J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Second main theorem and uniqueness theorem with moving targets on parabolic manifolds ☆

Qiming Yan

Department of Mathematics, Tongji University, Shanghai, 200092, PR China

ARTICLE INFO

Article history:
Received 16 April 2014
Available online xxxx
Submitted by Richard M. Aron

Keywords:
Second main theorem
Uniqueness theorem
Parabolic manifolds
Value distribution theory
Moving targets

ABSTRACT

In this paper, with the motivation from Diophantine approximation, a truncated second main theorem is established for meromorphic maps from M into $\mathbb{P}(V)$ with moving targets $g_j : M \to \mathbb{P}(V^*)$, $1 \leq j \leq q$, where M is a parabolic manifold and V is a Hermitian vector space. As an application of this second main theorem, a uniqueness theorem without counting multiplicities is given.

© 2014 Published by Elsevier Inc.

1. Introduction

The purpose of this paper is to give the second main theorem with truncated counting functions for meromorphic maps intersecting moving targets on parabolic manifolds. As the application, we also discuss the uniqueness problem with moving targets on parabolic manifolds.

In 1926, by using the second main theorem and Borel's lemma, R. Nevanlinna [13] proved the five-value theorem and four-value theorem for meromorphic functions. More precisely, let f_1 and f_2 be two nonconstant meromorphic functions on the complex plane $\mathbb{C}$, and let $a_1, \dots, a_q$ be q points in $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C})$. If $\operatorname{supp} \mu_{f_1,a_j} = \operatorname{supp} \mu_{f_2,a_j}$, $1 \leq j \leq q$, then $f_1 \equiv f_2$ for $q \geq 5$; if $\mu_{f_1,a_j} = \mu_{f_2,a_j}$, $1 \leq j \leq q$, then there exists a Möbius transformation L such that $f_2 = L \circ f_1$ for $q \geq 4$. Here μ_{f_t,a_j} is the pull-back divisor of a_j by f_t for $t = 1, 2$ and $j = 1, \dots, q$.

For the higher dimensional case, in 1975, H. Fujimoto considered the uniqueness problem for meromorphic maps with counting multiplicities. Let f_1 and f_2 be linearly nondegenerate meromorphic maps from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, and $H_j \in \mathbb{P}^n(\mathbb{C}^*)$, $1 \leq j \leq q$, such that $H_1, \dots, H_q$ are in general position. He [8] proved that if $\mu_{f_1,H_j} = \mu_{f_2,H_j}$, $1 \leq j \leq q$, then $f_1 \equiv f_2$ for $q \geq 3n + 2$, and there exists a projective linear transformation

☆ Project supported by the National Natural Science Foundation of China No. 11171255 and the Natural Science Foundation of Shanghai No. 13ZR1444100.

E-mail address: yan_qiming@hotmail.com.

http://dx.doi.org/10.1016/j.jmaa.2014.08.048
0022-247X/© 2014 Published by Elsevier Inc.

L of $\mathbb{P}^n(\mathbb{C})$ such that $f_2 = L \circ f_1$ for $q = 3n+1$. Here μ_{f_t,H_j}, $t = 1, 2$, $1 \le j \le q$, is the intersection divisor of f_t and H_j, which will be defined in Section 2.4. For the uniqueness problem without counting multiplicities, Smiley [18] showed that if $\operatorname{supp}\mu_{f_1,H_j} = \operatorname{supp}\mu_{f_2,H_j}$ and $\dim(\operatorname{supp}\mu_{f_1,H_i} \cap \operatorname{supp}\mu_{f_1,H_j}) \le m-2$ for $1 \le i < j \le q$, then $f_1 = f_2$ on $\bigcup_{j=1}^{q} \operatorname{supp}\mu_{f_1,H_j}$ implies $f_1 \equiv f_2$ for $q \ge 3n+2$. Ji [10] gave an algebraic dependence theorem for $q \ge 3n+1$.

In the proof of uniqueness theorems without counting multiplicities, the following Cartan's truncated second main theorem (see [2]) is an essential tool.

Theorem A. *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic map, and $H_j \in \mathbb{P}^n(\mathbb{C}^*)$, $1 \le j \le q$, such that $H_1, \dots, H_q$ are in general position. Then, for $s > 0$,*

$$(q-n-1)T_f(r,s) \mathrel{.\le.} \sum_{j=1}^{q} N_{f,H_j}^{(n)}(r,s) + o\big(T_f(r,s)\big),$$

where $.\le.$ means that the inequality holds for all r except for a finite measure subset $E \subset [s, +\infty)$.

Using a generalization of Theorem A, W. Stoll [22] studied the uniqueness problem for meromorphic maps by replacing $\mathbb{C}^m$ by parabolic covering spaces.

Let M be a connected complex manifold of dimension m with a surjective, proper holomorphic map $\pi : M \to \mathbb{C}^m$. Then $\tau = \|\pi\|^2$ is a parabolic exhaustion of M and (M, τ) is called a *parabolic covering space of* $\mathbb{C}^m$. Let V be a Hermitian vector space of dimension $n+1 > 1$.

Let $f_t : M \to \mathbb{P}(V)$ be a meromorphic map for $t = 1, \dots, \lambda$ and A be a nonempty subset of M. $f_1, \dots, f_\lambda$ are said to be *in p-special position on* A if, for any $x \in A$, there exist an open, connected neighborhood U_x of x and a reduced representation $F_t : U_x \to V$ (of f_t) such that, for any $1 \le t_1 < t_2 < \dots < t_p \le \lambda$,

$$F_{t_1}(x) \wedge \dots \wedge F_{t_p}(x) = 0.$$

If $A = M$ omit "on A". Also "special position" means "λ-special position". (For the definitions of meromorphic map and reduced representation, see Section 2.3.) Then f_1 and f_2 are in 2-special position if and only if $f_1 \equiv f_2$. If $f_1, \dots, f_\lambda$ are in λ-special position, then $f_1, \dots, f_\lambda$ are algebraically dependent, where $f_1, \dots, f_\lambda$ are said to be *algebraically dependent* if and only if there is a proper analytic subset S of $\mathbb{P}(V)^\lambda$ such that $(f_1, \dots, f_\lambda) \subset S$. (See Proposition 1.1 in [22].)

Stoll [22] extended the results given by Smiley [18] and Ji [10] as follows.

Theorem B. *Let (M, τ) be a parabolic covering space of $\mathbb{C}^m$ with branching divisor β of π. Let V be a Hermitian vector space of dimension $n+1 > 1$. Let l and λ be integers with $2 \le l \le \lambda \le n+1$. For $t = 1, \dots, \lambda$, let $f_t : M \to \mathbb{P}(V)$ be a linearly nondegenerate meromorphic map. Assume $N_\beta(r,s) = o(\sum_{t=1}^{\lambda} T_{f_t}(r,s))$ and $\log \frac{r}{s} = o(\sum_{t=1}^{\lambda} T_{f_t}(r,s))$, for $r \to \infty$. Let $H_1, \dots, H_q$ be in general position in $\mathbb{P}(V^*)$ with $q \ge n+1$. Assume that $A_j = \operatorname{supp}\mu_{f_1,H_j} = \operatorname{supp}\mu_{f_2,H_j} = \dots = \operatorname{supp}\mu_{f_\lambda,H_j}$ for each $j = 1, \dots, q$, and $\dim(A_i \cap A_j) \le m-2$ for $1 \le i < j \le q$. Define $A = \bigcup_{j=1}^{q} A_j$. Assume that $f_1, \dots, f_\lambda$ are in l-special position on A. If $q > \frac{n\lambda}{\lambda-l+1} + n + 1$, then $f_1, \dots, f_\lambda$ are in special position. In particular, $f_1, \dots, f_\lambda$ are algebraically dependent.*

Remark 1.1. Consider $M = \mathbb{C}^m$ with the identity $\pi : M \to \mathbb{C}^m$, and $V = \mathbb{C}^{n+1}$. Then Theorem B gives the result of Smiley [18] for $\lambda = 2$ and $q = 3n+2$, and extends the theorem of Ji [10] who considers the case $l = 2$, $\lambda = 3$ and $q = 3n+1$. For more results of related subjects, we refer readers to the references [3–6,9, 23,25,26].

Please cite this article in press as: Q. Yan, Second main theorem and uniqueness theorem with moving targets on parabolic manifolds, J. Math. Anal. Appl. (2015), http://dx.doi.org/10.1016/j.jmaa.2014.08.048

For $M = \mathbb{C}^m$ and $V = \mathbb{C}^{n+1}$, Ru [15] generalized Stoll's result to moving targets, in which the fixed target $H_j \in \mathbb{P}^n(\mathbb{C}^*)$ is replaced by meromorphic map $g_j : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C}^*)$ for $1 \le j \le q$. He firstly established the following truncated second main theorem for moving targets.

Theorem C. *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a nonconstant meromorphic map. Let $g_1, \ldots, g_q : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C}^*)$ be q meromorphic maps located in general position such that (f, g_j) is free for $1 \le j \le q$ (for definition, see Section 2.4). If $q \ge 2n + 1$, then*

$$\frac{q}{n(2n+1)} T_f(r,s) \mathrel{.\le.} \sum_{j=1}^{q} N^{(n)}_{f,g_j}(r,s) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big) + o\big(T_f(r,s)\big).$$

Applying Theorem C, Ru [15] gave the following theorem, which generalizes Theorem B for the case $M = \mathbb{C}^m$ and $V = \mathbb{C}^{n+1}$.

Theorem D. *Let $f_1, f_2, \ldots, f_\lambda : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be nonconstant meromorphic maps. Let $g_1, \ldots, g_q : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C}^*)$ be moving targets located in general position and $T_{g_j}(r,s) = o(\max_{1\le t\le\lambda}\{T_{f_t}(r,s)\})$ for $r \to \infty$, $1 \le j \le q$. Assume that (f_t, g_j) is free for $1 \le j \le q$, $1 \le t \le \lambda$. Assume that $A_j = \operatorname{supp}\mu_{f_1,g_j} = \operatorname{supp}\mu_{f_2,g_j} = \cdots = \operatorname{supp}\mu_{f_\lambda,g_j}$ for $1 \le j \le q$, and $\dim(A_i \cap A_j) \le m-2$ for $1 \le i < j \le q$. Let $A = \bigcup_{j=1}^{q} A_j$. Let l, $2 \le l \le \lambda$, be an integer such that $f_1, \ldots, f_\lambda$ are in l-special position on A. If $q > \frac{n^2(2n+1)\lambda}{\lambda-l+1}$, then $f_1, \ldots, f_\lambda$ are in special position.*

In this paper, we will extend the above theorems to more general parabolic manifolds. Throughout this paper, we shall use the standard notation in the value distribution theory of meromorphic maps on parabolic manifolds (see [20,24]). Some notation and definitions will be introduced in Section 2.

To establish the value distribution theory, we shall work on *admissible parabolic manifolds*, which satisfy the following assumptions:

(i) M is a connected complex manifold of dimension m.
(ii) There exists a parabolic exhaustion function τ on M.
(iii) For any positive integer n, let $\Psi : M \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic map. Then there is a holomorphic differential form B of degree $(m-1, 0)$ on M such that Ψ is general for B (see p. 144 of [20] or Section 1.3 of [12]) and

$$m i_{m-1} B \wedge \bar{B} \le Y(r) \upsilon^{m-1}$$

on $M[r]$ for some real positive valued function $Y(r)$ on M, which is independent of Ψ ($Y(r)$ is called a *majorant for* B). Here, for any positive integer m,

$$i_m := \left(\frac{\sqrt{-1}}{2\pi}\right)^m (-1)^{\frac{m(m-1)}{2}} m!.$$

We note that the manifold M with dimension m in Theorem B is an important class of admissible parabolic manifold. Let $\pi : M \to \mathbb{C}^m$ be a surjective, proper holomorphic map and β be the branching divisor of M. Then $\tau = \|\pi\|^2$ is a parabolic exhaustion of M. Let $\Psi : M \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic map. Stoll [20] showed that there exists a holomorphic differential form $\hat{B}$ of degree $(m-1, 0)$ on $\mathbb{C}^m$ such that the holomorphic form $B = \pi^*\hat{B}$ of degree $(m-1, 0)$ on M is majorized by τ with majorant $Y(r) \le 1 + r^{2n-2}$ for all $r > 0$ (cf. Theorem 10.2 in [20]). Further, we have

$$0 \le N_\beta(r,s) = \mathrm{Ric}_\tau(r,s),$$

where $\mathrm{Ric}_\tau(r,s)$ is the Ricci function (for definition, see p. 1048 of [24]), which depends only on the geometry (topology) of the manifold M.

An available technique to prove the second main theorem with moving targets is the combination of H. Cartan's and Steinmetz's technique [2,19,16]. In applying this technique to meromorphic maps on parabolic manifolds, the main difficulty lies in that it is hard to extend the logarithmic derivative lemma to meromorphic maps on parabolic manifolds. In [1], Ashline gave a version of the logarithmic derivative lemma on parabolic manifolds with an additional assumption that there exists a holomorphic m form $\Theta \not\equiv 0$ on M. Thus a second main theorem for meromorphic maps with moving targets was proved.

Motivated by the analogy between Nevanlinna theory and Diophantine approximation discovered by C. Osgood, P. Vojta and S. Lang, etc., Ru [14] got the following second main theorem with moving targets, and the logarithmic derivative lemma could be avoided.

Theorem E. *Let (M,τ) be an admissible parabolic manifold with dimension m. Let V be a Hermitian vector space with $\dim V = n+1 > 1$. Let $f : M \to \mathbb{P}(V)$ be a meromorphic map. Let $g_1,\dots,g_q : M \to \mathbb{P}(V^*)$ be meromorphic maps located in general position. Let $\mathcal{R}_{\mathcal{G}}$ be the smallest subfield containing all the coordinate functions of all g_j, $1 \le j \le q$. Assume that f is linearly nondegenerate over $\mathcal{R}_{\mathcal{G}}$ and $\log Y(r) = o(T_f(r,s))$ for $r \to \infty$, where $Y(r)$ is the majorant for B. Then, for $s > 0$ and for every $\varepsilon > 0$, there are constants $C_{1,\varepsilon} > 0$ and $C_{2,\varepsilon} > 0$ dependent on ε, such that*

$$(q-n-1-\varepsilon)T_f(r,s) \mathrel{.\le.} \sum_{j=1}^{q} N_{f,g_j}(r,s) + C_{1,\varepsilon}\,\mathrm{Ric}_\tau(r,s) + C_{2,\varepsilon}\Big(\log r + \max_{1\le j\le q} T_{g_j}(r,s)\Big).$$

We note that the counting functions in these theorems are not truncated. In [11], Liu obtained a generalized Schmidt's subspace theorem in Number Theory. Motivated by the technique shown in [11] and the analogy between Nevanlinna theory and Diophantine approximation, we can prove the following truncated second main theorem.

Theorem 1.1. *Let (M,τ) be an admissible parabolic manifold with dimension m. Let V be a Hermitian vector space with $\dim V = n+1 > 1$. Let $g_1,\dots,g_q : M \to \mathbb{P}(V^*)$ be meromorphic maps located in general position. Let $f : M \to \mathbb{P}(V)$ be a nonconstant meromorphic map such that $\mathrm{Ric}_\tau(r,s) = o(T_f(r,s))$ and $\log Y(r) = o(T_f(r,s))$ for $r \to \infty$. Assume that (f,g_j) is free for $1 \le j \le q$ and $\dim(\operatorname{supp}\mu_{f,g_i} \cap \operatorname{supp}\mu_{f,g_j}) \le m-2$ for $1 \le i < j \le q$. If $q \ge 2n+1$, then, for $s > 0$,*

$$\frac{q}{2n+1}T_f(r,s) \mathrel{.\le.} \sum_{j=1}^{q} N^{(n)}_{f,g_j}(r,s) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big) + o\big(T_f(r,s)\big). \tag{1}$$

Using this second main theorem, we obtain the following results for the uniqueness problem.

Theorem 1.2. *Let $f_1,\dots,f_\lambda : M \to \mathbb{P}(V)$ be nonconstant meromorphic maps with $\mathrm{Ric}_\tau(r,s) = o(T_{f_t}(r,s))$ and $\log Y(r) = o(T_{f_t}(r,s))$ for $r \to \infty, 1 \le t \le \lambda$. Let $g_1,\dots,g_q : M \to \mathbb{P}(V^*)$ be meromorphic maps located in general position and $T_{g_j}(r,s) = o(\max_{1\le t\le\lambda} T_{f_t}(r,s))$ for $r \to \infty$, $1 \le j \le q$. Assume that (f_t,g_j) is free for $1 \le j \le q$, $1 \le t \le \lambda$. Assume that $A_j = \operatorname{supp}\mu_{f_1,g_j} = \operatorname{supp}\mu_{f_2,g_j} = \cdots = \operatorname{supp}\mu_{f_\lambda,g_j}$ for each $j = 1,\dots,q$, and $\dim(A_i \cap A_j) \le m-2$ for $1 \le i < j \le q$. Define $A = \bigcup_{j=1}^q A_j$. Assume that $f_1,\dots,f_\lambda$ are in l-special position on A, where l is an integer with $2 \le l \le \lambda$. If $q > \frac{n(2n+1)\lambda}{\lambda-l+1}$, then $f_1,\dots,f_\lambda$ are in special position.*

For $M = \mathbb{C}^m$ and $V = \mathbb{C}^{n+1}$, we have the following corollary.

Corollary 1.1. *Let $f_1, \dots, f_\lambda : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be nonconstant meromorphic maps. Let $g_1, \dots, g_q : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C}^*)$ be moving targets located in general position and $T_{g_j}(r,s) = o(\max_{1 \le t \le \lambda}\{T_{f_t}(r,s)\})$ for $r \to \infty$, $1 \le j \le q$. Assume that (f_t, g_j) is free for $1 \le j \le q$, $1 \le t \le \lambda$. Assume that $A_j = \operatorname{supp} \mu_{f_1,g_j} = \operatorname{supp} \mu_{f_2,g_j} = \cdots = \operatorname{supp} \mu_{f_\lambda,g_j}$ for $1 \le j \le q$, and $\dim(A_i \cap A_j) \le m-2$ for $1 \le i < j \le q$. Let $A = \bigcup_{j=1}^q A_j$. Assume that $f_1, \dots, f_\lambda$ are in l-special position on A, where l is an integer with $2 \le l \le \lambda$. If $q > \frac{n(2n+1)\lambda}{\lambda - l + 1}$, then $f_1, \dots, f_\lambda$ are in special position.*

We organize our paper as follows: In Section 2, we recall some basic notation and definitions in the value distribution theory of meromorphic maps on parabolic manifolds. We give some truncated second main theorems for moving targets in Section 3, and prove Theorem 1.2 in Section 4.

2. Preliminaries

In this section, we list some fundamental notation, facts and results of meromorphic maps on parabolic manifolds. For references, see [20] or [24].

2.1. Parabolic manifolds

Let M be a connected, complex manifold of dimension m. Let τ be a nonnegative function of class C^∞ on M. For $r \ge 0$ and $S \subseteq M$, define

$$S[r] = \{x \in S \mid \tau(x) \le r^2\}, \qquad S(r) = \{x \in S \mid \tau(x) < r^2\},$$
$$S\langle r\rangle = \{x \in S \mid \tau(x) = r^2\}, \qquad S_* = \{x \in S \mid \tau(x) > 0\}.$$

Define

$$\upsilon = dd^c\tau \quad \text{on } M, \qquad \omega = dd^c \log \tau \quad \text{on } M_*, \qquad \sigma = d^c \log \tau \wedge \omega^{m-1} \quad \text{on } M_*.$$

Then τ is said to be a *parabolic exhaustion* and (M, τ) a *parabolic manifold* if and only if τ is unbounded, $M[r]$ is compact for all $r \ge 0$ and

$$\omega \ge 0, \qquad d\sigma = \omega^m \equiv 0, \qquad \upsilon^m \not\equiv 0 \quad \text{on } M_*.$$

Then $\upsilon \ge 0$ on M. Define $\hat{\mathbb{R}}_\tau = \{r \in \mathbb{R}^+ \mid d\tau(x) \ne 0 \text{ for all } x \in M\langle r\rangle\}$. Then $\mathbb{R}^+ \setminus \hat{\mathbb{R}}_\tau$ has measure zero. If $r \in \hat{\mathbb{R}}_\tau$, then $M\langle r\rangle$ is the boundary of $M(r)$ and $M\langle r\rangle$ is a differentiable, $(2m-1)$-dimensional submanifold of class C^∞ which we orient to the exterior of $M(r)$.

For all $r \in \hat{\mathbb{R}}_\tau$, $\int_{M\langle r\rangle} \sigma$ is a positive constant, independent of r (cf. p. 133 of [20]). Let $\kappa = \int_{M\langle r\rangle} \sigma$. In addition,

$$\int_{M[r]} \upsilon^m = \int_{M(r)} \upsilon^m = \kappa r^{2m}.$$

2.2. Divisor

A function $\nu : M \to \mathbb{Z}$ is called a *divisor* if for each point x in M there exists a connected, open neighborhood U of x and if there exist holomorphic functions $g \not\equiv 0$ and $h \not\equiv 0$ on U with $\nu|_U = \mu_g^0 - \mu_h^0$, where $\mu_g^0(x)$ and $\mu_h^0(x)$ denote the zero multiplicities of g and h at $x \in U$, respectively. If $\nu \equiv 0$, then ν is called the *null divisor*. For $\nu \not\equiv 0$, its support $S = \operatorname{supp} \nu$ is an analytic subset of M of pure dimension

$m-1$; if $\nu \equiv 0$, then $\operatorname{supp}\nu$ is empty. A divisor ν is nonnegative as a function if and only if for every x in M, there exist a connected open neighborhood U of x and a holomorphic function $g \not\equiv 0$ on U such that $\nu|_U = \mu_g^0$.

Let $f \not\equiv 0$ be a meromorphic function on M. For each $x \in M$, on a connected, open neighborhood U of x, there exist holomorphic functions $g \not\equiv 0$ and $h \not\equiv 0$ such that $f = \frac{g}{h}$ on U with $\dim g^{-1}(0) \cap h^{-1}(0) \leq m-2$. Then the *zero divisor* $\mu_f^0 (\geq 0)$ is defined by $\mu_f^0|_U = \mu_g^0$ and the *pole divisor* $\mu_f^\infty (\geq 0)$ is defined by $\mu_f^\infty|_U = \mu_h^0$. The *divisor* of f is given by $\mu_f|_U = \mu_g^0 - \mu_h^0$.

Let ν be a divisor on M with $S = \operatorname{supp}\nu$. The *counting function* of ν is defined to be

$$N_\nu(r,s) = \int_s^r n_\nu(t)\frac{dt}{t},$$

where

$$n_\nu(t) = t^{2-2m}\int_{S[t]} \nu \upsilon^{m-1}, \quad \text{if } m > 1,$$

$$n_\nu(t) = \sum_{z \in S[t]} \nu(z), \quad \text{if } m = 1.$$

Let $f \not\equiv 0$ be a meromorphic function. For r, s in $\hat{\mathbb{R}}_\tau$ with $0 < s < r$, *Jensen's formula* holds:

$$N_{\mu_f}(r,s) = \int_{M\langle r\rangle} \log|f|\sigma - \int_{M\langle s\rangle} \log|f|\sigma.$$

2.3. Meromorphic maps, reduced representation

Let M be a complex manifold with $\dim M = m$. Let $A \neq \emptyset$ be an open subset of M such that $S = M - A$ is analytic. Then A is dense in M. Let V be a complex vector space with dimension $n+1 > 1$. Let $f : A \to \mathbb{P}(V)$ be a holomorphic map on A. The closure Γ of the graph $\{(x, f(x)) \mid x \in A\}$ in $M \times \mathbb{P}(V)$ is called the *closed graph* of f. The map f is said to be *meromorphic* on M if (i) $\Gamma(f)$ is analytic in $M \times \mathbb{P}(V)$ and (ii) $\Gamma \cap (K \times \mathbb{P}(V))$ is compact for each compact subset $K \subseteq M$, i.e., the projection $\rho : \Gamma(f) \to M$ is proper. If f is meromorphic, then the set of *indeterminacy* $I_f = \{x \in M \mid \sharp\rho^{-1}(x) > 1\}$ is analytic with $\dim I_f \leq m-2$ and is contained in S. The holomorphic map $f : A \to \mathbb{P}(V)$ continues to a holomorphic map $f : M - I_f \to \mathbb{P}(V)$ such that we can assume that $S = I_f$. So a meromorphic function on M is a meromorphic map $f : M \to \mathbb{P}^1(\mathbb{C})$ that is not identically ∞.

Suppose that $f : A \to \mathbb{P}(V)$ is a holomorphic map as above. Also, suppose that U is a nonempty, open, connected subset of M. A holomorphic map $F : U \to V$ is called a *representation* of f on U if $F \not\equiv 0$ and if $f(x) = \mathbb{P}(F(x))$ for all $x \in U \cap A$ such that $F(x) \neq 0$. The representation is called *reduced* if $\dim F^{-1}(0) \leq m-2$. If $F : U \to V$ is a reduced representation, then $U \cap I_f = F^{-1}(0)$. Also, f is meromorphic if and only if for every point $x \in M$, there is a representation $F : U \to V$ of f with $x \in U$.

Next, $F : U \to V$ is said to be a *meromorphic representation* of a meromorphic map f if for all $x \in U$, there exist an open, connected neighborhood $U_x \subseteq U$ of x, a holomorphic function $0 \not\equiv h : U_x \to \mathbb{C}$, and a representation F' such that

$$F' = hF \quad \text{or} \quad F = \frac{F'}{h}.$$

Now, if $F : U \to V$ and $F' : U \to V$ are two meromorphic representations of f, then there exists a meromorphic function $h \not\equiv 0$ on U such that

$$F = hF'.$$

If F is a representation and F' is a reduced representation of f, then there exists a holomorphic function $h \not\equiv 0$ on U such that

$$F = hF'.$$

If F and F' are both reduced representations of f, then the above h will be nowhere zero.

Remark 2.1. There may be no global representation of f on M. However, if $M = \mathbb{C}^m$ and $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ is a meromorphic map, then f has a global reduced representation $F = (f_0, \ldots, f_n)$.

The *characteristic function* of a meromorphic map $f : M \to \mathbb{P}(V)$ is defined by

$$T_f(r, s) = \int_s^r \frac{dt}{t^{2m-1}} \int_{M[t]} f^*(\Omega) \wedge v^{m-1} (\geq 0) \quad \text{for } 0 < s < r,$$

where Ω is the Fubini–Study form on $\mathbb{P}(V)$.

2.4. Projective distance

Suppose that $f : M \to \mathbb{P}(V)$ and $g : M \to \mathbb{P}(V^*)$ are meromorphic maps. Let U be an open, connected subset of M. Let $F : U \to V$ be a reduced representation of f and $G : U \to V^*$ be a reduced representation of g. Let $\{v_0, \ldots, v_n\}$ be an orthonormal basis of V, and let $\{v_0^*, \ldots, v_n^*\}$ be the dual basis.

Take $a \in V^*$ and $b \in V^* \setminus \{0\}$, there is a unique meromorphic function $f_{a,b}$ called a *coordinate function* on M such that

$$f_{a,b}|_U = \frac{\langle F, a\rangle}{\langle F, b\rangle},$$

if $\langle F, b\rangle \not\equiv 0$.

Define

$$F \llcorner G = \sum_{i=0}^n \langle F, v_i\rangle \langle G, v_i^*\rangle,$$

$$\|F\|_1 = \left(\sum_{i=0}^n \left|\langle F, v_i\rangle\right|^2\right)^{1/2} \quad \Bigl(\text{or} \quad \|F\|_2 = \max_{0\leq i\leq n} \left|\langle F, v_i\rangle\right|\Bigr),$$

$$\|G\|_1 = \left(\sum_{i=0}^n \left|\langle G, v_i^*\rangle\right|^2\right)^{1/2} \quad \Bigl(\text{or} \quad \|G\|_2 = \max_{0\leq i\leq n} \left|\langle G, v_i^*\rangle\right|\Bigr).$$

We note that, by $\|\cdot\|_2 \leq \|\cdot\|_1 \leq (n+1)^{1/2}\|\cdot\|_2$, these two norms are equivalent.

Then the *projective distance* between f and g is defined by

$$\|f; g\||_U = \frac{|F \llcorner G|}{\|F\|\|G\|}.$$

Note that $\|f; g\|$ is a global function on M.

Now, (f, g) is called *free* if and only if there exist representations $F : U \to V$ of f and $G : U \to V^*$ of g such that $F \llcorner G \not\equiv 0$. Suppose that (f, g) is free. Define the *intersection divisor* of f and g by

$$\mu_{f,g}|_U = \mu_{F \llcorner G},$$

which is well-defined. The counting function for the intersection divisor of f and g is given by

$$N_{f,g}(r, s) = N_{\mu_{f,g}}(r, s)$$

and the *truncated counting function* (by a positive integer L) for the intersection divisor of f and g is

$$N^{(L)}_{f,g}(r, s) = N_{\mu^{(L)}_{f,g}}(r, s),$$

where $\mu^{(L)}_{f,g}(x) = \min\{L, \mu_{f,g}(x)\}$ for $x \in M$. For $r \in \hat{\mathbb{R}}_\tau$, define

$$m_{f,g}(r) = \int\limits_{M\langle r\rangle} \log \frac{1}{\|f; g\|} \sigma.$$

2.5. First main theorem for moving targets

For r, s in $\hat{\mathbb{R}}_\tau$ with $0 < s < r$, we have

$$T_f(r, s) + T_g(r, s) = N_{f,g}(r, s) + m_{f,g}(r) - m_{f,g}(s). \tag{2}$$

Since T_f, T_g and $N_{f,g}$ are all continuous in r and s, then $m_{f,g}$ will extend to a continuous function on $\mathbb{R}^+$ such that (2) holds for all r, s in $\mathbb{R}$ with $0 < s < r$. (See (1.2) of [1].)

2.6. Second main theorem for fixed targets

Let (M, τ) be an admissible parabolic manifold of dimension m. Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map which is linearly nondegenerate over $\mathbb{C}$. Let $\{H_j\}_{j=1}^q$ be a family of fixed targets in $\mathbb{P}^n(\mathbb{C}^*)$ located in general position. Then, for $s > 0$ and for $\varepsilon > 0$,

$$\begin{aligned}(q - n - 1)T_f(r, s) \mathrel{.\le.} &\sum_{j=1}^{q} N^{(n)}_{f,H_j}(r, s) + \frac{1}{2} n(n + 1) \operatorname{Ric}_\tau(r, s) + \varepsilon \log r \\ &+ n(n + 1)\kappa(1 + \varepsilon)^2 \big(\log T_f(r, s) + \log Y(r) + \log^+ \operatorname{Ric}_\tau(r, s)\big).\end{aligned}$$

(Cf. [20].)

3. Truncated second main theorem for moving targets

In [17], Ru and Wang proved a truncated second main theorem for meromorphic maps from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ intersecting a finite set of moving targets, in which the set of moving targets is assumed to be nondegenerate (see the definition below). Previously only general position or subgeneral position was considered.

Let g_j be a meromorphic map from $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C}^*)$ with reduced representation $G_j = (g_{j0}, \dots, g_{jn})$ for $1 \le j \le q$. For each j, there exists $\bar{j}$ with $0 \le \bar{j} \le n$ such that $g_{j\bar{j}} \not\equiv 0$, and put $\widetilde{G}_j = (\widetilde{g}_{j0}, \widetilde{g}_{j1}, \dots, \widetilde{g}_{jn})$ with $\widetilde{g}_{ji} = g_{ji}/g_{j\bar{j}}$ for $0 \le i \le n$. Denote by $\mathcal{M}$ the field of all meromorphic functions on $\mathbb{C}^m$.

Definition 3.1. The family $\mathcal{G} = \{G_1, \ldots, G_q\}$ is said to be *nondegenerate over* $\mathcal{M}$ if $\dim(\mathcal{G})_{\mathcal{M}} = n + 1$ and for each nonempty proper subset $\mathcal{G}_1$ of $\mathcal{G}$,

$$(\mathcal{G}_1)_{\mathcal{M}} \cap (\mathcal{G} \setminus \mathcal{G}_1)_{\mathcal{M}} \cap \mathcal{G} \neq \emptyset,$$

where $(\mathcal{G})_{\mathcal{M}}$ is the linear span of $\mathcal{G}$ over the field $\mathcal{M}$.

Denote $\widetilde{\mathcal{G}} = \{\widetilde{G}_1, \ldots, \widetilde{G}_q\}$ and $\mathcal{R}_{\mathcal{G}}$ the smallest subfield of $\mathcal{M}$ which contains $\mathbb{C}$ and all $\widetilde{g}_{ji}$ for all i, j.

Definition 3.2. The family $\widetilde{\mathcal{G}}$ is said to be *nondegenerate over* $\mathcal{R}_{\mathcal{G}}$ if $\dim(\widetilde{\mathcal{G}})_{\mathcal{R}_{\mathcal{G}}} = n+1$ and for each nonempty proper subset $\widetilde{\mathcal{G}}_1$ of $\widetilde{\mathcal{G}}$,

$$(\widetilde{\mathcal{G}}_1)_{\mathcal{R}_{\mathcal{G}}} \cap (\widetilde{\mathcal{G}} \setminus \widetilde{\mathcal{G}}_1)_{\mathcal{R}_{\mathcal{G}}} \cap \widetilde{\mathcal{G}} \neq \emptyset,$$

where $(\widetilde{\mathcal{G}})_{\mathcal{R}_{\mathcal{G}}}$ is the linear span of $\widetilde{\mathcal{G}}$ over the field $\mathcal{R}_{\mathcal{G}}$.

It is easy to show that if the family $\mathcal{G}$ is nondegenerate over $\mathcal{M}$, then $\widetilde{\mathcal{G}}$ is nondegenerate over $\mathcal{R}_{\mathcal{G}}$. In 2003, Ru and Wang [17] proved the following second main theorem.

Theorem F. *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a nonconstant meromorphic map. Let $g_1, \ldots, g_q$ be q meromorphic maps of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C}^*)$ such that (f, g_j) is free for $1 \leq j \leq q$. Assume that $\mathcal{G}$ is nondegenerate over $\mathcal{M}$. Then*

$$T_f(r,s) \mathrel{.\!\leq\!.} n \sum_{j=1}^{q} N^{(n)}_{f,g_j}(r,s) + O\Big(\max_{1\leq j\leq q} T_{g_j}(r,s)\Big) + o\big(T_f(r,s)\big).$$

In [7], Do and Si improved Theorem F as follows.

Theorem G. *Under the same assumptions as in Theorem F,*

$$T_f(r,s) \mathrel{.\!\leq\!.} \sum_{j=1}^{q} N^{(n)}_{f,g_j}(r,s) + O\Big(\max_{1\leq j\leq q} T_{g_j}(r,s)\Big) + o\big(T_f(r,s)\big).$$

In this section, we will extend Theorem G to meromorphic maps on parabolic manifolds. Unfortunately, the proof of Theorem G relies heavily on the lemma of logarithmic derivative. We note that, in [11], Liu proved the counterpart of Theorem G in Diophantine approximation, namely, a Schmidt's type theorem. Hence, we will use the technique shown in [11] to avoid using the logarithmic derivative lemma.

Firstly, we generalize the definition of nondegenerate to moving targets on parabolic manifolds.

Let $\mathcal{G} = \{g_1, \ldots, g_q\}$ be a family of q target meromorphic maps from M into $\mathbb{P}(V^*)$. For each j, let G_j be a reduced representation of g_j on U and $\bar{j}$ with $0 \leq \bar{j} \leq n$ such that $\langle G_j, v^*_{\bar{j}}\rangle \not\equiv 0$. Then $\widetilde{G}_j := \sum_{i=0}^{n} \frac{\langle G_j, v^*_i\rangle}{\langle G_j, v^*_{\bar{j}}\rangle} v^*_i$ is a global meromorphic representation of g_j. Denote $\widetilde{\mathcal{G}} = \{\widetilde{G}_1, \ldots, \widetilde{G}_q\}$. We note that $\frac{\langle G_j, v^*_i\rangle}{\langle G_j, v^*_{\bar{j}}\rangle}$ is meromorphic on M for $i = 0, \ldots, n$ and $j = 1, \ldots, q$. Denote by $\mathcal{R}_{\mathcal{G}}$ the smallest subfield containing $\mathbb{C}$ and all meromorphic functions $\frac{\langle G_j, v^*_i\rangle}{\langle G_j, v^*_{\bar{j}}\rangle}$ for all i, j.

Definition 3.3. The family $\widetilde{\mathcal{G}}$ is said to be *nondegenerate over* $\mathcal{R}_{\mathcal{G}}$ if $\dim(\widetilde{\mathcal{G}})_{\mathcal{R}_{\mathcal{G}}} = n+1$ and for each nonempty proper subset $\widetilde{\mathcal{G}}_1$ of $\widetilde{\mathcal{G}}$,

$$(\widetilde{\mathcal{G}}_1)_{\mathcal{R}_{\mathcal{G}}} \cap (\widetilde{\mathcal{G}} \setminus \widetilde{\mathcal{G}}_1)_{\mathcal{R}_{\mathcal{G}}} \cap \widetilde{\mathcal{G}} \neq \emptyset.$$

We have the following truncated second main theorem.

Theorem 3.1. *Let (M,τ) be an admissible parabolic manifold with dimension m. Let V be a Hermitian vector space with $\dim V = n+1 > 1$. Let $g_1,\dots,g_q : M \to \mathbb{P}(V^*)$ be q meromorphic maps such that $\widetilde{\mathcal{G}}$ is nondegenerate over $\mathcal{R}_{\mathcal{G}}$. Let $f : M \to \mathbb{P}(V)$ be a nonconstant meromorphic map such that $\mathrm{Ric}_\tau(r,s) = o(T_f(r,s))$ and $\log Y(r) = o(T_f(r,s))$ for $r \to \infty$. Assume that (f,g_j) is free for $1 \le j \le q$ and $\dim(\operatorname{supp}\mu_{f,g_i} \cap \operatorname{supp}\mu_{f,g_j}) \le m-2$ for $1 \le i < j \le q$. Then, for $s > 0$, we have*

$$T_f(r,s) \mathrel{.\le.} \sum_{j=1}^{q} N_{f,g_j}^{(n)}(r,s) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big) + o\big(T_f(r,s)\big).$$

We first prove the following two lemmas.

Let F and G_j, $1 \le j \le q$, be the reduced representations of f and g_j, $1 \le j \le q$, on U. Consider meromorphic functions (on U) $F \llcorner \widetilde{G}_j := \sum_{i=0}^{n} \frac{\langle G_j, v_i^*\rangle}{\langle G_j, v_j^*\rangle}\langle F, v_i\rangle = \frac{F\llcorner G_j}{\langle G_j, v_j^*\rangle}$, $1 \le j \le q$.

Lemma 3.1. *(Cf. [7,11].) Assume that $\widetilde{\mathcal{G}}$ is nondegenerate over $\mathcal{R}_{\mathcal{G}}$. There exist an integer $u \ge 1$ and subsets $I_1,\dots,I_u$ of $\widetilde{\mathcal{G}}$ with the following properties:*

(a) *$\{F\llcorner\widetilde{G}_j\}_{\widetilde{G}_j\in I_1}$ is minimal and $\{F\llcorner\widetilde{G}_j\}_{\widetilde{G}_j\in I_k}$ is linearly independent over $\mathcal{R}_{\mathcal{G}}$ for $2 \le k \le u$, where $\{F\llcorner\widetilde{G}_j\}_{\widetilde{G}_j\in I_1}$ is minimal means $\{F\llcorner\widetilde{G}_j\}_{\widetilde{G}_j\in I_1}$ is linearly dependent over $\mathcal{R}_{\mathcal{G}}$ but each nonempty proper subset of $\{F\llcorner\widetilde{G}_j\}_{\widetilde{G}_j\in I_1}$ is linearly independent over $\mathcal{R}_{\mathcal{G}}$.*

(b) *u is the minimal positive integer such that*

$$\left(\bigcup_{k=1}^{u}\{F\llcorner\widetilde{G}_j\}_{\widetilde{G}_j\in I_k}\right)_{\mathcal{R}_{\mathcal{G}}} = \big(\{F\llcorner\widetilde{G}_j\}_{\widetilde{G}_j\in\widetilde{\mathcal{G}}}\big)_{\mathcal{R}_{\mathcal{G}}}.$$

(c) *For each l with $2 \le l \le u$, there exist nonzero meromorphic functions $c_j \in \mathcal{R}_{\mathcal{G}} \setminus \{0\}$ such that*

$$\sum_{\widetilde{G}_j\in I_l} c_j F\llcorner\widetilde{G}_j \in \left(\bigcup_{k=1}^{l-1}\{F\llcorner\widetilde{G}_j\}_{\widetilde{G}_j\in I_k}\right)_{\mathcal{R}_{\mathcal{G}}}.$$

Proof. Since $\widetilde{G}_1 \in (\widetilde{\mathcal{G}} \setminus \{\widetilde{G}_1\})_{\mathcal{R}_{\mathcal{G}}}$, this implies that

$$F\llcorner\widetilde{G}_1 \in \big(\{F\llcorner\widetilde{G}_j\}_{j=2}^{q}\big)_{\mathcal{R}_{\mathcal{G}}}.$$

Choose a subset I_1 of $\widetilde{\mathcal{G}}$ containing $\widetilde{G}_1$ such that $\{F\llcorner\widetilde{G}_j\}_{\widetilde{G}_j\in I_1}$ is minimal. Assume that $I_1 = \{\widetilde{G}_1,\dots,\widetilde{G}_{t_1}\}$. Then there exist meromorphic functions $c_j \in \mathcal{R}_{\mathcal{G}} \setminus \{0\}$, $1 \le j \le t_1 - 1$, and $c_{t_1} = -1$ such that

$$\sum_{j=1}^{t_1} c_j F\llcorner\widetilde{G}_j = 0.$$

If $(\{F\llcorner\widetilde{G}_j\}_{\widetilde{G}_j\in I_1})_{\mathcal{R}_{\mathcal{G}}} = (\{F\llcorner\widetilde{G}_j\}_{\widetilde{G}_j\in\widetilde{\mathcal{G}}})_{\mathcal{R}_{\mathcal{G}}}$, by taking $u = 1$, then the proof is finished.

Otherwise, there exists $\widetilde{G} \in \widetilde{\mathcal{G}}$ such that $\widetilde{G} \in (I_1)_{\mathcal{R}_{\mathcal{G}}} \cap (\widetilde{\mathcal{G}} \setminus I_1)_{\mathcal{R}_{\mathcal{G}}}$. One of the following two cases holds:

(i) $\widetilde{G} \in \widetilde{\mathcal{G}} \setminus I_1$, we may assume that $\widetilde{G} = \widetilde{G}_{t_1+1} \in (I_1)_{\mathcal{R}_{\mathcal{G}}}$, i.e.,

$$F\llcorner\widetilde{G}_{t_1+1} \in \big(\{F\llcorner\widetilde{G}_j\}_{\widetilde{G}_j\in I_1}\big)_{\mathcal{R}_{\mathcal{G}}}.$$

Put $I_2 = \{\widetilde{G}_{t_1+1}\}$ and $c_{t_1+1} = 1$.

(ii) $\widetilde{G} \in I_1$, we may assume that $\widetilde{G} = \widetilde{G}_{t_1} \in (\widetilde{\mathcal{G}} \setminus I_1)_{\mathcal{R}_{\mathcal{G}}}$. Then there exists a subset of $\widetilde{\mathcal{G}} \setminus I_1$, we may assume that it is $\{\widetilde{G}_{t_1+1}, \ldots, \widetilde{G}_{t_2}\}$, and $c_j \in \mathcal{R}_{\mathcal{G}} \setminus \{0\}$, $t_1 + 1 \le j \le t_2$ such that

$$F \llcorner \widetilde{G}_{t_1} = \sum_{j=t_1+1}^{t_2} c_j F \llcorner \widetilde{G}_j$$

and $\{F \llcorner \widetilde{G}_j\}_{j=t_1+1}^{t_2}$ is independent over $\mathcal{R}_{\mathcal{G}}$. Set $I_2 = \{\widetilde{G}_{t_1+1}, \ldots, \widetilde{G}_{t_2}\}$.

If $(\{F \llcorner \widetilde{G}_j\}_{\widetilde{G}_j \in I_1 \cup I_2})_{\mathcal{R}_{\mathcal{G}}} = (\{F \llcorner \widetilde{G}_j\}_{\widetilde{G}_j \in \widetilde{\mathcal{G}}})_{\mathcal{R}_{\mathcal{G}}}$, then the proof is finished; otherwise, repeating the above argument, we would get another subset I_3. By continuing this process, since $\dim(\widetilde{\mathcal{G}})_{\mathcal{R}_{\mathcal{G}}} = n + 1$ is finite, there exist a u and subsets $I_1, \ldots, I_u$ satisfying the assertions (a)–(c) of the lemma. □

Remark 3.1. 1) Obviously, the construction of $I_1, \ldots, I_u$ and $\{c_j\}$ is independent of the choice of the reduced representation of f.

2) We may assume that the cardinality of I_l satisfies $\sharp I_1 \ge 3$ and $\sharp I_l \ge 2$ for $2 \le l \le u$.

(a) If $\sharp I_l = 1$, for some l with $2 \le l \le u$, i.e., $I_l = \{\widetilde{G}_{t_l}\}$, $\widetilde{G}_{t_l} \in \widetilde{\mathcal{G}}$, then $F \llcorner \widetilde{G}_{t_l} \in (\bigcup_{k=1}^{l-1} \{F \llcorner \widetilde{G}_j\}_{\widetilde{G}_j \in I_k})_{\mathcal{R}_{\mathcal{G}}}$, so

$$\left(\bigcup_{k=1}^{l} \{F \llcorner \widetilde{G}_j\}_{\widetilde{G}_j \in I_k} \right)_{\mathcal{R}_{\mathcal{G}}} = \left(\bigcup_{k=1}^{l-1} \{F \llcorner \widetilde{G}_j\}_{\widetilde{G}_j \in I_k} \right)_{\mathcal{R}_{\mathcal{G}}},$$

we can always delete I_l from $\{I_1, \ldots, I_u\}$. Hence $\sharp I_l \ge 2$ for any $l \in \{1, \ldots, u\}$.

(b) If $\sharp I_1 = 2$ and $(F \llcorner \widetilde{G}_1)_{\mathcal{R}_{\mathcal{G}}} = (\{F \llcorner \widetilde{G}_j\}_{\widetilde{G}_j \in \widetilde{\mathcal{G}}})_{\mathcal{R}_{\mathcal{G}}}$, then there exists $c_{\alpha,\beta} \in \mathcal{R}_{\mathcal{G}}$ such that

$$F \llcorner \widetilde{G}_\alpha = c_{\alpha,\beta} F \llcorner \widetilde{G}_\beta \tag{3}$$

for any $1 \le \alpha < \beta \le q$. Since $\dim(\widetilde{\mathcal{G}})_{\mathcal{R}_{\mathcal{G}}} = n + 1$, for any $a \in V^*$ and $b \in V^* \setminus \{0\}$, (3) implies the coordinate function $f_{a,b} \in \mathcal{R}_{\mathcal{G}}$. Hence, $T_f(r, s) \le O(\max_{1 \le j \le q} T_{g_j}(r, s))$ which implies Theorem 3.1. Otherwise, we replace I_1 by $\{\widetilde{G}_1\} \cup I_2$ which is minimal.

For an integer N with $1 \le N \le u$, put $I := \bigcup_{k=1}^{N} I_k = \{\widetilde{G}_1, \ldots, \widetilde{G}_{t_N}\}$, $\sharp I = t_N$.

For a reduced representation $F : U \to V$ of f and reduced representation $G_j : U \to V^*$ of g_j, the map $f_I = \mathbb{P}(F_I) : M \to \mathbb{P}^{\sharp I - 1}(\mathbb{C})$ is defined by

$$F_I|_U = (h_I F \llcorner \widetilde{G}_1, \ldots, h_I F \llcorner \widetilde{G}_{t_N}),$$

where h_I is a holomorphic function on U such that $(h_I F \llcorner \widetilde{G}_1, \ldots, h_I F \llcorner \widetilde{G}_{t_N})$ is a reduced representation of f_I on U.

Lemma 3.2. *Let $I = \bigcup_{k=1}^{N} I_k$ with $1 \le N \le u$. Then for $s > 0$, we have*

$$T_{f_I}(r, s) \mathrel{.\le.} \sum_{j=1}^{t_N} N_{f,g_j}^{(n)}(r, s) + O\Big(\max_{1 \le j \le q} T_{g_j}(r, s) \Big) + o\big(T_f(r, s)\big).$$

Proof. We now prove lemma by induction on N.

If $N = 1$, then $I = I_1 = \{\widetilde{G}_1, \dots, \widetilde{G}_{t_1}\} := I_0 \cup \{\widetilde{G}_{t_1}\}$. Since $\{F \llcorner \widetilde{G}_j\}_{\widetilde{G}_j \in I_1}$ is minimal, by the construction in Lemma 3.1,

$$F \llcorner \widetilde{G}_{t_1} = \sum_{j=1}^{t_1-1} c_j F \llcorner \widetilde{G}_j, \quad c_j \in \mathcal{R}_{\mathcal{G}}.$$

Take $f'_{I_0} : M \to \mathbb{P}^{t_1-2}(\mathbb{C})$ to be the meromorphic map with reduced representation

$$F'_{I_0}\big|_U = \big(h'_{I_0} c_1 F \llcorner \widetilde{G}_1, \dots, h'_{I_0} c_{t_1-1} F \llcorner \widetilde{G}_{t_1-1}\big),$$

where h'_{I_0} is a meromorphic function on U. It is easy to see that f'_{I_0} is linearly nondegenerate over $\mathbb{C}$. By the assumption $\dim(\operatorname{supp} \mu_{f,g_\alpha} \cap \operatorname{supp} \mu_{f,g_\beta}) \le m-2$ for any $\alpha \neq \beta$,

$$\mu^0_{h'_{I_0}} \le \sum_{j=1}^{t_1-1} \mu^0_{\langle G_j, v^*_j\rangle} + \sum_{j=1}^{t_1-1} \mu^\infty_{c_j} \quad \text{and} \quad \mu^\infty_{h'_{I_0}} \le \sum_{j=1}^{t_1-1} \mu^0_{c_j}.$$

Set $e_{I_0,l} = (0, \dots, 0, \overset{l\text{-th}}{1}, 0, \dots, 0) \in \mathbb{C}^{\sharp I_0}$, $1 \le l \le t_1 - 1$. Then $\{e_{I_0,l}\}_{l=1}^{t_1-1}$ is the orthonormal basis of $\mathbb{C}^{\sharp I_0}$, and let $\{e^*_{I_0,l}\}_{l=1}^{t_1-1}$ be the dual basis. Take $H_{I_0,l} \in \mathbb{P}^{t_1-2}(\mathbb{C}^*)$ with representation $\widetilde{H}_{I_0,l} = e^*_{I_0,l}$ for $1 \le l \le t_1 - 1$ and $H_{I_0,t_1} \in \mathbb{P}^{t_1-2}(\mathbb{C}^*)$ with representation $\widetilde{H}_{I_0,t_1} = \sum_{l=1}^{t_1-1} e^*_{I_0,l}$. Obviously, $H_{I_0,1}, \dots, H_{I_0,t_1}$ are in general position. Note that $F'_{I_0} \llcorner \widetilde{H}_{I_0,l} = h'_{I_0} c_l F \llcorner \widetilde{G}_l$, $1 \le l \le t_1 - 1$, and $F'_{I_0} \llcorner \widetilde{H}_{I_0,t_1} = h'_{I_0} F \llcorner \widetilde{G}_{t_1}$.

By the second main theorem for fixed targets, we have

$$\begin{aligned} T_{f'_{I_0}}(r,s) &\mathrel{.\le.} \sum_{j=1}^{t_1} N^{(t_1-2)}_{f'_{I_0}, H_{I_0,j}}(r,s) + n(n+1)\kappa(1+\varepsilon)^2 \log T_{f'_{I_0}}(r,s) + o\big(T_f(r,s)\big) \\ &\mathrel{.\le.} \sum_{j=1}^{t_1} N^{(n)}_{f'_{I_0}, H_{I_0,j}}(r,s) + n(n+1)\kappa(1+\varepsilon)^2 \log T_{f'_{I_0}}(r,s) + o\big(T_f(r,s)\big). \end{aligned}$$

Together with $N^{(n)}_{f'_{I_0}, H_{I_0,j}}(r,s) \le N^{(n)}_{f,g_j}(r,s) + O(\max_{1 \le j \le q} T_{g_j}(r,s))$, $1 \le j \le t_1$,

$$T_{f'_{I_0}}(r,s) \mathrel{.\le.} \sum_{j=1}^{t_1} N^{(n)}_{f,g_j}(r,s) + n(n+1)\kappa(1+\varepsilon)^2 \log T_{f'_{I_0}}(r,s) + O\Big(\max_{1 \le j \le q} T_{g_j}(r,s)\Big) + o\big(T_f(r,s)\big). \tag{4}$$

On the other hand, by $|c_j F \llcorner \widetilde{G}_j| \le (n+1)|c_j| \max_{0 \le i \le n} |\frac{\langle G_j, v^*_i\rangle}{\langle G_j, v^*_j\rangle} \langle F, v_i\rangle|$ for $j = 1, \dots, t_1 - 1$, it follows that

$$\max_{1 \le j \le t_1-1} |c_j F \llcorner \widetilde{G}_j| \le (n+1) \max_{1 \le j \le t_1-1} |c_j| \cdot \max_{\substack{1 \le j \le t_1-1 \\ 0 \le i \le n}} \left| \frac{\langle G_j, v^*_i\rangle}{\langle G_j, v^*_j\rangle} \right| \cdot \max_{0 \le i \le n} \big|\langle F, v_i\rangle\big|$$

and

$$\begin{aligned} \big\| f'_{I_0}; H_{I_0,1} \big\| &= \frac{|c_1 F \llcorner \widetilde{G}_1|}{\max_{1 \le j \le t_1-1} |c_j F \llcorner \widetilde{G}_j|} \\ &\ge \frac{\|f; g_1\|}{n+1} \cdot \frac{|c_1|}{\max_{1 \le j \le t_1-1} |c_j|} \cdot \frac{\max_{0 \le i \le n} |\langle G_1, v^*_i\rangle|}{|\langle G_1, v^*_1\rangle|} \cdot \frac{1}{\max_{\substack{1 \le j \le t_1-1 \\ 0 \le i \le n}} |\frac{\langle G_j, v^*_i\rangle}{\langle G_j, v^*_j\rangle}|}, \end{aligned}$$

which means

$$m_{f'_{I_0},H_{I_0,1}}(r) \le m_{f,g_1}(r) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big).$$

Thus,

$$\begin{aligned} T_{f'_{I_0}}(r,s) &= m_{f'_{I_0},H_{I_0,1}}(r) + N_{f'_{I_0},H_{I_0,1}}(r,s) + O(1) \\ &\le m_{f,g_1}(r) + N_{f,g_1}(r,s) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big) \\ &= T_f(r,s) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big). \end{aligned} \tag{5}$$

(4) and (5) yield

$$T_{f'_{I_0}}(r,s) \;.\!\le.\; \sum_{j=1}^{t_1} N^{(n)}_{f,g_j}(r,s) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big) + o\big(T_f(r,s)\big). \tag{6}$$

Consider $f_{I_1}: M \to \mathbb{P}^{t_1-1}(\mathbb{C})$ with reduced representation

$$F_{I_1}|_U = (h_{I_1} F \llcorner \widetilde{G}_1, \ldots, h_{I_1} F \llcorner \widetilde{G}_{t_1}),$$

where h_{I_1} is a holomorphic function on U with $\mu^0_{h_{I_1}} \le \sum_{j=1}^{t_1} \mu^0_{\langle G_j, v_j^*\rangle}$.

We now compare $T_{f_{I_1}}(r,s)$ and $T_{f'_{I_0}}(r,s)$.

Take $H_{I_1,1} \in \mathbb{P}^{t_1-1}(\mathbb{C}^*)$ with representation $\widetilde{H}_{I_1,1} = e^*_{I_1,1}$. Then

$$\|f_{I_1}; H_{I_1,1}\| = \frac{|F_{I_1} \llcorner \widetilde{H}_{I_1,1}|}{\|F_{I_1}\|\|\widetilde{H}_{I_1,1}\|} = \frac{|F \llcorner \widetilde{G}_1|}{\max_{1\le j\le t_1} |F \llcorner \widetilde{G}_j|}$$

and $F_{I_1} \llcorner \widetilde{H}_{I_1,1} = h_{I_1} F \llcorner \widetilde{G}_1$. Since

$$|F \llcorner \widetilde{G}_j| = \frac{1}{|c_j|} |c_j F \llcorner \widetilde{G}_j|, \quad 1 \le j \le t_1 - 1,$$

and

$$|F \llcorner \widetilde{G}_{t_1}| = \Big| \sum_{\widetilde{G}_j \in I_0} c_j F \llcorner \widetilde{G}_j \Big| \le \sum_{\widetilde{G}_j \in I_0} |c_j F \llcorner \widetilde{G}_j|,$$

we have

$$\max_{\widetilde{G}_j \in I_1} |F \llcorner \widetilde{G}_j| \le \sharp I_0 \cdot \max\Big\{ \max_{\widetilde{G}_j \in I_0} |1/c_j|, 1 \Big\} \cdot \max_{\widetilde{G}_j \in I_0} |c_j F \llcorner \widetilde{G}_j|$$

which yields

$$\|f_{I_1}; H_{I_1,1}\| \le \frac{\|f'_{I_0}; H_{I_0,1}\|}{\sharp I_0} \cdot \frac{|1/c_1|}{\max\{\max_{\widetilde{G}_j \in I_0} |1/c_j|, 1\}}.$$

Therefore

$$
\begin{aligned}
m_{f_{I_1},H_{I_1,1}}(r) &\le m_{f'_{I_0},H_{I_0,1}}(r) + \int_{M\langle r\rangle} \log \frac{\max\{\max_{\widetilde{G}_j\in I_0}|1/c_j|,1\}}{|1/c_1|}\sigma + O(1) \\
&\le m_{f'_{I_0},H_{I_0,1}}(r) + \int_{M\langle r\rangle} \log \frac{\max_{\widetilde{G}_j\in I_0}|1/c_j|}{|1/c_1|}\sigma + \int_{M\langle r\rangle} \log^+|c_1|\sigma + O(1) \\
&\le m_{f'_{I_0},H_{I_0,1}}(r) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big).
\end{aligned}
$$

By $F_{I_1}\llcorner\widetilde{H}_{I_1,1} = \frac{h_{I_1}}{h'_{I_0}c_1}F'_{I_0}\llcorner\widetilde{H}_{I_0,1}$, we have

$$
N_{f_{I_1},H_{I_1,1}}(r,s) \le N_{f'_{I_0},H_{I_0,1}}(r,s) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big).
$$

Then, we obtain that

$$
\begin{aligned}
T_{f_{I_1}}(r,s) &= m_{f_{I_1},H_{I_1,1}}(r) + N_{f_{I_1},H_{I_1,1}}(r,s) + O(1) \\
&\le m_{f'_{I_0},H_{I_0,1}}(r) + N_{f'_{I_0},H_{I_0,1}}(r,s) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big) \\
&= T_{f'_{I_0}}(r,s) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big).
\end{aligned}
\tag{7}
$$

Thus, (6) and (7) imply

$$
T_{f_{I_1}}(r,s) \le T_{f'_{I_0}}(r,s) \;.\!\le.\; \sum_{j=1}^{t_1} N^{(n)}_{f,g_j}(r,s) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big) + o\big(T_f(r,s)\big).
$$

Lemma 3.2 is proved for $N = 1$.

Let us assume that Lemma 3.2 holds for some integer N with $1 \le N \le u$. If $N+1 > u$, then Lemma 3.2 is done by induction. So, it is enough to consider the case $N+1 \le u$. Let $I^c_{N+1} := \bigcup_{k=1}^{N} I_k$, $I := \bigcup_{k=1}^{N+1} I_k = I^c_{N+1} \cup I_{N+1}$.

First of all, by induction, we obtain

$$
T_{f_{I^c_{N+1}}}(r,s) \;.\!\le.\; \sum_{\widetilde{G}_j\in I^c_{N+1}} N^{(n)}_{f,g_j}(r,s) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big) + o\big(T_f(r,s)\big). \tag{8}
$$

On the other hand, by the proof of Lemma 3.1,

$$
F\llcorner\widetilde{G}_{t_N} = \sum_{j=t_N+1}^{t_{N+1}} c_j F\llcorner\widetilde{G}_j, \quad \text{where } \widetilde{G}_{t_N} \in I^c_{N+1},\ c_j \in \mathcal{R}_{\mathcal{G}}.
$$

Let $f'_{I_{N+1}} : M \to \mathbb{P}^{\sharp I_{N+1}-1}(\mathbb{C})$ be the meromorphic map with reduced representation $F'_{I_{N+1}}|_U = (h'_{I_{N+1}}c_{t_N+1}F\llcorner\widetilde{G}_{t_N+1}, \ldots, h'_{I_{N+1}}c_{t_{N+1}}F\llcorner\widetilde{G}_{t_{N+1}})$, where $h'_{I_{N+1}}$ is a meromorphic function on U with

$$
\mu^0_{h'_{I_{N+1}}} \le \sum_{j=t_N+1}^{t_{N+1}} \mu^0_{\langle G_j, v^*_j\rangle} + \sum_{j=t_N+1}^{t_{N+1}} \mu^\infty_{c_j} \quad\text{and}\quad \mu^\infty_{h'_{I_{N+1}}} \le \sum_{j=t_N+1}^{t_{N+1}} \mu^0_{c_j}.
$$

Obviously, $f'_{I_{N+1}}$ is linearly nondegenerate over $\mathbb{C}$ and, by repeating the argument as in the proof of (5), $T_{f'_{I_{N+1}}}(r,s) \le T_f(r,s) + O(\max_{1\le j\le q} T_{g_j}(r,s))$.

Let $\{e_{I_{N+1},l}\}_{l=1}^{\sharp I_{N+1}}$ be the orthonormal basis of $\mathbb{C}^{\sharp I_{N+1}}$, and $\{e^*_{I_{N+1},l}\}_{l=1}^{\sharp I_{N+1}}$ be the dual basis. We can take $H_{I_{N+1},l} \in \mathbb{P}^{\sharp I_{N+1}-1}(\mathbb{C}^*)$ with representation $\widetilde{H}_{I_{N+1},l} = e^*_{I_{N+1},l}$ for $1 \le l \le \sharp I_{N+1}$ and $H_{I_{N+1},\sharp I_{N+1}+1} \in \mathbb{P}^{\sharp I_{N+1}-1}(\mathbb{C}^*)$ with representation $\widetilde{H}_{I_{N+1},\sharp I_{N+1}+1} = \sum_{l=1}^{\sharp I_{N+1}} e^*_{I_{N+1},l}$, which are in general position. Note that

$$F'_{I_{N+1}} \llcorner \widetilde{H}_{I_{N+1},l} = h'_{I_{N+1}} c_{t_N+l} F \llcorner \widetilde{G}_{t_N+l}, \quad 1 \le l \le \sharp I_{N+1},$$

and

$$F'_{I_{N+1}} \llcorner \widetilde{H}_{I_{N+1},\sharp I_{N+1}+1} = h'_{I_{N+1}} F \llcorner \widetilde{G}_{t_N}.$$

By the second main theorem for fixed targets, we have

$$\begin{aligned} T_{f'_{I_{N+1}}}(r,s) &\mathrel{.\le.} \sum_{j=1}^{\sharp I_{N+1}} N^{(\sharp I_{N+1}-1)}_{f'_{I_{N+1}},H_{I_{N+1},j}}(r,s) + N^{(\sharp I_{N+1}-1)}_{f'_{I_{N+1}},H_{I_{N+1},\sharp I_{N+1}+1}}(r,s) + o\big(T_f(r,s)\big) \\ &\mathrel{.\le.} \sum_{\widetilde{G}_j \in I_{N+1}} N^{(n)}_{f,g_j}(r,s) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big) + o\big(T_f(r,s)\big) + N_{f'_{I_{N+1}},H_{I_{N+1},\sharp I_{N+1}+1}}(r,s). \end{aligned}$$

By the first main theorem, it follows that

$$m_{f'_{I_{N+1}},H_{I_{N+1},\sharp I_{N+1}+1}}(r) \mathrel{.\le.} \sum_{\widetilde{G}_j \in I_{N+1}} N^{(n)}_{f,g_j}(r,s) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big) + o\big(T_f(r,s)\big). \tag{9}$$

So, by (8) and (9),

$$T_{f_{I^c_{N+1}}}(r,s) + m_{f'_{I_{N+1}},H_{I_{N+1},\sharp I_{N+1}+1}}(r) \mathrel{.\le.} \sum_{\widetilde{G}_j \in I} N^{(n)}_{f,g_j}(r,s) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big) + o\big(T_f(r,s)\big). \tag{10}$$

Next, we show that

$$m_{f'_{I_{N+1}},H_{I_{N+1},\sharp I_{N+1}+1}}(r) + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big) \ge T_{f_I}(r,s) - T_{f_{I^c_{N+1}}}(r,s). \tag{11}$$

Claim.

$$T_{f_I}(r,s) - T_{f_{I^c_{N+1}}}(r,s) \le \int_{M\langle r\rangle} \log \frac{\max_{\widetilde{G}_j \in I} |F \llcorner \widetilde{G}_j|}{\max_{\widetilde{G}_j \in I^c_{N+1}} |F \llcorner \widetilde{G}_j|} \sigma + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big). \tag{12}$$

Proof of the claim. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open covering of M, and let $F_\lambda : U_\lambda \to V$ and $G_{j,\lambda} : U_\lambda \to V^*$ be the reduced representations of f and g_j on U_λ.

Let

$$F_{I,\lambda} := F_I|_{U_\lambda} = (\ldots, h_{I,\lambda} F_\lambda \llcorner \widetilde{G}_j, \ldots)_{\widetilde{G}_j \in I}$$

be the reduced representation of f_I on U_λ, where $h_{I,\lambda}$ is a holomorphic function on U_λ with $\mu^0_{h_{I,\lambda}} \le \sum_{\widetilde{G}_j \in I} \mu^0_{\langle G_{j,\lambda}, v^*_j\rangle}$. Then

$$\Phi(F_{I,\lambda}) := (\ldots, h_{I,\lambda} F_\lambda \llcorner \widetilde{G}_j, \ldots)_{\widetilde{G}_j \in I^c_{N+1}}$$

is a representation of $f_{I^c_{N+1}}$ on U_λ. ($\Phi(F_{I,\lambda})$ may not be a reduced representation.)

Since $F_{I,\lambda} : U_\lambda \to \mathbb{C}^{\sharp I}$ is a reduced representation of f_I on U_λ, there exists a holomorphic function $\mathbf{g}_{\lambda\mu} : U_\lambda \cap U_\mu \to \mathbb{C} \setminus \{0\}$ such that

$$F_{I,\lambda} = \mathbf{g}_{\lambda\mu} F_{I,\mu} \quad \text{on } U_\lambda \cap U_\mu.$$

Let L_{f_I} be the holomorphic line bundle on M with transition function $\{\mathbf{g}_{\lambda\mu}\}$. Let $\{U_\lambda, s_\lambda\}_{\lambda\in\Lambda}$ be the holomorphic frame such that

$$s_\mu = \mathbf{g}_{\lambda\mu} s_\lambda \quad \text{on } U_\lambda \cap U_\mu.$$

Define $\check{F}_{I,\lambda} \in \Gamma(U_\lambda, M \times \mathbb{C}^{\sharp I})$ by $\check{F}_{I,\lambda} = (z, F_{I,\lambda}(z))$ for $z \in U_\lambda$. By $\check{F}_{I,\lambda} \otimes s_\lambda = \mathbf{g}_{\lambda\mu}\check{F}_{I,\mu} \otimes s_\lambda = \check{F}_{I,\mu} \otimes \mathbf{g}_{\lambda\mu} s_\lambda = \check{F}_{I,\mu} \otimes s_\mu$ on $U_\lambda \cap U_\mu$, we can define a holomorphic section $F_{f_I} \in \Gamma((M \times \mathbb{C}^{\sharp I}) \otimes L_{f_I})$ by $F_{f_I}|_{U_\lambda} = \check{F}_{I,\lambda} \otimes s_\lambda$. Let ℓ be the standard Hermitian metric along the fibers of the trivial bundle $M \times \mathbb{C}^{\sharp I}$ and ρ be a Hermitian metric along the fibers of L_{f_I}. Then, by p. 140 of [21], we hold

$$T_{f_I}(r,s) = \int_s^r \frac{dt}{t^{2m-1}} \int_{M[t]} c_1(L_{f_I}, \rho) \wedge \upsilon^{m-1} + \int_{M\langle r\rangle} \log \|F_{f_I}\|_{\ell\otimes\rho}\sigma - \int_{M\langle s\rangle} \log \|F_{f_I}\|_{\ell\otimes\rho}\sigma.$$

On the other hand, we have

$$\Phi(F_{I,\lambda}) = \mathbf{g}_{\lambda\mu}\Phi(F_{I,\mu}) \quad \text{on } U_\lambda \cap U_\mu.$$

Define $\check{\Phi}(F_{I,\lambda}) \in \Gamma(U_\lambda, M \times \mathbb{C}^{\sharp I^c_{N+1}})$ by $\check{\Phi}(F_{I,\lambda}) = (z, \Phi(F_{I,\lambda})(z))$ for $z \in U_\lambda$. Since

$$\check{\Phi}(F_{I,\lambda}) \otimes s_\lambda = \mathbf{g}_{\lambda\mu}\check{\Phi}(F_{I,\mu}) \otimes s_\lambda = \check{\Phi}(F_{I,\mu}) \otimes \mathbf{g}_{\lambda\mu} s_\lambda = \check{\Phi}(F_{I,\mu}) \otimes s_\mu$$

on $U_\lambda \cap U_\mu$, there exists a holomorphic section $F_{f_{I^c_{N+1}}} \in \Gamma((M \times \mathbb{C}^{\sharp I^c_{N+1}}) \otimes L_{f_I})$ with $F_{f_{I^c_{N+1}}}|_{U_\lambda} = \check{\Phi}(F_{I,\lambda}) \otimes s_\lambda$. Then, $F_{f_{I^c_{N+1}}}$ is a representation section of $f_{I^c_{N+1}}$ because $\Phi(F_{I,\lambda})$ is a representation of $f_{I^c_{N+1}}$. We define a divisor

$$\mu_{F_{f_{I^c_{N+1}}}}|_{U_\lambda} = \mu_{\check{\Phi}(F_{I,\lambda})} = \mu_{\Phi(F_{I,\lambda})}.$$

Let $\frac{1}{h_\lambda}\Phi(F_{I,\lambda})$ be the reduced representation of $f_{I^c_{N+1}}$ on U_λ, where h_λ is a holomorphic function. By $\Phi(F_{I,\lambda}) = \mathbf{g}_{\lambda\mu}\Phi(F_{I,\mu})$ on $U_\lambda \cap U_\mu$, we have

$$\frac{1}{h_\lambda}\Phi(F_{I,\lambda}) = \frac{h_\mu}{h_\lambda}\mathbf{g}_{\lambda\mu} \cdot \frac{1}{h_\mu}\Phi(F_{I,\mu}) \quad \text{on } U_\lambda \cap U_\mu.$$

Since $\frac{1}{h_\lambda}\Phi(F_{I,\lambda})$ and $\frac{1}{h_\mu}\Phi(F_{I,\mu})$ are reduced, then $\frac{h_\mu}{h_\lambda}\mathbf{g}_{\lambda\mu}$ is a nowhere zero holomorphic function. Hence, $\frac{h_\mu}{h_\lambda}$ is also holomorphic without zeros. That means $\mu_{h_\mu} = \mu_{h_\lambda}$ on $U_\lambda \cap U_\mu$. Thus

$$\mu_{F_{f_{I^c_{N+1}}}}|_{U_\lambda} = \mu_{\check{\Phi}(F_{I,\lambda})} = \mu_{\Phi(F_{I,\lambda})} = \mu_{h_\lambda}|_{U_\lambda} \geq 0.$$

By p. 140 of [21], we have

$$\begin{aligned} T_{f_{I^c_{N+1}}}(r,s) = &\int_s^r \frac{dt}{t^{2m-1}} \int_{M[t]} c_1(L_{f_I}, \rho) \wedge \upsilon^{m-1} - N_{\mu_{\Phi(F_{I,\lambda})}}(r,s) \\ &+ \int_{M\langle r\rangle} \log \|F_{f_{I^c_{N+1}}}\|_{\ell\otimes\rho}\sigma - \int_{M\langle s\rangle} \log \|F_{f_{I^c_{N+1}}}\|_{\ell\otimes\rho}\sigma. \end{aligned}$$

We also note that, on U_λ,

$$\|F_{f_I}\|_{\ell\otimes\rho} = \|F_{I,\lambda}\| \cdot \|s_\lambda\|_\rho \quad \text{and} \quad \|F_{f_{I^c_{N+1}}}\|_{\ell\otimes\rho} = \big\|\Phi(F_{I,\lambda})\big\| \cdot \|s_\lambda\|_\rho,$$

where $\|F_{I,\lambda}\| = \max_{\widetilde{G}_j\in I} |h_{I,\lambda}F_\lambda \llcorner \widetilde{G}_j|$ and $\|\Phi(F_{I,\lambda})\| = \max_{\widetilde{G}_j\in I^c_{N+1}} |h_{I,\lambda}F_\lambda \llcorner \widetilde{G}_j|$.

Consequently,

$$\begin{aligned} T_{f_I}(r,s) - T_{f_{I^c_{N+1}}}(r,s) &= \int\limits_{M\langle r\rangle} \log \frac{\|F_{f_I}\|_{\ell\otimes\rho}}{\|F_{f_{I^c_{N+1}}}\|_{\ell\otimes\rho}}\sigma + N_{\mu_{\Phi(F_{I,\lambda})}}(r,s) + O(1) \\ &\le \int\limits_{M\langle r\rangle} \log \frac{\max_{\widetilde{G}_j\in I}|F_\lambda\llcorner\widetilde{G}_j|}{\max_{\widetilde{G}_j\in I^c_{N+1}}|F_\lambda\llcorner\widetilde{G}_j|}\sigma + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big), \end{aligned}$$

where the last inequality provided by $\mu_{h_\lambda} \le \sum_{\widetilde{G}_j\in I_{N+1}} \mu_{\langle G_{j,\lambda}, v_j^*\rangle}$, and

$$\frac{\max_{\widetilde{G}_j\in I}|F_\lambda\llcorner\widetilde{G}_j|}{\max_{\widetilde{G}_j\in I^c_{N+1}}|F_\lambda\llcorner\widetilde{G}_j|}$$

is a global function on M independent to λ. This finishes the proof of the claim. □

Now, it suffices to show

$$\int\limits_{M\langle r\rangle} \log \frac{\max_{\widetilde{G}_j\in I_{N+1}}|c_jF\llcorner\widetilde{G}_j|}{|F\llcorner\widetilde{G}_{t_N}|}\sigma + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big) \ge \int\limits_{M\langle r\rangle} \log \frac{\max_{\widetilde{G}_j\in I}|F\llcorner\widetilde{G}_j|}{\max_{\widetilde{G}_j\in I^c_{N+1}}|F\llcorner\widetilde{G}_j|}\sigma. \tag{13}$$

Firstly, we have

$$\max_{\widetilde{G}_j\in I_{N+1}} |F\llcorner\widetilde{G}_j| \le \max_{\widetilde{G}_j\in I_{N+1}} \frac{1}{|c_j|} \cdot \max_{\widetilde{G}_j\in I_{N+1}} |c_jF\llcorner\widetilde{G}_j|$$

or

$$\max_{\widetilde{G}_j\in I_{N+1}} |c_jF\llcorner\widetilde{G}_j| \ge \frac{|1/c_{t_{N+1}}|}{\max_{\widetilde{G}_j\in I_{N+1}}|1/c_j|} \cdot \max_{\widetilde{G}_j\in I_{N+1}} |F\llcorner\widetilde{G}_j| \cdot |c_{t_{N+1}}|.$$

Hence,

$$\int\limits_{M\langle r\rangle} \log \frac{\max_{\widetilde{G}_j\in I_{N+1}}|c_jF\llcorner\widetilde{G}_j|}{|F\llcorner\widetilde{G}_{t_N}|}\sigma + O\Big(\max_{1\le j\le q} T_{g_j}(r,s)\Big) \ge \int\limits_{M\langle r\rangle} \log \frac{\max_{\widetilde{G}_j\in I_{N+1}}|F\llcorner\widetilde{G}_j|}{|F\llcorner\widetilde{G}_{t_N}|}\sigma. \tag{14}$$

We estimate the last integration in (14). Set

$$M^1\langle r\rangle := \left\{ z\in M\langle r\rangle \;\middle|\; \frac{\max_{\widetilde{G}_j\in I}|F\llcorner\widetilde{G}_j|}{\max_{\widetilde{G}_j\in I^c_{N+1}}|F\llcorner\widetilde{G}_j|}(z) = 1 \right\}$$

and $M^2\langle r\rangle := M\langle r\rangle \setminus M^1\langle r\rangle$. It is easy to see that $M^1\langle r\rangle$ and $M^2\langle r\rangle$ are measurable sets. We see that

$$\frac{\max_{\widetilde{G}_j\in I}|F\llcorner\widetilde{G}_j|}{\max_{\widetilde{G}_j\in I^c_{N+1}}|F\llcorner\widetilde{G}_j|}(z)=\frac{\max_{\widetilde{G}_j\in I_{N+1}}|F\llcorner\widetilde{G}_j|}{\max_{\widetilde{G}_j\in I^c_{N+1}}|F\llcorner\widetilde{G}_j|}(z)\quad\text{for } z\in M^2\langle r\rangle. \tag{15}$$

On the other hand, by $F\llcorner\widetilde{G}_{t_N}=\sum_{\widetilde{G}_j\in I_{N+1}}c_jF\llcorner\widetilde{G}_j$, we have

$$|F\llcorner\widetilde{G}_{t_N}|=\sharp I_{N+1}\cdot\frac{\max_{t_N+1\le j\le t_{N+1}}|c_j|}{|c_{t_{N+1}}|}\cdot\max_{\widetilde{G}_j\in I_{N+1}}|F\llcorner\widetilde{G}_j|\cdot|c_{t_{N+1}}|. \tag{16}$$

Since $\widetilde{G}_{t_N}\in I^c_{N+1}$, we have

$$|F\llcorner\widetilde{G}_{t_N}|\le\max_{\widetilde{G}_j\in I^c_{N+1}}|F\llcorner\widetilde{G}_j|. \tag{17}$$

Hence, by using (16) for $z\in M^1\langle r\rangle$ and (17) for $z\in M^2\langle r\rangle$,

$$\begin{aligned}&\int\limits_{M\langle r\rangle}\log\frac{\max_{\widetilde{G}_j\in I_{N+1}}|F\llcorner\widetilde{G}_j|}{|F\llcorner\widetilde{G}_{t_N}|}\sigma+\int\limits_{M^1\langle r\rangle}\log\frac{\max_{t_N+1\le j\le t_{N+1}}|c_j|}{|c_{t_{N+1}}|}\sigma+\int\limits_{M^1\langle r\rangle}\log|c_{t_{N+1}}|\sigma\\&\ge\int\limits_{M^2\langle r\rangle}\log\frac{\max_{\widetilde{G}_j\in I_{N+1}}|F\llcorner\widetilde{G}_j|}{\max_{\widetilde{G}_j\in I^c_{N+1}}|F\llcorner\widetilde{G}_j|}\sigma+O(1)\\&=\int\limits_{M\langle r\rangle}\log\frac{\max_{\widetilde{G}_j\in I}|F\llcorner\widetilde{G}_j|}{\max_{\widetilde{G}_j\in I^c_{N+1}}|F\llcorner\widetilde{G}_j|}\sigma+O(1),\end{aligned} \tag{18}$$

where the last equation provided by (15). Together with

$$\begin{aligned}&\int\limits_{M^1\langle r\rangle}\log\frac{\max_{t_N+1\le j\le t_{N+1}}|c_j|}{|c_{t_{N+1}}|}\sigma+\int\limits_{M^1\langle r\rangle}\log|c_{t_{N+1}}|\sigma\\&\le\int\limits_{M\langle r\rangle}\log\frac{\max_{t_N+1\le j\le t_{N+1}}|c_j|}{|c_{t_{N+1}}|}\sigma+\int\limits_{M\langle r\rangle}\log^+|c_{t_{N+1}}|\sigma\le O\Big(\max_{1\le j\le q}T_{g_j}(r,s)\Big),\end{aligned}$$

(14) and (18) imply (13). Then (11) is derived from (12) and (13).

Hence, by (10) and (11), we have proved

$$T_{f_I}(r,s)\le\sum_{\widetilde{G}_j\in I}N^{(n)}_{f,g_j}(r,s)+O\Big(\max_{1\le j\le q}T_{g_j}(r,s)\Big)+o\big(T_f(r,s)\big).\qquad\square$$

Proof of Theorem 3.1. From Lemma 3.2, we can pick $N=u$, $I=\bigcup_{k=1}^u I_k$,

$$T_{f_I}(r,s)\ .\!\le\!.\ \sum_{\widetilde{G}_j\in I}N^{(n)}_{f,g_j}(r,s)+O\Big(\max_{1\le j\le q}T_{g_j}(r,s)\Big)+o\big(T_f(r,s)\big). \tag{19}$$

We now compare $T_{f_I}(r,s)$ and $T_f(r,s)$.

Let F be the reduced representation of f on U and

$$F_I|_U=(h_IF\llcorner\widetilde{G}_1,\dots,h_IF\llcorner\widetilde{G}_{t_u})$$

be the reduced representation of f_I.

By the fact $\dim(\widetilde{\mathcal{G}})_{\mathcal{R}_{\mathcal{G}}} = n+1$, there exist $n+1$ vectors of $\widetilde{\mathcal{G}}$, then we may assume they are $\widetilde{G}_{j_1}, \ldots, \widetilde{G}_{j_{n+1}}$, which are linearly independent. By solving the linear system

$$\begin{pmatrix} \frac{\langle G_{j_1}, v_0^*\rangle}{\langle G_{j_1}, v_{\overline{j_1}}^*\rangle} & \cdots & \frac{\langle G_{j_1}, v_n^*\rangle}{\langle G_{j_1}, v_{\overline{j_1}}^*\rangle} \\ \vdots & \vdots & \vdots \\ \frac{\langle G_{j_{n+1}}, v_0^*\rangle}{\langle G_{j_{n+1}}, v_{\overline{j_{n+1}}}^*\rangle} & \cdots & \frac{\langle G_{j_{n+1}}, v_n^*\rangle}{\langle G_{j_{n+1}}, v_{\overline{j_{n+1}}}^*\rangle} \end{pmatrix} \begin{pmatrix} \langle F, v_0\rangle \\ \vdots \\ \langle F, v_n\rangle \end{pmatrix} = \begin{pmatrix} F \llcorner \widetilde{G}_{j_1} \\ \vdots \\ F \llcorner \widetilde{G}_{j_{n+1}} \end{pmatrix},$$

we obtain

$$\langle F, v_i\rangle = c_{i,j_1} F \llcorner \widetilde{G}_{j_1} + \cdots + c_{i,j_{n+1}} F \llcorner \widetilde{G}_{j_{n+1}}, \quad 0 \le i \le n,$$

where $c_{i,j_k} \in \mathcal{R}_{\mathcal{G}}$. On the other hand, by (b) of Lemma 3.1, we have

$$\big(\{F \llcorner \widetilde{G}_j\}_{\widetilde{G}_j \in I}\big)_{\mathcal{R}_{\mathcal{G}}} = \big(\{F \llcorner \widetilde{G}_j\}_{\widetilde{G}_j \in \widetilde{\mathcal{G}}}\big)_{\mathcal{R}_{\mathcal{G}}},$$

which implies that $\langle F, v_i\rangle = \sum_{\widetilde{G}_j \in I} c'_{i,j} F \llcorner \widetilde{G}_j$, $0 \le i \le n$, with $c'_{i,j} \in \mathcal{R}_{\mathcal{G}}$. Hence, we get

$$\max_{0 \le i \le n} \big|\langle F, v_i\rangle\big| \le \sharp I \cdot \frac{\max_{0 \le i \le n, \widetilde{G}_j \in I} |c'_{i,j}|}{|c'_{0,1}|} \cdot \max_{\widetilde{G}_j \in I} |F \llcorner \widetilde{G}_j| \cdot \big|c'_{0,1}\big|$$

and

$$\|f; g_1\| \ge \frac{\|f_I; H_{I,1}\|}{\sharp I |c'_{0,1}|} \cdot \frac{|\langle G_1, v_{\overline{1}}^*\rangle|}{\max_{0 \le i \le n} |\langle G_1, v_i^*\rangle|} \cdot \frac{|c'_{0,1}|}{\max_{0 \le i \le n, \widetilde{G}_j \in I} |c'_{i,j}|},$$

which means

$$m_{f,g_1}(r) \le m_{f_I, H_{I,1}}(r) + O\Big(\max_{1 \le j \le q} T_{g_j}(r,s)\Big).$$

By $F \llcorner G_1 = \sum_{i=0}^n \langle G_1, v_i^*\rangle \langle F, v_i\rangle$ and $F_I \llcorner \widetilde{H}_{I,1} = h_I \sum_{i=0}^n \frac{\langle G_1, v_i^*\rangle}{\langle G_1, v_{\overline{1}}^*\rangle} \langle F, v_i\rangle$, we have

$$N_{f,g_1}(r,s) \le N_{f_I, H_{I,1}}(r,s) + O\Big(\max_{1 \le j \le q} T_{g_j}(r,s)\Big).$$

Thus,

$$T_f(r,s) \le T_{f_I}(r,s) + O\Big(\max_{1 \le j \le q} T_{g_j}(r,s)\Big). \tag{20}$$

From (19) and (20), we derive that

$$\begin{aligned} T_f(r,s) &\mathrel{.\le.} \sum_{\widetilde{G}_j \in I} N_{f,g_j}^{(n)}(r,s) + O\Big(\max_{1 \le j \le q} T_{g_j}(r,s)\Big) + o\big(T_f(r,s)\big) \\ &\mathrel{.\le.} \sum_{j=1}^q N_{f,g_j}^{(n)}(r,s) + O\Big(\max_{1 \le j \le q} T_{g_j}(r,s)\Big) + o\big(T_f(r,s)\big). \end{aligned}$$

This completes the proof of Theorem 3.1. $\square$

The $\mathcal{G}$ (or $\widetilde{\mathcal{G}} = \{\widetilde{G}_1, \ldots, \widetilde{G}_q\}$) are said to be *in general position* if any $n+1$ vectors in $\widetilde{\mathcal{G}}$ are linearly independent over $\mathcal{R}_{\mathcal{G}}$. Obviously, if $\widetilde{G}_1, \ldots, \widetilde{G}_q$ are located in general position and $q \geq 2n+1$, then $\widetilde{\mathcal{G}}$ is nondegenerate over $\mathcal{R}_{\mathcal{G}}$. In this case, we have a stronger result (i.e. Theorem 1.1).

Proof of Theorem 1.1. We can prove (1) by induction on q.

When $q = 2n+1$, (1) is just Theorem 3.1.

For $q > 2n+1$, we assume that (1) holds for $q-1$ and verify (1) for q.

In fact, choose $q-1$ moving targets at a time and apply (1). This gives q inequalities as follows.

$$\frac{q-1}{2n+1} T_f(r,s) \mathrel{.\!\leq\!.} \sum_{j \neq l} N^{(n)}_{f,g_j}(r,s) + O\Big(\max_{1 \leq j \leq q} T_{g_j}(r,s)\Big) + o\big(T_f(r,s)\big)$$

for $l = 1, \ldots, q$. Summing up these q inequalities, we have

$$\frac{q(q-1)}{2n+1} T_f(r,s) \mathrel{.\!\leq\!.} (q-1) \sum_{j=1}^{q} N^{(n)}_{f,g_j}(r,s) + O\Big(\max_{1 \leq j \leq q} T_{g_j}(r,s)\Big) + o\big(T_f(r,s)\big),$$

which proves (1). $\square$

4. Proof of Theorem 1.2

Assume that $f_1, \ldots, f_\lambda$ are not in special position. Let $F_t : U \to V$ be a reduced representation of f_t on U for $t = 1, \ldots, \lambda$. Then $F_1 \wedge \cdots \wedge F_\lambda : U \to \bigwedge_\lambda V$ is not identically zero, there exists one and only one divisor defined by

$$\mu_{f_1 \dot\wedge \cdots \dot\wedge f_\lambda}|_U = \mu_{F_1 \wedge \cdots \wedge F_\lambda}.$$

Obviously $\mu_{f_1 \dot\wedge \cdots \dot\wedge f_\lambda} \geq 0$. Also, we can define a meromorphic map $f_1 \wedge \cdots \wedge f_\lambda : M \to \mathbb{P}(\bigwedge_\lambda V)$ by $f_1 \wedge \cdots \wedge f_\lambda = \mathbb{P}(F_1 \wedge \cdots \wedge F_\lambda)$ on U.

Set $S = \operatorname{supp} \mu_{f_1 \dot\wedge \cdots \dot\wedge f_\lambda}$. Then $A \subseteq S$. Denote by $\mathcal{R}(A)$ and $\mathcal{R}(S)$ the sets of regular points of A and S, and denote by $\Sigma(A)$ and $\Sigma(S)$ the sets of singular points of A and S. Define

$$I = \Sigma(S) \cup \Sigma(A) \cup I(f_1 \wedge \cdots \wedge f_\lambda) \cup \left(\bigcup_{t=1}^{\lambda} I(f_t) \right),$$

which has at most dimension $m-2$.

There exists one and only one divisor ν_A on M with $\nu_A(x) = 1$ for all $x \in \mathcal{R}(A)$ and $\nu_A(x) = 0$ for all $x \in M - A$. We now prove

$$(\lambda - l + 1)\nu_A \leq \mu_{f_1 \dot\wedge \cdots \dot\wedge f_\lambda}$$

at every point $x_0 \in A - I$.

By the definition of I, we can find an open, connected neighborhood U of x_0 and a holomorphic map

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) : U \to P \times Q$$

with $\alpha(x_0) = 0$ such that

(1) $U \cup I = \emptyset$ and $U \cap A = U \cap S = \alpha_m^{-1}(0) = U \cap \mathcal{R}(A) = U \cap \mathcal{R}(S)$,

(2) there exist a reduced representation $F_t : U \to V$ of f_t for $t = 1, \ldots, \lambda$ and a holomorphic function h on U with

$$\mu_{f_1 \dot{\wedge} \cdots \dot{\wedge} f_\lambda}|_U = \mu_{F_1 \wedge \cdots \wedge F_\lambda} = \mu_h^0,$$

where P is a ball centered at $0 \in \mathbb{C}^{m-1}$ and Q is a disc centered at $0 \in \mathbb{C}$.

Let $\pi_1 : P \times Q \to P$ and $\pi_2 : P \times Q \to Q$ be the projections. Then $\pi_2 \circ \alpha = \alpha_m$, $\beta := \pi_1 \circ \alpha = (\alpha_1, \ldots, \alpha_{m-1}) : U \to P$. Let $\iota : U \cap A \to U$ be the inclusion map. Then $\gamma = \beta \circ \iota : U \cap A \to P$ is biholomorphic. Let $\delta = \gamma^{-1} : P \to U \cap A$ be the inverse map. Observe that $\alpha_m \circ \iota \equiv 0$. For $t \in \{1, \ldots, \lambda\}$, the Hartogs series development of F_t on U uniquely defines holomorphic vector functions $W_{tk} : P \to V$ for all $k \geq 0$ such that

$$F_t = \sum_{k=0}^{\infty} (\alpha_m)^k W_{tk} \circ \beta$$

where the convergence is uniform on every compact subset of U. Then

$$F_t = W_t + \alpha_m U_t,$$

where $W_t := W_{t0}$ and $U_t := \sum_{k=0}^{\infty} (\alpha_m)^k W_{tk+1} \circ \beta$. Denote by $T[\theta, \lambda]$ the set of all increasing injective maps from $\{1, 2, \ldots, \theta\}$ to $\{1, 2, \ldots, \lambda\}$. For each $\eta \in T[\theta, \lambda]$, there exists a unique $\hat{\eta} \in T[\lambda - \theta, \lambda]$ such that $(\operatorname{Im} \eta) \cap (\operatorname{Im} \hat{\eta}) = \emptyset$. Abbreviate $\epsilon_\eta = \operatorname{sing} \eta$. $f_1, \ldots, f_\lambda$ being in l-special position on A implies that, for any $\eta \in T[l, \lambda]$,

$$W_{\eta(1)} \wedge \cdots \wedge W_{\eta(l)} = 0.$$

Thus

$$F_1 \wedge \cdots \wedge F_\lambda = \sum_{\theta=1}^{l-1} (\alpha_m)^{\lambda-\theta} \sum_{\eta \in T[\theta, \lambda]} \epsilon_\eta \left(\bigwedge_{u=1}^{\theta} W_{\eta(u)} \circ \beta \right) \wedge \left(\bigwedge_{v=1}^{\lambda-\theta} U_{\hat{\eta}(v)} \right) + (\alpha_m)^\lambda U_1 \wedge \cdots \wedge U_\lambda.$$

The lowest exponent of α_m is $\lambda - l + 1 > 0$. Hence, $(\lambda - l + 1)\nu_A \leq \mu_{f_1 \dot{\wedge} \cdots \dot{\wedge} f_\lambda}$.

We have, for every $1 \leq t \leq \lambda$,

$$\sum_{j=1}^{q} N_{f_t, g_j}^{(n)}(r, s) \leq \frac{n}{\lambda - l + 1} N_{\mu_{f_1 \dot{\wedge} \cdots \dot{\wedge} f_\lambda}}(r, s). \tag{21}$$

By the first main theorem of the exterior product (cf. (3.28) of [22]),

$$N_{\mu_{f_1 \dot{\wedge} \cdots \dot{\wedge} f_\lambda}}(r, s) \leq \sum_{i=1}^{\lambda} T_{f_i}(r, s) + O(1). \tag{22}$$

Combining (21) and (22) yields

$$\sum_{j=1}^{q} N_{f_t, g_j}^{(n)}(r, s) \leq \frac{n}{\lambda - l + 1} \sum_{i=1}^{\lambda} T_{f_i}(r, s) + O(1).$$

By Theorem 1.1, for $1 \leq t \leq \lambda$,

$$\frac{q}{2n+1}T_{f_t}(r,s) \mathrel{.{\leq}.} \frac{n}{\lambda-l+1}\sum_{i=1}^{\lambda}T_{f_i}(r,s) + O\Big(\max_{1\leq j\leq q}T_{g_j}(r,s)\Big) + o\big(T_{f_t}(r,s)\big).$$

Thus

$$\frac{q}{2n+1}\sum_{t=1}^{\lambda}T_{f_t}(r,s) \mathrel{.{\leq}.} \frac{n\lambda}{\lambda-l+1}\sum_{t=1}^{\lambda}T_{f_t}(r,s) + o\left(\sum_{t=1}^{\lambda}T_{f_t}(r,s)\right),$$

which gives a contradiction under the assumption that

$$q > \frac{n(2n+1)\lambda}{\lambda-l+1}.$$

References

[1] G. Ashline, The defect relation of meromorphic maps on parabolic manifolds, Mem. Amer. Math. Soc. 139 (1999), No. 665, x+78 pp.

[2] H. Cartan, Sur les zeros des combinaisions linearires de p fonctions holomorpes donnees, Mathematica (Cluj) 7 (1933) 80–103.

[3] Z. Chen, M. Ru, A uniqueness theorem for moving targets with truncated multiplicities, Houston J. Math. 32 (2006) 589–601.

[4] Z. Chen, Q. Yan, Uniqueness theorem of meromorphic mappings into $\mathbb{P}^N(\mathbb{C})$ sharing $2N+3$ hyperplanes regardless of multiplicities, Internat. J. Math. 20 (2009) 717–726.

[5] G. Dethloff, V.T. Tran, Uniqueness problem for meromorphic mappings with truncated multiplicities and moving targets, Nagoya Math. J. 181 (2006) 75–101.

[6] D.T. Do, D.Q. Si, Uniqueness problem with truncated multiplicities of meromorphic mappings in several complex variables for moving targets, Internat. J. Math. 16 (2005) 903–939.

[7] D.T. Do, D.Q. Si, Second main theorem with truncated counting function in several complex variables for moving targets, Forum Math. 20 (2008) 163–179.

[8] H. Fujimoto, The uniqueness problem of meromorphic maps into the complex projective space, Nagoya Math. J. 58 (1975) 1–23.

[9] H. Fujimoto, Uniqueness problem with truncated multiplicities in value distribution theory, Nagoya Math. J. 152 (1998) 131–152.

[10] S. Ji, Uniqueness problem without multiplicities in value distribution theory, Pacific J. Math. 135 (1988) 323–348.

[11] Y. Liu, On the problem of integer solutions to decomposable form inequalities, Int. J. Number Theory 4 (2008) 859–872.

[12] Y. Liu, M. Ru, A defect relation for meromorphic maps on parabolic manifolds intersecting hypersurfaces, Illinois J. Math. 49 (2005) 237–257.

[13] R. Nevanlinna, Einige Eindeutigkeitssätze in der Theorie der Meromorphen Funktionen, Acta Math. 48 (1926) 367–391.

[14] M. Ru, The second main theorem with moving targets on parabolic manifolds, Indiana Univ. Math. J. 46 (1997) 299–318.

[15] M. Ru, A uniqueness theorem with moving targets without counting multiplicity, Proc. Amer. Math. Soc. 129 (2001) 2701–2707.

[16] M. Ru, W. Stoll, The second main theorem for moving targets, J. Geom. Anal. 1 (1991) 99–138.

[17] M. Ru, J.T.-Y. Wang, Truncated second main theorem with moving targets, Trans. Amer. Math. Soc. 356 (2004) 557–571.

[18] L. Smiley, Geometric conditions for unicity of holomorphic curves, in: Contemp. Math., vol. 25, 1983, pp. 149–154.

[19] N. Steinmetz, Eine Verallgemeinerung des zweiten Nevanlinnaschen Hauptsatzes, J. Reine Angew. Math. 368 (1986) 134–141.

[20] W. Stoll, The Ahlfors–Weyl theory of meromorphic maps on parabolic manifolds, in: Lecture Notes in Math., vol. 981, Springer, Berlin, 1983, pp. 101–219.

[21] W. Stoll, Value Distribution Theory for Meromorphic Maps, Aspects Math., vol. E7, Friedr. Vieweg & Sohn, Braunschweig, 1985.

[22] W. Stoll, On the propagation of dependences, Pacific J. Math. 139 (1989) 311–337.

[23] Z.-H. Tu, Uniqueness problem of meromorphic mappings in several complex variables for moving targets, Tohoku Math. J. 54 (2002) 567–579.

[24] P.-M. Wong, W. Stoll, Second main theorem of Nevanlinna theory for nonequidimensional meromorphic maps, Amer. J. Math. 116 (1994) 1031–1071.

[25] Q. Yan, Uniqueness theorem of meromorphic mappings with few moving hyperplanes, Houston J. Math. 36 (2010) 653–664.

[26] Q. Yan, Z. Chen, A degeneracy theorem for meromorphic mappings with truncated multiplicities, Acta Math. Sci. Ser. B Engl. Ed. 31 (2011) 549–560.

Please cite this article in press as: Q. Yan, Second main theorem and uniqueness theorem with moving targets on parabolic manifolds, J. Math. Anal. Appl. (2015), http://dx.doi.org/10.1016/j.jmaa.2014.08.048