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The algebraic equalities and their topological consequences in weighted spaces

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A R T I C L E I N F O

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A B S T R A C T

We study algebraic equalities and their topological consequences in weighted Banach, Fréchet, or (LB) spaces of holomorphic-like functions on a locally compact and σ-compact Hausdorff space X. Our main results are the following: (1) The algebraic equality $\mathcal{V}A(X) = \mathcal{V}_0A(X)$ for (LB)-spaces with O- and o-growth conditions given by a weight sequence $\mathcal{V} = (v_n)_n$ always implies that these spaces are (DFS). The converse statement is valid under the additional condition (CD) which is a weakened version of the typical biduality condition for the steps $A_{v_n}(X)$ and $A_{(v_n)0}(X)$ generating $\mathcal{V}A(X)$ and $\mathcal{V}_0A(X)$, respectively; (2) Under the same condition (CD), the algebraic equality $A\overline{V}(X) = A\overline{V}_0(X)$ between the projective hulls of $\mathcal{V}A(X)$ and $\mathcal{V}_0A(X)$ is equivalent to $A\overline{V}(X)$ semi-Montel. Thus, we completely remove or significantly weaken some stringent conditions used before in many papers studying the similar problems (see, e.g., Bierstedt and Bonet, 2006 [5] and references therein).

1. Introduction

Weighted spaces of continuous and holomorphic functions with O- and o-growth conditions play an important role in approximation and spectral theories, complex and Fourier analysis, convolution and partial differential equations, as well as distribution theory. Also, they are themselves of a great interest for mathematical research. For these reasons, they were studied intensively by many authors, especially after the seminal paper [8] of Bierstedt, Meise and Summers.

In case of spaces of continuous functions the situation was clarified completely at the end of the 1980s, at least what concerns algebraic and topological properties of (LB)-spaces and their projective hulls and the projective description problem (in addition to [8] see also Bierstedt and Bonet [3]). The research then

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concentrated on holomorphic function spaces. But, in spite of more than 30 years of efforts, not so many results of complete type have been obtained and several important problems have remained open (see, e.g., Bierstedt's survey [2]). In this connection, one of the most fruitful ideas to get the desired results and investigate the open problems is to find out some conditions under which weighted spaces of holomorphic functions behave similarly to the corresponding spaces of continuous functions.

In particular, Bierstedt and Bonet [5, Sections 1 and 2] (see also Bierstedt and Bonet [4] and Bierstedt, Bonet and Galbis [6]) pointed out that, under some additional assumptions, algebraic equalities between spaces of the same type (both (LB) or projective hulls) with O- and o-growth conditions imply some strong topological properties of the spaces involved into these equalities.

It should be noted that some of the assumptions used in [5] look rather restrictive. Namely, in Sections 1 and 2 of [5] it is supposed that all the steps $H_{v_n}(G)$ constituting the (LB)-space $\mathcal{V}H(G)$ satisfy the biduality condition from [9] and/or the interpolation property (see hypothesis (1) in the beginning of [5, Section 2]).

Starting from Bierstedt and Bonet [5] and using some new ideas, we show that similar as well as much stronger results are true without any additional conditions or under conditions which are weaker than in [5]. Our technique suits not only to spaces of holomorphic functions $H_v(G)$, $\mathcal{V}H(G)$, $H\overline{V}(G)$, etc. on an open set in $\mathbb{C}^N$ but, as in [8], to more general spaces $A_v(X)$, $\mathcal{V}A(X)$, $A\overline{V}(X)$, etc. of functions on a locally compact and σ-compact set X.

The present paper is organized as follows. In Section 2 we fix our notation, recall some well-known facts on weighted function spaces and establish some simple auxiliary results from functional analysis which play an important role in the next Section 3.

Section 3 is divided into two subsections. The first one is devoted to the topological consequences of algebraic equalities between weighted spaces of the same type with O- and o-growth conditions. We establish (Theorem 3.3) that the equality of such a type for Banach, Fréchet, or (LB)-spaces implies that the corresponding spaces are always finite dimensional, Montel, or (DFS), respectively. Thus, we remove all restrictions used before in the results of such a kind for spaces of holomorphic functions. In addition, in case of Fréchet spaces our result is finer than the previous one (see Remark 3.4 below).

Similarly, studying in the second subsection the equivalence between the algebraic equalities for weighted (LB)-spaces as well as their projective hulls, we use, instead of the biduality property for the steps as in [5, Section 2], weaker condition (CD) and obtain much stronger results. In particular, it is shown (Theorem 3.8) that, whenever (CD) holds, the algebraic equality $\mathcal{V}A(X) = \mathcal{V}_0A(X)$ or $AV(X) = A\overline{V}_0(X)$ is equivalent to $\mathcal{V}A(X)$ (DFS) or, respectively, $A\overline{V}(X)$ semi-Montel. Combining this with [5, Proposition 14], we deduce that, provided all the steps $H_{v_n}(G)$ satisfy the biduality property, the space $H\overline{V}_0(G)$ is semi-Montel if and only if it is semireflexive. This answers the question posed in [5, p. 759]. In addition, at the end of this subsection it is proved (Theorem 3.12) that, for a domain in $\mathbb{C}$ whose complement has no one-point component or an absolutely convex bounded domain in $\mathbb{C}^N$, the condition (CD) can be removed and the algebraic equality $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ or $H\overline{V}(G) = H\overline{V}_0(G)$ is always equivalent to $\mathcal{V}H(G)$ (DFS) or $H\overline{V}(G)$ semi-Montel, respectively.

2. Preliminaries

In this section we collect notation, definitions and preliminary facts which will be used in the sequel.

2.1. Weighted spaces

Let X be a locally compact and σ-compact Hausdorff space and $C(X)$ the space of all continuous complex-valued functions on X endowed with the compact–open topology co defined by the system of seminorms

$$\|f\|_K := \sup_{x\in K}\bigl|f(x)\bigr|,$$

where K runs over all compact sets in X. Let $A(X)$ be some predetermined subspace of $C(X)$ which is supposed to be *semi-Montel* (i.e. each bounded subset of this space is relatively compact). In particular, if X is an open set G in $\mathbb{C}^N$, we can take $A(G) = H(G)$, the space of all holomorphic functions on G.

A continuous and strictly positive real-valued function v on X will be called a *weight*. For a weight v on X, define the following weighted Banach spaces:

$$A_v(X) := \left\{f \in A(X) : \|f\|_v := \sup_{x\in X}\frac{|f(x)|}{v(x)} < \infty\right\},$$
$$A_{v0}(X) := \left\{f \in A(X) : \frac{f(x)}{v(x)} \text{ vanishes at infinity on } X\right\},$$

endowed with the norm $\|\cdot\|_v$. As usual, we say that a function g vanishes at infinity on X if for every $\varepsilon > 0$ there exists a compact subset K of X such that $|g(x)| < \varepsilon$ for all $x \in X \setminus K$. Obviously, $A_{v0}(X)$ is a closed subspace of $A_v(X)$.

Notice that our definition of the weight norm $\|\cdot\|_v$ differs slightly from many other papers, where it is defined by $\|f\|_v := \sup_{x\in X} v(x)|f(x)|$. Certainly, this is not essential but, in our opinion, more convenient for some reasons, especially when one uses associated weights (see below Section 2.2). Thus, when we refer to papers using another definition of weight norms we often reformulate their results in accordance with our notation.

For a decreasing sequence $V = (v_n)_n$ of weights v_n on X, we define

$$AV(X) := \bigcap_n A_{v_n}(X) \quad \text{and} \quad AV_0(X) := \bigcap_n A_{(v_n)0}(X)$$

and equip both of these spaces with the locally convex topology induced by $\{\|\cdot\|_{v_n};\ n \in \mathbb{N}\}$. Clearly, $AV(X)$ and $AV_0(X)$ are both Fréchet spaces and $AV_0(X)$ is a closed topological subspace of $AV(X)$.

Given an increasing sequence $\mathcal{V}$ of weights v_n on X, we consider the following weighted inductive limits:

$$\mathcal{V}A(X) := \operatorname{ind}_n A_{v_n}(X) \quad \text{and} \quad \mathcal{V}_0A(X) := \operatorname{ind}_n A_{(v_n)0}(X);$$

that is, we take the increasing union of all Banach spaces $A_{v_n}(X)$, respectively, $A_{(v_n)0}(X)$, and endow it with the strongest locally convex topology for which the injections $A_{v_n}(X) \to \mathcal{V}A(X)$, respectively, $A_{(v_n)0}(X) \to \mathcal{V}_0A(X)$ become continuous for all $n \in \mathbb{N}$. It is clear that $\mathcal{V}_0A(X)$ is a linear subspace of $\mathcal{V}A(X)$ (and the inclusion operator is continuous), but it is not known in general whether $\mathcal{V}_0A(X)$ is also a topological subspace of $\mathcal{V}A(X)$. Since the unit balls of $A_{v_n}(X)$ are closed in $\mathcal{V}A(X)$, the inductive limit $\mathcal{V}A(X) = \operatorname{ind}_n A_{v_n}(X)$ is regular, i.e., each its bounded set is contained and bounded in some step $A_{v_n}(X)$ (see Makarov [11]).

In order to describe the inductive limit topology of $\mathcal{V}A(X)$ in terms of weighted sup-norms, the following family of weights on X, associated with $\mathcal{V}$, was introduced in Bierstedt, Meise and Summers [8]:

$$\overline{V} = \overline{V}(\mathcal{V}) := \left\{\bar{v} \text{ weight on } X : \sup_{x\in X}\frac{v_n(x)}{\bar{v}(x)} < \infty,\ \forall n\right\}.$$

This family generates the corresponding associated weighted spaces

$$A\overline{V}(X) := \left\{f \in A(X) : \|f\|_{\bar{v}} = \sup_{x\in X}\frac{|f(x)|}{\bar{v}(x)} < \infty,\ \forall \bar{v} \in \overline{V}\right\}$$

and

$$A\overline{V}_0(X) := \left\{ f \in A(X) : \frac{f(x)}{\bar{v}(x)} \text{ vanishes at infinity on } X,\ \forall \bar{v} \in \overline{V} \right\},$$

endowed with the Hausdorff locally convex topology defined by the norm system $\{\|\cdot\|_{\bar{v}};\ \bar{v} \in \overline{V}\}$. These spaces $A\overline{V}(X)$ and $A\overline{V}_0(X)$ are called *the projective hulls* of the inductive limits $\mathcal{V}A(X)$ and $\mathcal{V}_0A(X)$, respectively.

It is easy to see that $A\overline{V}_0(X)$ is a closed topological subspace of $A\overline{V}(X)$ and both spaces are complete. Moreover, there are continuous injections $\mathcal{V}A(X) \to A\overline{V}(X)$ and $\mathcal{V}_0A(X) \to A\overline{V}_0(X)$. By Bierstedt, Meise and Summers [8, Theorem 1.13], spaces $\mathcal{V}A(X)$ and $A\overline{V}(X)$ coincide as sets and have the same bounded sets, but their topologies can differ one from another. Next, it is not known whether $\mathcal{V}_0A(X)$ is always a topological subspace of $A\overline{V}_0(X)$. The last two observations lead to the *projective description problem* (see, e.g., Bierstedt [2]).

2.2. Auxiliary facts from functional analysis

We start with two simple results from functional analysis. Perhaps, they are known and stated here for the reader's convenience. Throughout below in this subsection, E, F, and G are some Hausdorff locally convex spaces (l.c.s.). For a l.c.s. E, we will denote by τ_E and $\mathcal{O}_E$ its topology and a base of absolutely convex neighborhoods of the origin, respectively.

Lemma 2.1. *Let $E \hookrightarrow F$ and B be an absolutely convex subset of E which is relatively compact in F. Consider the following assertions:*

(i) *B is relatively compact in E.*
(ii) *$\tau_E = \tau_F$ on B.*
(iii) *Each net $(x_\lambda)_{\lambda\in\Lambda}$ of B converging to 0 in F converges (to 0) in E.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii). *If E is complete, then* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

In case F is metrizable, condition (iii) *can be replaced with:*

(iii)′ *Each sequence $(x_n)_{n=1}^{\infty}$ of B converging to 0 in F converges (to 0) in E.*

Proof. It is easy to see that, for F metrizable, (iii) $\Leftrightarrow$ (iii)′. Always (ii) $\Rightarrow$ (iii) and, since B is absolutely convex, (iii) $\Rightarrow$ (ii). So (ii) $\Leftrightarrow$ (iii).

(i) $\Rightarrow$ (ii): Suppose by contradiction that B is relatively compact in E but τ_E is strictly finer than τ_F on B. Then, using that B is absolutely convex, we can find a neighborhood $U_0 \in \mathcal{O}_E$ such that $V \cap B \nsubseteq U_0 \cap B$ for every $V \in \mathcal{O}_F$. Taking $x_V \in V \cap B$ with $x_V \notin U_0$, we get the net $(x_V)_{V\in\mathcal{O}_F}$ which converges to 0 in F. Since B is relatively compact in E, there exists a subnet $(y_V)_{V\in\mathcal{O}'_F}$ of $(x_V)_{V\in\mathcal{O}_F}$ which converges in E. Obviously, $y_V \to 0$ in E which contradicts to $y_V \notin U_0$ for all $V \in \mathcal{O}'_F$.

To finish the proof, it remains to note that (ii) implies that B is precompact in E and, for E complete, this is equivalent to B relatively compact in E. $\square$

A linear operator $L : E \to F$ between two l.c.s. is said to be *compact* if there exists a neighborhood $U \in \mathcal{O}_E$ such that $L(U)$ is relatively compact in F. Next, L is said to be *Montel* if it maps bounded sets in E into relatively compact sets in F. Obviously, each compact operator is Montel and for a Banach space E the converse is also true.

Lemma 2.2. *Let $E \hookrightarrow F \hookrightarrow G$ and G be semi-Montel. Consider the following assertions:*

(i) *The inclusion operator $id : E \to F$ is Montel.*
(ii) *Every bounded net of E which converges to 0 in G is also convergent (to 0) in F.*

Always (i) $\Rightarrow$ (ii) *and* (i) $\Leftrightarrow$ (ii)*, whenever F is complete.*

In case G is metrizable, the last condition can be replaced with:

(ii)′ *Each bounded sequence of E which converges to 0 in G is also convergent (to 0) in F.*

Proof. (i) $\Rightarrow$ (ii): Let $(x_\lambda)_{\lambda\in\Lambda}$ be a bounded net in E which converges to 0 in G. By (i), the set $B := \{x_\lambda : \lambda \in \Lambda\}$ is relatively compact in F and then the absolutely convex envelope $\Gamma(B)$ of B is the same. Since $E \hookrightarrow G$, B and $\Gamma(B)$ are bounded sets in G. Using that G is semi-Montel, it then follows that $\Gamma(B)$ is relatively compact in G. Hence, by Lemma 2.1, $\tau_F = \tau_G$ on $\Gamma(B)$. Consequently, $x_\lambda \to 0$ in F.

(ii) $\Rightarrow$ (i), whenever F is complete: If this is not true, then there exists a bounded set B in E which is not relatively compact in F. Consequently, the absolutely convex envelope $\Gamma(B)$ of B is also bounded in E and not relatively compact in F. Since G is semi-Montel, $\Gamma(B)$ is relatively compact in G. Applying Lemma 2.1 to F, G and $\Gamma(B)$, we can find a net $(x_\lambda)_{\lambda\in\Lambda}$ of $\Gamma(B)$ which converges to 0 in G but does not converge in F. This contradicts (ii) and completes the proof. □

Applying Lemma 2.2 with $G = A(X)$ we have

Corollary 2.3. *Let $E \hookrightarrow F \hookrightarrow A(X)$ and F be complete. The following assertions are equivalent:*

(i) *The inclusion operator $id : E \to F$ is Montel.*
(ii) *Each bounded sequence of E which converges to 0 in $A(X)$ is also convergent (to 0) in F.*

This corollary has evident consequences for weighted spaces defined above. Note that by the closed graph theorem, for two spaces E, F of such a type, the inclusion $E \subset F$ is always continuous. In particular, we have the following characterization of compact embedding for Banach weighted spaces.

Corollary 2.4. *The inclusion of $A_v(X)$ into $A_w(X)$ (or, $A_{v0}(X)$ into $A_{w0}(X)$) is compact if and only if every bounded sequence $(f_k)_k \subset A_v(X)$ (respectively, $A_{v0}(X)$) converging to 0 with respect to the co topology, also converges to 0 in $A_w(X)$ (respectively, $A_{w0}(X)$).*

3. The algebraic equalities between weighted spaces with o- and O-growth conditions

In this section we study the algebraic equalities between weighted spaces with o- and O-growth conditions of the same type.

Starting with the topological consequences of these equalities for Banach, Fréchet, and (LB)-spaces, we establish that they imply that the corresponding spaces are always finite dimensional, Montel, and (DFS), respectively (see Theorem 3.3). Thus, we remove all restrictions used before in the results of such a type for spaces of holomorphic functions. In addition, in case of Fréchet spaces our result is finer than the previous one (in this connection see Remark 3.4 below).

Similarly, studying the converse statements (or the equivalence between algebraic equalities and topological structures of the corresponding spaces), we remove or weaken the assumptions used in the previous papers.

3.1. Topological consequences of algebraic equalities

For two decreasing sequences V and W we will write that $V \leq W$ if for every $m \in \mathbb{N}$ there exist $n \in \mathbb{N}$ and $C > 0$ such that $v_n \leq Cw_m$ on X. Clearly, in this case $AV(X) \hookrightarrow AW(X)$.

Proposition 3.1. *Suppose that V and W are two decreasing weight sequences and $V \leq W$. If $AV(X) \subset AW_0(X)$, then the inclusion of $AV_0(X)$ into $AW_0(X)$ is Montel.*

Proof. We proceed by contradiction and assume that the inclusion of $AV_0(X)$ into $AW_0(X)$ is not Montel. Using Corollary 2.3 with $E = AV_0(X)$ and $F = AW_0(X)$, we can find a bounded sequence $(f_k)_{k=1}^{\infty}$ in $AV_0(X)$ satisfying the following conditions:

(a) $(f_k)_{k=1}^{\infty}$ converges to 0 with respect to the *co* topology;
(b) $(f_k)_{k=1}^{\infty}$ does not converge to 0 in $AW_0(X)$, i.e., there is $m \in \mathbb{N}$ so that $(f_k)_{k=1}^{\infty}$ does not converge to 0 by the norm $\|\cdot\|_{w_m}$. W.l.o.g., we may assume that, for some $c > 0$,

$$\|f_k\|_{w_m} \geq c \quad \text{for all } k \in \mathbb{N}.$$

Since $(f_k)_{k=1}^{\infty}$ is bounded in $AV_0(X)$,

$$M_n := \sup_{k \in \mathbb{N}} \|f_k\|_{v_n} < \infty \quad \text{for all } n \in \mathbb{N}.$$

From $V \leq W$ it follows that, for some $n_0 \in \mathbb{N}$ and $M > 1$, $v_{n_0} \leq Mw_m$ on X. Hence, $v_n \leq Mw_m$ on X for all $n \geq n_0$. W.l.o.g., we may assume that $v_n \leq Mw_m$ for all $n \in \mathbb{N}$.

Let $(Q_k)_{k=1}^{\infty}$ be a fundamental sequence of compact sets of X. We set $K_1 := Q_1$ and take $b \in (0, c)$. By condition (a), there is $k_1 \in \mathbb{N}$ such that

$$\|f_k\|_{K_1} \leq \frac{b}{2M} \inf_{x \in K_1} v_1(x), \quad \text{for all } k \geq k_1.$$

Setting $g_1 := f_{k_1}$ and using the condition (b), we can find a point $x_1 \notin K_1$ with $|g_1(x_1)| \geq bw_m(x_1)$.

Suppose that, for some $j \in \mathbb{N}$, K_s, g_s, x_s are already defined for all $1 \leq s \leq j$ and choose K_{j+1}, g_{j+1}, x_{j+1} in the following way. Take K_{j+1} from $(Q_k)_{k=1}^{\infty}$ so that:

(i) $x_j \in K_{j+1}$;
(ii) $|g_1(x)| + |g_2(x)| + ... + |g_j(x)| \leq \frac{b}{2M} v_{j+1}(x)$ for all $x \notin K_{j+1}$.

Next, define g_{j+1} and x_{j+1} in just the same way as we already chose the function g_1 and point x_1:

(iii) By condition (a), there exists $k_{j+1} \in \mathbb{N}$ so that

$$\|f_k\|_{K_{j+1}} \leq \frac{b}{2^{j+1}M} \inf_{x \in K_{j+1}} v_{j+1}(x), \quad \forall k \geq k_{j+1}.$$

(iv) Setting $g_{j+1} := f_{k_{j+1}}$ and using the condition (b), we find a point $x_{j+1} \notin K_{j+1}$ with $|g_{j+1}(x_{j+1})| \geq bw_m(x_{j+1})$.

Put $f := \sum_j g_j$ and prove that $f \in AV(X)$, but $f \notin AW_0(X)$.

Given a compact set K in X, find $j_0 \in \mathbb{N}$ with $K \subset K_{j_0}$. Then by condition (iii) we have, for $j > j_0$,

$$\|g_j\|_K \le \frac{b}{2^j M} \inf_{x \in K_j} v_j(x) \le \frac{b}{2^j M} \inf_{x \in K_{j_0}} v_{j_0}(x).$$

Hence, the series $\sum_j g_j$ converges absolutely in $(A(X), co)$ and, consequently, $f \in A(X)$.

Let $n \in \mathbb{N}$. For any $x \notin K_n$, there exists $j_0 \ge n$ with $x \in K_{j_0+1} \setminus K_{j_0}$. Then from (ii) and (iii) we have

$$\begin{aligned} |f(x)| &\le \sum_{j=1}^{j_0-1} |g_j(x)| + |g_{j_0}(x)| + \sum_{j=j_0+1}^{\infty} |g_j(x)| \\ &\le \frac{b}{2M} v_{j_0}(x) + M_n v_n(x) + \sum_{j=j_0+1}^{\infty} \frac{b}{2^j M} v_j(x) \le (b + M_n) v_n(x). \end{aligned}$$

Thus, $f \in A_{v_n}(X)$ for every $n \in \mathbb{N}$. That is, $f \in AV(X)$.

Using (ii), (iii), and (iv), we have, for each $s \ge 2$,

$$\begin{aligned} |f(x_s)| &\ge |g_s(x_s)| - \sum_{j=1}^{s-1} |g_j(x_s)| - \sum_{j=s+1}^{\infty} |g_j(x_s)| \\ &\ge b w_m(x_s) - \frac{b}{2M} v_s(x_s) - \sum_{j=s+1}^{\infty} \frac{b}{2^j M} v_j(x_s) \ge \frac{b}{4} w_m(x_s). \end{aligned}$$

Consequently, $f(x)/w_m(x)$ does not vanish at infinity on X and $f \notin AW_0(X)$. This completes the proof. □

Corollary 3.2. *(Cf. [5, Proposition 3].) Suppose that v and w are two weights on X such that, for some $C > 0$, $v \le Cw$ on X. If $A_v(X) \subset A_{w0}(X)$, then the inclusion of $A_{v0}(X)$ into $A_{w0}(X)$ is compact.*

The main result of this subsection is as follows.

Theorem 3.3. *The following statements are true:*

(1) *The algebraic (or, topological) equality $A_v(X) = A_{v0}(X)$ implies that $A_v(X)$ and $A_{v0}(X)$ are finite-dimensional spaces.*
(2) *The algebraic (or, topological) equality $AV(X) = AV_0(X)$ implies that $AV(X)$ and $AV_0(X)$ are Montel spaces.*
(3) *The algebraic equality $\mathcal{V}A(X) = \mathcal{V}_0A(X)$ implies that $\mathcal{V}A(X)$ and $\mathcal{V}_0A(X)$ are (DFS)-spaces.*

Proof. Statements (1) and (2) are immediate consequences of Corollary 3.2 and Proposition 3.1. To see this, it is enough to consider $w = v$ and $W = V$, respectively.

Let us prove (3). Always $\mathcal{V}_0A(X) \hookrightarrow \mathcal{V}A(X)$. Then, by the open mapping theorem, the algebraic equality $\mathcal{V}A(X) = \mathcal{V}_0A(X)$ implies that these two spaces coincide topologically. Thus, it is sufficient to check that $\mathcal{V}_0A(X)$ is (DFS).

The equality $\mathcal{V}A(X) = \mathcal{V}_0A(X)$ and Grothendieck's factorization theorem (see [12, Theorem 24.33]) imply that for each $n \in \mathbb{N}$ there is $m > n$ with $A_{v_n}(X) \subset A_{(v_m)0}(X)$. Then, by Corollary 3.2, we have that the inclusion of $A_{(v_n)0}(X)$ into $A_{(v_m)0}(X)$ is compact. That is, $\mathcal{V}_0A(X)$ is a (DFS)-space. □

Remark 3.4. (1) In many papers (see, e.g., [4,6,9]) several results on the canonical equalities between weighted spaces of holomorphic functions with O-growth conditions and the biduals of spaces with o-growth conditions of the same type were obtained. In case when weighted spaces with O- and o-growth conditions coincide, one can deduce from them that these spaces have some topological properties (concerning mainly their reflexivity).

In particular, Bierstedt and Summers [9, Corollary 1.2] proved that $H_v(G)$ is isometrically isomorphic to the bidual $H_{v0}(G)''$ provided the *biduality condition* holds (that is, the *co*-closure of the unit ball $B_{v0}(G)$ coincides with the unit ball $B_v(G)$). Combining this with Bonet and Wolf [10, Corollary 2], it then follows that $H_v(G)$ and $H_{v0}(G)$ are finite-dimensional whenever $H_v(G) = H_{v0}(G)$ algebraically and the biduality condition holds. Our statement (1) in Theorem 3.3 shows that the biduality condition is superfluous here.

Next, from Bierstedt and Bonet [4, Section 3.A] and Bierstedt, Bonet and Galbis [6, Theorem 1.5(d)] it follows that, for balanced domains and rapidly varying radial weights, the equality $HV(G) = HV_0(G)$ implies that $HV(G)$, as well as $HV_0(G)$, is reflexive. Recall that a domain G is called *balanced* if $\lambda z \in G$ for every $z \in G$ and all $|\lambda| = 1$, while a weight v on a balanced domain G is called *radial* if $v(\lambda z) = v(z)$ for all $z \in G$ and $|\lambda| = 1$. Theorem 3.3(2) establishes that the equality $HV(G) = HV_0(G)$ guarantees that $HV(G)$ and $HV_0(G)$ are Montel (consequently, reflexive) without any restrictions on domains and weights.

(2) Assuming that $\mathcal{V}$ consists of radial weights on a balanced domain G and $H_{(v_1)0}(G)$ contains all the polynomials, Bierstedt and Bonet [4, Section 3.B] and Bierstedt, Bonet and Galbis [6, Theorem 1.6(d)] proved that the algebraic equality $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ implies that $\mathcal{V}_0H(G)$ is reflexive. Later, Bierstedt and Bonet [5] established the following much stronger consequence of this equality provided the *interpolation property* holds for each step $H_{v_n}(G)$. Recall that a sequence $(z_j)_j \subset G$ is said to be *interpolating* for $H_v(G)$ if the restriction operator $R : f \in H_v(G) \mapsto (f(z_j))_j$ maps $H_v(G)$ onto

$$\ell_\infty(v) := \left\{(c_j)_j \in \mathbb{C}^{\mathbb{N}} :\ \sup \frac{|c_j|}{v(z_j)} < \infty\right\}.$$

By [5, Theorem 4(a)], if for each $n \in \mathbb{N}$ every discrete sequence in G contains an interpolating subsequence for $H_{v_n}(G)$, then the equality $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ implies that $\mathcal{V}H(G)$ and $\mathcal{V}_0H(G)$ are (DFS)-spaces. We showed that this result (see Theorem 3.3(3)) is valid without additional assumptions.

3.2. Equivalence between algebraic equalities and topological structures

Let $\mathcal{V} = (v_n)$ be an increasing weight sequence. In the previous subsection we have shown that the algebraic equality $\mathcal{V}A(X) = \mathcal{V}_0A(X)$ implies that $\mathcal{V}A(X)$ is a (DFS) space. In [5] Bierstedt and Bonet studied also when the converse statements are true for spaces of holomorphic functions. Assuming that the step spaces $H_{v_n}(G)$ satisfy the biduality condition and interpolation property, they proved (see [5, Theorem 4(c)]) that the algebraic equalities $\mathcal{V}H(G) = \mathcal{V}_0H(G)$ and $H\overline{V}(G) = H\overline{V}_0(G)$ hold if and only if $\mathcal{V}H(G)$ is (DFS) and $H\overline{V}(G)$ is semi-Montel, respectively.

In this subsection we consider weighted function spaces on a locally compact and σ-compact space X and show that in assertions of such a type the condition on the interpolating property is superfluous and the biduality condition can be replaced with the following weaker one:

(CD) *for each $n \in \mathbb{N}$ there are $m \geq n$ and $M \geq 1$ such that the unit ball $B_{v_n}(X)$ of $A_{v_n}(X)$ is contained in $\overline{MB_{v_m0}(X)}^{co}$, the co-closure of M times the unit ball $B_{v_m0}(X)$ of $A_{(v_m)0}(X)$.*

Remark 3.5. 1. Let $v_n(z) = (1 + |z|)^n e^{\operatorname{Re} z}$, $z \in \mathbb{C}$, $n \in \mathbb{N}$. Then every space $H_{v_n}(\mathbb{C})$ is $(n + 1)$-dimensional and coincides with $\operatorname{Span}\{z^k e^z :\ 0 \leq k \leq n\}$ (see, e.g., [1, Theorem 2.7]). Evidently, $H_{v_n0}(\mathbb{C}) = H_{v_{n-1}}(\mathbb{C})$ and $B_{v_n}(\mathbb{C}) \subset B_{v_{n+1}0}(\mathbb{C})$. Consequently, none of the steps satisfy the biduality property, while $\mathcal{V}H(G)$ satisfies (CD).

2. Note also that by Bierstedt and Bonet [4, Proposition 13], in case of holomorphic functions, (CD) is also the biduality condition of a special kind, but for the (LB)-spaces $\mathcal{V}H(G)$ and $\mathcal{V}_0H(G)$ instead of their steps.

To continue, we need some preparation.

Following Bierstedt, Bonet and Taskinen [7, Definition 1.1], the functions

$$\widetilde{v}(x) := \sup\{|f(x)| : f \in B_v(X)\},$$
$$\widetilde{v}_0(x) := \sup\{|f(x)| : f \in B_{v0}(X)\}, \quad x \in X,$$

are called the *associated* weights (with v). Similarly to the case of holomorphic functions the next assertions hold.

Lemma 3.6. *Associated weights have the following properties:*

(a) $\widetilde{v}_0 \le \widetilde{v} \le v$ *on* X*;*
(b) $A_v(X) = A_{\widetilde{v}}(X)$ *isometrically;*
(c) *for each* $x \in X$ *there is* $f = f_x \in B_v(X)$ *with* $|f(x)| = \widetilde{v}(x)$*;*
(d) *if* $B_v(X) \subset \overline{MB_{w0}(X)}^{co}$*, then* $\widetilde{v} \le M\widetilde{w}_0$ *on* X*;*
(e) *if* $\widetilde{v}/w$ *vanishes at infinity on* X*, then the canonical injection* $A_v(X) \to A_w(X)$ *is compact;*
(f) *if the canonical injection* $A_{v0}(X) \to A_{w0}(X)$ *is compact, then* $\widetilde{v}_0/w$ *vanishes at infinity on* X*.*

Using some ideas in the proof of Theorem 4(b) from Bierstedt and Bonet [5], we have the following characterization for the space $A\overline{V}(X)$ to be semi-Montel.

Lemma 3.7. *The space* $A\overline{V}(X)$ *is semi-Montel if and only if the inclusion of* $A_{v_n}(X)$ *into* $A_{\bar{v}}(X)$ *is compact for all* $n \in \mathbb{N}$ *and* $\bar{v} \in \overline{V}$*.*

Proof. Since $A\overline{V}(X)$ and $\mathcal{V}A(X)$ have the same bounded subsets and $\mathcal{V}A(X)$ is regular, $A\overline{V}(X)$ is semi-Montel if and only if the unit balls $B_{v_n}(X)$ are all relatively compact in $A\overline{V}(X)$, or, using that $A\overline{V}(X)$ is complete, precompact in $A\overline{V}(X)$. Clearly, the ball $B_{v_n}(X)$ is precompact in $A\overline{V}(X)$ if and only if it is precompact (or relatively compact) in $A_{\bar{v}}(X)$ for all $\bar{v} \in \overline{V}$. Thus, $A\overline{V}(X)$ is semi-Montel if and only if the unit ball $B_{v_n}(X)$ is relatively compact in $A_{\bar{v}}(X)$ for all $n \in \mathbb{N}$ and $\bar{v} \in \overline{V}$ which gives the proof. □

Theorem 3.8. *Suppose that* $\mathcal{V}A(X)$ *satisfies (CD). The following statements are true:*

(1) *The algebraic equality* $\mathcal{V}A(X) = \mathcal{V}_0A(X)$ *is equivalent to* $\mathcal{V}A(X)$ *(DFS).*
(2) *The algebraic equality* $A\overline{V}(X) = A\overline{V}_0(X)$ *is equivalent to* $A\overline{V}(X)$ *semi-Montel.*

Proof. (1) The direct statement was proved in Theorem 3.3(3).

Now we suppose that $\mathcal{V}A(X)$ is a (DFS)-space. Fix some function $f \in \mathcal{V}A(X)$. Then $f \in A_{v_n}(X)$ for some $n \in \mathbb{N}$ and, without loss of generality, we may assume that $f \in B_{v_n}(X)$. Condition (CD) implies that there exist numbers $m \ge n$ and $M > 1$ and a sequence $(f_k)_k$ in $MB_{(v_m)0}(X)$ which converges to f in the *co* topology.

Since $\mathcal{V}A(X)$ is a (DFS)-space, the inclusion of $A_{v_m}(X)$ into $A_{v_p}(X)$ is compact for some $p \ge m$. By Corollary 2.4, it then follows that the sequence $(f_k)_k$ converges to f in $A_{v_p}(X)$. Hence, $f \in A_{(v_p)0}(X)$ and consequently $f \in \mathcal{V}_0A(X)$. Thus, $\mathcal{V}A(X) = \mathcal{V}_0A(X)$.

(2) Suppose that $A\overline{V}(X) = A\overline{V}_0(X)$ holds algebraically and fix some $n \in \mathbb{N}$ and $\bar{v} \in \overline{V}$. Using (CD), find $m \ge n$ and $M > 1$ so that $B_{v_n}(X) \subset M\overline{B_{(v_m)0}(X)}^{co}$. Then $\widetilde{v}_n \le M\widetilde{v}_{m0}$ on X.

The hypothesis implies that $A_{v_m}(X) \subset A_{\bar{v}0}(X)$. Hence, by Corollary 3.2, the inclusion $A_{(v_m)0}(X) \to A_{\bar{v}0}(X)$ is compact. By Lemma 3.6(f), this yields that the quotient $\widetilde{v}_{m0}/\bar{v}$ vanishes at infinity on X. Consequently, so does the quotient $\widetilde{v}_n/\bar{v}$ which implies that the inclusion of $A_{v_n}(X)$ into $A_{\bar{v}}(X)$ is compact. Thus, by Lemma 3.7, the space $A\overline{V}(X)$ is semi-Montel.

To prove the converse, let $A\overline{V}(X)$ be semi-Montel. Given f in $A\overline{V}(X)$, find $n \in \mathbb{N}$ so that $f \in A_{v_n}(X)$. W.l.o.g. we can assume that $f \in B_{v_n}(X)$. Again using (CD), choose $m \geq n$ and $M \geq 1$ so that $B_{v_n}(X) \subset M\overline{B_{(v_m)0}(X)}^{co}$. Then there exists a sequence $(f_k)_k \subset MB_{(v_m)0}(X)$ which converges to f in the co topology.

Since $A\overline{V}(X)$ is semi-Montel, by Lemma 3.7 the inclusion of $A_{v_m}(X)$ into $A_{\bar{v}}(X)$ is compact for every $\bar{v} \in \overline{V}$. Then, by Corollary 2.4, $f_k \to f$ as $k \to \infty$ in $A_{\bar{v}}(X)$. Therefore, $f \in A_{\bar{v}0}(X)$ because of $f_k \in A_{\bar{v}0}(X)$ for all k. Consequently, $f \in A\overline{V}_0(X)$ and then $A\overline{V}(X) = A\overline{V}_0(X)$. □

Comparing Proposition 14 and Theorem 4 in [5, p. 759], Bierstedt and Bonet noted that, in case all the steps $H_{v_n}(G)$ satisfy the interpolation property and biduality condition, the space $H\overline{V}_0(G)$ is semi-Montel if and only if it is semireflexive. They also asked whether this equivalence is true without the interpolation property. Obviously, Theorem 3.8 and [5, Proposition 14] give the following answer to this question.

Corollary 3.9. *Let $\mathcal{V} = (v_n)_n$ consist of weights on an open set $G \subset \mathbb{C}^N$ such that all the steps $H_{v_n}(G)$ satisfy the biduality condition. The following assertions are equivalent:*

(1) *The algebraic equality $H\overline{V}(G) = H\overline{V}_0(G)$ holds.*
(2) *$H\overline{V}_0(G)$ is semi-Montel.*
(3) *$H\overline{V}_0(G)$ is semireflexive.*

Remark 3.10. Corollary 3.9 is also true in the general case, for spaces $A\overline{V}(X)$ and $A\overline{V}_0(X)$. Indeed, (1) ⇒ (2) by Theorem 3.8 while (2) always implies (3). To see that in this case (3) ⇒ (1), it is sufficient to repeat arguments of the proof in [5, Proposition 14(a)].

Under some additional assumptions the case of holomorphic functions admits further refinements given below.

From Bierstedt, Bonet and Galbis [6, Theorem 1.5(d)] it follows easily that the condition (CD) holds (even, $B_{v_n}(G) = \overline{B_{(v_n)0}(G)}^{co}$ for all n) when $\mathcal{V} = (v_n)_n$ is an increasing sequence of radial weights on a balanced domain $G \subset \mathbb{C}^N$ such that $H_{v_1 0}(G)$ contains all the polynomials. If G is bounded, the last condition means that the weight v_1 can be extended continuously up to $\overline{G}$ with $v_1|_{\partial G} \equiv \infty$, while for $G = \mathbb{C}^N$ it means that v_1 is rapidly increasing at infinity, i.e., $\log|z| = o(\log v_1(z))$ as $z \to \infty$.

For some classes of domains, results like Theorem 3.8 are true without condition (CD). Namely, we suppose that G is either a domain in $\mathbb{C}$ whose complement has no one-point component or an absolutely convex bounded domain in $\mathbb{C}^N$. In Bierstedt and Bonet [5, Propositions 3, 7 and 12] it was proved that for domains of such a type the inclusion of $H_v(G)$ into $H_{w0}(G)$ is always compact. In this case we have also the following criteria for $\mathcal{V}H(G)$ and $H\overline{V}(G)$ to be (DFS) and semi-Montel, respectively.

Proposition 3.11. *Suppose that G is either a domain in $\mathbb{C}$ whose complement has no one-point component or an absolutely convex open bounded domain in $\mathbb{C}^N$. The following criteria are valid:*

(1) *The space $\mathcal{V}H(G)$ is (DFS) if and only if for each $n \in \mathbb{N}$ there exists $m > n$ such that $\widetilde{v}_n/v_m$ vanishes at infinity on G.*
(2) *The space $H\overline{V}(G)$ is semi-Montel if and only if $\widetilde{v}_n/\bar{v}$ vanishes at infinity on G for all $n \in \mathbb{N}$ and $\bar{v} \in \overline{V}$.*

Proof. For a domain G in $\mathbb{C}$ with the complement having no one-point component it was proved in [1, Theorems 4.3 and 4.5].

Furthermore, using [5, Lemma 11] and an evident modification of the proofs of [1, Theorems 3.13, 4.3 and 4.5] we get the same criteria in the case of an absolutely convex bounded domain $G \subset \mathbb{C}^N$. $\square$

As an immediate consequence, we get the following sharpening of Theorem 3.8.

Theorem 3.12. *Suppose that G is either a domain in $\mathbb{C}$ whose complement has no one-point component or an absolutely convex bounded domain in $\mathbb{C}^N$. Then the algebraic equalities $\mathcal{V}H(G) = \mathcal{V}_0 H(G)$ and $H\overline{V}(G) = H\overline{V}_0(G)$ are equivalent to $\mathcal{V}H(G)$ (DFS) and $H\overline{V}(G)$ semi-Montel, respectively.*

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