^2\frac{l_n}{d_n}$. Rescaling the magnetizations $\acute{m}^n(x,y,z) = m^n(\lambda_n x, l_n y, d_n z)$, we can rewrite the exchange energy for all $m^n(x) \in A_n$ as

$$E_{ex}\left(m^n(x)\right) = \frac{l_n d_n}{\lambda_n}\int\limits_{\Omega(1,1)}\left(\left|\partial_x \acute{m}^n(x)\right|^2 + \frac{\lambda_n^2}{l_n^2}\left|\partial_y \acute{m}^n(x)\right|^2 + \frac{\lambda_n^2}{d_n^2}\left|\partial_z \acute{m}^n(x)\right|^2\right)\mathrm{d}x,$$

and it is clear that $\acute{m}^n\colon \Omega(1,1) \to \mathbb{S}^2$. Thus one would expect that for sufficiently big n the approximation holds

$$E_{ex}\left(m^n(x)\right) \approx \frac{4 l_n d_n}{\lambda_n}\int\limits_{\mathbb{R}}\left|\partial_x m^n(x)\right|^2\mathrm{d}x$$

and

$$E_s\left(m^n(x)\right) \approx \frac{4}{\pi} l_n d_n c_n|\ln c_n|\lambda_n\int\limits_{\mathbb{R}}\left(\left|m_2^n(x)\right|^2 + \frac{\pi}{c_n|\ln c_n|}\left|m_3^n(x)\right|^2\right)\mathrm{d}x.$$

This calculation suggests that the coefficients $\frac{l_n d_n}{\lambda_n}$ and $l_n d_n c_n|\ln c_n|\lambda_n$ should be taken equal and they will both be the scaling of $E(m^n)$. This leads to $\lambda_n = \frac{1}{\sqrt{c_n|\ln c_n|}}$.

Proof of Theorem 2.2. Lower semicontinuity. One can without loss of generality assume that $\acute{E}(\acute{m}^n) \leq M$ for some $M > 0$ and all $n \in \mathbb{N}$. Following Kohn and Slastikov [15] let us prove that

$$\liminf_{n\to\infty} \frac{E_{mag}(m^n)}{\mu_n} = \liminf_{n\to\infty} \frac{E_{mag}(\bar{m}^n)}{\mu_n}. \tag{4.1}$$

By the Poincaré inequality, we have

$$\int_{\Omega} |m - \bar{m}|^2 \leq C\big(d^2 + l^2\big) \int_{\Omega} |\nabla m|^2 \leq C\big(d^2 + l^2\big) E(m).$$

Owing now to Lemma A.1 and the above inequality, we have

$$\big|E_{mag}\big(m^n\big) - E_{mag}\big(\bar{m}^n\big)\big| \leq M_1 \mu_n \sqrt{l_n^2 + d_n^2}, \tag{4.2}$$

for some M_1, which implies (4.1). Let now $\{q_n\}$ be a sequence with $0 < q_n < 1$ yet to be defined. We have by the Plancherel equality,

$$\begin{aligned}
q_n \frac{E_{ex}(m^n)}{\mu_n} &\geq \frac{q_n}{\mu_n} \int_{\Omega_n} \big|\partial_x \bar{m}^n(\xi)\big|^2 \, d\xi \\
&= 4\frac{q_n l_n d_n}{\mu_n} \int_{\mathbb{R}} \big|\widehat{\partial_x \bar{m}^n}(x)\big|^2 \, dx \\
&= 4\frac{q_n l_n d_n}{\mu_n} \int_{\mathbb{R}} \big|x \cdot \widehat{\bar{m}^n}(x)\big|^2 \, dx \\
&\geq \frac{4 q_n d_n}{l_n \mu_n} \int_{\mathbb{R}\setminus[-\frac{1}{l_n}, \frac{1}{l_n}]} \big(\big|\widehat{\bar{m}_2^n}(x)\big|^2 + \big|\widehat{\bar{m}_3^n}(x)\big|^2\big) \, dx,
\end{aligned}$$

and according to part (iii) of Lemma 3.2, we have for big n as well,

$$\frac{E_s(\bar{m}^n)}{\mu_n} \geq \frac{4}{\pi \mu_n} l_n d_n c_n |\ln c_n| \left(1 - \frac{5}{\sqrt{|\ln c_n|}}\right) \int_{-\frac{1}{l_n}}^{\frac{1}{l_n}} \left(\frac{1}{|\ln c_n|} \big|\widehat{\bar{m}_2^n}(x)\big|^2 + \big|\widehat{\bar{m}_3^n}(x)\big|^2\right) dx.$$

Now choose q_n so that

$$\frac{4}{\pi \mu_n} l_n d_n c_n |\ln c_n| \left(1 - \frac{5}{\sqrt{|\ln c_n|}}\right) = \frac{4 q_n d_n}{l_n \mu_n},$$

or

$$q_n = \frac{1}{\pi} l_n d_n c_n |\ln c_n| \left(1 - \frac{5}{\sqrt{|\ln c_n|}}\right),$$

and it is clear that $q_n \to 0$. Applying now the obtained inequalities, (4.1) and the convergence $\nabla \acute{m}^n \rightharpoonup \nabla m^0$ in $L^2(\Omega(1,1))$, we obtain

$$\begin{aligned}
\liminf_{n\to\infty}\frac{E(m^n)}{\mu_n} &\geq \liminf_{n\to\infty}(1-q_n)\int\limits_{\Omega(1,1)}\left|\partial_x\acute{m}^n\right|^2 \mathrm{d}\xi + \liminf_{n\to\infty} q_n\frac{E_{ex}(m^n)}{\mu_n} + \liminf_{n\to\infty}\frac{E_{mag}(m^n)}{\mu_n} \\
&= \liminf_{n\to\infty}(1-q_n)\int\limits_{\Omega(1,1)}\left|\partial_x\acute{m}^n\right|^2 \mathrm{d}\xi + \liminf_{n\to\infty} q_n\frac{E_{ex}(m^n)}{\mu_n} + \liminf_{n\to\infty}\frac{E_{mag}(\bar{m}^n)}{\mu_n} \\
&\geq \liminf_{n\to\infty}(1-q_n)\int\limits_{\Omega(1,1)}\left|\partial_x\acute{m}^n\right|^2 \mathrm{d}\xi + \liminf_{n\to\infty} q_n\frac{E_{ex}(m^n)}{\mu_n} + \liminf_{n\to\infty}\frac{E_{s}(\bar{m}^n)}{\mu_n} \\
&\geq 4\int\limits_{\mathbb{R}}\left|\partial_x m^0\right|^2 + \liminf_{n\to\infty}\frac{4}{\pi\mu_n}l_n d_n c_n|\ln c_n|\left(1-\frac{5}{\sqrt{|\ln c_n|}}\right)\int\limits_{\mathbb{R}}\left(\left|\bar{m}_2^n\right|^2+|\ln c_n|\left|\bar{m}_3^n\right|^2\right) \\
&= 4\int\limits_{\mathbb{R}}\left|\partial_x m^0\right|^2 \mathrm{d}x + \frac{4}{\pi}\liminf_{n\to\infty}\frac{1}{\lambda_n}\int\limits_{\mathbb{R}}\left(\left|\bar{m}_2^n(x)\right|^2+|\ln c_n|\left|\bar{m}_3^n(x)\right|^2\right)\mathrm{d}x.
\end{aligned}$$

It is then standard to prove that the convergence $\acute{m}^n \to m^0$ in $L^2_{loc}(\Omega(1,1))$ implies

$$\liminf_{n\to\infty}\frac{1}{\lambda_n}\int\limits_{\mathbb{R}}\left|\bar{m}_2^n(x)\right|^2 \geq \int\limits_{\mathbb{R}}\left|m_2^0(x)\right|^2 \quad \text{and} \quad \liminf_{n\to\infty}\frac{1}{\lambda_n}\int\limits_{\mathbb{R}}\left|\bar{m}_3^n(x)\right|^2 \geq \int\limits_{\mathbb{R}}\left|m_3^0(x)\right|^2,$$

thus since $|\ln c_n| \to \infty$, we conclude that

$$\liminf_{n\to\infty}\frac{E(m^n)}{\mu_n} \geq E_0\left(m^0\right).$$

Recovery sequence. Let us prove that the sequence $m^n(x)$, where

$$m^n(\lambda_n x, y, z) = m^0(x), \quad \text{if } \xi \in \Omega(l_n, d_n) \quad \text{and} \quad m^n(\xi) = 0, \quad \text{if } \xi \in \mathbb{R}^3 \setminus \Omega(l_n, d_n),$$

satisfies the required condition. If m_3^0 is not identically zero, then $E_0(m^0) = \infty$ and due to the *lower semi-continuity* part of the foregoing theorem, we have that $E_0(m^0) \leq \liminf_{n\to\infty}\acute{E}_n(m^n)$, thus the proof follows. Assume now that $m_3^0 \equiv 0$. It remains to only prove the reverse inequality $\limsup_{n\to\infty}\acute{E}_n(\acute{m}^n) \leq E_0(m^0)$. It is clear that

$$E\left(m^n\right) = 4\mu_n\int\limits_{\mathbb{R}}\left|\partial_x m^0\right|^2 \mathrm{d}x + E_{mag}\left(m^n\right).$$

Due to Lemma 3.2 and the Plancherel equality, we have

$$E_s\left(m^n\right) \leq \frac{4}{\pi}l_n d_n c_n\left(|\ln c_n|+3\right)\int\limits_{\mathbb{R}}\left|m_2^0(x)\right|^2 \mathrm{d}x,$$

thus

$$\limsup_{n\to\infty}\frac{E_s(m^n)}{\mu_n} \leq \frac{4}{\pi}\int\limits_{\mathbb{R}}\left|m_2^0(x)\right|^2 \mathrm{d}x.$$

We have furthermore by Lemma 3.4 that

$$\limsup_{n\to\infty}\frac{E_v(m^n)}{\mu_n} = 0,$$

thus combining all the obtained inequalities for the energy summands, we discover

$$\limsup_{n\to\infty} \acute{E}_n\big(\acute{m}^n\big) \leq E_0\big(m^0\big).$$

Compactness. The inequality $E(m^n) \leq C\mu_n$ implies

$$\int\limits_{\Omega(1,1)} \big|\partial_x \acute{m}^n\big|^2 \leq C, \qquad \int\limits_{\Omega(1,1)} \big|\partial_y \acute{m}^n\big|^2 \leq C\frac{l_n^2}{\lambda_n^2} \quad \text{and} \quad \int\limits_{\Omega(1,1)} \big|\partial_z \acute{m}^n\big|^2 \leq C\frac{d_n^2}{\lambda_n^2},$$

thus a subsequence (not relabeled) of $\{\nabla \acute{m}^n\}$ has a weak limit $f = f(x)$ in $L^2(\Omega(1,1))$. On the other hand $\acute{m}^n$ has a unit length pointwise, thus a subsequence (not relabeled) of $\{\acute{m}^n\}$ has a strong local limit m^0 in $L^2(\Omega(1,1))$. It is then straightforward to show that m^0 is weakly differentiable with $f = \nabla m^0$, thus $\{\acute{m}^n\}$ converges to m^0 in the sense of Definition 2.1. It has been proven in [11], that actually one can translate the subsequence $\{\acute{m}^n\}$ in the x direction so that the limit m^0 satisfies $m^0(\pm\infty) = \pm 1$. Finally owing to the *lower semi-continuity* part of the lemma, we discover $E_0(m^0) \leq \liminf \acute{E}(\acute{m}^n) \leq C < \infty$, thus $m_3^0 \equiv 0$, i.e., $m^0 \in A_0^3$. $\square$

5. The rate of convergence

Recall that for any $\alpha > 0$ one can explicitly determine the minima of the energy functional e.g., [16,12,11],

$$E_\alpha(m) = \int\limits_{\mathbb{R}} \big|\partial_x m(x)\big|^2\,\mathrm{d}x + \alpha \int\limits_{\mathbb{R}} \big(\big|m_y(x)\big|^2 + \big|m_z(x)\big|^2\big)\,\mathrm{d}x$$

in the admissible set

$$A_0 = \big\{m: \mathbb{R} \to \mathbb{R}^3 \;:\; |m| = 1,\ m(\pm\infty) = \pm 1\big\}.$$

The minimizer is given by the formula

$$m = m^{\alpha,\beta} = \left(\frac{e^{2\sqrt{\alpha}x}\cdot\beta^2 - 1}{e^{2\sqrt{\alpha}x}\cdot\beta^2 + 1},\ \frac{2\beta e^{\sqrt{\alpha}x}}{e^{2\sqrt{\alpha}x}\cdot\beta^2 + 1}\cos\theta,\ \frac{2\beta e^{\sqrt{\alpha}x}}{e^{2\sqrt{\alpha}x}\cdot\beta^2 + 1}\sin\theta\right), \tag{5.1}$$

where $\beta \in \mathbb{R}$. Note that the minimal value of the energy does not depend on θ, i.e., it is invariant under rotations in the cross section plane, and for a fixed θ any minimizer can be obtained from $m^\alpha := m^{\alpha,1}$ by a translation in the x direction. The minimizer m^α satisfies $m_x^\alpha(0) = 0$. The minimal value of E_α in A_0 will be $4\sqrt{\alpha}$. Therefore, due to the fact $m^0 \in A_0^3$, the minimizers m^0 of E_0 have the form

$$m^0 = \left(\frac{e^{\frac{2x}{\sqrt{\pi}}}\cdot\beta^2 - 1}{e^{\frac{2x}{\sqrt{\pi}}}\cdot\beta^2 + 1},\ \frac{2\beta e^{\frac{2x}{\sqrt{\pi}}}}{e^{\frac{2x}{\sqrt{\pi}}}\cdot\beta^2 + 1},\ 0\right). \tag{5.2}$$

The minimal value of E_0 is $\frac{16}{\sqrt{\pi}}$.

Proof of Theorem 2.4. We need to get accurate lower and upper bounds on $E(m)$. For an upper bound, we choose the recovery sequence $m(x,y,z) = m^0(\frac{x}{\lambda_n})$, where $m_3 \equiv 0$ and m^0 is a minimizer of the energy functional

$$E_0(m) = 4\int\limits_{\mathbb{R}} |\partial_x m|^2\,\mathrm{d}x + \frac{4}{\pi}\int\limits_{\mathbb{R}} \big(\big|m_2(x)\big|^2 + \big|m_3(x)\big|^2\big)\,\mathrm{d}x.$$

Due to Lemma 3.2, we have for big n,

$$E(m^0) \le \frac{4l_n d_n}{\lambda_n} \int_{\mathbb{R}} |\partial_x m^0|^2 \,\mathrm{d}x + \frac{4l_n d_n c_n(3 - \ln c_n)}{\pi} \int_{\mathbb{R}} |m_2^0(x)|^2 \,\mathrm{d}x + E_v(m^0).$$

Next, due to Lemma 3.4, we get for big n,

$$\begin{aligned}\frac{E(m)}{\mu_n} &\le 4E_0(m) + \frac{12}{\pi|\ln c_n|} \int_{\mathbb{R}} |m_z^0(x)|^2 \,\mathrm{d}x + 2M_{m^0} d_n \lambda_n (1 - \ln c_n) \\ &\le \frac{16}{\sqrt{\pi}} + \frac{10}{|\ln c_n|} + 2\sqrt{l_n d_n |\ln c_n|},\end{aligned}$$

thus the minimal energy satisfies the inequality

$$\frac{\min_{m\in\tilde{A}_n} E(m)}{\mu_n} - \frac{16}{\sqrt{\pi}} \le \frac{10}{|\ln c_n|} + 2\sqrt{l_n d_n |\ln c_n|}. \tag{5.3}$$

Assume now $m \in \tilde{A}_n$ is an energy minimizer in Ω_n. We have that $I(l_n, d_n, x) \ge I(d_n, l_n, x)$, thus by Lemma 3.2, we have

$$E_{mag}(\bar{m}) \ge \frac{4}{\pi} l_n d_n c_n |\ln c_n| \left(1 - \frac{5}{\sqrt{|\ln c_n|}}\right) \int_{-\frac{1}{l_n}}^{\frac{1}{l_n}} \left(|\widehat{\bar{m}_2}|^2 + |\widehat{\bar{m}_3}|^2\right) \mathrm{d}x.$$

According to (5.3), we have for big n,

$$\frac{\min_{m\in\tilde{A}_n} E(m)}{\mu_n} \le \frac{16}{\sqrt{\pi}} + 1 < 11.$$

We have furthermore for big n, that

$$\begin{aligned}\int_{\mathbb{R}\setminus[-\frac{1}{l_n}, \frac{1}{l_n}]} \left(|\widehat{\bar{m}_2}|^2 + |\widehat{\bar{m}_3}|^2\right) \mathrm{d}x &\le l_n^2 \int_{\mathbb{R}} \left(|x \cdot \widehat{\bar{m}_2}|^2 + |x \cdot \widehat{\bar{m}_3}|^2\right) \mathrm{d}x \\ &= l_n^2 \int_{\mathbb{R}} \left(|\partial_x \bar{m}_2|^2 + |\partial_x \bar{m}_3|^2\right) \mathrm{d}x \\ &\le \frac{l_n}{4d_n} \int_{\Omega_n} \left(|\partial_x m_2|^2 + |\partial_x m_3|^2\right) \mathrm{d}x \\ &\le \frac{l_n E_{ex}(m)}{4d_n} \\ &\le \frac{11 l_n \mu_n}{4d_n},\end{aligned}$$

thus

$$\frac{4}{\pi} l_n d_n c_n |\ln c_n| \int_{\mathbb{R}\setminus[-\frac{1}{l_n}, \frac{1}{l_n}]} \left(|\widehat{\bar{m}_2}|^2 + |\widehat{\bar{m}_3}|^2\right) \mathrm{d}x \le \frac{11}{\pi} l_n^2 c_n |\ln c_n| \mu_n,$$

therefore, we obtain

$$E_{mag}(\bar{m}) \geq \frac{4}{\pi} l_n d_n c_n |\ln c_n| \left(1 - \frac{5}{\sqrt{|\ln c_n|}}\right) \int\limits_{\mathbb{R}} \left(|\bar{m}_2|^2 + |\bar{m}_3|^2\right) \mathrm{d}x - \frac{11}{\pi} l_n^2 c_n |\ln c_n| \mu_n. \tag{5.4}$$

It is straightforward to see using the definition of the average that

$$\int\limits_{\Omega_n} \left(|m_2|^2 + |m_3|^2\right) - \int\limits_{\Omega_n} \left(|\bar{m}_2|^2 + |\bar{m}_3|^2\right) = \int\limits_{\Omega_n} \left(|m_2 - \bar{m}_2|^2 + |m_3 - \bar{m}_3|^2\right),$$

thus by the Poincaré inequality, we get for big n,

$$\int\limits_{\Omega_n} \left(|m_2|^2 + |m_3|^2\right) \leq \int\limits_{\Omega_n} \left(|\bar{m}_2|^2 + |\bar{m}_3|^2\right) + 11C\mu_n\left(l_n^2 + d_n^2\right). \tag{5.5}$$

Next, due to estimate (4.1), we have for big n, that

$$E_{mag}(m) \geq E_{mag}(\bar{m}) - M_1 \mu_n \sqrt{d_n^2 + l_n^2},$$

where $M_1 = 11C$ and C is the Poinceré constant for $R(l_n, d_n)$. Combining now the last inequality with (5.4) and (5.5), for bin n, we discover

$$\begin{aligned} E_{mag}(m) \geq{} & \frac{4}{\pi} l_n d_n c_n |\ln c_n| \left(1 - \frac{5}{\sqrt{|\ln c_n|}}\right) \int\limits_{\mathbb{R}} \left(|m_2|^2 + |m_3|^2\right) \mathrm{d}x \\ & - \frac{11}{\pi} l_n^2 c_n |\ln c_n| \mu_n - 12C\mu_n \sqrt{l_n^2 + d_n^2}. \end{aligned} \tag{5.6}$$

For the whole energy, we obtain for big n,

$$\frac{E(m)}{\mu_n} \geq 4\left(1 - \frac{5}{\sqrt{|\ln c_n|}}\right)\left(\int\limits_{\Omega(1,1)} |\partial_x \acute{m}|^2 \, \mathrm{d}\xi + \frac{1}{\pi} \int\limits_{\Omega(1,1)} \left(|\acute{m}_2|^2 + |\acute{m}_3|^2\right) \mathrm{d}\xi\right) - 20Cl_n.$$

It has been shown in [11, Lemma 3.3], that if $m \in A_n$ then $\bar{m}(\pm\infty) = \pm 1$, thus we have $\acute{m}(\pm\infty, y, z) = \pm 1$ on a full measure subset Q of $R(1,1)$. Therefore, we have for any $(y, z) \in Q$ that

$$\int\limits_{\mathbb{R}} \left|\partial_x \acute{m}(x,y,z)\right|^2 \mathrm{d}x + \frac{1}{\pi} \int\limits_{\mathbb{R}} \left(\left|\acute{m}_2(x,y,z)\right|^2 + \left|\acute{m}_3(x,y,z)\right|^2\right) \mathrm{d}x \geq \frac{4}{\sqrt{\pi}},$$

which gives

$$\int\limits_{\Omega(1,1)} |\partial_x \acute{m}|^2 \, \mathrm{d}\xi + \frac{1}{\pi} \int\limits_{\Omega(1,1)} \left(|\acute{m}_2|^2 + |\acute{m}_3|^2\right) \mathrm{d}\xi \geq \frac{16}{\sqrt{\pi}}.$$

Finally we get for the energies,

$$\frac{E(m)}{\mu_n} - \frac{16}{\sqrt{\pi}} \geq -\frac{200}{\sqrt{|\ln c_n|}} - 20Cl_n.$$

A combination of the last inequality and (5.3) completes the proof. In conclusion, let us mention that for sufficiently small d and l the minimizer m must have almost the shape of $m^{\alpha,\beta}$ i.e., must be a transverse wall. $\square$

Acknowledgments

The present results are part of the author's PhD thesis. The author is thankful to his supervisor Dr. Prof. S. Müller for suggesting the topic and many good advices. This work was supported by scholarships by Max-Planck Institute for Mathematics in the Sciences in Leipzig, Germany, and HCM for Mathematics in Bonn, Germany.

Appendix A

In this section we recall a key inequality and study the function a_c.

Lemma A.1. *For any vector fields $m_1, m_2 \in M(\Omega)$ with finite energies there holds*

$$\left|E_{mag}(m_1) - E_{mag}(m_2)\right| \le \|m_1 - m_2\|^2_{L^2(\Omega)} + 2\|m_1 - m_2\|_{L^2(\Omega)}\sqrt{E_{mag}(m_1)}.$$

Proof. The proof is trivial and can be found in [16]. □

Consider now $c \to a_c$ as a map from $(0, +\infty)$ to $(0, +\infty)$.

Lemma A.2. *The function a_c has the following properties:*

(i) a_c *increases in* $(0, +\infty)$,
(ii) $\lim_{c\to 0} \frac{a_c}{c|\ln c|} = \frac{1}{2}$,
(iii) $\lim_{c\to +\infty} a_c = \frac{\pi}{2}$.

Proof. The first property follows from the fact that the function $\frac{1-e^{-t}}{t}$ decreases over $(0, +\infty)$. The second property is Corollary 3.3. Assume now $c \ge 4$. It is clear that

$$\frac{1 - e^{-\frac{2t}{c}}}{\frac{2t}{c}} \ge 1 - \frac{t}{c} \quad \text{if } t \in \left[0, \frac{c}{2}\right],$$

thus taking into account the inequality $\sqrt{c} \le \frac{c}{2}$, we discover

$$\frac{1 - e^{-\frac{2t}{c}}}{\frac{2t}{c}} \ge 1 - \frac{t}{c} \ge 1 - \frac{1}{\sqrt{c}}, \quad \text{if } t \in [0, \sqrt{c}].$$

Therefore, for a_c we have on one hand

$$\liminf_{c\to\infty} a_c \ge \liminf_{c\to\infty}\left(1 - \frac{1}{\sqrt{c}}\right)\int_0^{\sqrt{c}} \frac{\sin^2 t}{t^2}\,\mathrm{d}t = \int_0^{+\infty} \frac{\sin^2 t}{t^2}\,\mathrm{d}t = \frac{\pi}{2},$$

but on the other hand

$$a_c \le \int_0^{+\infty} \frac{\sin^2 t}{t^2}\,\mathrm{d}t = \frac{\pi}{2} \quad \text{for any } c > 0,$$

which achieves the proof. □

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