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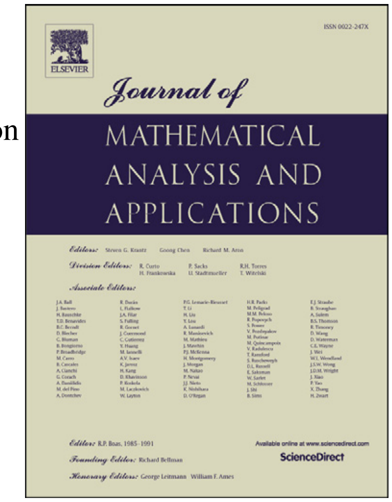

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On an inverse boundary value problem of a nonlinear elliptic equation in three dimensions

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Abstract

This work considers an inverse boundary value problem for a 3D nonlinear elliptic partial differential equation in a bounded domain. In general, the problem is severely ill-posed. The formal solution can be written as a hyperbolic cosine function in terms of the 2D elliptic operator via its eigenfunction expansion, and it is shown that the solution is stabilized or regularized if the large eigenvalues are cut off. In a theoretical framework, a truncation approach is developed to approximate the solution of the ill-posed problem in a regularization manner. Under some assumptions on regularity of the exact solution, we obtain several explicit error estimates including an error estimate of Hölder type. A local Lipschitz case of source term for this nonlinear problem is obtained. For numerical illustration, two examples on the elliptic sine-Gordon and elliptic Allen-Cahn equations are constructed to demonstrate the feasibility and efficiency of the proposed methods.

Keywords and phrases: Nonlinear elliptic equation, Ill-posed problem, Regularization, Truncation method.
Mathematics subject Classification 2000: 35K05, 35K99, 47J06, 47H10.

1. Introduction

In this paper, we consider the problem of reconstructing the temperature of a body from interior measurements. In fact, in many engineering contexts (see, e.g., [4]), we cannot attach a temperature sensor at the surface of a body (e.g., the skin of a missile). Hence, to get the temperature distribution on the surface, we have to use the temperature measured inside the body. Let L be a positive real number and $\Omega = (0,\pi)\times(0,\pi)$. We are interested in the following inverse boundary value problem: Find $u(x,y,0)$ for $(x,y)\in\Omega$ where $u(x,y,z)$ satisfies the following nonlinear elliptic equation:

$$\Delta u = F(x,y,z,u(x,y,z)), \quad (x,y,z)\in\Omega\times(0,+\infty), \tag{1.1}$$

subject to the conditions

$$\begin{cases} u(x,y,z) = 0, & (x,y,z)\in\partial\Omega\times(0,+\infty),\\ u(x,y,L) = \varphi(x,y), & (x,y)\in\Omega,\\ u_z(x,y,L) = 0, & (x,y)\in\Omega. \end{cases} \tag{1.2}$$

Here $\Delta u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2$, the function $\varphi\in L^2(\Omega)$ is known, and F is called the source function to be defined later. Having found $u(x,y,0)$ a forward problem can be solved to find $u(x,y,z)$ for all $(x,y,z)\in\Omega\times(0,L)$.

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It is widely recognized nowadays that Cauchy problems for the Poisson equation, and more generally for elliptic equations, has a central position in all inverse boundary value problems which are encountered in many practical applications such as electrocardiography [25], astrophysics [9] and plasma physics [3, 21]. These problems are also closely related to inverse source problems arising from, e.g., electroencephalography and magnetoencephalography [26]. The continued interest in this kind of problems is evidenced by the number of publications on this topic. We refer to the monograph [25] for further reading on Cauchy problems for elliptic equations.

It is well-known that inverse boundary value problems are exponentially ill-posed in the sense of Hadamard. Existence of solutions and their stability with respect to given data do not hold even if the data are very smooth. In fact, the problems are extremely sensitive to measurement errors; hence, even in the case of existence, a solution does not depend continuously on the given data. This, of course, implies that a properly designed numerical treatment is required.

Inverse boundary problems for *linear* elliptic equations have been studied extensively, see, e.g., [1, 3]. Indeed, in the case $F = 0$ in (1.1) with the following conditions

$$\begin{cases} u(x,y,z) = 0, & (x,y,z) \in \partial\Omega \times (0,+\infty), \\ u(x,y,L) = \varphi(x,y), & (x,y) \in \Omega, \\ \lim\limits_{z\to\infty} u_z(x,y,L) = 0, & (x,y) \in \Omega, \end{cases} \tag{1.3}$$

the problem is studied in [10, 26, 33]. In these studies, the algebraic invertibility of the inverse problem is established. However, regularization is not investigated. In [24], the authors apply the nonlocal boundary value method to solve an abstract Cauchy problem for the homogeneous elliptic equation. Eldén et al develope useful numerical methods to solve the homogeneous problem; see for example [14, 15, 16, 17]. Level set type methods are also proposed [32] for Cauchy problems for linear elliptic equations.

Although there are many works on Cauchy problems for linear elliptic equations, to the best of our knowledge the literature on the *nonlinear* case is very few. In the abstract framework of operators on Hilbert spaces, regularization techniques are developed by B. Kaltenbacher and her coauthors in [2, 27, 28, 29]. The present paper serves to develop necessary theoretical bases for a regularization of problem (1.1)–(1.2).

Our approach can be summarized as follows. Let φ and φ^ϵ be the exact and measured data at $z = L$, respectively, which satisfy $\|\varphi - \varphi^\epsilon\|_{L^2(\Omega)} \le \epsilon$. Assume that problem (1.1)–(1.2) has a unique solution $u(x,y,z)$. By using the method of separation of variables, one can show that

$$u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\cosh\left((L-z)\sqrt{m^2+n^2}\right)\varphi_{mn} + \int_z^L \frac{\sinh\left((\tau-z)\sqrt{m^2+n^2}\right)}{\sqrt{m^2+n^2}} F_{mn}(u)(\tau)d\tau\right]\phi_{mn}(x,y). \tag{1.4}$$

Indeed, let $u(x,y,z) = \sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty} u_{mn}(z)\phi_{mn}(x,y)$ be the Fourier series in $L^2(\Omega)$ with $\phi_{mn}(x,y) = \frac{2}{\pi}\sin(mx)\sin(ny)$. From (1.1)–(1.2) , we can obtain the following ordinary differential equation with given data at $z = L$

$$\begin{cases} \dfrac{d^2}{dz^2}u_{mn}(z) - \sqrt{m^2+n^2}u_{mn}(z) = F_{mn}(u)(z), \quad z \in (0,+\infty), \\ u_{mn}(L) = \varphi_{mn}, \\ \dfrac{d}{dz}u_{mn}(L) = 0, \end{cases} \tag{1.5}$$

where $F_{mn}(u)(z) = \int_\Omega F(x,y,z,u(x,y,z))\phi_{mn}(x,y)dxdy$.

It is easy to see that the solution of problem (1.5) is given by

$$u_{mn}(z) = \cosh\left((L-z)\sqrt{m^2+n^2}\right)\varphi_{mn} + \int_z^L \frac{\sinh\left((\tau-z)\sqrt{m^2+n^2}\right)}{\sqrt{m^2+n^2}} F_{mn}(u)(\tau)d\tau. \tag{1.6}$$

Then the solution of (1.1)–(1.2) satisfies the integral equation (1.4). Since $z < L$, we know from (1.4) that, when m, n become large, the terms

$$\cosh\left((L-z)\sqrt{m^2+n^2}\right) \quad \text{and} \quad \frac{\sinh\left((\tau-z)\sqrt{m^2+n^2}\right)}{\sqrt{m^2+n^2}}$$

increase rather quickly. Thus, these terms are the cause for instability. In order to regularize problem (1.1)–(1.2) , we stabilize problem (1.4) by filtering the high frequencies with a suitable method. The essence of our regularization method is to eliminate all high frequencies from the solution, and consider (1.4) only for m, n satisfying $\sqrt{m^2+n^2} \leq C_\epsilon$. Here C_ϵ is a constant which will be selected appropriately as a regularization parameter which satisfies $\lim_{\epsilon\to 0} C_\epsilon = \infty$. Such a method is the called **Fourier truncated method.** We shall use the following well-posed problem

$$\begin{cases} \Delta u^\epsilon = P_{C_\epsilon} F(x,y,z,u^\epsilon(x,y,z)), & (x,y,z) \in \Omega \times (0,+\infty), \\ u^\epsilon(x,y,z) = 0, & (x,y,z) \in \partial\Omega \times (0,+\infty), \\ u^\epsilon(x,y,L) = \varphi^\epsilon(x,y), \quad u^\epsilon_z(x,y,L) = 0, & (x,y) \in \Omega, \end{cases} \tag{1.7}$$

where P_{C_ϵ} is the orthogonal projection from $L^2(\Omega)$ onto the eigenspace $\text{span}\{\phi_{mn}(x,y) | \sqrt{m^2+n^2} \leq C_\epsilon\}$, i.e.

$$P_{C_\epsilon} w = \sum_{\substack{m,n\geq 1 \\ \sqrt{m^2+n^2}\leq C_\epsilon}} \langle w, \phi_{mn}\rangle \, \phi_{mn} \quad \text{for all } w \in L^2(\Omega).$$

The Fourier truncation method is useful and convenient for dealing with ill-posed problems. The method is effective for linear backward problems; see e.g. [19, 36]. It has also been successfully applied to some other ill-posed problems [15, 39]. In the present paper, by using the truncation regularization method, we show that the approximate solution u^ϵ of problem (1.7) satisfies the following integral equation

$$u^\epsilon(x,y,z) = \sum_{\substack{m,n\geq 1 \\ \sqrt{m^2+n^2}\leq C_\epsilon}} \left[\cosh\left((L-z)\sqrt{m^2+n^2}\right)\varphi^\epsilon_{mn} + \int_z^L \frac{\sinh\left((\tau-z)\sqrt{m^2+n^2}\right)}{\sqrt{m^2+n^2}} F_{mn}(u^\epsilon)(\tau)d\tau\right]\phi_{mn}(x,y) \tag{1.8}$$

where

$$\varphi^\epsilon_{mn} = \int_\Omega \varphi^\epsilon(x,y)\phi_{mn}(x,y)dxdy \quad \text{and} \quad F_{mn}(u^\epsilon)(\tau) = \int_\Omega F(x,y,\tau,u^\epsilon(x,y,\tau))\phi_{mn}(x,y)dxdy.$$

We then prove that under some suitable conditions of the exact solution u, the approximate solution u^ϵ converges to u as $\epsilon \to 0$. To the best of our knowledge, this is the first result on convergence rate when a regularization method is used to solve inverse boundary value problems for elliptic equations with locally Lipschitz source terms. Moreover, error estimates in higher Sobolev space is presented for the first time in this paper. In particular, in this paper we will present regularized solutions for two cases of the source function F:

Case 1. F is a global Lipschitz function.

Case 2. F is a locally Lipschitz function.

Our method is in principle not restricted to the Poisson equation on a rectangular domain. In fact, the method works for more general operators defined on any bounded Euclidean domain. Indeed, the analysis presented in this paper will be particularly derived from a general abstract problem in a Hilbert space H

$$\begin{cases} u_{zz} = Au + F(z,u(z)), & z \in (0,+\infty), \\ u(L) = \varphi, \quad u_z(L) = 0, \end{cases} \tag{1.9}$$

where u is a mapping from $[0,+\infty)$ to H and $A : D(A) \to H$ is a positive self-adjoint unbounded operator.

The paper is organized as follows. In Section 2, we present the main results on regularization theory for both cases: global and local Lipschitz source functions. Section 3 is devoted to numerical experiments which show the efficacy of the proposed methods. We finish the paper with some concluding remarks in Section 4.

2. The main results

Definition 2.1 (Gevrey-type space (see [7, 35])). *The Gevrey class of functions of order $s > 0$ and index $\sigma > 0$ is denoted by $G_\sigma^{s/2}$ and is defined as*

$$G_\sigma^{s/2} := \left\{ f \in L^2(\Omega) : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m^2+n^2)^{\frac{s}{2}} \exp\left(2\sigma\sqrt{m^2+n^2}\right) |\langle f, \phi_{mn}\rangle|^2 < \infty \right\}.$$

It is a Hilbert space equipped with the norm

$$\|f\|_{G_\sigma^{s/2}} = \sqrt{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m^2+n^2)^{\frac{s}{2}} \exp\left(2\sigma\sqrt{m^2+n^2}\right) |\langle f, \phi_{mn}\rangle|^2 < \infty \Big\}}.$$

For a Hilbert space B, we denote

$$L^\infty(0,L;B) = \left\{ f : [0,L] \to B \,\Big|\, \operatorname*{ess\,sup}_{0\le z\le L} \|f(z)\|_B < \infty \right\}$$

and

$$\|f\|_{L^\infty(0,L;B)} = \operatorname*{ess\,sup}_{0\le z\le L} \|f(z)\|_B.$$

First, we consider some assumptions on the exact solution

$$\operatorname*{ess\,sup}_{0\le z\le L} \sqrt{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \exp\left(2z\sqrt{m^2+n^2}\right) u_{mn}^2(z)} \le I_1, \tag{2.10}$$

$$\operatorname*{ess\,sup}_{0\le z\le L} \sqrt{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (m^2+n^2)^\beta \exp\left(2z\sqrt{m^2+n^2}\right) u_{mn}^2(z)} \le I_2 \tag{2.11}$$

$$\operatorname*{ess\,sup}_{0\le z\le L} \sqrt{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \exp\left(2(z+\alpha)\sqrt{m^2+n^2}\right) u_{mn}^2(z)} \le I_3, \tag{2.12}$$

$$\|u\|_{L^\infty(0,L;G_L^0)} = \operatorname*{ess\,sup}_{0\le z\le L} \|u(z)\|_{G_L^0} = \operatorname*{ess\,sup}_{0\le z\le L} \sqrt{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \exp\left(2L\sqrt{m^2+n^2}\right) u_{mn}^2(z)} \le I_4, \tag{2.13}$$

for all $z \in [0,L]$, where $\alpha, \beta, I_1, I_2, I_3, I_4$ are positive constants and $u_{mn}(z) = \int_\Omega u(x,y,z)\phi_{mn}(x,y)dxdy$.

The following lemma will be useful in the subsequent analysis.

Lemma 2.1. *For any $w \in G_\sigma^k$ we have following inequality*

$$\|w - P_{C_\epsilon} w\|_{L^2(\Omega)} \le C_\epsilon^{-k} e^{-\sigma C_\epsilon} \|w\|_{G_\sigma^k}.$$

Proof. For $w \in G_\sigma^k$, we have

$$\begin{aligned}
\|w - P_{C_\epsilon} w\|^2_{L^2(\Omega)} &= \sum_{\substack{m,n\geq 1 \\ \sqrt{m^2+n^2}\leq C_\epsilon}} (m^2+n^2)^{-k} \exp\Big(-2\sigma\sqrt{m^2+n^2}\Big)(m^2+n^2)^k \exp\Big(2\sigma\sqrt{m^2+n^2}\Big)\Big|\Big\langle w, \phi_{mn}\Big\rangle\Big|^2 \\
&\leq C_\epsilon^{-2k} e^{-2\sigma C_\epsilon} \sum_{\substack{m,n\geq 1 \\ \sqrt{m^2+n^2}\leq C_\epsilon}} (m^2+n^2)^k \exp\Big(2\sigma\sqrt{m^2+n^2}\Big)\Big|\Big\langle w, \phi_{mn}\Big\rangle\Big|^2 \\
&\leq C_\epsilon^{-2k} e^{-2\sigma C_\epsilon} \|w\|^2_{G_\sigma^k}.
\end{aligned}$$

This completes the proof. □

2.1. Results for global Lipschitz source functions

Assume in this section that F is a global Lipschitz function, i.e. $F \in L^\infty([0,\pi]\times[0,\pi]\times[0,L]\times\mathbb{R})$, satisfying the following condition

$$|F(x,y,z,w) - F(x,y,z,v)| \leq K_F |w - v| \quad \forall x,y,z,w,v, \tag{2.14}$$

for some $K_F > 0$ independent of x, y, z, w, v.

In this paper, we shall write $u(z) = u(\cdot,\cdot,z)$ for short. The following theorem provides an error estimate in the L^2-norm when the exact solution belongs to the Gevrey space.

Theorem 2.1. *Let $\epsilon > 0$ and let F satisfy* (2.14)*. Then the problem* (1.7) *has a unique solution $u^\epsilon \in C([0,L]; L^2(\Omega))$.*

1. *Assume that u satisfies* (2.10)*. If $C_\epsilon > 0$ is chosen such that ϵe^{LC_ϵ} is bounded, then we obtain*

$$\|u^\epsilon(z) - u(z)\|_{L^2(\Omega)} \leq \sqrt{2I_1^2 + 4\epsilon^2 e^{2LC_\epsilon}} e^{2K_F^2 L(L-z)} e^{-zC_\epsilon}.$$

Moreover, for ϵ sufficiently small, there exists $z_\epsilon \in [0,L]$ such that $\lim_{\epsilon\to 0} z_\epsilon = 0$ and

$$\|u^\epsilon(z_\epsilon) - u(0)\|_{L^2(\Omega)} \leq \Big[\sqrt{2I_1^2 + 4\epsilon^2 e^{2LC_\epsilon}} e^{2K_F^2 L(L-z)} + \|u_z\|_{L^\infty(0,L;L^2(\Omega))}\Big]\frac{\ln(C_\epsilon)}{C_\epsilon}.$$

As a consequence, if we choose $C_\epsilon = \frac{1}{L}\ln(\frac{1}{\epsilon})$ then

$$\begin{cases} \|u^\epsilon(z) - u(z)\|_{L^2(\Omega)} \leq \sqrt{2I_1^2 + 4} e^{2K_F^2 L^2} \epsilon^{\frac{z}{L}}, \ \ \textit{for} \ \ z \in (0,L] \\ \|u^\epsilon(z_\epsilon) - u(0)\|_{L^2(\Omega)} \leq \Big[\sqrt{2I_1^2 + 4} e^{2K_F^2 L^2} + I_1\Big]\dfrac{L\ln\Big(\frac{1}{L}\ln(\frac{1}{\epsilon})\Big)}{\ln(\frac{1}{\epsilon})}. \end{cases} \tag{2.15}$$

2. *Assume that u satisfies* (2.11)*. If C_ϵ is chosen such that $\lim_{\epsilon\to+\infty} \epsilon e^{LC_\epsilon} = 0$, then we have*

$$\|u^\epsilon(z) - u(z)\|_{L^2(\Omega)} \leq \sqrt{2C_\epsilon^{-2\beta} I_2^2 + 4\epsilon^2 e^{2LC_\epsilon}} e^{2K_F^2 L(L-z)} e^{-zC_\epsilon}.$$

3. *Assume that u satisfies* (2.12)*. If C_ϵ is chosen such that $\lim_{\epsilon\to+\infty} \epsilon e^{LC_\epsilon} = 0$, then we have*

$$\|u^\epsilon(z) - u(z)\|_{L^2(\Omega)} \leq \sqrt{2e^{-2\alpha C_\epsilon} I_3^2 + 4\epsilon^2 e^{2LC_\epsilon}} e^{2K_F^2 L(L-z)} e^{-zC_\epsilon}.$$

Remark 2.1. 1. *In part 1, Theorem 2.1, if we choose $C_\epsilon = \frac{1}{L}\ln(\frac{1}{\epsilon})$ then the error is of order $\epsilon^{\frac{z}{L}}$. This error gives no information on the continuous dependence of the solution on the data at $z = 0$. To improve this, we need a stronger condition of u as in Part 2, Part 3.*

2. *In part 2, Theorem 2.1, if we choose* $C_\epsilon = \frac{\gamma}{L}\ln\left(\frac{1}{\epsilon}\right), \quad 0 < \gamma < 1$, *then the error is of logarithmic order* $\left[\ln(\frac{1}{\epsilon})\right]^{-\beta}$.

3. *In part 3, Theorem 2.1, if we choose* $C_\epsilon = \frac{1}{L+\alpha}\ln\left(\frac{1}{\epsilon}\right)$, *then the error is of Hölder order* $\epsilon^{\frac{\alpha}{L+\alpha}}$.

Before proving Theorem 2.1, we prove the following lemmas.

Lemma 2.2. *The problem* (1.8) *has unique a weak solution* $u^\epsilon(x,y,z)$ *which is in* $C([0,L];L^2(\Omega)\cap L^2(0,L;H_0^1(\Omega))\cap C^1(0,L;H_0^1(\Omega))$.

Proof. Put

$$H(u^\epsilon)(x,y,z) = \Psi(x,y,z) + \sum_{\substack{m,n\geq 1\\ \sqrt{m^2+n^2}\leq C_\epsilon}} \left(\int_z^L \frac{\sinh\left((\tau-z)\sqrt{m^2+n^2}\right)}{\sqrt{m^2+n^2}} F_{mn}(u^\epsilon)(\tau)d\tau\right)\phi_{mn}(x,y),$$

where

$$\Psi(x,y,z) = \sum_{\substack{m,n\geq 1\\ \sqrt{m^2+n^2}\leq C_\epsilon}} \cosh\left((L-z)\sqrt{m^2+n^2}\right)\varphi^\epsilon_{mn}\phi_{mn}(x,y).$$

We claim that

$$\left\|H^p(v^\epsilon)(z) - H^p(w^\epsilon)(z)\right\|_{L^2(\Omega)} \leq \sqrt{\frac{\left(K_F^2 L\exp(2LC_\epsilon)\max\{L,1\}\right)^p}{p!}}|||v^\epsilon - w^\epsilon|||, \tag{2.16}$$

for $p \geq 1$, where $|||.|||$ is the sup norm in $C([0,L];L^2(\Omega))$. We shall prove the above inequality by induction.

For $p = 1$, using the inequality

$$\int_z^L \left[\frac{\sinh\left((\tau-z)\sqrt{m^2+n^2}\right)}{\sqrt{m^2+n^2}}\right]^2 d\tau \leq \int_z^L \exp\left(2\sqrt{m^2+n^2}(\tau-z)\right)d\tau \leq \exp(2\sqrt{m^2+n^2}L)L,$$

and the Lipschitz property of F, we have

$$\begin{aligned}
\left\|H(v^\epsilon)(z) - H(w^\epsilon)(z)\right\|^2_{L^2(\Omega)} &= \sum_{\substack{m,n\geq 1\\ \sqrt{m^2+n^2}\leq C_\epsilon}} \left[\int_z^L \frac{\sinh\left((\tau-z)\sqrt{m^2+n^2}\right)}{\sqrt{m^2+n^2}}(F_{mn}(v^\epsilon)(\tau) - F_{mn}(w^\epsilon)(\tau))d\tau\right]^2 \\
&\leq \sum_{\substack{m,n\geq 1\\ \sqrt{m^2+n^2}\leq C_\epsilon}} \int_z^L \left[\frac{\sinh\left((\tau-z)\sqrt{m^2+n^2}\right)}{\sqrt{m^2+n^2}}\right]^2 d\tau \int_z^L \left|F_{mn}(v^\epsilon)(\tau) - F_{mn}(w^\epsilon)(\tau)\right|^2 d\tau \\
&\leq \exp(2LC_\epsilon)L\int_z^L \left[\sum_{\substack{m,n\geq 1\\ \sqrt{m^2+n^2}\leq C_\epsilon}} \left|F_{mn}(v^\epsilon)(\tau) - F_{mn}(w^\epsilon)(\tau)\right|^2\right]d\tau \\
&\leq \exp(2LC_\epsilon)L\int_z^L \left[\sum_{m=1}^\infty\sum_{m=1}^\infty \left|F_{mn}(v^\epsilon)(\tau) - F_{mn}(w^\epsilon)(\tau)\right|^2\right]d\tau \\
&= \exp(2LC_\epsilon)L\int_z^L \left\|F(x,y,\tau,v^\epsilon(x,y,\tau)) - F(x,y,\tau,u^\epsilon(x,y,\tau))\right\|^2 d\tau \\
&\leq \max\{L,1\}K_F^2\exp(2LC_\epsilon)L^2|||v^\epsilon - w^\epsilon|||^2.
\end{aligned}$$

Thus (2.16) holds for $p = 1$. Suppose that (2.16) holds for $p = j$. We prove that (2.16) holds for $p = j+1$. We have

$$\begin{aligned}
&\left\|H^{j+1}(v^\epsilon)(z) - H^{j+1}(w^\epsilon)(z)\right\|^2_{L^2(\Omega)} \\
&\quad= \sum_{\substack{m,n\geq 1\\ \sqrt{m^2+n^2}\leq C_\epsilon}} \left[\int_z^L \frac{\sinh\left((\tau - z)\sqrt{m^2+n^2}\right)}{\sqrt{m^2+n^2}}\Big(F_{mn}(H^j(v^\epsilon))(\tau) - F_{mn}(H^j(w^\epsilon))(\tau)\Big)d\tau\right]^2 \\
&\quad\leq \exp(2LC_\epsilon)L\int_z^L \left(\sum_{\substack{m,n\geq 1\\ \sqrt{m^2+n^2}\leq C_\epsilon}} \left|F_{mn}(H^j(v^\epsilon))(\tau) - F_{mn}(H^j(w^\epsilon))(\tau)\right|^2\right)d\tau \\
&\quad\leq \exp(2LC_\epsilon)L\int_z^L \left\|F(\tau, H^j(v^\epsilon)(\tau)) - F(\tau, H^j(w^\epsilon)(\tau))\right\|^2 d\tau \\
&\quad\leq \exp(2LC_\epsilon)LK_F^2\int_z^L \|H^j(v^\epsilon)(\tau) - H^j(w^\epsilon)(\tau)\|^2 d\tau \\
&\quad\leq \exp(2LC_\epsilon)LK_F^2K_F^{2j}\exp(2LjC_\epsilon)\left(\int_z^L \frac{(L-\tau)^j}{j!}d\tau\right)\Big(\max\{A,1\}\Big)^j|||v^\epsilon - w^\epsilon|||^2 \\
&\quad\leq K_F^{2(j+1)}\exp(2L(j+1)C_\epsilon)\frac{(L-z)^{j+1}}{(j+1)!}\Big(\max\{L,1\}\Big)^{j+1}|||v^\epsilon - w^\epsilon|||^2.
\end{aligned}$$

Therefore, we get

$$\left\|H^p(v^\epsilon)(z) - H^p(w^\epsilon)(z)\right\|_{L^2(\Omega)} \leq \sqrt{\frac{\left(K_F^2L\exp(2LC_\epsilon)\max\{L,1\}\right)^p}{p!}}|||v^\epsilon - w^\epsilon||| \tag{2.17}$$

for all $v^\epsilon, w^\epsilon \in C([0,L];L^2(\Omega))$. Now we consider $H : C([0,L];L^2(\Omega)) \to C([0,L];L^2(\Omega))$. It can be shown that

$$\lim_{p\to\infty}\sqrt{\frac{\left(K_F^2Le^{2LC_\epsilon}\max\{L,1\}\right)^p}{p!}} = 0.$$

As a consequence, there exists a positive integer number p_0 such that H^{p_0} is a contraction. It follows that the equation $H^{p_0}(u) = u$ has a unique solution $u^\epsilon \in C([0,L];L^2(\Omega))$ We claim that $H(u^\epsilon) = u^\epsilon$. In fact, one has $H(G^{p_0}(u^\epsilon)) = H(u^\epsilon)$. Hence $H^{p_0}(H(u^\epsilon)) = H(u^\epsilon)$. By the uniqueness of the fixed point of H^{p_0}, one has $H(u^\epsilon) = u^\epsilon$, i.e., the equation $H(u^\epsilon) = u^\epsilon$ has a unique solution $u^\epsilon \in C([0,L];L^2(\Omega))$. □

Lemma 2.3. *Let u be an exact solution to problem (1.1) and let u^ϵ be as* (1.8). *Then we have the following estimate*

$$\begin{aligned}
\|u^\epsilon(z) - P_{C_\epsilon}u(z)\|^2_{L^2(\Omega)} &\leq 2\exp\Big(2(L-z)C_\epsilon\Big)\|\varphi^\epsilon - \varphi\|^2_{L^2(\Omega)} \\
&\quad + 2K_F^2(L-z)\int_z^L \exp\Big(2(\tau - z)C_\epsilon\Big)\|u^\epsilon(\tau) - u(\tau)\|^2_{L^2(\Omega)}d\tau.
\end{aligned}$$

Proof. By using the Lipschitz property of F, we obtain

$$
\begin{aligned}
\|u^\epsilon - P_{C_\epsilon} u\|^2_{L^2(\Omega)} \le\ & 2 \sum_{\substack{m,n\ge 1\\ \sqrt{m^2+n^2}\le C_\epsilon}} \left| \cosh\Big((L-z)\sqrt{m^2+n^2}\Big)(\varphi^\epsilon_{mn} - \varphi_{mn}) \right|^2 \\
& + 2 \sum_{\substack{m,n\ge 1\\ \sqrt{m^2+n^2}\le C_\epsilon}} \left[\int_z^L \frac{\sinh\Big((\tau - z)\sqrt{m^2+n^2}\Big)}{\sqrt{m^2+n^2}} \Big(F_{mn}(u^\epsilon)(\tau) - F_{mn}(u)(\tau)\Big) d\tau \right]^2 \\
\le\ & 2 \sum_{\substack{m,n\ge 1\\ \sqrt{m^2+n^2}\le C_\epsilon}} \left| \exp\Big((L-z)\sqrt{m^2+n^2}\Big)(\varphi^\epsilon_{mn} - \varphi_{mn}) \right|^2 \\
& + 2 \sum_{\substack{m,n\ge 1\\ \sqrt{m^2+n^2}\le C_\epsilon}} \left[\int_z^L \exp\Big((\tau - z)\sqrt{m^2+n^2}\Big) \Big(F_{mn}(u^\epsilon)(\tau) - F_{mn}(u)(\tau)\Big) d\tau \right]^2 \\
\le\ & 2 \exp\Big(2(L-z)C_\epsilon\Big) \|\varphi^\epsilon - \varphi\|^2_{L^2(\Omega)} \\
& + 2K_F^2 (L-z) \int_z^L \exp\Big(2(\tau - z)C_\epsilon\Big) \|u^\epsilon(\tau) - u(\tau)\|^2_{L^2(\Omega)} d\tau.
\end{aligned}
$$

This completes the proof of lemma. □

We now prove Theorem 2.1.

Proof. **Proof of Part 1:** Since $u \in G^0_z$, Lemma 2.1 gives

$$\|u(z) - P_{C_\epsilon} u(z)\|^2_{L^2(\Omega)} \le e^{-2zC_\epsilon} \|u(z)\|^2_{G^0_z}.$$

Lemma 2.2 and the triangle inequality yield

$$
\begin{aligned}
\|u^\epsilon(z) - u(z)\|^2_{L^2(\Omega)} &\le 2\|u^\epsilon(z) - P_{C_\epsilon} u(z)\|^2_{L^2(\Omega)} + 2\|u(z) - P_{C_\epsilon} u(z)\|^2_{L^2(\Omega)} \\
&\le 2e^{-2zC_\epsilon} \|u(z)\|^2_{G^0_z} + 4\exp\Big(2(L-z)C_\epsilon\Big) \|\varphi^\epsilon - \varphi\|^2_{L^2(\Omega)} \\
&\qquad + 4K_F^2 (L-z) \int_z^L \exp\Big(2(\tau - z)C_\epsilon\Big) \|u^\epsilon(\tau) - u(\tau)\|^2_{L^2(\Omega)} d\tau.
\end{aligned}
$$

This implies that

$$e^{2zC_\epsilon} \|u^\epsilon(z) - u(z)\|^2_{L^2(\Omega)} \le 2 \sup_{0\le z\le L} \|u(z)\|^2_{G^0_z} + 4e^{2LC_\epsilon}\epsilon^2 + 4K_F^2 L \int_z^L e^{2\tau C_\epsilon} \|u^\epsilon(\tau) - u(\tau)\|^2_{L^2(\Omega)} d\tau.$$

Applying Gronwall's inequality, we obtain

$$e^{2zC_\epsilon} \|u^\epsilon(z) - u(z)\|^2_{L^2(\Omega)} \le \Big[2 \sup_{0\le z\le L} \|u(z)\|^2_{G^0_z} + 4e^{2LC_\epsilon}\epsilon^2\Big] e^{4K_F^2 L(L-z)}.$$

Therefore

$$\|u^\epsilon(z) - u(z)\|_{L^2(\Omega)} \le \sqrt{2I_1^2 + 4\epsilon^2 e^{2LC_\epsilon}}\, e^{2K_F^2 L(L-z)} e^{-zC_\epsilon}. \tag{2.18}$$

If ϵ is sufficiently small then $C_\epsilon > \frac{1}{L} e^{\frac{1}{L}}$. Consider the following equation

$$e^{-zC_\epsilon} = z.$$

The solution to this equation satisfies another equation $h(z) = 0$ where $h(z) = \ln(z) + zC_\epsilon$. The function h is strictly increasing. Moreover, $\lim_{z\to 0^+} h(z) = -\infty$ and

$$h\Big(\frac{\ln(C_\epsilon)}{C_\epsilon}\Big) = \ln\Big(\ln(C_\epsilon)\Big) - \ln(C_\epsilon) + \ln(C_\epsilon) = \ln\Big(\ln(C_\epsilon)\Big) > 0$$

for ϵ small enough. Thus the equation $h(z) = 0$ has a unique solution $z_\epsilon > 0$ which satisfies

$$z_\epsilon \le \frac{\ln(C_\epsilon)}{C_\epsilon}.$$

The continuity of u_z gives, for sufficiently small ϵ,

$$\|u(z_\epsilon) - u(0)\|_{L^2(\Omega)} = \Big\| \int_0^{z_\epsilon} u_z(\xi) d\xi \Big\|_{L^2(\Omega)} \le z_\epsilon \sup_{0\le z\le L} \|u_z(z)\|_{L^2(\Omega)}. \tag{2.19}$$

Combining (2.18) and (2.19) and noting that $e^{-z_\epsilon C_\epsilon} = z_\epsilon \le \frac{\ln(C_\epsilon)}{C_\epsilon}$, we obtain

$$\begin{aligned}
\|u^\epsilon(z_\epsilon) - u(0)\|_{L^2(\Omega)} &\le \|u^\epsilon(z_\epsilon) - u(z_\epsilon)\|_{L^2(\Omega)} + \|u(z_\epsilon) - u(0)\|_{L^2(\Omega)} \\
&\le \sqrt{2 \sup_{0\le z\le L} \|u(z)\|^2_{G^0_z} + 4\epsilon^2 e^{2LC_\epsilon}} e^{2K_F^2 L(L-z)} e^{-z_\epsilon C_\epsilon} + z_\epsilon \sup_{0\le z\le L} \|u_z(z)\|_{L^2(\Omega)} \\
&= \left[\sqrt{2I_1^2 + 4\epsilon^2 e^{2LC_\epsilon}} e^{2K_F^2 L(L-z)} + \|u_z\|_{L^\infty(0,L;L^2(\Omega))} \right] \frac{\ln(C_\epsilon)}{C_\epsilon}.
\end{aligned}$$

Proof of Part 2: Since $u \in G_z^\beta$ Lemma 2.1 gives

$$\|u(z) - P_{C_\epsilon} u(z)\|^2_{L^2(\Omega)} \le C_\epsilon^{-2\beta} e^{-2zC_\epsilon} \|u(z)\|^2_{G_z^\beta}.$$

Lemma 2.2 and triangle inequality lead to

$$\begin{aligned}
\|u^\epsilon(z) - u(z)\|^2_{L^2(\Omega)} &\le 2\|u^\epsilon - P_{C_\epsilon} u\|^2_{L^2(\Omega)} + 2\|u - P_{C_\epsilon} u\|^2_{L^2(\Omega)} \\
&\le 2C_\epsilon^{2\beta} e^{-2zC_\epsilon} \|u(z)\|^2_{G_z^\beta} + 4\exp\Big(2(L-z)C_\epsilon\Big) \|\varphi^\epsilon - \varphi\|^2_{L^2(\Omega)} \\
&\qquad + 4K_F^2 (L-z) \int_z^L \exp\Big(2(\tau - z)C_\epsilon\Big) \|u^\epsilon(\tau) - u(\tau)\|^2_{L^2(\Omega)} d\tau.
\end{aligned}$$

This implies that

$$e^{2zC_\epsilon} \|u^\epsilon(z) - u(z)\|^2_{L^2(\Omega)} \le 2C_\epsilon^{-2\beta} \sup_{0\le z\le L} \|u(z)\|^2_{G_z^\beta} + 4e^{2LC_\epsilon}\epsilon^2 + 4K_F^2 L \int_z^L e^{2\tau C_\epsilon} \|u^\epsilon(\tau) - u(\tau)\|^2_{L^2(\Omega)} d\tau.$$

Applying Gronwall's inequality, we obtain

$$e^{2zC_\epsilon} \|u^\epsilon(z) - u(z)\|^2_{L^2(\Omega)} \le \Big[2C_\epsilon^{-2\beta} \sup_{0\le z\le L} \|u(z)\|^2_{G_z^\beta} + 4e^{2LC_\epsilon}\epsilon^2 \Big] e^{4K_F^2 L(L-z)}.$$

Therefore

$$\|u^\epsilon(z) - u(z)\|_{L^2(\Omega)} \le \sqrt{2C_\epsilon^{-2\beta} I_2^2 + 4\epsilon^2 e^{2LC_\epsilon}} e^{2K_F^2 L(L-z)} e^{-zC_\epsilon}.$$

Part 3 can be proved by using the same technique. The proof of which is omitted. $\square$

Corresponding results to Theorem 2.1 in two dimensions can be summarized in following corollary.

Corollary 2.1. *Let u satisfy the 2-D nonlinear problem*

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = F(x, y, u(x, y)), & (x, y) \in \Omega \times (0, +\infty), \\ u(x, y) = 0, & (x, y) \in \partial\Omega \times (0, +\infty), \\ u(x, 0) = h(x), & x \in \Omega, \end{cases}$$

where $\Omega = (0, \pi)$ and h is the unknown boundary condition determined from given interior data

$$\begin{cases} u(x, L) = \varphi(x), & x \in \Omega, \\ u_y(x, L) = 0, & x \in \Omega. \end{cases}$$

If $F \in L^\infty([0, \pi] \times [0, L] \times \mathbb{R})$ satisfies the condition

$$|F(x, y, w) - F(x, y, v)| \le K_F |w - v| \quad \forall x, y, w, v,$$

for some $K_F > 0$ independent of x, y, w, v, then the following well-posed problem

$$\begin{cases} \Delta u^\epsilon = P_{C_\epsilon} F(x, y, u^\epsilon(x, y)), & (x, y) \in \Omega \times (0, +\infty), \\ u^\epsilon(x, y) = 0, & (x, y) \in \partial\Omega \times (0, +\infty), \\ u^\epsilon(x, L) = \varphi^\epsilon(x), \quad u^\epsilon_y(x, L) = 0, & x \in \Omega, \end{cases}$$

has a unique solution $u^\epsilon \in C([0, L]; L^2(0, \pi))$. Here P_{C_ϵ} is the orthogonal projection from $L^2(\Omega)$ onto the eigenspace $\mathrm{span}\{\phi_m(x) \mid m \le C_\epsilon\}$ *, i.e.*

$$P_{C_\epsilon} w = \sum_{m=1}^{[C_\epsilon]} \left\langle w, \phi_m \right\rangle \phi_m \quad \forall w \in L^2(\Omega).$$

1. *Assume that u satisfies*

$$\sup_{0 \le y \le L} \sqrt{\sum_{m=1}^{\infty} \exp\left(2my\right) u_m^2(y)} \le J_1.$$

   If $C_\epsilon > 0$ is chosen such that ϵe^{LC_ϵ} is bounded, then we obtain

$$\|u^\epsilon(y) - u(y)\|_{L^2(\Omega)} \le \sqrt{2J_1^2 + 4\epsilon^2 e^{2LC_\epsilon}}\, e^{2K_F^2 L(L-y)} e^{-yC_\epsilon}.$$

   Moreover, for ϵ sufficiently small, there exists $y_\epsilon \in [0, L]$ such that $\lim_{\epsilon \to 0} y_\epsilon = 0$ and

$$\|u^\epsilon(y_\epsilon) - u(0)\|_{L^2(\Omega)} \le \left[\sqrt{2J_1^2 + 4\epsilon^2 e^{2LC_\epsilon}}\, e^{2K_F^2 L(L-y)} + \|u_y\|_{L^\infty(0,L;L^2(\Omega))} \right] \frac{\ln(C_\epsilon)}{C_\epsilon}.$$

2. *Assume that u satisfies*

$$\sup_{0 \le y \le L} \sqrt{\sum_{m=1}^{\infty} m^{2\beta} \exp\left(2ym\right) u_m^2(y)} \le J_2.$$

   If C_ϵ is chosen such that $\lim_{\epsilon \to +\infty} \epsilon e^{LC_\epsilon} = 0$, then we have

$$\|u^\epsilon(y) - u(y)\|_{L^2(\Omega)} \le e^{2K_F^2 L(L-y)} \sqrt{2C_\epsilon^{-2\beta} J_2^2 + 4\epsilon^2 e^{2LC_\epsilon}}\, e^{-yC_\epsilon}.$$

3. *Assume that u satisfies*

$$\sup_{0\le y\le L}\sqrt{\sum_{m=1}^{\infty}\exp\Big(2(z+\alpha)m\Big)u_m^2(y)}\le J_3.$$

If C_ϵ is chosen such that $\lim_{\epsilon\to+\infty}\epsilon e^{LC_\epsilon}=0$, then we have

$$\|u^\epsilon(y)-u(y)\|_{L^2(\Omega)}\le\sqrt{2e^{-2\alpha C_\epsilon}I_3^2+4\epsilon^2e^{2LC_\epsilon}}e^{2K_F^2L(L-y)}e^{-yC_\epsilon}.$$

In the next theorem, we establish the Hölder estimate.

Theorem 2.2. *Let $\epsilon>0$ and let F be the function defined in* (2.14). *If u satisfies* (2.13) *then by choosing C_ϵ such that $\lim_{\epsilon\to+\infty}\epsilon e^{LC_\epsilon}=0$, we can construct a regularized solution U^ϵ such that*

$$\|U^\epsilon(z)-u(z)\|_{L^2(\Omega)}\le\begin{cases}R_1(z,u)e^{-zC_\epsilon}, & z\in[\frac{L}{2},L],\\ R_2(z,u)e^{\frac{-zC_\epsilon}{2}}e^{\frac{-LC_\epsilon}{4}}, & z\in[0,\frac{L}{2}),\end{cases}\tag{2.20}$$

where

$$R_1(z,u)=\sqrt{2\|u\|^2_{L^\infty(0,L;G_L^0)}+4\epsilon^2e^{2LC_\epsilon}}e^{2K_F^2L(L-z)}$$

$$R_2(z,u)=2\sqrt{2\|u\|^2_{L^\infty(0,L;G_L^0)}+4\epsilon^2e^{2LC_\epsilon}}e^{2K_F^2L(L-hL)}\exp\Big(K_F^2hL(hL-z)\Big)$$
$$+2\|u\|_{L^\infty(0,L;G_L^0)}\exp\Big(K_F^2hL(L-z)\Big).$$

Proof. We define a new regularized solution as follows

$$U^\epsilon(x,y,z)=\begin{cases}u^\epsilon(x,y,z), & (x,y,z)\in\Omega\times[hL,L],\\ w^\epsilon(x,y,z), & (x,y,z)\in\Omega\times[0,hL),\end{cases}\tag{2.21}$$

where

$$u^\epsilon(x,y,z)=$$
$$\sum_{\substack{m,n\ge1\\ \sqrt{m^2+n^2}\le C_\epsilon}}\left[\cosh\Big((L-z)\sqrt{m^2+n^2}\Big)\varphi^\epsilon_{mn}+\int_z^L\frac{\sinh\Big((\tau-z)\sqrt{m^2+n^2}\Big)}{\sqrt{m^2+n^2}}F_{mn}(u^\epsilon)(\tau)d\tau\right]\phi_{mn}(x,y)$$

and

$$w^\epsilon(x,y,z)=$$
$$\sum_{\substack{m,n\ge1\\ \sqrt{m^2+n^2}\le C_\epsilon}}\left[\cosh\Big((hL-z)\sqrt{m^2+n^2}\Big)u^\epsilon_{mn}(hL)+\int_z^{hL}\frac{\sinh\Big((\tau-z)\sqrt{m^2+n^2}\Big)}{\sqrt{m^2+n^2}}F_{mn}(w^\epsilon)(\tau)d\tau\right]\phi_{mn}(x,y).$$

The proof is divided into two steps.
Step 1. Estimate of $\|u^\epsilon-u\|_{L^2(\Omega)}$ for $hL\le z\le L$.

Using Lemma 2.1, we have

$$\Big\|u(z)-P_{C_\epsilon}u(z)\Big\|^2_{L^2(\Omega)}\le\exp(-2LC_\epsilon)\|u(z)\|^2_{G_L^0}.$$

Lemma 2.3 and triangle inequality lead to

$$\begin{aligned}\|u^\epsilon(z)-u(z)\|^2_{L^2(\Omega)} &\le 2\|u^\epsilon - P_{C_\epsilon}u\|^2_{L^2(\Omega)} + 2\|u-P_{C_\epsilon}u\|^2_{L^2(\Omega)}\\ &\le 2e^{-2LC_\epsilon}\|u(z)\|^2_{G^0_L} + 4\exp\Big(2(L-z)C_\epsilon\Big)\|\varphi^\epsilon-\varphi\|^2_{L^2(\Omega)}\\ &\quad + 4K_F^2(L-z)\int\limits_z^L \exp\Big(2(\tau-z)C_\epsilon\Big)\|u^\epsilon(\tau)-u(\tau)\|^2_{L^2(\Omega)}d\tau.\end{aligned}$$

This implies that

$$e^{2zC_\epsilon}\|u^\epsilon(z)-u(z)\|^2_{L^2(\Omega)} \le 2e^{2(z-L)C_\epsilon}\sup_{0\le z\le L}\|u(z)\|^2_{G^0_L} + 4e^{2LC_\epsilon}\epsilon^2 + 4K_F^2L\int\limits_z^L e^{2\tau C_\epsilon}\|u^\epsilon(\tau)-u(\tau)\|^2_{L^2(\Omega)}d\tau.$$

Applying Gronwall's inequality, we obtain

$$e^{2zC_\epsilon}\|u^\epsilon(z)-u(z)\|^2_{L^2(\Omega)} \le \Big[2\sup_{0\le z\le L}\|u(z)\|^2_{G^0_L} + 4e^{2LC_\epsilon}\epsilon^2\Big]e^{4K_F^2L(L-z)}.$$

Therefore for all $z\in[hL,L]$

$$\|u^\epsilon(z)-u(z)\|_{L^2(\Omega)} \le \sqrt{2\sup_{0\le z\le L}\|u(z)\|^2_{G^0_L} + 4\epsilon^2e^{2LC_\epsilon}}\,e^{2K_F^2L(L-z)}e^{-zC_\epsilon} = R_1(z,u)e^{-zC_\epsilon}.$$

Step 2. Estimate of $\|w^\epsilon(z)-u(z)\|_{L^2(\Omega)}$ for $0\le z\le hL$.

Define W^ϵ by

$$W^\epsilon(x,y,z) = \sum_{\substack{m,n\ge1\\ \sqrt{m^2+n^2}\le C_\epsilon}}\left[\cosh\Big((hL-z)\sqrt{m^2+n^2}\Big)u_{mn}(hL) + \int_z^{hL}\frac{\sinh\Big((\tau-z)\sqrt{m^2+n^2}\Big)}{\sqrt{m^2+n^2}}F_{mn}(W^\epsilon)(\tau)d\tau\right]\phi_{mn}(x,y). \tag{2.22}$$

First, we estimate $\|w^\epsilon(z)-W^\epsilon(z)\|_{L^2(\Omega)}$. By using the Lipschitz property of F, we obtain

$$\begin{aligned}\|w^\epsilon(z)-W^\epsilon(z)\|^2_{L^2(\Omega)} &\le 2\sum_{\substack{m,n\ge1\\ \sqrt{m^2+n^2}\le C_\epsilon/2}}\left|\cosh\Big((hL-z)\sqrt{m^2+n^2}\Big)\big(u^\epsilon_{mn}(hL)-u_{mn}(hL)\big)\right|^2\\ &\quad + 2\sum_{\substack{m,n\ge1\\ \sqrt{m^2+n^2}\le C_\epsilon/2}}\left[\int_z^{hL}\frac{\sinh\Big((\tau-z)\sqrt{m^2+n^2}\Big)}{\sqrt{m^2+n^2}}\Big(F_{mn}(w^\epsilon)(\tau)-F_{mn}(W^\epsilon)(\tau)\Big)d\tau\right]^2\\ &\le 2\exp((hL-z)C_\epsilon)\|u^\epsilon(hL)-u(hL)\|^2_{L^2(\Omega)}\\ &\quad + 2K_F^2(hL-z)\int\limits_z^{hL}\exp((\tau-z)C_\epsilon)\|w^\epsilon(\tau)-W^\epsilon(\tau)\|^2_{L^2(\Omega)}d\tau.\end{aligned}$$

Hence

$$\begin{aligned}\exp(zC_\epsilon)\|w^\epsilon(z)-W^\epsilon(z)\|^2_{L^2(\Omega)} &\le 2\exp(hLC_\epsilon)\|u^\epsilon(hL)-u(hL)\|^2_{L^2(\Omega)}\\ &\quad + 2K_F^2hL\int\limits_z^{hL}\exp(\tau C_\epsilon)\|w^\epsilon(\tau)-W^\epsilon(\tau)\|^2_{L^2(\Omega)}d\tau.\end{aligned}$$

Using Gronwall's inequality we have

$$\exp(zC_\epsilon)\|w^\epsilon(z) - W^\epsilon(z)\|^2_{L^2(\Omega)} \le 2\exp\Big(2K_F^2 hL(hL-z)\Big)\exp(hLC_\epsilon)\|u^\epsilon(hL) - u(hL)\|^2_{L^2(\Omega)}.$$

This implies that

$$\begin{aligned}\|w^\epsilon(z) - W^\epsilon(z)\|_{L^2(\Omega)} &\le 2\exp\Big(K_F^2 hL(hL-z)\Big)\exp(\frac{(hL-z)C_\epsilon}{2})\|u^\epsilon(hL) - u(hL)\|_{L^2(\Omega)}\\ &\le 2\sqrt{2\sup_{0\le z\le L}\|u(z)\|^2_{G_L^0} + 4\epsilon^2 e^{2LC_\epsilon}e^{2K_F^2 L(L-hL)}}\exp\Big(K_F^2 hL(hL-z)\Big)\exp\Big(\frac{(-hL-z)C_\epsilon}{2}\Big),\end{aligned} \tag{2.23}$$

for all $z \in [0, hL]$.

Next we estimate estimate the error between u and W^ϵ in the L^2 norm. Combining (1.4) and (2.22), we get

$$\begin{aligned} u(z) - W^\epsilon(z) = &\\ \sum_{\substack{m,n\ge1\\ \sqrt{m^2+n^2}>C_\epsilon/2}} &\left[\cosh\Big((hL-z)\sqrt{m^2+n^2}\Big)u_{mn}(L) + \int_z^{hL}\frac{\sinh\Big((\tau-z)\sqrt{m^2+n^2}\Big)}{\sqrt{m^2+n^2}}F_{mn}(u)(\tau)d\tau\right]\phi_{mn}(x,y)\\ + \sum_{\substack{m,n\ge1\\ \sqrt{m^2+n^2}\le C_\epsilon/2}} &\left[\int_z^{hL}\frac{\sinh\Big((\tau-z)\sqrt{m^2+n^2}\Big)}{\sqrt{m^2+n^2}}\Big(F_{mn}(u)(\tau) - F_{mn}(W^\epsilon)(\tau)\Big)d\tau\right]\phi_{mn}(x,y). \end{aligned} \tag{2.24}$$

Putting

$$A_2 = \sum_{\substack{m,n\ge1\\ \sqrt{m^2+n^2}>C_\epsilon/2}}\left[\cosh\Big((hL-z)\sqrt{m^2+n^2}\Big)u_{mn}(L) + \int_z^{hL}\frac{\sinh\Big((\tau-z)\sqrt{m^2+n^2}\Big)}{\sqrt{m^2+n^2}}F_{mn}(u)(\tau)d\tau\right]\phi_{mn}(x,y), \tag{2.25}$$

$$B_2 = \sum_{\substack{m,n\ge1\\ \sqrt{m^2+n^2}\le C_\epsilon/2}}\left[\int_z^{hL}\frac{\sinh\Big((\tau-z)\sqrt{m^2+n^2}\Big)}{\sqrt{m^2+n^2}}\Big(F_{mn}(u)(\tau) - F_{mn}(W^\epsilon)(\tau)\Big)d\tau\right]\phi_{mn}(x,y). \tag{2.26}$$

We have

$$\begin{aligned}\|A_2\|^2_{L^2(\Omega)} &= \sum_{\substack{m,n\ge1\\ \sqrt{m^2+n^2}>C_\epsilon/2}}\left[\cosh\Big((hL-z)\sqrt{m^2+n^2}\Big)u_{mn}(L) + \int_z^{hL}\frac{\sinh\Big((\tau-z)\sqrt{m^2+n^2}\Big)}{\sqrt{m^2+n^2}}F_{mn}(u)(\tau)d\tau\right]^2\\ &\le \sum_{\substack{m,n\ge1\\ \sqrt{m^2+n^2}>C_\epsilon/2}}\exp\Big(-2L\sqrt{m^2+n^2}\Big)\exp\Big(2L\sqrt{m^2+n^2}\Big)u^2_{mn}(z)\\ &\le \exp(-2LC_\epsilon)\sup_{0\le z\le L}\|u(z)\|^2_{G_L^0}.\end{aligned}$$

We deduce by using the Lipschitz property of F

$$\|B_2\|^2_{L^2(\Omega)} = \sum_{\substack{m,n\geq 1\\ \sqrt{m^2+n^2}\leq C_\epsilon/2}} \left[\int_z^{hL} \frac{\sinh\Big((\tau-z)\sqrt{m^2+n^2}\Big)}{\sqrt{m^2+n^2}}\Big(F_{mn}(u)(\tau)-F_{mn}(W^\epsilon)(\tau)\Big)d\tau\right]^2$$

$$\leq \sum_{\substack{m,n\geq 1\\ \sqrt{m^2+n^2}\leq C_\epsilon/2}} \int_z^{hL}\left[\frac{\sinh\Big((\tau-z)\sqrt{m^2+n^2}\Big)}{\sqrt{m^2+n^2}}\right]^2 ds \int_z^{L}\Big|F_{mn}(W^\epsilon)(\tau)-F_{mn}(u)(\tau)\Big|^2 d\tau$$

$$\leq \sum_{\substack{m,n\geq 1\\ \sqrt{m^2+n^2}\leq C_\epsilon/2}} \exp((\tau-z)C_\epsilon)(hL-z)\int_z^{hL}\Big|F_{mn}(W^\epsilon)(\tau)-F_{mn}(u)(\tau)\Big|^2 d\tau$$

$$\leq K_F^2(hL-z)\exp\Big((\tau-z)C_\epsilon\Big)\int_z^{hL}\Big\|W^\epsilon(\tau)-u(\tau)\Big\|^2_{L^2(\Omega)}d\tau.$$

It follows from (2.24) that

$$\begin{aligned}\|u(z)-W^\epsilon(z)\|^2_{L^2(\Omega)} &\leq 2\Big(\|A_2\|^2_{L^2(\Omega)}+\|B_2\|^2_{L^2(\Omega)}\Big)\\ &\leq 2\exp(-LC_\epsilon)\sup_{0\leq z\leq L}\|u(z)\|^2_{G_L^0}+2K_F^2hL\int_z^{hL}\exp((\tau-z)C_\epsilon)\|u(\tau)-W^\epsilon(\tau)\|^2_{L^2(\Omega)}d\tau.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\exp(zC_\epsilon)\|u(z)-W^\epsilon(z)\|^2_{L^2(\Omega)} &\leq 2\exp((z-L)C_\epsilon)\sup_{0\leq z\leq L}\|u(z)\|^2_{G_L^0}\\ &\quad+2K_F^2hL\int_z^{hL}\exp(\tau C_\epsilon)\|u(\tau)-W^\epsilon(\tau)\|^2_{L^2(\Omega)}d\tau\\ &\leq 2\exp((hL-L)C_\epsilon)\sup_{0\leq z\leq L}\|u(z)\|^2_{G_L^0}\\ &\quad+2K_F^2hL\int_z^{hL}\exp(\tau C_\epsilon)\|u(\tau)-W^\epsilon(\tau)\|^2_{L^2(\Omega)}d\tau.\end{aligned}$$

Applying Gronwall's inequality, we have

$$\|u(z)-W^\epsilon(z)\|_{L^2(\Omega)} \leq 2\sup_{0\leq z\leq L}\|u(z)\|_{G_L^0}\exp\Big(K_F^2hL(L-z)\Big)\exp\Big(\frac{hLC_\epsilon-LC_\epsilon-zC_\epsilon}{2}\Big). \tag{2.27}$$

Combining (2.23) and (2.27), we obtain for $0\leq z\leq hL$

$$\begin{aligned}\|u(z)-w^\epsilon(z)\|_{L^2(\Omega)} &\leq \|u(z)-W^\epsilon(z)\|_{L^2(\Omega)}+\|w^\epsilon(z)-W^\epsilon(z)\|_{L^2(\Omega)}\\ &\leq 2\sqrt{2\sup_{0\leq z\leq L}\|u(z)\|^2_{G_L^0}+4\epsilon^2e^{2LC_\epsilon}}\,e^{2K_F^2L(L-hL)}\exp\Big(K_F^2hL(hL-z)\Big)\exp\Big(\frac{(-hL-z)C_\epsilon}{2}\Big)\\ &\quad+2\sup_{0\leq z\leq L}\|u(z)\|_{G_L^0}\exp\Big(K_F^2hL(L-z)\Big)\exp\Big(\frac{hLC_\epsilon-LC_\epsilon-zC_\epsilon}{2}\Big).\end{aligned}$$

For ϵ fixed, it is easy to see that

$$\max_{h\in[0,1]}\Big\{e^{\frac{-hLC_\epsilon}{2}},e^{-\frac{(1-h)LC_\epsilon}{2}}\Big\}=e^{\frac{-LC_\epsilon}{4}}.$$

This leads to

$$\|u(z) - w^{\epsilon}(z)\|_{L^2(\Omega)} \le R_2(z,u) e^{\frac{-zC_\epsilon}{2}} e^{\frac{-LC_\epsilon}{4}}.$$

□

Remark 2.2. 1. *If the same technique as in Theorem 2.1 is used, the order of convergence will be e^{-zC_ϵ}. In particular, when $z = 0$ there is no convergence.*

2. *If $C_\epsilon = \frac{1}{L}$, then* (2.20) *becomes*

$$\|U^{\epsilon}(z) - u(z)\|_{L^2(\Omega)} \le \begin{cases} R_1(z,u)\epsilon^{\frac{z}{L}}, & z \in [\frac{L}{2}, L], \\ R_2(z,u)\epsilon^{\frac{z}{L}}\epsilon^{\frac{1}{4}}, & z \in [0, \frac{L}{2}). \end{cases} \tag{2.28}$$

It is obvious that (2.28) *is sharp and this error may be better than the previous one in Theorem 2.1.*

The next theorem provides an error estimate in the Sobolev space $H^p(\Omega)$ which is equipped with a norm defined by

$$\|g\|^2_{H^p(\Omega)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(m^2+n^2\right)^p\left|\left\langle g, \phi_{mn}\right\rangle\right|^2.$$

Theorem 2.3. 1. *If u satisfies* (2.12)*, then with $C_\epsilon = \frac{1}{L+\alpha}\ln\left(\frac{1}{\epsilon}\right)$ we have*

$$\|u^{\epsilon}(z) - u(z)\|_{H^p(\Omega)} \le \left[R_3(z)e^{-\alpha C_\epsilon} + 2e^{K_F^2 L(L-z)}\epsilon e^{LC_\epsilon}\right]C_\epsilon^p e^{-zC_\epsilon},\ z \in [0, L],$$

where

$$R_3(z) = \sqrt{2}e^{K_F^2 L(L-z)} I_3^2.$$

2. *If u satisfies* (2.13)*, then we can construct a regularized solution U^ϵ satisfying*

$$\|U^{\epsilon}(z) - u(z)\|\|_{H^p(\Omega)} \le \begin{cases} R_1(z,u)C_\epsilon^p e^{-zC_\epsilon} + \|u\|_{L^\infty(0,L;G_L^0)}C_\epsilon^p e^{-LC_\epsilon}, & z \in [\frac{L}{2}, L], \\ R_2(z,u)C_\epsilon^p e^{\frac{-zC_\epsilon}{2}} e^{\frac{-LC_\epsilon}{4}} + \|u\|_{L^\infty(0,L;G_L^0)}C_\epsilon^p e^{-LC_\epsilon}, & z \in [0, \frac{L}{2}). \end{cases}$$

Proof. **Proof of Part 1:** First, we have

$$\begin{aligned}
\|u^{\epsilon}(z) - P_{C_\epsilon}u(z)\|^2_{H^p(\Omega)} &= \sum_{\substack{m,n\ge 1\\ \sqrt{m^2+n^2}\le C_\epsilon}} \left(m^2+n^2\right)^p\left|\left\langle u^\epsilon(z) - u(z), \phi_{mn}(x,y)\right\rangle\right|^2 \\
&\le C_\epsilon^{2p} \sum_{\substack{m,n\ge 1\\ \sqrt{m^2+n^2}\le C_\epsilon}} \left|\left\langle u^\epsilon(z) - u(z), \phi_{mn}(x,y)\right\rangle\right|^2 \\
&\le C_\epsilon^{2p}\|u(z) - u^\epsilon(z)\|^2_{L^2(\Omega)}.
\end{aligned}$$

It follows from Theorem 2.1 that

$$\|u^{\epsilon}(z) - P_{C_\epsilon}u(z)\|_{H^p(\Omega)} \le e^{2K_F^2 L(L-z)}\sqrt{2e^{-2\alpha C_\epsilon}\sup_{0\le z\le L}\|u(z)\|^2_{G^0_{z+\alpha}} + 4\epsilon^2 e^{2LC_\epsilon}}\,C_\epsilon^p e^{-zC_\epsilon}. \tag{2.29}$$

On the other hand, consider the function

$$G(\xi) = \xi^p e^{-D\xi}, \quad D > 0.$$

Since $G'(\xi) = \xi^{p-1}e^{-D\xi}(p - D\xi)$, it follows that G is strictly decreasing when $\xi \ge p$. Thus if $\epsilon \le e^{\frac{-p(L+\alpha)}{2\alpha}}$ i.e, $2(z+\alpha)C_\epsilon \ge p$, then for $m^2+n^2 \ge C_\epsilon^2$

$$\left(m^2+n^2\right)^p \exp\left(-2(z+\alpha)\sqrt{m^2+n^2}\right) \le C_\epsilon^{2p}e^{-2(z+\alpha)C_\epsilon},$$

and

$$\begin{aligned}
&\|u(z) - P_{C_\epsilon} u(z)\|^2_{H^p(\Omega)} \\
&\quad = \sum_{\substack{m,n\geq 1\\ \sqrt{m^2+n^2}>C_\epsilon}} \Big(m^2+n^2\Big)^p \exp\Big(-2(z+\alpha)\sqrt{m^2+n^2}\Big)\exp\Big(2(z+\alpha)\sqrt{m^2+n^2}\Big)\Big|\Big\langle u(z),\phi_{mn}(x,y)\Big\rangle\Big|^2 \\
&\quad \leq C_\epsilon^{2p}\exp(-2(z+\alpha)C_\epsilon) \sum_{\substack{m,n\geq 1\\ \sqrt{m^2+n^2}>C_\epsilon}} \exp\Big(2(z+\alpha)\sqrt{m^2+n^2}\Big)\Big|\Big\langle u(z),\phi_{mn}(x,y)\Big\rangle\Big|^2 \\
&\quad \leq \sup_{0\leq z\leq L}\|u(z)\|^2_{G^0_{z+\alpha}} C_\epsilon^{2p} e^{-2(z+\alpha)C_\epsilon}.
\end{aligned}$$

Therefore

$$\|u(z) - P_{C_\epsilon} u(z)\|_{H^p(\Omega)} \leq \sup_{0\leq z\leq L}\|u(z)\|_{G^0_{z+\alpha}} C_\epsilon^{p} e^{-(z+\alpha)C_\epsilon}. \tag{2.30}$$

Combining (2.29) and (2.30), we get

$$\begin{aligned}
\|u^\epsilon(z) - u(z)\|_{H^p(\Omega)} &\leq \|u^\epsilon(z) - P_{C_\epsilon}u(z)\|_{H^p(\Omega)} + \|u(z) - P_{C_\epsilon}u(z)\|_{H^p(\Omega)} \\
&\leq \left[e^{2K_F^2 L(L-z)}\sqrt{2e^{-2\alpha C_\epsilon}\sup_{0\leq z\leq L}\|u(z)\|^2_{G^0_{z+\alpha}} + 4\epsilon^2 e^{2LC_\epsilon}} + \sup_{0\leq z\leq L}\|u(z)\|_{G^0_{z+\alpha}} e^{-\alpha C_\epsilon}\right] C_\epsilon^p e^{-zC_\epsilon}.
\end{aligned}$$

The inequality $\sqrt{a^2+b^2}\leq a+b$ for $a,b\geq 0$ leads to

$$\|u^\epsilon(z) - u(z)\|_{H^p(\Omega)} \leq \Big[R_3(z)e^{-\alpha C_\epsilon} + 2e^{K_F^2 L(L-z)}\epsilon e^{LC_\epsilon}\Big] C_\epsilon^p e^{-zC_\epsilon}.$$

Proof of Part 2: If $\epsilon \leq e^{\frac{-p}{2}}$ i.e, $2LC_\epsilon \geq p$, then for $m^2+n^2\geq C_\epsilon$

$$\Big(m^2+n^2\Big)^p\exp\Big(-2L\sqrt{m^2+n^2}\Big) \leq C_\epsilon^p e^{-2LC_\epsilon}$$

and

$$\begin{aligned}
&\|u(z) - P_{C_\epsilon} u(z)\|^2_{H^p(\Omega)} \\
&\quad = \sum_{\substack{m,n\geq 1\\ \sqrt{m^2+n^2}>C_\epsilon}} \Big(m^2+n^2\Big)^p \exp\Big(-2L\sqrt{m^2+n^2}\Big)\exp\Big(2L\sqrt{m^2+n^2}\Big)\Big|\Big\langle u(.,z),\phi_{mn}(x,y)\Big\rangle\Big|^2 \\
&\quad \leq C_\epsilon^{2p}\exp(-2LC_\epsilon) \sum_{\substack{m,n\geq 1\\ \sqrt{m^2+n^2}>C_\epsilon}} \exp\Big(2L\sqrt{m^2+n^2}\Big)\Big|\Big\langle u(z),\phi_{mn}(x,y)\Big\rangle\Big|^2 \\
&\quad \leq \sup_{0\leq z\leq L}\|u(z)\|^2_{G^0_L} C_\epsilon^{2p} e^{-2LC_\epsilon}.
\end{aligned}$$

Therefore, we obtain

$$\|u(z) - P_{C_\epsilon} u(z)\|_{H^p(\Omega)} \leq \sup_{0\leq z\leq L}\|u(z)\|_{G^0_L} C_\epsilon^{p} e^{-LC_\epsilon}. \tag{2.31}$$

Combining (2.29) and (2.31), we claim that

$$\begin{aligned}
\|U^\epsilon(z) - u(z)\|_{H^p(\Omega)} &\leq \|U^\epsilon(z) - P_{C_\epsilon}u(z)\|_{H^p(\Omega)} + \|u(z) - P_{C_\epsilon}u(z)\|_{H^p(\Omega)} \\
&\leq C_\epsilon^p\|u(z) - U^\epsilon(z)\|_{L^2(\Omega)} + \sup_{0\leq z\leq L}\|u(z)\|_{G^0_L} C_\epsilon^p e^{-LC_\epsilon}.
\end{aligned}$$

It follows from

$$\|U^\epsilon(z) - u(z)\|_{L^2(\Omega)} \leq \begin{cases} R_1(z,u)e^{-zC_\epsilon}, & z\in[\frac{L}{2},L], \\ R_2(z,u)e^{\frac{-zC_\epsilon}{2}}e^{\frac{-LC_\epsilon}{4}}, & z\in[0,\frac{L}{2}), \end{cases}$$

that

$$\|U^\epsilon(z)-u(z)\|\|_{H^p(\Omega)} \le \begin{cases} R_1(z,u)C_\epsilon^p e^{-zC_\epsilon} + \|u\|_{L^\infty(0,L;G_L^0)} C_\epsilon^p e^{-LC_\epsilon}, & z\in[\frac{L}{2},L],\\ R_2(z,u)C_\epsilon^p e^{\frac{-zC_\epsilon}{2}} e^{\frac{-LC_\epsilon}{4}} + \|u\|_{L^\infty(0,L;G_L^0)} C_\epsilon^p e^{-LC_\epsilon}, & z\in[0,\frac{L}{2}). \end{cases}$$

□

2.2. Results for locally Lipschitz source functions

In this subsection, we assume that the function $F : [0,\pi]\times[0,\pi]\times[0,L]\times\mathbb{R}\to\mathbb{R}$, $F = F(x,y,z,u)$ satisfies: for each $M > 0$ and for any u, v satisfying $|u|, |v| \le M$, there holds

$$|F(x,y,z,u) - F(x,y,z,v)| \le K_F(M)\,|u-v|\,, \tag{2.32}$$

where $(x,y,z)\in[0,\pi]\times[0,\pi]\times[0,L]$ and

$$K_F(M) := \sup\left\{\left|\frac{F(x,y,z,u)-F(x,y,z,v)}{u-v}\right| : |u|\,,|v|\le M, u\ne v, (x,y,z)\in[0,\pi]\times[0,\pi]\times[0,L]\right\} < +\infty.$$

We note that $K_F(M)$ is increasing and $\lim\limits_{M\to+\infty} K_F(M) = +\infty$. Now, we outline our ideas to construct a regularization for problem (1.1)–(1.2).

For all $M > 0$, we approximate F by F_M defined by

$$F_M\,(x,y,z,u(x,y,z)) = \begin{cases} F(x,y,z,M), & u(x,y,z) > M,\\ F(x,y,z,u(x,y,z)), & -M\le u(x,y,z)\le M,\\ F(x,y,z,-M), & u(x,y,z) < -M. \end{cases}$$

For each $\epsilon > 0$, we consider a parameter $M_\epsilon \to +\infty$ as $\epsilon\to 0$. Let $u_{\epsilon,\varphi^\epsilon}$ be a solution of the following problem

$$\begin{cases} \Delta v = P_{C_\epsilon} F_{M_\epsilon}(x,y,z,v(x,y,z)), & (x,y,z)\in\Omega\times(0,+\infty),\\ v(x,y,z) = 0, & (x,y,z)\in\partial\Omega\times(0,+\infty),\\ v(x,y,L) = \varphi^\epsilon(x,y), \quad v_z(x,y,L) = 0, & (x,y)\in\Omega. \end{cases} \tag{2.33}$$

The following theorem provides some error estimates in the L^2-norm when the exact solution belongs to the Gevrey space.

Theorem 2.4. *Let $\epsilon > 0$ and let F be the function defined in* (2.32)*. Then the problem* (2.33) *has a unique solution $u_{\epsilon,\varphi^\epsilon}\in C([0,L];L^2(\Omega))$.*

1. *Assume that u satisfies* (2.10)*. If C_ϵ and M_ϵ are chosen such that ϵe^{LC_ϵ} is bounded and*

$$\begin{cases} \lim_{\epsilon\to 0}\exp\left(2K_F^2(M_\epsilon)L^2\right)\frac{\ln(C_\epsilon)}{C_\epsilon} = 0,\\ \lim_{\epsilon\to 0}\exp\left(2K_F^2(M_\epsilon)L^2\right)e^{-zC_\epsilon} = 0, \quad z>0, \end{cases}$$

then we obtain

$$\|u_{\epsilon,\varphi^\epsilon}(z)-u(z)\|_{L^2(\Omega)} \le \sqrt{2I_1^2+4\epsilon^2e^{2LC_\epsilon}}\exp\left(2K_F^2(M_\epsilon)L^2\right)e^{-zC_\epsilon}. \tag{2.34}$$

Moreover, for ϵ sufficiently small, there exists $z_\epsilon\le\min\left\{L,\frac{\ln(C_\epsilon)}{C_\epsilon}\right\}$ such that

$$\|u_{\epsilon,\varphi^\epsilon}(z_\epsilon)-u(0)\|_{L^2(\Omega)} \le \left[\sqrt{2I_1^2+4\epsilon^2e^{2LC_\epsilon}}\exp\left(2K_F^2(M_\epsilon)L^2\right)+\|u\|_{L^\infty(0,L;L^2(\Omega))}\right]\frac{\ln(C_\epsilon)}{C_\epsilon}. \tag{2.35}$$

2. *Assume that u satisfies* (2.11). *If C_ϵ and M_ϵ are chosen such that* $\lim_{\epsilon\to+\infty} \epsilon e^{LC_\epsilon} = 0$ *and*

$$\lim_{\epsilon\to 0} \exp\left(2K_F^2(M_\epsilon)L^2\right)C_\epsilon^{-\beta} = \lim_{\epsilon\to 0} \exp\left(2K_F^2(M_\epsilon)L^2\right)\epsilon e^{LC_\epsilon} = 0,$$

then we have

$$\|u_{\epsilon,\varphi^\epsilon}(z) - u(z)\|_{L^2(\Omega)} \le \sqrt{2C_\epsilon^{-2\beta} I_2^2 + 4\epsilon^2 e^{2LC_\epsilon}} \exp\left(2K_F^2(M_\epsilon)L^2\right)e^{-zC_\epsilon}. \tag{2.36}$$

3. *Assume that u satisfies* (2.12). *If C_ϵ and M_ϵ are chosen such that* $\lim_{\epsilon\to+\infty} \epsilon e^{LC_\epsilon} = 0$ *and*

$$\lim_{\epsilon\to 0} \exp\left(2K_F^2(M_\epsilon)L^2\right)e^{-\alpha C_\epsilon} = \lim_{\epsilon\to 0} \exp\left(2K_F^2(M_\epsilon)L^2\right)\epsilon e^{LC_\epsilon} = 0,$$

then we have

$$\|u_{\epsilon,\varphi^\epsilon}(z) - u(z)\|_{L^2(\Omega)} \le \sqrt{2e^{-2\alpha C_\epsilon} I_3^2 + 4\epsilon^2 e^{2LC_\epsilon}} \exp\left(2K_F^2(M_\epsilon)L^2\right)e^{-zC_\epsilon}.$$

Before proving the theorem, we show the following lemmas.

Lemma 2.4. *For $u_1(x,y,z)$, $u_2(x,y,z)$, we have*

$$\Big|F_M(x,y,z,u_2(x,y,z)) - F_M(x,y,z,u_1(x,y,z))\Big| \le K_F(M)\Big|u_2(x,y,z) - u_1(x,y,z)\Big|.$$

Proof. If $u_1(x,y,z) < -M$ and $u_2(x,y,z) < -M$ then

$$\Big|F_M(x,y,z,u_2(x,y,z)) - F_M(x,y,z,u_1(x,y,z))\Big| = 0.$$

If $u_1(x,y,z) < -M \le u_2(x,y,z) \le M$ then

$$\begin{aligned}\Big|F_M(x,y,z,u_2(x,y,z)) - F_M(x,y,z,u_1(x,y,z))\Big| &= \Big|F_M(x,y,z,u_2(x,y,z)) - F_M(x,y,z,-M)\Big| \\ &\le K_F(M)\Big|M + u_2(x,y,z)\Big| \\ &\le K_F(M)\Big|u_2(x,y,z) - u_1(x,y,z)\Big|.\end{aligned}$$

If $u_1(x,y,z) < -M < M < u_2(x,y,z)$ then

$$\begin{aligned}\Big|F_M(x,y,z,u_2(x,y,z)) - F_M(x,y,z,u_1(x,y,z))\Big| &= \Big|F_M(x,y,z,M) - F_M(x,y,z,-M)\Big| \\ &\le 2MK_F(M) \\ &\le K_F(M)\Big|u_2(x,y,z) - u_1(x,y,z)\Big|.\end{aligned}$$

If $-M \le u_1(x,y,z)$, $u_2(x,y,z) \le M$ then

$$\begin{aligned}\Big|F_M(x,y,z,u_2(x,y,z)) - F_M(x,y,z,u_1(x,y,z))\Big| &= \Big|F(x,y,z,u_2(x,y,z)) - F(x,y,z,u_1(x,y,z))\Big| \\ &\le K_F(M)\Big|u_2(x,y,z) - u_1(x,y,z)\Big|.\end{aligned}$$

This completes the proof. □

Lemma 2.5. *Let u be exact solution to problem* (1.1)–(1.2). *Then we have the following estimate*

$$\begin{aligned}\|u_{\epsilon,\varphi^\epsilon}(z) - P_{C_\epsilon}u(z)\|_{L^2(\Omega)}^2 &\le 2\exp\left(2(L-z)C_\epsilon\right)\|\varphi^\epsilon - \varphi\|_{L^2(\Omega)}^2 \\ &\quad + 2K_F^2(M_\epsilon)(L-z)\int_z^L \exp\left(2(\tau - z)C_\epsilon\right)\|u_{\epsilon,\varphi^\epsilon}(\tau) - u(\tau)\|_{L^2(\Omega)}^2 d\tau.\end{aligned}$$

Proof. From the definitions of $u_{\epsilon,\varphi^\epsilon}$ and u, we obtain

$$
\begin{aligned}
&\|u_{\epsilon,\varphi^\epsilon}(z) - P_{C_\epsilon} u(z)\|^2_{L^2(\Omega)} \\
&\le 2 \sum_{\substack{m,n\ge 1\\ \sqrt{m^2+n^2}\le C_\epsilon}} \left| \cosh\Big((L-z)\sqrt{m^2+n^2}\Big)(\varphi^\epsilon_{mn} - \varphi_{mn}) \right|^2 \\
&\quad + 2 \sum_{\substack{m,n\ge 1\\ \sqrt{m^2+n^2}\le C_\epsilon}} \left[\int_z^L \frac{\sinh\Big((\tau-z)\sqrt{m^2+n^2}\Big)}{\sqrt{m^2+n^2}} \Big((F_{M_\epsilon})_{mn}(u_{\epsilon,\varphi^\epsilon})(\tau) - F_{mn}(u)(\tau)\Big) d\tau \right]^2 \\
&\le 2 \sum_{\substack{m,n\ge 1\\ \sqrt{m^2+n^2}\le C_\epsilon}} \left| \exp\Big((L-z)\sqrt{m^2+n^2}\Big)(\varphi^\epsilon_{mn} - \varphi_{mn}) \right|^2 \\
&\quad + 2 \sum_{\substack{m,n\ge 1\\ \sqrt{m^2+n^2}\le C_\epsilon}} \left[\int_z^L \exp\Big((\tau-z)\sqrt{m^2+n^2}\Big) \Big((F_{M_\epsilon})_{mn}(u_{\epsilon,\varphi})(\tau) - F_{mn}(u)(\tau)\Big) d\tau \right]^2 \\
&\le 2\exp\Big(2(L-z)C_\epsilon\Big)\|\varphi^\epsilon - \varphi\|^2_{L^2(\Omega)} \\
&\quad + 2(L-z)\int_z^L \exp\Big(2(\tau-z)C_\epsilon\Big) \Big\| F_{M_\epsilon}\Big(x,y,\tau,u_{\epsilon,\varphi^\epsilon}(\tau)\Big) - F(x,y,\tau,u(\tau)) \Big\|^2_{L^2(\Omega)} d\tau. \qquad (2.37)
\end{aligned}
$$

Since $\lim_{\epsilon\to 0^+} M_\epsilon = +\infty$, for a sufficiently small $\epsilon > 0$, there is an M_ϵ such that $M_\epsilon \ge \|u\|_{L^\infty([0,L],L^2(\Omega))}$. For this value of M_ϵ we have $F_{M_\epsilon}(x,y,z,u(x,y,z)) = F(x,y,z,u(x,y,z))$. Using the Lipschitz property of F_M as in Lemma 2.4, we get

$$
\Big\| F_{M_\epsilon}\Big(x,y,\tau,u_{\epsilon,\varphi^\epsilon}(\tau)\Big) - F(x,y,\tau,u(\tau)) \Big\|^2_{L^2(\Omega)} \le K_F(M_\epsilon) \Big\| u_{\epsilon,\varphi^\epsilon}(\tau) - u(\tau) \Big\|_{L^2(\Omega)}. \qquad (2.38)
$$

Combining (2.37) and (2.38), we complete the proof of Lemma 2.5. □

We now prove Theorem 2.4.

Proof. **Proof of Part 1:** Since $u \in G^0_z$ then using Lemma 2.1, we get

$$
\|u(z) - P_{C_\epsilon}u(z)\|^2_{L^2(\Omega)} \le e^{-2zC_\epsilon}\|u(z)\|^2_{G^0_z}.
$$

Lemma 2.3 and the triangle inequality lead to

$$
\begin{aligned}
\Big\| u_{\epsilon,\varphi^\epsilon}(z) - u(z) \Big\|^2_{L^2(\Omega)} &\le 2\|u_{\epsilon,\varphi^\epsilon}(z) - P_{C_\epsilon}u(z)\|^2_{L^2(\Omega)} + 2\|u(z) - P_{C_\epsilon}u(z)\|^2_{L^2(\Omega)} \\
&\le 2e^{-2zC_\epsilon}\|u(z)\|^2_{G^0_z} + 4\exp\Big(2(L-z)C_\epsilon\Big)\|\varphi^\epsilon - \varphi\|^2_{L^2(\Omega)} \\
&\quad + 4K_F^2(M_\epsilon)(L-z)\int_z^L \exp\Big(2(\tau-z)C_\epsilon\Big)\Big\| u_{\epsilon,\varphi^\epsilon}(\tau) - u(\tau) \Big\|^2_{L^2(\Omega)} d\tau.
\end{aligned}
$$

This implies that

$$
e^{2zC_\epsilon}\|u_{\epsilon,\varphi^\epsilon}(z) - u(z)\|^2_{L^2(\Omega)} \le 2\sup_{0\le z\le L}\|u(z)\|^2_{G^0_z} + 4e^{2LC_\epsilon}\epsilon^2 + 2K_F^2(M_\epsilon)L\int_z^L e^{2\tau C_\epsilon}\Big\| u_{\epsilon,\varphi^\epsilon}(\tau) - u(\tau) \Big\|^2_{L^2(\Omega)} d\tau.
$$

Applying Gronwall's inequality, we obtain

$$e^{2zC_\epsilon}\|u_{\epsilon,\varphi^\epsilon}(z)-u(z)\|^2_{L^2(\Omega)} \le \Big[2\sup_{0\le z\le L}\|u(z)\|^2_{G_z^0} + 4e^{2LC_\epsilon}\epsilon^2\Big]\exp\Big(4K_F^2(M_\epsilon)L(L-z)\Big),$$

which leads to the desired result

$$\|u_{\epsilon,\varphi^\epsilon}(z)-u(z)\|_{L^2(\Omega)} \le \sqrt{2\sup_{0\le z\le L}\|u(z)\|^2_{G_z^0} + 4\epsilon^2 e^{2LC_\epsilon}}\exp\Big(2K_F^2(M_\epsilon)L^2\Big)e^{-zC_\epsilon}.$$

Part 2 and Part 3 can be proved by using the same technique. The proof of which is omitted.

□

Remark 2.3. 1. *In part 1, Theorem 2.4, let us choose $C_\epsilon = \frac{1}{L}\ln(\frac{1}{\epsilon})$. We can find M_ϵ such that*

$$K_F(M_\epsilon) = \frac{1}{2L}\sqrt{\ln\Big(\frac{1}{L}\ln(\frac{1}{\epsilon})\Big)}. \tag{2.39}$$

Then (2.34) *becomes*

$$\|u_{\epsilon,\varphi^\epsilon}(z)-u(z)\|_{L^2(\Omega)} \le \frac{1}{2L}\sqrt{4+2\sup_{0\le z\le L}\|u(z)\|^2_{G_z^0}}\ \sqrt{\frac{1}{L}\ln(\frac{1}{\epsilon})}\epsilon^{\frac{z}{L}}. \tag{2.40}$$

and (2.35) *becomes*

$$\|u_{\epsilon,\varphi^\epsilon}(z_\epsilon)-u(0)\|_{L^2(\Omega)} \le \sqrt{2\sup_{0\le z\le L}\|u(z)\|^2_{G_z^0} + 4\frac{\ln\Big(\frac{1}{L}\ln(\frac{1}{\epsilon})\Big)}{\sqrt{\frac{1}{L}\ln(\frac{1}{\epsilon})}}} + \sup_{0\le z\le L}\|u_z(z)\|_{L^2(\Omega)}\frac{\ln\Big(\frac{1}{L}\ln(\frac{1}{\epsilon})\Big)}{\frac{1}{L}\ln(\frac{1}{\epsilon})}. \tag{2.41}$$

If $F(u) = u - u^3$, equation (1.1) *becomes the elliptic Allen-Cahn type equation which models phase transitions. The elliptic Allen-Cahn equation appears in the study of differential geometry related to, for example, interface area, interface curvature and minimal hypersurface. A simple computation reveals*

$$\begin{aligned} K_F(M_\epsilon) &= \sup\left\{\left|\frac{F(u)-F(v)}{u-v}\right| : |u|,|v|\le M_\epsilon, u\ne v,\right\} \\ &= \sup\left\{\left|\frac{u-v-u^3+v^3}{u-v}\right| : |u|,|v|\le M_\epsilon, u\ne v,\right\} \\ &= 1+3M_\epsilon^2 \end{aligned}$$

and togeter with (2.39)

$$K_F(M_\epsilon) = 1+3M_\epsilon^2 = \frac{1}{2L}\sqrt{\ln\Big(\frac{1}{L}\ln(\frac{1}{\epsilon})\Big)}.$$

This implies

$$M_\epsilon = \sqrt{\frac{\frac{1}{L}\sqrt{\ln\Big(\frac{1}{L}\ln(\frac{1}{\epsilon})\Big)}-1}{6}}.$$

2. *In part 2, Theorem 2.4, let us choose $C_\epsilon = \frac{\gamma}{L}\ln(\frac{1}{\epsilon})$, for $\gamma\in(0,1)$. We can take M_ϵ such that*

$$K_F(M_\epsilon) = \frac{1}{2L}\sqrt{\frac{1}{\beta-r}\ln\Big(\frac{\gamma}{L}\ln(\frac{1}{\epsilon})\Big)},$$

for $r \in (0, \beta)$. Then (2.36) becomes

$$\|u_{\epsilon,\varphi^\epsilon}(z) - u(z)\|_{L^2(\Omega)} \le \sqrt{2C_\epsilon^{-2\beta} \sup_{0\le z\le L} \|u(z)\|^2_{G_z^\beta} + 4\epsilon^2 e^{2LC_\epsilon}} \exp\left(2K_F^2(M_\epsilon)L^2\right) e^{-zC_\epsilon}$$
$$= \sqrt{2 \sup_{0\le z\le L} \|u(z)\|^2_{G_z^\beta} + 4\epsilon^{2-2\gamma}\Big[\frac{\gamma}{L}\ln(\frac{1}{\epsilon})\Big]^\beta}\Big[\frac{\gamma}{L}\ln(\frac{1}{\epsilon})\Big]^{-r}\epsilon^{\frac{z\gamma}{L}}.$$

3. *In part 3, Theorem 2.4, let us choose $C_\epsilon = \frac{1}{L+\alpha}\ln(\frac{1}{\epsilon})$. We can take M_ϵ such that*

$$K_F(M_\epsilon) = \frac{1}{2L}\sqrt{\frac{r-\alpha}{L+\alpha}\ln(\frac{1}{\epsilon})}, \quad 0 < r < \alpha$$

Then

$$\|u_{\epsilon,\varphi^\epsilon}(z) - u(z)\|_{L^2(\Omega)} \le \sqrt{2 \sup_{0\le z\le L} \|u(z)\|^2_{G_z^{L+\alpha}} + 4}\,\epsilon^{\frac{r+\alpha}{L+\alpha}}.$$

Remark 2.4. *Similar techniques as in Theorem 2.4 can be used to derive an error estimate in the Sobolev space $H^p(\Omega)$.*

3. Numerical examples

In this section, we present two examples in order to illustrate the efficiency of the proposed methods.

3.1. Example 1: Elliptic sine-Gordon equation

One of the examples of nonlinear equations we are interested in is the elliptic sine-Gordon equation. This equation comes from several areas of mathematical physics including the theory of Josephson effects, superconductors and spin waves in ferromagnets; see e.g. [11]. The equation has recently been studied by, for example G. Chen et al [12] and A.S. Fokas et al [18]. The elliptic sine-Gordon equation originates from the static case of the hyperbolic sine-Gordon equation modelling the Josephson junction in superconductivity. In this example, we choose the regularization parameter $C_\epsilon = \ln\left(\frac{1}{\epsilon(1-\ln(\epsilon))^2}\right)$ which implies $L = 1$ and $r = 2$. We observe that $\epsilon = c10^{-r}$ where $c > 0$ and $r \in \mathbb{N}$. To define the measured data φ^ϵ, we take a perturbation of the size ϵrand in the exact data φ, where the random generator takes values in $[-1, 1]$. More precisely, we define

$$\varphi^\epsilon(x,y) = \varphi(x,y)\left(1 + \frac{\epsilon\cdot\text{rand}}{\|\varphi\|_{L^2(\Omega)}}\right).$$

The approximate solutions are expected to converge to the exact under proper discretizations. The l_2-norm errors and the relative root mean square errors are computed. We take

$$F(x,y,z,u(x,y,z)) = \sin u + G(x,y,z), \quad \varphi(x,y) = x^2 y(x-\pi)(\pi-y)^2,$$

where

$$G(x,y,z) = \sin\left(x^2 y(x-\pi)(\pi-y)^2\cos(\pi z)\right)$$
$$+ \left[2(\pi-3x)y(\pi-y)^2 + 2x^2(\pi-x)(3y-2\pi) + \pi^2 x^2 y(x-\pi)(\pi-y)^2\right]\cos(\pi z).$$

Then the exact solution is $u(x,y,z) = x^2 y(\pi-x)(\pi-y)^2\cos(\pi z)$. By putting $\alpha(m,n) = \sqrt{m^2+n^2}$, the approximate solution is given by

$$u^\epsilon(x,y,z) = \sum_{\substack{m,n\ge 1\\ \sqrt{m^2+n^2}\le C_\epsilon}} \Big[\cosh((1-z)\alpha(m,n))\varphi^\epsilon_{mn}$$
$$+ \int_z^1 \frac{\sinh((\tau-z)\alpha(m,n))}{\alpha(m,n)} f_{mn}(u^\epsilon)(\tau)\,d\tau\Big]\sin(mx)\sin(ny),$$

where

$$\varphi^{\epsilon}_{mn} = \frac{4}{\pi^2}\left(1 + \frac{\epsilon \cdot \text{rand}}{\|\varphi\|}\right)\int_0^{\pi}\int_0^{\pi} \varphi(x,y)\sin(mx)\sin(ny)\,dxdy.$$

Due to the presence of the function G in the nonlinear term, we compute $F_{mn}(u^{\epsilon})$ as

$$F_{mn}(u^{\epsilon})(\tau) = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} \sin(u^{\epsilon}(x,y,\tau))\sin(mx)\sin(ny)\,dxdy + H_{mn}(\tau) + \cos(\pi\tau)\,I_{mn},$$

where

$$H_{mn}(\tau) = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} \sin\left(x^2 y(x-\pi)(\pi-y)^2\cos(\pi\tau)\right)\sin(mx)\sin(ny)\,dxdy$$

and

$$\begin{aligned} I_{mn} = {} & \frac{8}{\pi^2}\int_0^{\pi}\int_0^{\pi}(\pi-3x)\,y\,(\pi-y)^2\sin(mx)\sin(ny)\,dxdy \\ & + \frac{8}{\pi^2}\int_0^{\pi}\int_0^{\pi} x^2(\pi-x)(3y-2\pi)\sin(mx)\sin(ny)\,dxdy \\ & + 4\int_0^{\pi}\int_0^{\pi} x^2 y(x-\pi)(\pi-y)^2\sin(mx)\sin(ny)\,dxdy. \end{aligned}$$

We choose $u^{\epsilon}_{L_1}(x,y) \equiv \varphi^{\epsilon}(x,y)$ for the partition $z_i = i\Delta z$ where $\Delta z = \dfrac{1}{L_1}, i = \overline{0, L_1}$. The following iterative scheme deals with the nonlinear term. For $i = 0, \ldots, L_1 - 1$, we get

$$u^{\epsilon}(x,y,z_i) = \sum_{\substack{m,n\ge 1 \\ \sqrt{m^2+n^2}\le C_{\epsilon}}} \left(R_{mn}(\epsilon,z_i) + W_{mn}(\epsilon,z_i) + P_{mn}(z_i)\right)\sin(mx)\sin(ny),$$

where

$$R_{mn}(\epsilon,z_i) = \frac{16}{m^3n^3}\left(1 + 105\frac{\epsilon\cdot\text{rand}(m,n)}{\pi^7}\right)\cosh((1-z_i)\,\alpha(m,n))\,(1+2(-1)^m)\,(2+(-1)^n),$$

$$W_{mn}(\epsilon,z_i) = \frac{4}{\pi^2}\sum_{h=i+1}^{L_1}\int_{t_{h-1}}^{t_h}\frac{\sinh((\tau-z_i)\,\alpha(m,n))}{\alpha(m,n)}d\tau\int_0^{\pi}\int_0^{\pi}\sin\left(u^{\epsilon}_h(x,y)\right)\sin(mx)\sin(ny)\,dxdy,$$

$$P_{mn}(z_i) = \int_{z_i}^{1}\frac{\sinh((\tau-z_i)\,\alpha(m,n))}{\alpha(m,n)}\left(H_{mn}(\tau) + \cos(\pi\tau)\,F_{mn}\right)d\tau.$$

These integrals are computed by Gauss-Legendre quadratures. Particularly, we shall approximate the following integrations.

$$J_1(h) = \int_0^{\pi}\int_0^{\pi}\sin\left(u^{\epsilon}_h(x,y)\right)\sin(mx)\sin(ny)\,dxdy,$$

$$J_2(i) = \int_{z_i}^{1}\sinh((\tau-z_i)\,\alpha(m,n))\,H_{mn}(\tau)\,d\tau.$$

Given $(j_0, l_0, p_0) \in \mathbb{N}^3$, we approximate these integrations by

$$J_1(h) \approx \sum_{j=0}^{j_0}\sum_{l=0}^{l_0} w_j w_l \sin\left(u^{\epsilon}_h\left(x_j, y_l\right)\right)\sin\left(mx_j\right)\sin(ny_l),$$

ϵ	1.0E-03	1.0E-05	1.0E-07	1.0E-09
E_1	1.32766286E+00	9.23534307E-02	2.98111525E-02	1.43603493E-02
E_2	1.47903472E-01	1.02882994E-02	3.32100347E-03	1.59976270E-03

Table 1: Error (3.42)-(3.43) between regularized solution and exact solution in Example 1 at $z = 0.9998$ in the meshsize $\Delta z = 1/5000$.

and

$$J_2(i) \approx \frac{4}{\pi^2}\sum_{p=0}^{p_0} w_p \sinh\left(\left(z_p - z_i\right)\alpha(m,n)\right)$$
$$\times \sum_{j=0}^{j_0}\sum_{l=0}^{l_0} w_j w_l \sin\left(x_j^2 y_l\left(x_j-\pi\right)(\pi-y_l)^2\cos\left(\pi z_p\right)\right)\sin\left(m x_j\right)\sin\left(n y_l\right),$$

where x_j, y_l and z_p are abscissae in $[0,\pi]$ and $[z_i, 1]$, and w_j, w_l and w_p are corresponding weights.

Remark 3.1. *In general, the whole process of computation is performed in four steps.*

Step 1. Choose L_1, L_2 and $L_3 \in \mathbb{Z}^+$ and define, for $i = \overline{0, L_1}$, $j = \overline{0, L_2}$, $l = \overline{0, L_3}$,

$$z_i = i\Delta z, \quad \Delta z = \frac{1}{L_1}, \qquad x_j = j\Delta x, \quad \Delta x = \frac{\pi}{L_2}, \qquad y_l = l\Delta y, \quad \Delta y = \frac{\pi}{L_3}.$$

Step 2. Put $u_i^\epsilon(x,y) \equiv u^\epsilon(x,y,z_i)$, $i = \overline{0,L_1}$ and set $u_{L_1}^\epsilon(x,y) = \varphi^\epsilon(x,y)$. Then, we compute a vector of components $u_i^\epsilon(x,y)$

$$U^\epsilon(x,y) = \left[u_0^\epsilon(x,y) \quad u_1^\epsilon(x,y) \quad \ldots \quad u_{L_1}^\epsilon(x,y)\right]^T \in \mathbb{R}^{L_1+1}.$$

Step 3. For $i = \overline{0,L_1}$, $j = \overline{0,L_2}$, $l = \overline{0,L_3}$, put $u_i^\epsilon\left(x_j, y_l\right) = u_{i,j,l}^\epsilon$ and $u\left(x_j, y_l, z_i\right) = u_{i,j,l}$, $i = \overline{0,L_1}$. Construct the following matrices of size $\mathbb{R}^{L_2+1} \times \mathbb{R}^{L_3+1}$

$$U_i^\epsilon = \begin{bmatrix} u_{i,0,0}^\epsilon & u_{i,0,1}^\epsilon & \cdots & u_{i,0,L_3}^\epsilon \\ u_{i,1,0}^\epsilon & u_{i,1,1}^\epsilon & \cdots & u_{i,1,L_3}^\epsilon \\ \vdots & \vdots & \ddots & \vdots \\ u_{i,L_2,0}^\epsilon & u_{i,L_2,1}^\epsilon & \ldots & u_{i,L_2,L_3}^\epsilon \end{bmatrix} \quad \textit{and} \quad U_i = \begin{bmatrix} u_{i,0,0} & u_{i,0,1} & \cdots & u_{i,0,L_3} \\ u_{i,1,0} & u_{i,1,1} & \cdots & u_{i,1,L_3} \\ \vdots & \vdots & \ddots & \vdots \\ u_{i,L_2,0} & u_{i,L_2,1} & \ldots & u_{i,L_2,L_3} \end{bmatrix}.$$

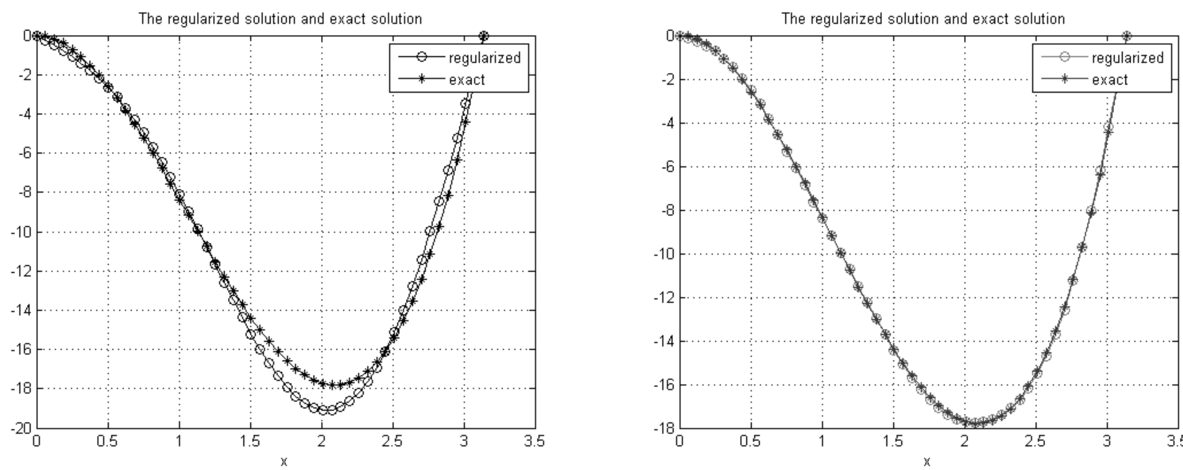
Step 4. For $i = \overline{0,L_1}$, the errors are computed by

$$E_1(z_i) = \sqrt{\frac{1}{(L_2+1)(L_3+1)}\sum_{j=0}^{L_2}\sum_{l=0}^{L_3}\left|u_{i,j,l}^\epsilon - u_{i,j,l}\right|^2}, \tag{3.42}$$

$$E_2(z_i) = \frac{\sqrt{\sum_{j=0}^{L_2}\sum_{l=0}^{L_3}\left|u_{i,j,l}^\epsilon - u_{i,j,l}\right|^2}}{\sqrt{\sum_{j=0}^{L_2}\sum_{l=0}^{L_3}\left|u_{i,j,l}\right|^2}}. \tag{3.43}$$

In this computation, we compute the numerical solution at z very near $L = 1$, namely, $z = 0.9998$ in the meshsize $\Delta z = 1/5000$ by choosing $L_1 = 5000$, $L_2 = L_3 = 50$. We also take the constants $p_0 = j_0 = l_0 = 9$.

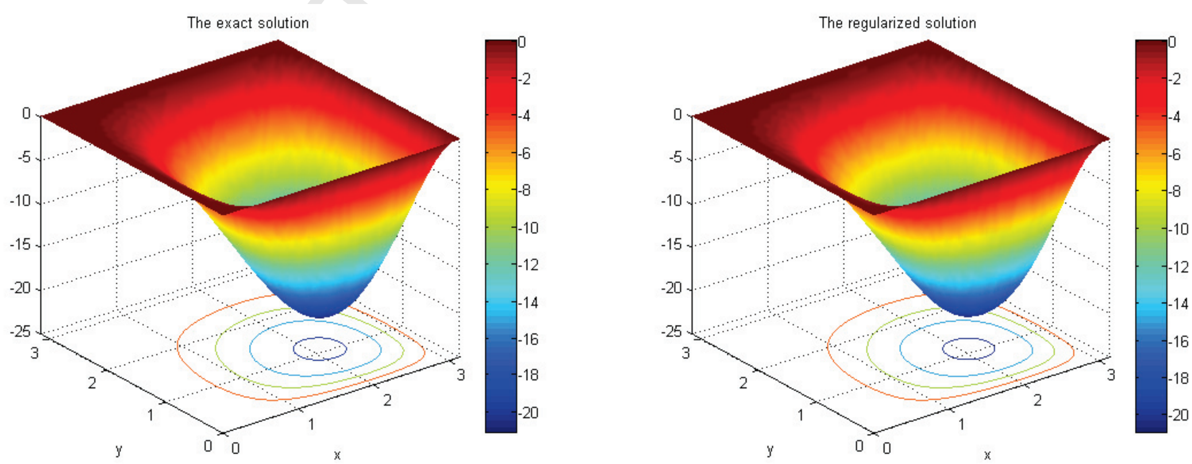
In Table 1, we show the errors E_1 and E_2 (defined in (3.42)-(3.43)) between the exact solution and regularized solution. It is observed that the computed errors agree with our theoretical results. In Figure 1, we plot the exact and regularized solutions against the x-variable, when $\epsilon = 10^{-3}$ and $\epsilon = 10^{-5}$. Convergence is also observed. Furthermore, in order to give a global visual comparison, in Figure 2 we plot the 3-D graphs of the exact and regularized solutions for $\epsilon = 10^{-5}$.



(a) $\epsilon = 10^{-3}$

(b) $\epsilon = 10^{-5}$

Figure 1: Regularized solution and exact solution of Example 1 at $y = \dfrac{\pi}{2}, z = 0.9998$ with $\epsilon = 10^{-3}$ and 10^{-5}.



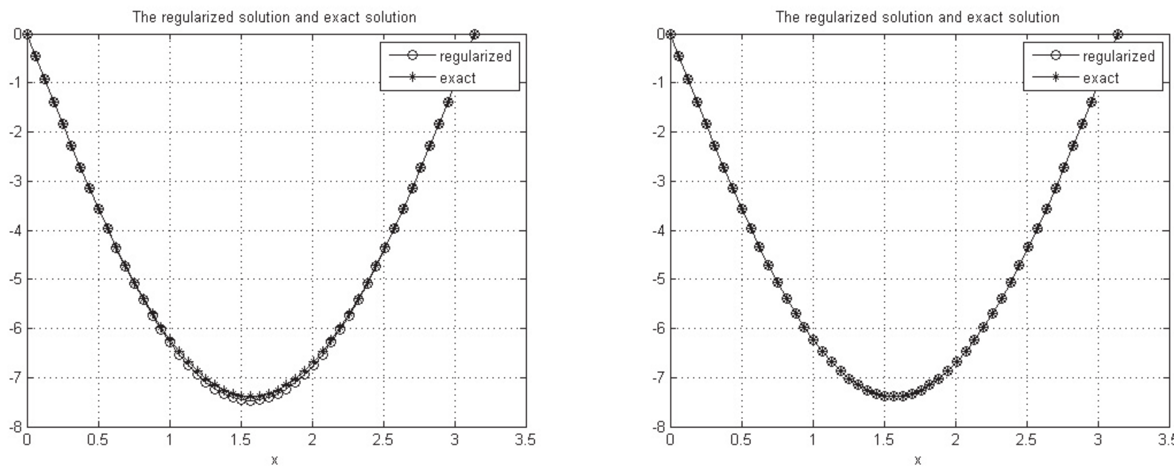
(a) Exact solution

(b) Regularized solution

Figure 2: 3-D graphs of exact solution at $y = \dfrac{\pi}{2}, z = 0.9998$ and its regularized solution for $\epsilon = 10^{-5}$ in Example 1.

ϵ	1.0E-02	1.0E-03	1.0E-04	1.0E-05
E_1	2.72033670E-02	2.16776832E-02	1.49835328E-02	9.06238139E-03
E_2	7.80256493E-03	6.89209117E-03	3.11245256E-03	8.23713802E-04

Table 2: Errors (3.42)–(3.43) between regularized solution in recursive step $q = 10$ and exact solution in Example 2 at $z = 0.98$ in the meshsize $\Delta z = 1/50$.



(a) $\epsilon = 10^{-3}$ (b) $\epsilon = 10^{-5}$

Figure 3: Regularized solution and exact solution of Example 2 at $y = \frac{\pi}{2}, z = 0.98$ with $\epsilon = 10^{-3}$ and 10^{-5}.

3.2. Example 2: Elliptic Allen-Cahn equation

For this example, we take

$$F(x, y, z, u(x, y, z)) = u - u^3 + G(x, y, z) \quad \text{and} \quad \varphi(x, y) = -e^2 \sin x \sin y,$$

where

$$G(x, y, z) = u_{ex}^3 + 3e^{4-2z} \sin x \sin y.$$

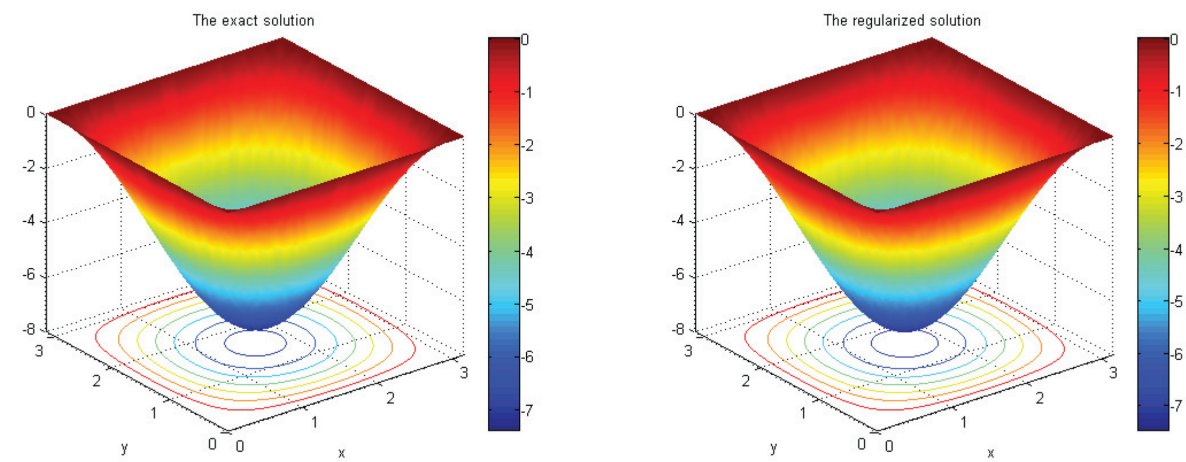
The exact solution is then $u_{ex}(x, y, z) = e^3 \left(e^{1-2z} - 2e^{-z}\right) \sin x \sin y$.

As in Example 1, we compute the approximate solution and choose the regularization parameter $C_\epsilon = \ln\left(\frac{1}{\epsilon}\right)$. We compute

$$\begin{aligned} u_{\epsilon,\varphi^\epsilon}(x, y, z) &= \sum_{m,n=1}^{\alpha(m,n)\leq[C_\epsilon]} \Big[\cosh\left((1-z)\,\alpha(m,n)\right)\varphi_{mn}^\epsilon \\ &\quad + \int_z^1 \frac{\sinh\left((\tau - z)\,\alpha(m,n)\right)}{\alpha(m,n)} F_{mn}\left(u_{\epsilon,\varphi^\epsilon}\right)(\tau)\, d\tau\Big] \sin(mx)\sin(ny), \end{aligned}$$

where

$$F_{mn}\left(u_{\epsilon,\varphi^\epsilon}\right)(\tau) = \begin{cases} F_{mn}(M_\epsilon)(\tau), & u_{\epsilon,\varphi^\epsilon} > M_\epsilon, \\ F_{mn}\left(u_{\epsilon,\varphi^\epsilon}\right)(\tau), & u_{\epsilon,\varphi^\epsilon} \in [-M_\epsilon, M_\epsilon], \\ F_{mn}(-M_\epsilon)(\tau), & u_{\epsilon,\varphi^\epsilon} < -M_\epsilon, \end{cases} \qquad M_\epsilon = \sqrt{\frac{\sqrt{\ln(-\ln(\epsilon))} - 1}{3}}.$$



(a) Exact solution

(b) Regularized solution

Figure 4: 3-D graphs of exact solution at $y = \frac{\pi}{2}, z = 0.98$ and its regularized solution for $\epsilon = 10^{-3}$ in Example 2.

The term φ^{ϵ}_{mn} can be directly calculated. For each $q \in \mathbb{N}$, we define the sequence $\left\{u^q_{\epsilon,\varphi^\epsilon}\right\}_{q\geq 0}$ by recurrence, starting with $u^0_{\epsilon,\varphi^\epsilon} \equiv -exp(2)\sin x \sin y$. The sequence satisfies

$$u^{q+1}_{\epsilon,\varphi^\epsilon}(x,y,z) = \sum_{m,n=1}^{\alpha(m,n)\leq[C_\epsilon]} \Big[\cosh\left((1-z)\,\alpha(m,n)\right)\varphi^{\epsilon}_{mn} + \int_z^1 \frac{\sinh\left((\tau - z)\,\alpha(m,n)\right)}{\alpha(m,n)} F_{mn}\left(u^q_{\epsilon,\varphi^\epsilon}\right)(\tau)\,d\tau\Big]\sin(mx)\sin(ny).$$

Similarly to Example 1, we present the numerical results in Table 2 and Figure 3 for the case $L_1 = L_2 = L_3 = 50$. The 3D graphs of the solutions are presented in Figure 4.

In comparison with the Example 1, it can be predicted that the smoother the function, the better the convergence rate. However, sufficient recurrence steps (e.g. $q = 10$) are required to obtain fast convergence. In the case $\epsilon = 10^{-5}$ and $q = 2$ the errors are $E_1 = 2.4461E - 02$ and $E_2 = 7.0160E - 03$. Of course, a larger value for q incurs high computational costs. Moreover, the numerical results of points far from the point $z = L$ may be obtained when the meshsizes and ϵ are sufficiently small. A better approximation is required, and this is a topic for future research.

4. Conclusion

In this paper, we propose a regularization method based on the cut-off method for solving inverse boundary value problems of nonlinear elliptic equations. Error estimates are provided. Numerical experiments with the elliptic sine-Gordon equation and elliptic Allen-Cahn equation are carried out to corroborate the efficiency of the method.

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