J. Math. Anal. Appl. 438 (2016) 395–443

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Existence of global strong solution for Korteweg system with large infinite energy initial data

Boris Haspot

Université Paris Dauphine, PSL Research University, Ceremade, Umr Cnrs 7534, Place du Maréchal De Lattre De Tassigny, 75775 Paris cedex 16, France

A R T I C L E   I N F O

Article history:
Received 20 December 2014
Available online 27 January 2016
Submitted by D. Wang

Keywords:
Korteweg system
Strong solution
Besov spaces
Critical spaces
Paraproduct

A B S T R A C T

This work is devoted to the study of the initial boundary value problem for a general isothermal model of capillary fluids derived by J.E. Dunn and J. Serrin (1985) (see [18]), which can be used as a phase transition model. We aim at proving the existence of local and global (under a condition of smallness on the initial data) strong solutions with initial density $\ln \rho_0$ belonging to the Besov space $B^{\frac{N}{2}}_{2,\infty}$. It implies in particular that some classes of discontinuous initial density generate strong solutions. The proof relies on the fact that the density can be written as the sum of the solution ρ_L of the associated linear system and a remainder term $\bar{\rho}$; this last term is more regular than ρ_L provided that we have regularizing effects induced on the bilinear convection term. The main difficulty consists in obtaining new estimates of maximum principle type for the associated linear system; this is based on a characterization of the Besov space in terms of the semi-group associated with this linear system. We show in particular the existence of global strong solution for small initial data in $(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty} \cap L^\infty) \times B^{\frac{N}{2}-1}_{2,\infty}$; it allows us to exhibit a family of large energy initial data when $N = 2$ providing global strong solution. In conclusion we introduce the notion of quasi-solutions for the Korteweg's system (a tool which has been developed in the framework of the compressible Navier–Stokes equations [31,30,32,26,27]) which enables to obtain the existence of global strong solution with a smallness condition which is subcritical. Indeed we can deal with large initial velocity in $B^{\frac{N}{2}-1}_{2,1}$. As a corollary, we get global strong solution for highly compressible Korteweg system when $N \geq 2$. It means that for any large initial data (under an irrotational condition on the initial velocity) we have the existence of global strong solution provided that the Mach number is sufficiently large.

1. Introduction

We are concerned with compressible fluids endowed with internal capillarity. The model we consider originates from the XIXth century work by J.F. Van der Waals and D.J. Korteweg [44,37] and was actually

E-mail address: haspot@ceremade.dauphine.fr.

http://dx.doi.org/10.1016/j.jmaa.2016.01.047

derived in its modern form in the 1980s using the second gradient theory (see [18,35,43]). The first investigations begin with the Young–Laplace theory which claims that the phases are separated by a hypersurface and that the jump in the pressure across the hypersurface is proportional to the curvature of the hypersurface. The main difficulty consists in describing the location and the movement of the interfaces. Another major problem is to understand whether the interface behaves as a discontinuity in the state space (sharp interface) or whether the phase boundary corresponds to a more regular transition (diffuse interface, DI). The diffuse interface models have the advantage to consider only one set of equations in a single spatial domain (the density takes into account the different phases) which considerably simplifies the mathematical and numerical study (indeed in the case of sharp interfaces, we have to treat a problem with free boundary). Another approach corresponds to determining equilibrium solutions which classically consists of the minimization of the free energy functional. Unfortunately this minimization problem has an infinity of solutions, and many of them are physically irrelevant. In order to overcome this difficulty, J.F. Van der Waals in the XIX-th century was the first to add a term of capillarity in order to select the physically correct solutions. This theory is widely accepted as a thermodynamically consistent model for equilibria.

Korteweg-type models are based on an extended version of nonequilibrium thermodynamics, which assumes that the energy of the fluid not only depends on standard variables but also on the gradient of the density. Alternatively, another way to penalize the high density variations consists in applying a zero order but non-local operator to the density gradient (see [42,41,40]). For more results on non-local Korteweg system, we refer also to [10–13,23,24].

Let us now consider a fluid of density $\rho \geq 0$, velocity field $u \in \mathbb{R}^N$, we are now going to consider the so-called local Korteweg system which is a compressible capillary fluid model, it can be derived from a Cahn–Hilliard like free energy (see the pioneering work by J.E. Dunn and J. Serrin in [18] and also [1,8,21]). The conservation of mass and of momentum write:

$$\begin{cases} \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho u) = 0, \\ \dfrac{\partial}{\partial t}(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho)D(u)) - \nabla\big(\lambda(\rho))\operatorname{div}u\big) + \nabla P(\rho) = \operatorname{div}K, \end{cases} \tag{1.1}$$

where the Korteweg tensor reads as:

$$\operatorname{div}K = \nabla\big(\rho\kappa(\rho)\Delta\rho + \frac{1}{2}(\kappa(\rho) + \rho\kappa'(\rho))|\nabla\rho|^2\big) - \operatorname{div}\big(\kappa(\rho)\nabla\rho \otimes \nabla\rho\big). \tag{1.2}$$

κ is the coefficient of capillarity and is a regular function. The term $\operatorname{div}K$ allows to describe the variation of density at the interfaces between two phases, generally a mixture liquid-vapor. P is a general increasing pressure. $D(u) = \frac{1}{2}(\nabla u +^t \nabla u)$ defines the stress tensor, μ and λ are the two Lamé viscosity coefficients depending on the density ρ and satisfying:

$$\mu > 0 \ \text{ and } \ 2\mu + N\lambda \geq 0.$$

We briefly recall the classical energy estimates for the system (1.1); let $\bar{\rho} > 0$ be a constant reference density (in what follows, we shall assume that $\bar{\rho} = 1$) and let Π be defined by:

$$\Pi(s) = s\Big(\int_{\bar{\rho}}^{s} \frac{P(z)}{z^2}dz - \frac{P(\bar{\rho})}{\bar{\rho}}\Big),$$

so that $P(s) = s\Pi'(s) - \Pi(s)$, $\Pi'(\bar{\rho}) = 0$. Multiplying the equation of momentum conservation in the system (1.1) by u and integrating by parts over $(0,t) \times \mathbb{R}^N$, we get the following estimate:

$$\int_{\mathbb{R}^N}\big(\frac{1}{2}\rho|u|^2+(\Pi(\rho)-\Pi(\bar{\rho}))+\frac{1}{2}\kappa(\rho)|\nabla\rho|^2\big)(t)dx+2\int_0^t\int_{\mathbb{R}^N}\mu(\rho)|D(u)|^2dxdt$$
$$+\int_0^t\int_{\mathbb{R}^N}\lambda(\rho)(\mathrm{div}u)^2dxdt\le\int_{\mathbb{R}^N}\big(\rho_0|u_0|^2+(\Pi(\rho_0)-\Pi(\bar{\rho}))+\frac{1}{2}\kappa(\rho_0)|\nabla\rho_0|^2\big)dx. \tag{1.3}$$

It follows that assuming that the initial total energy is finite:

$$\mathcal{E}_0=\int_{\mathbb{R}^N}\big(\rho_0|u_0|^2+(\Pi(\rho_0)-\Pi(\bar{\rho}))+\frac{\kappa(\rho_0)}{2}|\nabla\rho_0|^2\big)dx<+\infty\,, \tag{1.4}$$

then we have the following a priori bounds when $P(\rho)=a\rho^\gamma$ with $\gamma>1$ (see [39] for the definition of the Orlicz spaces $L^\gamma_2(\mathbb{R}^N)$):

$$(\rho-1)\in L^\infty(L^\gamma_2(\mathbb{R}^N))\ \text{ and }\ \rho|u|^2\in L^\infty((0,+\infty),L^1(\mathbb{R}^N)),$$
$$\sqrt{\kappa(\rho)}\nabla\rho\in L^\infty((0,+\infty),L^2(\mathbb{R}^N))^N\ \text{ and }\ \sqrt{\mu(\rho)}Du\in L^2((0,+\infty)\times\mathbb{R}^N)^{N^2}.$$

In what follows, we are interested in investigating the existence of global strong solution for the system (1.1) with initial density not necessary continuous (in particular admitting jump in the pressure across the interfaces). In order to realize this program, it seems natural to work with initial data belonging to critical Besov space (it means in spaces as large as possible) for the scaling of the equations. Let us now recall the notion of scaling for the Korteweg's system (1.1). Such an approach is now classical for incompressible Navier–Stokes equation and yields local well-posedness (or global well-posedness for small initial data) in spaces with minimal regularity. In our situation we can easily check that, if (ρ,u) solves (1.1), then (ρ_λ,u_λ) solves also this system:

$$\rho_\lambda(t,x)=\rho(\lambda^2t,\lambda x),\ \ u_\lambda(t,x)=\lambda u(\lambda^2t,\lambda x)$$

provided the pressure laws P have been changed to λ^2P.

Definition 1.1. We say that a function space is critical with respect to the scaling of the equation if the associated norm is invariant under the transformation:

$$(\rho,u)\longrightarrow(\rho_\lambda,u_\lambda)$$

(up to a constant independent of λ).

This suggests us to choose initial data (ρ_0,u_0) in spaces whose norm is invariant for all $\lambda>0$ by the transformation $(\rho_0,u_0)\longrightarrow(\rho_0(\lambda\cdot),\lambda u_0(\lambda\cdot))$. A natural candidate is the Besov space $B^{N/2}_{2,\infty}\times(B^{N/2-1}_{2,\infty})^N$ (see the section 2 for some definitions of Besov spaces). However since $B^{N/2}_{2,\infty}$ is not included in L^∞, we cannot expect a priori L^∞ estimate on the density, in particular it seems delicate to deal with the nonlinear term such as the pressure since it is then impossible to use composition theorems. Another difficulty concerns the control of the vacuum or more precisely the L^∞ norm of $\frac{1}{\rho}$; indeed it is crucial to avoid vacuum if we want to take into account the parabolic effects of the momentum equation.

That is why in the sequel we will work with the critical Besov spaces $(B^{N/2}_{2,\infty}\cap L^\infty)\times(B^{N/2-1}_{2,\infty})^N$. However estimating the L^∞ norm of the density all along the time for general physical coefficients is generally a hard task in fluid mechanics, even if the Korteweg system allows to obtain regularizing effects on the density. This is the reason why in the literature the authors consider initial density which are in Banach space X

embedded in L^∞; it suffices then to propagate the regularity of X all along the time via the parabolic structure of the system in order to have L^∞ estimate on the density ρ. In our case we shall proceed in a different way and we will explain in the sequel how to overcome this type of difficulty.

Let us briefly mention that the existence of strong solutions for $N \geq 2$ is known since the works by H. Hattori and D. Li [33,34]. R. Danchin and B. Desjardins in [17] improve this result by working for the first time in critical Besov spaces for the scaling of the equations. More precisely the initial data $(\rho_0 - 1, \rho_0 u_0)$ belong to $B^{\frac{N}{2}}_{2,1} \times B^{\frac{N}{2}1}_{2,1}$ (it is important to point out that $B^{\frac{N}{2}}_{2,1}$ is embedded in L^∞ which allows to control the vacuum and the L^∞ norm of the density). In [38], M. Kotschote showed the existence of strong solution for the isothermal model in bounded domain by using Doreâ–Venni Theory and $\mathcal{H}^\infty$ calculus. In [22], we generalize the results of [17] in the case of non-isothermal Korteweg system with physical coefficients depending on the density and the temperature.

1.1. Mathematical results

We now are going to state our main results. As we explained previously, one of the main difficulty in order to obtain strong solutions for Korteweg system in very general Besov spaces consists in dealing with the L^∞ control on $\frac{1}{\rho}$ and on ρ. To do this, it is essential to understand precisely the structure of the equations in order to use in a suitable way a maximum principle argument. We are going to consider at first the particular case of the capillary coefficient $\kappa(\rho) = \frac{\kappa}{\rho}$ with $\kappa > 0$. Indeed it is possible to rewrite the system (1.1) in a simple way by introducing an effective velocity which allows to highlight the parabolic structure of the Korteweg system (this effective velocity has been introduced by A. Jüngel in [36], we refer also to [7,36,29,20,28] where the authors prove the existence of global weak solutions).

Remark 1. Let us give some explanations on this choice of capillarity $\kappa(\rho) = \frac{\kappa}{\rho}$, indeed this regime flows exhibits particular phenomena in the case of the compressible Korteweg Euler system (which is called Euler system with quantum pressure when $\kappa(\rho) = \frac{\kappa}{\rho}$). At least heuristically, the system is equivalent via the Madelung transform to the Gross–Pitaevskii equations which are globally well-posed for large initial data in dimension $N = 1, 2, 3$ (we refer to [19]). One of the main difficulty to pass from Gross–Pitaevskii equations to the Euler system with quantum pressure consists in dealing with the vacuum (we refer to [3,4] for the existence of global strong "dispersive" solution with small initial irrotational data). We would also like to mention very interesting results of global weak solutions for the compressible Euler system with quantum pressure due to P. Antonelli and P. Marcati (see [2]). To finish, we mention that there exist global strong solutions with large initial data in one dimension for the system (1.1) when $\kappa(\rho) = \frac{\mu^2}{\rho}$ and $\mu(\rho) = \mu\rho$, $\lambda(\rho) = 0$ (we refer to [14]). Furthermore these solutions converge to a global weak entropy solution of the compressible Euler system when μ goes to 0. It shows in particular that the Korteweg system is relevant to select the physical solution of the compressible Euler system via a viscosity-capillarity vanishing process.

When $\kappa(\rho) = \frac{\kappa}{\rho}$ with $\kappa > 0$, we can rewrite the capillarity tensor as follows (see the appendix for more details on the computations):

$$K(\rho) = \kappa\rho(\nabla\Delta(\ln\rho) + \frac{1}{2}\nabla(|\nabla\ln\rho|^2)).$$

The system (1.1) reads as follows (at least if we assume that there is no vacuum):

$$\begin{cases} \partial_t \ln\rho + u\cdot\nabla\ln\rho + \mathrm{div} u = 0, \\ \partial_t u + u\cdot\nabla u - \frac{1}{\rho}\mathrm{div}(2\mu(\rho)Du) - \frac{1}{\rho}\nabla(\lambda(\rho)\mathrm{div} u) + \nabla F(\rho) = \kappa\nabla\Delta(\ln\rho) + \frac{\kappa}{2}\nabla(|\nabla\ln\rho|^2), \\ (\ln\rho, u)_{\ t=0} = (\ln\rho_0, u_0), \end{cases} \tag{1.5}$$

with $F(\rho)$ defined by $\frac{F'(\rho)}{\rho} = P'(\rho)$. In the sequel we will use the following definition.

Definition 1.2. We now set:

$$q = \ln \rho.$$

In the first theorem of this paper we are going to prove the existence of global strong solution for (1.5) with *small* initial data and of strong solution in finite time with large initial data when we deal with specific viscosity coefficients and pressure terms. More precisely we shall deal with the shallow water viscosity coefficients:

$$\mu(\rho) = \mu\rho, \ \lambda(\rho) = \lambda\rho,$$

with $\mu > 0$ and $2\mu + \lambda > 0$. It leads to the following system:

$$\begin{cases} \partial_t q + u \cdot \nabla q + \mathrm{div} u = 0, \\ \partial_t u + u \cdot \nabla u - \mu\Delta u - 2\mu\nabla q \cdot D(u) - (\lambda + \mu)\nabla\mathrm{div} u - \lambda \mathrm{div} u \nabla q + \nabla F(\rho) \\ \quad = \kappa\nabla\Delta q + \frac{\kappa}{2}\nabla(|\nabla q|^2), \\ (q, u)_{\ t=0} = (\ln \rho_0, u_0). \end{cases} \tag{1.6}$$

The interest to restrict our attention to this type of physical coefficients is to show the existence of strong solution in very large Besov space of initial data. Indeed the system (1.6) has essentially no nonlinear terms in the sense that we can deal with all the nonlinearities without using composition theorem; it implies in particular that we do not require any L^∞ control on the density q in order to obtain existence of strong solution.

Let us give a definition on the space in which we are going to work.

Definition 1.3. We set $X_0^{\frac{N}{2}}$ and $X_0^{\frac{N}{2}-1}$ the space which corresponds to:

$$X_0^{\frac{N}{2}} = \overline{\mathcal{S}_0 \cap B^{\frac{N}{2}}_{2,\infty}}^{B^{\frac{N}{2}}_{2,\infty}},$$
$$X_0^{\frac{N}{2}-1} = \overline{\mathcal{S}_0 \cap B^{\frac{N}{2}-1}_{2,\infty}}^{B^{\frac{N}{2}}_{2,\infty}}.$$

Remark 2. Here $\mathcal{S}_0$ defines functions in the Schwartz space whose Fourier transforms are supported away from 0 and we consider the closure of $\mathcal{S}_0 \cap B^{\frac{N}{2}}_{2,\infty}$ in $B^{\frac{N}{2}}_{2,\infty}$ and of $\mathcal{S}_0 \cap B^{\frac{N}{2}-1}_{2,\infty}$ in $B^{\frac{N}{2}-1}_{2,\infty}$. We can observe in particular that if $u \in X_0^{\frac{N}{2}}$ then we have:

$$\lim_{j \to \pm\infty} 2^{j\frac{N}{2}} \|\Delta_j u\|_{L^2} = 0.$$

It is necessary to work in these spaces if we wish to prove the existence of strong solution in finite time (indeed it requires that $\lim_{T\to 0} \|u_L\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,\infty})} = 0$ with u_L solution of the heat equation for the initial velocity u_0 in $X_0^{\frac{N}{2}-1}$).

Concerning the existence of global strong solution with small initial data, we will take only initial data in $(B^{\frac{N}{2}}_{2,\infty} \cap B^{\frac{N}{2}-1}_{2,\infty}) \times (B^{\frac{N}{2}\,1}_{2,\infty})^N$.

Let us now state our main results, we refer to the section 2 for the definitions of the Besov space and the Hybrid Besov spaces.

Theorem 1.1. *Let $N \geq 2$. Assume that $\mu(\rho) = \mu\rho$, $\lambda(\rho) = \lambda\rho$ with $\mu > 0$, $2\mu + \lambda > 0$ and $P(\rho) = K\rho$ with $K > 0$. We also suppose that $q_0 \in X_0^{\frac{N}{2}}$ and $u_0 \in X_0^{\frac{N}{2}-1}$. There exists a time T such that (1.6) has a unique solution (q, u) on $(0, T)$ with:*

$$q \in \widetilde{L}^\infty_T(B^{\frac{N}{2}}_{2,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{2}+2}_{2,\infty}), \ \ \text{and} \ \ u \in \widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty}).$$

Furthermore there exists ε_0 such that if in addition:

$$\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty}} + \|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}} \leq \varepsilon_0, \tag{1.7}$$

then the solution (q, u) is global with:

$$q \in \widetilde{L}^\infty(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty}) \cap \widetilde{L}^1(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty}), \ u \in \widetilde{L}^\infty(\mathbb{R}^+, B^{\frac{N}{2}-1}_{2,\infty}) \cap \widetilde{L}^1(\mathbb{R}^+, B^{\frac{N}{2}+1}_{2,\infty}).$$

If in addition to the hypothesis (1.7) we assume that (q_0, u_0) belongs to $\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,2} \times B^{\frac{N}{2}-1}_{2,2}$ and that $q_0 \in L^\infty$ then there exists a unique solution (ρ, u) of the system (1.1) with $q = e\rho$. Furthermore for any $T > 0$ there exists $C_T > 0$ depending on T such that:

$$\|\frac{1}{\rho}\|_{L^\infty_T(L^\infty)} + \|\rho\|_{L^\infty_T(L^\infty)} \leq C_T.$$

In addition for any $T > 0$ we have:

$$q \in \widetilde{L}^\infty_T(B^{\frac{N}{2}}_{2,2}) \cap \widetilde{L}^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,2}) \ \ \text{and} \ \ u \in \widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,2}) \cap \widetilde{L}^1(B^{\frac{N}{2}+1}_{2,2}). \tag{1.8}$$

Remark 3. It is worth pointing point out that in the first part of the Theorem 1.1, we solve the system (1.6) and not the system (1.1). Indeed we do not assume any control on q_0 in L^∞, it means that we have no information on the vacuum of the density. It is then not clear that a solution of (1.6) is also a solution of (1.1) when there is vacuum. This result proves in a certain sense that the good variable to consider is not the density ρ but rather $\ln\rho$. Let us emphasize on the fact that this result allows to deal with general critical initial data $u_0 \in B^{\frac{N}{2}-1}_{2,\infty}$ which is not classical for compressible systems (the most of the time u_0 belongs to $B^{\frac{N}{2}-1}_{2,1}$ unlike the incompressible Navier–Stokes equations, see [9]). This is obviously due to the fact that the Korteweg system provides a parabolic effect on the density ρ (it is of course not the case for the compressible Navier–Stokes system, see [25]).

In the second part of the theorem we consider initial data with additional hypotheses of regularity, indeed now $q_0 = \ln\rho_0$ belongs also to $L^\infty \cap B^{\frac{N}{2}}_{22}$ and u_0 to $B^{\frac{N}{2}-1}_{22}$. It allows us to show that with such initial data we have existence of strong solution for the "real" Korteweg system (1.1). The main difficulty consists in estimating the L^∞ norm of the density $q = \ln\rho$; to do this we decompose the solution q as the sum of the solution of the linearized system q_L and a remainder term $\bar{q}$ which takes into account the nonlinear terms. By combining some maximum principle results on q_L and regularizing effects on $\bar{q}$ for the third index of the Besov space, we can show that q is well bounded in L^∞. We obtain our maximum principle result via a precise characterization of Besov spaces in terms of the semi-group associated with the linear system related to (1.1).

In addition we observe that in this theorem we can deal with discontinuous initial density which is not the case in [17]. However for any discontinuous initial density $q_0 \in B^{\frac{N}{2}}_{2,\infty}$, we remark that the density is immediately regularized in the sense that ρ is in $C^\infty((0, T), \mathbb{R}^N)$ for any $T > 0$. This is due to the fact that the interfaces are diffuse.

Remark 4. In this theorem we assume that (q_0, u_0) belongs to $(\widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}} \cap L^\infty) \times B_{2,2}^{\frac{N}{2}-1}$ but we just require a smallness assumption in $\widetilde{B}_{2,\infty}^{\frac{N}{2}-1,\frac{N}{2}} \times B_{2,\infty}^{\frac{N}{2}-1}$. In particular it allows us to prove the existence of global strong solution with large initial data in the energy space when $N = 2$, that is up to our knowledge something of new for the Korteweg system when $N = 2$.

We give an example of such initial data in the Corollary 3 where u_{0,ε,l_0} can be chosen as small as possible in $B_{2,\infty}^{\frac{N}{2}-1}$ but very *large* in $B_{2,2}^{\frac{N}{2}-1}$. Let us recall that when $N = 2$, $B_{2,2}^{\frac{N}{2}-1}$ corresponds to L^2 which is the energy space for u_0 (see (1.3)).

Another interesting point is that compared with [17], we can choose initial density with a large L^∞ norm.

Remark 5. We now want to point out the specificity of the different physical coefficients, it is typically the case for the pressure and the viscosity coefficients where $P(\rho) = K\rho$ and $\mu(\rho) = \mu\rho$. They correspond to the "magic" situation where there is no nonlinear terms to estimate, which require a L^∞ control on the density ρ.

Remark 6. We would like to mention that we could easily extend the result of strong solution in finite time to the framework of Besov spaces constructed on general L^p spaces when the initial data verify:

$$q_0 \in B_{p,\infty}^{\frac{N}{p}} \text{ and } u_0 \in B_{p,\infty}^{\frac{N}{p}-1}.$$

Concerning the existence of global strong solution we would have to assume that q_0 is in $\widetilde{B}_{2,p,\infty}^{\frac{N}{2}-1\,\frac{N}{p}}$ and $u_0 \in \widetilde{B}_{2,p,\infty}^{\frac{N}{2}-1,\frac{N}{p}-1}$.

The previous theorem uses in a crucial way the structure of the viscosity, capillary and pressure coefficients; indeed it allows to obtain global strong solution with a smallness assumption concerning only the space $\widetilde{B}_{2,\infty}^{\frac{N}{2}-1,\frac{N}{2}} \times B_{2,\infty}^{\frac{N}{2}-1}$. We would like to extend this result to general physical coefficients and in particular dealing with the case of the constant capillary coefficient in order to generalize the results of [17]. We have then the following result.

Theorem 1.2. *Let $N \geq 2$. Assume that $\mu(\rho) = \mu\rho$ or μ, $\lambda(\rho) = \lambda\rho$ or λ with $\mu > 0$, $2\mu + \lambda > 0$, $\kappa(\rho) = \frac{\kappa}{\rho}$ or κ with $\kappa > 0$ and P a regular function such that $P'(1) > 0$. Furthermore we suppose that $\rho_0 = 1 + h_0$:*

$$h_0 \in \widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}} \cap B_{2,1}^{\frac{N}{2}-2} \cap L^\infty \ \text{ and } \ u_0 \in B_{2,1}^{\frac{N}{2}-2} \cap B_{2,2}^{\frac{N}{2}-1}.$$

There exists ε_0 such that if:

$$\|h_0\|_{\widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}} \cap B_{2,1}^{\frac{N}{2}-2} \cap L^\infty} + \|u_0\|_{B_{2,1}^{\frac{N}{2}-2} \cap B_{2,2}^{\frac{N}{2}-1}} \leq \varepsilon_0,$$

then there exists a global unique solution (ρ, u) of the system (1.1) with $\rho = 1 + h$ and:

$$\begin{aligned} h \in \widetilde{L}^\infty(\mathbb{R}^+, \widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}} \cap B_{2,1}^{\frac{N}{2}-2}) \cap \widetilde{L}^1(\mathbb{R}^+, \widetilde{B}_{2,2}^{\frac{N}{2}+1,\frac{N}{2}+2} \cap B_{2,1}^{\frac{N}{2}}) \cap L^\infty(\mathbb{R}^+, L^\infty) \\ \text{and } \ u \in \widetilde{L}^\infty(\mathbb{R}^+, B_{2,2}^{\frac{N}{2}-1} \cap B_{2,1}^{\frac{N}{2}-2}) \cap \widetilde{L}^1(\mathbb{R}^+, B_{2,2}^{\frac{N}{2}+1} \cap B_{2,1}^{\frac{N}{2}}). \end{aligned} \tag{1.9}$$

Remark 7. We would like to mention that this theorem extend the results of [17] in terms of rough regularity on the initial data. Here we only assume that u_0 belongs to $B_{2,2}^{\frac{N}{2}-1}$ instead $B_{2,1}^{\frac{N}{2}-1}$, the main task as in the previous Theorem 1.1 consists in getting control on the L^∞ norm of the density ρ without assuming $h_0 \in B_{2,1}^{\frac{N}{2}}$ as in [17].

As in the previous theorem we can choose initial density which are not continuous.

Remark 8. Let us point out that in the previous Theorem 1.2 we ask additional regularity in low frequencies on (h_0, u_0); indeed we assume that (h_0, u_0) are in $B^{\frac{N}{2}-2}_{2,1} \times (B^{\frac{N}{2}-2}_{2,1})^N$. This is essentially due to the fact that we have shown *maximum principle* for the system (N) p. 411 (which does not take into account precisely the low frequency behavior of the Korteweg system) and not for the system $(N1)$ p. 413 (see the Proposition 3.10). It is probably possible to extend the Proposition 3.10 to the system $(N1)$. In particular it will allow to avoid this additional regularity on the initial data in low frequencies. For more explanations we refer to the Remark 20.

We are now interested in dealing with the specific case $\kappa(\rho) = \frac{\mu^2}{\rho}$ and $\mu(\rho) = \mu\rho$, $\lambda(\rho) = 0$ which corresponds to compressible Navier–Stokes system with quantum pressure for an intermediary regime (see [14] for more explanations on this notion). Setting $v = u + \mu\nabla \ln \rho$ we can rewrite the system (1.6) as follows (we refer to the appendix for more details on the computations or [36,29]):

$$\begin{cases} \partial_t \rho - \mu\Delta\rho = -\mathrm{div}(\rho v), \\ \rho\partial_t v + \rho u \cdot \nabla v - \mathrm{div}(\mu\rho\,\nabla v) + \nabla P(\rho) = 0, \end{cases} \tag{1.10}$$

which is equivalent to the following system (at least if we control the vacuum) with $q = \ln \rho$:

$$\begin{cases} \partial_t q - \mu\Delta q + v \cdot \nabla q = -\mathrm{div} v + \mu|\nabla q|^2, \\ \partial_t v + u \cdot \nabla v - \mu\Delta v - \mu\nabla q \cdot \nabla v + \nabla F(\rho) = 0. \end{cases} \tag{1.11}$$

In this particular case we are able to use a new tool developed in [26,27,31,30,32] called the quasi-solutions. More precisely we can check that there exists a particular solution of the following system (where we have canceled out the pressure P):

$$\begin{cases} \partial_t \rho - \mu\Delta\rho = -\mathrm{div}(\rho v), \\ \rho\partial_t v + \rho u \cdot \nabla v - \mathrm{div}(\mu\rho\,\nabla v) = 0. \end{cases} \tag{1.12}$$

Indeed we verify that $(\rho_1, -\mu\nabla \ln \rho_1)$ is a particular solution of (1.10) if the density ρ_1 verifies the heat equation:

$$\partial_t \rho_1 - \mu\Delta\rho_1 = 0. \tag{1.13}$$

The idea now consists in working around this quasi-solution, it will allow us to prove the existence of global strong solution with small initial data in subcritical norms. We are going to search solution under the form:

$$q = \ln \rho = \ln \rho_1 + h_2 \;\text{ with } \rho_1 = 1 + h_1 \;\text{ and }\; u = -\mu\nabla \ln \rho_1 + u_2.$$

We deduce from (1.11) that (h_2, u_2) verifies the following system when $P(\rho) = K\rho$ (this last choice is only a way to simplify the computations in the sequel):

$$\begin{cases} \partial_t h_2 + \mathrm{div} u_2 - \mu\nabla \ln \rho_1 \cdot \nabla h_2 + u_2 \cdot \nabla \ln \rho_1 = F(h_2, u_2), \\ \partial_t u_2 - \mu\Delta u_2 - \mu\nabla \mathrm{div} u_2 - \kappa\nabla\Delta h_2 + K\nabla h_2 - 2\mu\nabla \ln \rho_1 \cdot Du_2 - 2\mu\nabla h_2 \cdot Du_1 \\ \qquad + u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 - \mu^2\nabla(\nabla \ln \rho_1 \cdot \nabla h_2) = G(h_2, u_2), \\ (h_2(0,\cdot), u_2(0,\cdot)) = (h_0^2, u_0^2), \end{cases} \tag{1.14}$$

with:

$$\begin{aligned}&F(h_2,u_2)=-u_2\cdot\nabla h_2,\\&G(h_2,u_2)=-u_2\cdot\nabla u_2+2\mu\nabla h_2\cdot Du_2-K\nabla\ln\rho_1+\frac{\mu^2}{2}\nabla(|\nabla h_2|^2).\end{aligned}\tag{1.15}$$

Let us state our main theorem for the system (1.14).

Theorem 1.3. *Let $N\geq 2$. Assume that $\mu(\rho)=\mu\rho$, $\kappa(\rho)=\frac{\mu^2}{\rho}$ and $\lambda(\rho)=0$ with $\mu>0$ and $P(\rho)=K\rho$ with $K>0$. Furthermore we suppose that there exists $c_1>0$ such that $\rho_1^0\geq c_1>0$ with $u_0=-\mu\nabla[\ln\rho_1^0]+u_2^0$ and $\ln\rho_0=\ln(\rho_1^0)+h_2^0$ such that $\rho_1^0=1+h_1^0$. In addition we assume that:*

$$h_1^0\in\widetilde{B}_{2,1}^{\frac{N}{2}-2,\frac{N}{2}},\;\; h_2^0\in\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}}\;\;\text{and}\;\;u_2^0\in B_{2,1}^{\frac{N}{2}-1}.$$

Furthermore there exist $C>0$, ε_0 (depending on h_1^0), and two regular functions g, g_1 such that if:

$$\begin{aligned}&Cg(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B_{2,1}^{\frac{N}{2}-2}}\exp\big(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B_{2,1}^{\frac{N}{2}}}\big)\leq\frac12,\\&\|h_2^0\|_{\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}}}+\|u_2^0\|_{B_{2,1}^{\frac{N}{2}-1}}\leq\varepsilon_0,\end{aligned}\tag{1.16}$$

then there exists a global unique solution (ρ,u) of the system (1.1) with: $u=-\mu\nabla\ln\rho_1+u_2$, $\ln\rho=\ln\rho_1+h_2$ and $\rho_1=1+h_1$ verifying the following heat equation:

$$\begin{cases}\partial_t\rho_1-\mu\Delta\rho_1=0,\\ \rho_1(0,\cdot)=\rho_1^0=1+h_1^0.\end{cases}$$

Furthermore we have:

$$h_2\in\widetilde{L}^\infty(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})\cap\widetilde{L}^1(\widetilde{B}_{2,1}^{\frac{N}{2}+1,\frac{N}{2}+2})\;\;\text{and}\;\;u_2\in\widetilde{L}^\infty(B_{2,1}^{\frac{N}{2}-1})\cap\widetilde{L}^1(B_{2,1}^{\frac{N}{2}+1}).\tag{1.17}$$

Remark 9. This theorem ensures the existence of global strong solution with large initial data for the scaling of the equations which is new up to our knowledge. Indeed it suffices to consider $h_1^0(x)=\varphi(\lambda x)$ with $\varphi\in\widetilde{B}_{2,1}^{\frac{N}{2}-2,\frac{N}{2}}$ such that $1+\varphi\geq c>0$. We observe in particular that:

$$\begin{cases}\|h_1^0\|_{B_{2,1}^{\frac{N}{2}}}=\|\varphi\|_{B_{2,1}^{\frac{N}{2}}},\\ \|\rho_1^0\|_{L^\infty}=\|1+\varphi\|_{L^\infty},\\ \|\frac{1}{\rho_1^0}\|_{L^\infty}=\|\frac{1}{1+\varphi}\|_{L^\infty},\\ \|h_1^0\|_{B_{2,1}^{\frac{N}{2}-2}}=\frac{1}{\lambda^2}\|\varphi\|_{B_{2,1}^{\frac{N}{2}-2}}.\end{cases}\tag{1.18}$$

It implies that h_1^0 verifies (1.16) by choosing λ large enough. Thus if we take φ large in $B_{2,\infty}^{\frac{N}{2}}$, our initial density h_1^0 is large in the critical Besov space $B_{2,\infty}^{\frac{N}{2}}$. In particular when $N=2$, it is possible to choose φ large in $B_{2,2}^1$; it turns out that there is existence of global strong solution for large initial data in the energy space (we refer to (1.3) for the energy inequality). This theorem provides then a first answer to the question of the existence of global strong solution with large energy initial data in dimension $N=2$ (at least for a class of initial data). This question remains obviously open for general initial data.

It is also possible to choose $h_1^0(x)=\ln(\lambda)\varphi(\lambda x)$ with $\lambda>0$, it improves again the *size of the large* initial data in $B_{2,\infty}^{\frac{N}{2}}$.

Remark 10. We would like also to point out that nonlinear condition of smallness as (1.16) have been proved also in some works of J.-Y. Chemin and I. Gallagher (see [15,16]) for incompressible Navier–Stokes equations. Indeed the authors show the existence of global strong solution for large initial data in $B^{-1}_{\infty,\infty}$ which is the largest critical space for the Navier–Stokes equations. However our proof is really different of [15,16] since our initial data are completely irrotational (that is of course not the case for incompressible equation). In addition we work around the quasi-solutions which naturally absorb the convection term, it ensures better results in term of smallness assumption (1.16) compared with [15,16] (indeed in these papers the authors work around the solution of the Stokes equation, and the process of smallness is related to a smallness assumption on the term of convection). This is obviously due to the fact that our system is compressible which allows us to deal with irrotational data.

Remark 11. We think that we could improve the condition on initial density $h_1^0 \in B^{\frac{N}{2}-2}_{2,1}$ by working around the quasi-solution only in high frequencies and by dealing directly with $\rho - 1$ in low frequencies in the spirit of [25].

It may be also interesting to work with general regular pressure, it would make the proof more technical.

We are going to finish by presenting a result of global strong solution with large initial data when we assume that the system (1.1) is highly compressible. In the sequel we will work with $P(\rho) = K\rho$, by *highly compressible* we mean that $K = \frac{1}{\mathrm{Mr}^2}$ goes to zero or in other words that the Mach number $\mathrm{Mr} > 0$ goes to $+\infty$.

Corollary 1. *Let $N \geq 2$. Assume that $\mu(\rho) = \mu\rho$, $\kappa(\rho) = \frac{\mu^2}{\rho}$ and $\lambda(\rho) = 0$ with $\mu > 0$ and $P(\rho) = K\rho$ with $K > 0$. Furthermore we suppose that $u_0 = -\mu\nabla[\ln \rho_1^0] + u_2^0$ and $\ln \rho_0 = \ln(\rho_1^0) + h_2^0$ such that $\rho_1^0 = 1 + h_1^0$ and there exists $c_1 > 0$ such that $\rho_1^0 \geq c_1 > 0$. In addition we suppose that:*

$$h_1^0 \in \widetilde{B}^{\frac{N}{2}-2,\frac{N}{2}}_{2,1}, \quad h_2^0 \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1} \quad \text{and} \quad u_2^0 \in B^{\frac{N}{2}-1}_{2,1}.$$

Furthermore there exists $\varepsilon_0 > 0$ (depending on h_1^0 and the viscosity coefficient μ) such that for any $K \leq \varepsilon_0$, there exists $\varepsilon_1 > 0$ such that if

$$\|h_2^0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}} + \|u_2^0\|_{B^{\frac{N}{2}-1}_{2,1}} \leq \varepsilon_1, \tag{1.19}$$

then there exists a global unique solution (ρ, u) of the system (1.1) with: $u = -\mu\nabla \ln \rho_1 + u_2$ and $\ln \rho = \ln \rho_1 + h_2$ with $\rho_1 = 1 + h_1$ verifying the following system:

$$\begin{cases} \partial_t \rho_1 - \mu\Delta\rho_1 = 0, \\ \rho_1(0,\cdot) = \rho_1^0 = 1 + h_1^0. \end{cases}$$

Furthermore we have:

$$h_2 \in \widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,N}_{2,1}) \cap \widetilde{L}^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1}) \ \text{and} \ u_2 \in \widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,1}) \cap \widetilde{L}^1(B^{\frac{N}{2}+1}_{2,1}). \tag{1.20}$$

Remark 12. This theorem shows the existence of global strong solution for any large initial data of the form $u_0 = -\mu\nabla \ln \rho_0$ provided that K is sufficiently small with $P(\rho) = K\rho$. In other terms we get global existence (and uniqueness) for highly compressible fluids in any dimension $N \geq 2$. Roughly speaking $K = \frac{1}{Mr^2}$ tends to be very small when $\|h_1^0\|_{\widetilde{B}^{\frac{N}{2}-2,\frac{N}{2}}_{2,1}}$ is very large (in other words it means that the Mach number Mr goes to $+\infty$ which corresponds to a highly compressible limit, see [30,32]).

This article is structured in the following way, first of all we recall in the section 2 some definitions and theorems related to the Littlewood–Paley theory. Next in the section 3 we prove Theorems 1.1 and 1.2. In section 4, we show the Theorem 1.3 by introducing the notion of quasi-solution which will play a crucial role. In section 5 we finish with the proof of the Corollary 1. We postpone in appendix (see Appendix A) some technical computation on the capillarity tensor.

2. Littlewood–Paley theory and Besov spaces

Throughout the paper, C stands for a constant whose exact meaning depends on the context. The notation $A \lesssim B$ means that $A \leq CB$ with $C > 0$. For all Banach space X, we denote by $C([0,T],X)$ the set of continuous functions on $[0,T]$ with values in X. For $p \in [1,+\infty]$, the notation $L^p(0,T,X)$ or $L^p_T(X)$ stands for the set of measurable functions on $(0,T)$ with values in X such that $t \to \|f(t)\|_X$ belongs to $L^p(0,T)$. Littlewood–Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. We can use for instance any $\varphi \in C^\infty(\mathbb{R}^N)$ and $\chi \in C^\infty(\mathbb{R}^N)$, supported respectively in $\mathcal{C} = \{\xi \in \mathbb{R}^N / \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and $B(0, \frac{4}{3})$ such that:

$$\sum_{l\in\mathbb{Z}} \varphi(2^{-l}\xi) = 1 \ \text{ if } \ \xi \neq 0,$$

and:

$$\chi(\xi) + \sum_{l\in\mathbb{N}} \varphi(2^{-l}\xi) = 1 \ \ \forall \xi \in \mathbb{R}^N.$$

Denoting $h = \mathcal{F}^{-1}\varphi$, we then define the dyadic blocks by:

$$\Delta_l u = \varphi(2^{-l}D)u = 2^{lN} \int_{\mathbb{R}^N} h(2^l y)u(x-y)dy \ \text{ and } \ S_l u = \sum_{k\leq l-1} \Delta_k u\,.$$

Formally, one can write that:

$$u = \sum_{k\in\mathbb{Z}} \Delta_k u\,.$$

This decomposition is called homogeneous Littlewood–Paley decomposition. Let us observe that the above formal equality does not hold in $\mathcal{S}'(\mathbb{R}^N)$ for two reasons:

1. The right hand-side does not necessarily converge in $\mathcal{S}'(\mathbb{R}^N)$.
2. Even if it does, the equality is not always true in $\mathcal{S}'(\mathbb{R}^N)$ (consider the case of the polynomials).

2.1. Homogeneous Besov spaces and first properties

Definition 2.1. We denote by $\mathcal{S}'_h$ the space of tempered distribution u such that:

$$\lim_{j\to-\infty} S_j u = 0 \ \text{ in } \ \mathcal{S}'.$$

Definition 2.2. For $s \in \mathbb{R}$, $p \in [1,+\infty]$, $q \in [1,+\infty]$, and $u \in \mathcal{S}'(\mathbb{R}^N)$ we set:

$$\|u\|_{B^s_{p,q}} = \big(\sum_{l\in\mathbb{Z}} (2^{ls}\|\Delta_l u\|_{L^p})^q\big)^{\frac{1}{q}}.$$

The homogeneous Besov space $B^s_{p,q}$ is the set of distribution u in $\mathcal{S}'_h$ such that $\|u\|_{B^s_{p,q}} < +\infty$.

Remark 13. The above definition is a natural generalization of the homogeneous Sobolev and Hölder spaces: one can show that $B^s_{\infty,\infty}$ is the homogeneous Hölder space C^s and that $B^s_{2,2}$ is the homogeneous space H^s.

Proposition 2.1. *The following properties hold:*

1. *There exists a universal constant C such that:*

$$C^{-1}\|u\|_{B^s_{p,r}} \leq \|\nabla u\|_{B^{s-1}_{p,r}} \leq C\|u\|_{B^s_{p,r}}.$$

2. *If $p_1 < p_2$ and $r_1 \leq r_2$ then $B^s_{p_1,r_1} \hookrightarrow B^{s-N(1/p_1-1/p_2)}_{p_2,r_2}$.*
3. *Moreover we have the following interpolation inequalities, there exists $C > 0$ such that for any $\theta \in]0,1[$ and $s < \widetilde{s}$ we have:*

$$\|u\|_{B^{\theta s+(1-\theta)\widetilde{s}}_{p,r}} \leq \|u\|^{\theta}_{B^s_{p,r}} \|u\|^{1-\theta}_{B^{\widetilde{s}}_{p,r}},$$

$$\|u\|_{B^{\theta s+(1-\theta)\widetilde{s}}_{p,1}} \leq \frac{C}{\theta(1-\theta)(\widetilde{s}-s)} \|u\|^{\theta}_{B^s_{p,\infty}} \|u\|^{1-\theta}_{B^{\widetilde{s}}_{p,\infty}}.$$

We now recall a few product laws in Besov spaces coming directly from the paradifferential calculus of J.-M. Bony (see [6,5]).

Proposition 2.2. *We have the following laws of product:*

- *For all $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$ we have:*

$$\|uv\|_{B^s_{p,r}} \leq C(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{B^s_{p,r}}). \tag{2.21}$$

- *Let $(p,p_1,p_2,r,\lambda_1,\lambda_2) \in [1,+\infty]^2$ such that: $\frac{1}{p} \leq \frac{1}{p_1}+\frac{1}{p_2}$, $p_1 \leq \lambda_2$, $p_2 \leq \lambda_1$, $\frac{1}{p} \leq \frac{1}{p_1}+\frac{1}{\lambda_1}$ and $\frac{1}{p} \leq \frac{1}{p_2}+\frac{1}{\lambda_2}$. We have then the following inequalities:*

  If $s_1+s_2+N\inf(0,1-\frac{1}{p_1}-\frac{1}{p_2}) > 0$, $s_1+\frac{N}{\lambda_2} < \frac{N}{p_1}$ and $s_2+\frac{N}{\lambda_1} < \frac{N}{p_2}$ then:

$$\|uv\|_{B^{s_1+s_2-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,r}} \lesssim \|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}, \tag{2.22}$$

  when $s_1+\frac{N}{\lambda_2} = \frac{N}{p_1}$ (resp. $s_2+\frac{N}{\lambda_1} = \frac{N}{p_2}$) we replace $\|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}$ (resp. $\|v\|_{B^{s_2}_{p_2,\infty}}$) by $\|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,r}}$ (resp. $\|v\|_{B^{s_2}_{p_2,\infty}\cap L^\infty}$), if $s_1+\frac{N}{\lambda_2} = \frac{N}{p_1}$ and $s_2+\frac{N}{\lambda_1} = \frac{N}{p_2}$ we take $r=1$.
  If $s_1+s_2 = 0$, $s_1 \in (\frac{N}{\lambda_1}-\frac{N}{p_2}, \frac{N}{p_1}-\frac{N}{\lambda_2}]$ and $\frac{1}{p_1}+\frac{1}{p_2} \leq 1$ then:

$$\|uv\|_{B^{-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,\infty}} \lesssim \|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,\infty}}. \tag{2.23}$$

  If $|s| < \frac{N}{p}$ for $p \geq 2$ and $-\frac{N}{p'} < s < \frac{N}{p}$ else, we have:

$$\|uv\|_{B^s_{p,r}} \leq C\|u\|_{B^s_{p,r}}\|v\|_{B^{\frac{N}{p}}_{p,\infty}\cap L^\infty}. \tag{2.24}$$

Remark 14. In the sequel p will be either p_1 or p_2 and in this case $\frac{1}{\lambda} = \frac{1}{p_1} - \frac{1}{p_2}$ if $p_1 \leq p_2$, respectively $\frac{1}{\lambda} = \frac{1}{p_2} - \frac{1}{p_1}$ if $p_2 \leq p_1$.

Corollary 2. *Let $r \in [1, +\infty]$, $1 \le p \le p_1 \le +\infty$ and s such that:*

- $s \in (-\frac{N}{p_1}, \frac{N}{p_1})$ *if* $\frac{1}{p} + \frac{1}{p_1} \le 1$,
- $s \in (-\frac{N}{p_1} + N(\frac{1}{p} + \frac{1}{p_1} - 1), \frac{N}{p_1})$ *if* $\frac{1}{p} + \frac{1}{p_1} > 1$,

then we have if $u \in B^s_{p,r}$ and $v \in B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty$:

$$\|uv\|_{B^s_{p,r}} \le C\|u\|_{B^s_{p,r}} \|v\|_{B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty}.$$

The study of non-stationary PDE's requires space of type $L^\rho(0, T, X)$ for appropriate Banach spaces X. In our case, we expect X to be a Besov space, so that it is natural to localize the equation through Littlewood–Paley decomposition. That is why we are going to define the spaces of Chemin–Lerner which are a refinement of the spaces $L^\rho_T(B^s_{p,r})$.

Definition 2.3. Let $\rho \in [1, +\infty]$, $T \in [1, +\infty]$ and $s_1 \in \mathbb{R}$. We set:

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})} = \big(\sum_{l\in\mathbb{Z}} 2^{lrs_1} \|\Delta_l u(t)\|^r_{L^\rho(L^p)}\big)^{\frac{1}{r}}.$$

We then define the space $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ as the set of tempered distribution u over $(0, T) \times \mathbb{R}^N$ such that $\lim_{q\to-\infty} S_q u = 0$ in $\mathcal{S}'((0, T) \times \mathbb{R}^N)$ and $\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})} < +\infty$.

We set $\widetilde{C}_T(\widetilde{B}^{s_1}_{p,r}) = \widetilde{L}^\infty_T(B^{s_1}_{p,r}) \cap \mathcal{C}([0, T], B^{s_1}_{p,r})$. Let us emphasize that, according to Minkowski inequality, we have:

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})} \le \|u\|_{L^\rho_T(B^{s_1}_{p,r})} \text{ if } r \ge \rho, \quad \|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})} \ge \|u\|_{L^\rho_T(B^{s_1}_{p,r})} \text{ if } r \le \rho. \tag{2.25}$$

Remark 15. It is easy to generalize Proposition 2.2 and Corollary 2 to $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ spaces. The indices s_1, p, r behave just as in the stationary case whereas the time exponent ρ behaves according to Hölder inequality.

In the sequel we will need composition lemma in $\widetilde{L}^\rho_T(B^s_{p,r})$ spaces (we refer to [5] for a proof).

Proposition 2.3. *Let $s > 0$, $(p, r) \in [1, +\infty]$ and $u \in \widetilde{L}^\rho_T(B^s_{p,r}) \cap L^\infty_T(L^\infty)$.*

1. *Let $F \in W^{[s]+2,\infty}_{loc}(\mathbb{R}^N)$ such that $F(0) = 0$. Then $F(u) \in \widetilde{L}^\rho_T(B^s_{p,r})$. More precisely there exists a function C depending only on s, p, r, N and F such that:*

$$\|F(u)\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \le C(\|u\|_{L^\infty_T(L^\infty)})\|u\|_{\widetilde{L}^\rho_T(B^s_{p,r})}.$$

2. *Let $F \in W^{[s]+3,\infty}_{loc}(\mathbb{R}^N)$ such that $F(0) = 0$. Then $F(u) - F'(0)u \in \widetilde{L}^\rho_T(B^s_{p,r})$. More precisely there exists a function C depending only on s, p, r, N and F such that:*

$$\|F(u) - F'(0)u\|_{\widetilde{L}^\rho_T(B^s_{p,r})} \le C(\|u\|_{L^\infty_T(L^\infty)})\|u\|^2_{\widetilde{L}^\rho_T(B^s_{p,r})}.$$

Let us now give some estimates for the heat equation.

Proposition 2.4. *Let $s \in \mathbb{R}$, $(p, r) \in [1, +\infty]^2$ and $1 \le \rho_2 \le \rho_1 \le +\infty$. Assume that $u_0 \in B^s_{p,r}$ and $f \in \widetilde{L}^{\rho_2}_T(B^{s-2+2/\rho_2}_{p,r})$. Let u be a solution of:*

$$\begin{cases} \partial_t u - \mu\Delta u = f \\ u_{t=0} = u_0 \,. \end{cases}$$

Then there exists $C > 0$ depending only on N, μ, ρ_1 and ρ_2 such that:

$$\|u\|_{\widetilde{L}^{\rho_1}_T(\widetilde{B}^{s+2/\rho_1}_{p,r})} \le C\big(\|u_0\|_{B^s_{p,r}} + \mu^{\frac{1}{\rho_2}-1}\|f\|_{\widetilde{L}^{\rho_2}_T(B^{s-2+2/\rho_2}_{p,r})}\big)\,.$$

If in addition r is finite then u belongs to $C([0, T], B^s_{p,r})$.

Hybrid Besov spaces

The homogeneous Besov spaces fail to have nice inclusion properties: owing to the low frequencies, the embedding $B^s_{p,1} \hookrightarrow B^t_{p,1}$ does not hold for $s > t$. Still, the functions of $B^s_{p,1}$ are locally more regular than those of $B^t_{p,1}$: for any $\phi \in C^\infty_0$ and $u \in B^s_{p,1}$, the function $\phi u \in B^t_{p,1}$. This motivates the definition of Hybrid Besov spaces introduced by R. Danchin (see a definition in [5], see also [25]) where the growth conditions satisfied by the dyadic blocks and the coefficient of integrability are not the same for low and high frequencies. Hybrid Besov spaces have been used by R. Danchin in order to prove global well-posedness for compressible gases in critical spaces (we refer to [5] for an elegant proof of this result).

Definition 2.4. Let $l_0 \in \mathbb{N}$, $s, t, \in \mathbb{R}$, $(r, r_1) \in [1, +\infty]^2$ and $(p, q) \in [1, +\infty]$. We set:

$$\|u\|_{\widetilde{B}^{s,t}_{p,q,1}} = \sum_{l \le l_0} 2^{ls}\|\Delta_l u\|_{L^p} + \sum_{l > l_0} 2^{lt}\|\Delta_l u\|_{L^q},$$

and:

$$\|u\|_{\widetilde{B}^{s,t}_{(p,r),(q,r_1)}} = \big(\sum_{l \le l_0}(2^{ls}\|\Delta_l u\|_{L^p})^r\big)^{\frac{1}{r}} + \big(\sum_{l > l_0}(2^{lt}\|\Delta_l u\|_{L^q})^{r_1}\big)^{\frac{1}{r_1}}.$$

Remark 16. When $p = q$ and $r = r_1$ we will note to simplify $\widetilde{B}^{s,t}_{(p,r),(p,r)} = \widetilde{B}^{s,t}_{p,r}$.

Notation 1. *We will often use the following notation:*

$$u_{BF} = \sum_{l \le l_0} \Delta_l u \quad \text{and} \quad u_{HF} = \sum_{l > l_0} \Delta_l u.$$

Remark 17. We have the following properties:

- We have $\widetilde{B}^{s,s}_{p,p,1} = B^s_{p,1}$.
- If $s_1 \ge s_3$ and $s_2 \ge s_4$ then $\widetilde{B}^{s_3,s_2}_{p,q,1} \hookrightarrow \widetilde{B}^{s_1,s_4}_{p,q,1}$.

Remark 18. In the sequel we shall often use this hybrid Besov space in order to distinguish the behavior of our solution in low and high frequencies, in particular we would like to mention that we can prove results analogous to Proposition 2.2 and Corollary 2 (see [25]).

We shall conclude this section by some example of initial data verifying the Theorem 1.1 with large energy initial data when $N = 2$. More precisely we are interested in defining initial data which are small in

$\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty}$ but large in $B^{\frac{N}{2}}_{2,2}$ (it improves in particular the results of [17] where the initial density is assumed small in $B^{\frac{N}{2}}_{2,1}$ and then in $B^{\frac{N}{2}}_{2,2}$). It implies in particular that your initial data $\nabla\sqrt{\rho_0}$ is large in L^2 when $N=2$.

Let us start by recalling a classical example of function in $B^s_{p,\infty}$, by sake of completeness we are going to recall the proof (see also [5]).

Proposition 2.5. *Let $\sigma\in]0,N[$. For any $p\in[1,+\infty]$, the function $|\cdot|^{-\sigma}$ belongs to $B^{\frac{N}{p}-\sigma}_{p,\infty}$.*

Proof. By Proposition 2.1 it suffices to show that $u_\sigma=|\cdot|^{-\sigma}$ belongs to $B^{N-\sigma}_{1,\infty}$. Let us introduce a smooth compactly supported function χ which is identically equal to 1 near the unit ball and such that u is splitting as follows:

$$u_\sigma=u_0+u_1\quad\text{with}\;\;u_0(x)=\chi(x)|x|^{-\sigma}\;\;\text{and}\;\;u_1(x)=(1-\chi(x))|x|^{-\sigma}.$$

Clearly u_0 is in L^1 and u_1 belongs to L^q whenever $q>\frac{N}{\sigma}$. The homogeneity of the function u_σ gives via a change of variable:

$$\begin{aligned}\Delta_j u_\sigma&=2^{jN}u_\sigma*h(2^j\cdot)\\&=2^{j(N+\sigma)}u_\sigma(2^j\cdot)*h(2^j\cdot)\\&=2^{j\sigma}(\Delta_0u_\sigma)(2^j\cdot).\end{aligned}\tag{2.26}$$

Therefore, $2^{j(N-\sigma)}\|\Delta_ju_\sigma\|_{L^1}=\|\Delta_0u_\sigma\|_{L^1}$, it remains then to show that Δ_0u_σ is in L^1. As u_0 is in L^1, Δ_0u_0 is also in L^1 according to the continuity of the operator Δ_0 on Lebesgue spaces. By Bernstein inequalities, we have:

$$\|\Delta_0u_1\|_{L^1}\le C_k\|D^k\Delta_0u_1\|_{L^1}\le C_k\|D^ku_1\|_{L^1}.$$

Leibniz's formula ensures that $D^ku_1-(1-\chi)D^ku_\sigma$ is a smooth compactly supported function. We then complete the proof by choosing k such that $k>N-\sigma$. $\square$

We can now deduce from the previous proposition suitable functions verifying the Theorem 1.1.

Corollary 3. *Let us consider:*

$$u_{0,\varepsilon,l_0}(x)=\mathcal{S}(\frac{1}{|x|^{1-\varepsilon}}),$$

with $\mathcal{S}$ defined by $\mathcal{F}(\mathcal{S}u)(\xi)=1_{\mathbb{R}^N\setminus B(0,2^{l_0})}(\xi)\;\hat{u}(\xi)$. Then for all $r\in[1,+\infty[$, for all $M>0$, for all $\varepsilon_1>0$ there exist $\varepsilon>0$ and $l_0>0$ such that:

$$\begin{aligned}&\|u_{0,\varepsilon,l_0}\|_{B^{\frac{N}{2}-1}_{2,\infty}}\le\varepsilon_1,\\&\|u_{0,\varepsilon,l_0}\|_{B^{\frac{N}{2}-1}_{2,r}}\ge M.\end{aligned}\tag{2.27}$$

Proof. In the sequel in order to simplify the notation we shall write u_0 for u_{0,ε,l_0}. Let us denote by u_σ the function $\frac{1}{|x|^\sigma}$ with $\sigma\in]0,N[$. By (2.26) we observe that:

$$\|\Delta_lu_\sigma\|_{L^2}=2^{l(\sigma-\frac{N}{2})}\|\Delta_0u_\sigma\|_{L^2}.\tag{2.28}$$

It implies that for $l \geq l_0$ we have for M_0 independent on ε when $\varepsilon \leq \frac{1}{10}$:

$$2^{l(\frac{N}{2}-1+\varepsilon)}\|\Delta_l u_0\|_{L^2} = \|\Delta_0 u_{1-\varepsilon}\|_{L^2} \leq M_0.$$

Indeed using the same arguments than in the proof of the Proposition 2.5 we have by Bernstein inequality for $C > 0$:

$$\begin{aligned}\|\Delta_0 u_{1-\varepsilon}\|_{L^2} &\leq C\|\Delta_0 u_{1-\varepsilon}\|_{L^1}\\ &\leq C(\|\chi u_{1-\varepsilon}\|_{L^1} + C^k\|D^k((1-\chi)u_{1-\varepsilon})\|_{L^1})\\ &\leq M_0.\end{aligned}$$

According to (2.28) we deduce that for all $l \geq l_0$:

$$2^{l(\frac{N}{2}-1)}\|\Delta_l u_0\|_{L^2} = 2^{-l\varepsilon}\|\Delta_0 u_\sigma\|_{L^2} \leq M_0 2^{-l_0\varepsilon}.$$

It implies in particular since $\Delta_l u_0 = 0$ for $l < l_0$ that:

$$\|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}} \leq M_0 2^{-l_0\varepsilon}. \tag{2.29}$$

We now fix l_0 such that:

$$M_0 2^{-l_0\varepsilon} = \varepsilon_1. \tag{2.30}$$

It yields that:

$$\|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}} \leq \varepsilon_1. \tag{2.31}$$

Let us now estimate the norm of u_0 in $B^{\frac{N}{2}-1}_{2,r}$ with $r \in [1, +\infty[$ and $M_\varepsilon = \|\Delta_0 u_{1-\varepsilon}\|_{L^2}$:

$$\begin{aligned}\|u_0\|_{B^{\frac{N}{2}-1}_{2,r}} &= (\sum_{l\in\mathbb{Z}} 2^{rl(\frac{N}{2}-1)}\|\Delta_l u_0\|^r_{L^2})^{\frac{1}{r}}\\ &= (\sum_{l\geq l_0} 2^{-rl\varepsilon}M_\varepsilon^r)^{\frac{1}{r}}\\ &= M_\varepsilon 2^{-l_0\varepsilon}(\frac{1}{1-2^{-r\varepsilon}})^{\frac{1}{r}}.\end{aligned} \tag{2.32}$$

For all $M' > 0$, for all $r < +\infty$ there exists $\varepsilon > 0$ small enough such that:

$$(\frac{1}{1-2^{-r\varepsilon}})^{\frac{1}{r}} \geq M'. \tag{2.33}$$

Let us prove now that when $\varepsilon \leq \frac{1}{10}$ there exists $\alpha > 0$ such that $M_\varepsilon \geq \alpha > 0$. Assume by the absurd that this is wrong. It implies that there exists a sequel $(\varepsilon_n)_{n\in\mathbb{N}}$ which converges to 0 when n goes to infinity such that $M_{\varepsilon_n} = \|\Delta_0 u_{1-\varepsilon_n}\|_{L^2}$ goes to 0 when n goes to infinity. By Plancherel theorem and the fact that we know the Fourier transform of $|\cdot|^{-1+\varepsilon}$ (see [5], p. 23) it implies that:

$$\|\mathcal{F}\Delta_0 u_{1-\varepsilon_n}\|_{L^2} = c_{N,1-\varepsilon_n}\|\varphi|\cdot|^{1-\varepsilon_n-N}\|_{L^2} \to_{n\to+\infty} 0.$$

Since we can bound by below $\| |\varphi| \cdot |^{1-\varepsilon_n - N}\|_{L^2}$ independently of n, it implies that $c_{N,1-\varepsilon_n}$ goes to 0 when n goes to infinity and this is absurd.

We finally obtain from (2.31), (2.33) and (2.32) that:

$$\begin{aligned} &\|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}} \leq \varepsilon_1, \\ &\|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}} \geq \frac{M'\varepsilon_1\alpha}{M_0}. \end{aligned} \tag{2.34}$$

It concludes the proof of the corollary by taking $M' = \frac{MM_0}{\alpha\varepsilon_1}$. □

3. Proof of Theorems 1.1 and 1.2

In this part we are interested in proving the Theorems 1.1 and 1.2 concerning the existence of strong solutions in critical space for the scaling of the equations. We would like to point out that in the Theorem 1.1 the viscosity and the capillarity coefficients allow to exhibit a particular structure for the equations. This fact will be crucial in order to obtain estimates on the density without assuming any control on the vacuum or on the L^∞ norm of the density. Indeed when $\mu(\rho) = \mu\rho$ and $\kappa(\rho) = \frac{\kappa}{\rho}$ with $\kappa > 0$, the system depends only on the unknown $\ln\rho$ “in a linear way” in the sense that we do need to use any composition lemma for the nonlinear terms.

Next in order to prove the second part of the Theorem 1.1, we will have to estimate the L^∞ norm of $\frac{1}{\rho}$ and ρ. To do this, we are going to consider solution (q, u) under the following form:

$$(q, u) = (q_L, u_L) + (\bar{q}, \bar{u}),$$

with (q_L, u_L) the solution of the linearized part of the system (1.6). In order to bound q in L^∞, we will combine maximum principle arguments on q_L and regularizing effect on the third index of Besov space for $\bar{q}$. More precisely we will prove that $\bar{q}$ is in $\widetilde{L}^\infty_T(B^{\frac{N}{2}}_{2,1})$ for any $T > 0$ which is embedded in $L^\infty_T(L^\infty)$. Let us mention that in order to prove that q_L is bounded in L^∞ norm we shall prove an accurate characterization of the Besov space in term of the semi-group associated with the linearized part of the system (1.6) (see the Proposition 3.10). For more details on this part which is the main difficulty of the proof we refer to the subsection 3.4.1 and 3.4.2.

As a first step of the proof of Theorems 1.1 and 1.2, let us start by studying the linear part of the system (1.6) which corresponds to the following system (with F and G source terms):

$$\begin{cases} \partial_t q + \mathrm{div} u = F, \\ \partial_t u - a\Delta u - b\nabla\mathrm{div} u - c\nabla\Delta q = G, \\ (q, u)(0, \cdot) = (q_0, u_0). \end{cases} \tag{N}$$

3.1. Study of the linearized equation

We want to prove a priori estimates in Chemin–Lerner spaces for system (N) with the following hypotheses on a, b, c which are constant:

$$0 < a < \infty,\ 0 < a + b < \infty \ \text{ and } \ 0 < c < \infty.$$

This system has been studied by R. Danchin and B. Desjardins (see [17]) in the framework of the Besov space $B^s_{2,1}$, the following proposition uses exactly the same type of arguments used in [17] excepted that we extend the result to general Besov spaces $B^s_{2,r}$ with $r \in [1,+\infty]$. By sake of completeness we are going to show the following proposition.

Proposition 3.6. *Let $1 \le r \le +\infty$, $s \in \mathbb{R}$, and we assume that (q_0, u_0) belongs to $B^{\frac{N}{2}+s}_{2,r} \times (B^{\frac{N}{2}-1+s}_{2,r})^N$ with the source terms (F,G) in $\widetilde{L}^1_T(B^{\frac{N}{2}+s}_{2,r}) \times (\widetilde{L}^1_T(B^{\frac{N}{2}-1+s}_{2,r}))^N$.*

Let $(q,u) \in (\widetilde{L}^1_T(B^{\frac{N}{2}+s+2}_{2,r}) \cap \widetilde{L}^\infty_T(B^{\frac{N}{2}+s}_{2,r})) \times \big(\widetilde{L}^1_T(B^{\frac{N}{2}+s+1}_{2,r}) \cap \widetilde{L}^\infty_T(B^{\frac{N}{2}+s-1}_{2,r})\big)^N$ be a solution of the system (N), then there exists a universal constant C such that for any $T > 0$:

$$\|(\nabla q, u)\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1+s}_{2,r}) \cap \widetilde{L}^\infty_T(B^{\frac{N}{2}-1+s}_{2,r})} \le C(\|(\nabla q_0, u_0)\|_{B^{\frac{N}{2}-1+s}_{2,2}} + \|(\nabla F, G)\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1+s}_{2,r})}).$$

Proof. As we mentioned previously, we are going to follow the arguments developed in [17]. Let us apply to the system (N) the operator Δ_l which gives:

$$\partial_t q_l + \mathrm{div} u_l = F_l \tag{3.35}$$

$$\partial_t u_l - \mathrm{div}(a\nabla u_l) - \nabla(b\,\mathrm{div} u_l) - c\nabla\Delta q_l = G_l \tag{3.36}$$

Performing integrations by parts and using (3.35) we have:

$$\begin{aligned} -c\int_{\mathbb{R}^N} u_l \cdot \nabla\Delta q_l dx &= c\int_{\mathbb{R}^N} \mathrm{div} u_l\, \Delta q_l dx \\ &= -c\int_{\mathbb{R}^N} \partial_t q_l\, \Delta q_l dx + c\int_{\mathbb{R}^N} F_l\, \Delta q_l dx \\ &= c\int_{\mathbb{R}^N} \partial_t \nabla q_l \cdot \nabla q_l dx - c\int_{\mathbb{R}^N} \nabla F_l \cdot \nabla q_l dx \\ &= \frac{c}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |\nabla q_l|^2 dx - c\int_{\mathbb{R}^N} \nabla q_l.\nabla F_l\, dx. \end{aligned}$$

Next, we take the inner product of (3.36) with u_l and using the previous equality, it yields:

$$\frac{1}{2}\frac{d}{dt}(\|u_l\|^2_{L^2} + c\int_{\mathbb{R}^N} |\nabla q_l|^2 dx) + \int_{\mathbb{R}^N} (a|\nabla u_l|^2 + b|\mathrm{div} u_l|^2) dx = \int_{\mathbb{R}^N} G_l.u_l\, dx + c\int_{\mathbb{R}^N} \nabla q_l.\nabla F_l\, dx\,. \tag{3.37}$$

In order to recover some terms in Δq_l we take the inner product of the gradient of (3.35) with u_l, the inner product scalar of (3.36) with ∇q_l and we sum, we obtain then:

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^N} \nabla q_l.u_l dx + c\int_{\mathbb{R}^N} (\Delta q_l)^2 dx = \int_{\mathbb{R}^N} (&G_l.\nabla q_l + |\mathrm{div} u_l|^2 + u_l.\nabla F_l \\ &- a\nabla u_l : \nabla^2 q_l - b\Delta q_l \mathrm{div} u_l) dx. \end{aligned} \tag{3.38}$$

Let $\alpha > 0$ small enough. We define k_l by:

$$k_l^2 = \|u_l\|^2_{L^2} + c\|\nabla q_l\|^2_{L^2} + 2\alpha\int_{\mathbb{R}^N} \nabla q_l.u_l dx\,. \tag{3.39}$$

By using (3.37), (3.38) and the Young inequalities, we get provided that α is chosen small enough:

$$\frac{1}{2}\frac{d}{dt}k_l^2+\frac{1}{2}\int_{\mathbb{R}^N}(a|\nabla u_l|^2+b|\mathrm{div}u_l|^2+2\alpha c|\Delta q_l|^2)dx\lesssim\|G_l\|_{L^2}(2\alpha\|\nabla q_l\|_{L^2}+\|u_l\|_{L^2})$$
$$+\|\nabla F_l\|_{L^2}(2\alpha\|u_l\|_{L^2}+\|\nabla q_l\|_{L^2}). \tag{3.40}$$

For α small enough we have according to (3.39):

$$\frac{1}{2}k_l^2\leq\|u_l\|^2+c\|\nabla q_l\|_{L^2}^2\leq\frac{3}{2}k_l^2\,. \tag{3.41}$$

Hence from (3.40) and (3.41) we deduce that there exists $K>0$ small enough, $C>0$ such that:

$$\frac{1}{2}\frac{d}{dt}k_l^2+K2^{2l}k_l^2\leq C\,k_l\,(\|G_l\|_{L^2}+\|\nabla F_l\|_{L^2}).$$

By integrating with respect to the time, we obtain:

$$k_l(t)\leq e^{-K2^{2l}t}k_l(0)+C\int_0^t e^{-K2^{2l}(t-\tau)}(\|\nabla F_l(\tau)\|_{L^2}+\|G_l(\tau)\|_{L^2})d\tau\,.$$

Using convolution inequalities, it yields for $1\leq\rho_1\leq\rho\leq+\infty$:

$$\|k_l\|_{L^\rho([0,T])}\lesssim\big(2^{-\frac{2l}{\rho}}k_l(0)+2^{-2l(1+\frac{1}{\rho}-\frac{1}{\rho_1})}\|(\nabla F_l,G_l)\|_{L_T^{\rho_1}(L^2)}\big). \tag{3.42}$$

Moreover since we have:

$$C^{-1}\,k_l\leq\|\nabla q_l\|_{L^2}+\|u_l\|_{L^2}\leq C\,k_l,$$

multiplying by $2^{(\frac{N}{2}-1+s+\frac{2}{\rho})l}$, taking the l^r norm and using (3.41), we end up with:

$$\|(\nabla q,u)\|_{L_T^\rho(B_{2,r}^{\frac{N}{2}-1+s+\frac{2}{\rho}})}\leq\|(\nabla F,G)\|_{\widetilde{L}_T^{\rho_1}(B_{2,r}^{\frac{N}{2}-3+s+\frac{2}{\rho_1}})}+\|(\nabla q_0,u_0)\|_{B_{2,r}^{\frac{N}{2}-1+s}}.$$

It conclude the proof of the proposition. □

Let us extend the result of the Proposition 3.6 to the case where we include the pressure term inside of the linearized system. This is necessary when we are interested in dealing with the existence of global strong solution with small initial data. Indeed in this case it is very important to take into account the behavior of low frequencies. More precisely we will consider the following linear system:

$$\begin{cases}\partial_t q+\mathrm{div}u=F,\\ \partial_t u-a\Delta u-b\nabla\mathrm{div}u-c\nabla\Delta q+d\nabla q=G,\\ (q,u)(0,\cdot)=(q_0,u_0).\end{cases} \tag{$N1$}$$

We now want to prove a priori estimates in Chemin–Lerner spaces for system $(N1)$ with the following hypotheses on a,b,c,d which are constant:

$$0<a<\infty,\ 0<a+b<\infty,\ 0<c<\infty\ \text{and}\ 0<d<\infty.$$

This system has also been studied by Danchin and Desjardins in [17] in the framework of Besov spaces of the type $B^s_{2,1}$ with $s \in \mathbb{R}$, let us generalize this study to the case of general Besov spaces $B^s_{2,r}$ with $r \in [1, +\infty]$ by using similar arguments.

Proposition 3.7. *Let $1 \le r \le +\infty$, $s \in \mathbb{R}$, and we assume that (q_0, u_0) belongs to $\widetilde{B}^{\frac{N}{2}-1+s,\frac{N}{2}+s}_{2,r} \times (B^{\frac{N}{2}-1+s}_{2,r})^N$. Furthermore we suppose that the source terms (F, G) are in $\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1+s,\frac{N}{2}+s}_{2,r}) \times (\widetilde{L}^1_T(B^{\frac{N}{2}-1+s}_{2,r}))^N$.*

Let $(q, u) \in (\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+s+1,\frac{N}{2}+s+2}_{2,r}) \cap \widetilde{L}^\infty_T(\widetilde{B}^{\frac{N}{2}-1+s,\frac{N}{2}+s}_{2,r})) \times \big(\widetilde{L}^1_T(B^{\frac{N}{2}+s+1}_{2,r}) \cap \widetilde{L}^\infty_T(B^{\frac{N}{2}+s-1}_{2,r})\big)^N$ be a solution of the system $(N1)$, then there exists a universal constant C such that for any $T > 0$:

$$\begin{aligned}
&\|q\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1+s,\frac{N}{2}+2+s}_{2,r})} + \|q\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1+s,\frac{N}{2}+s}_{2,r})} + \|u\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1+s}_{2,r})} + \|u\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1+s}_{2,r})} \\
&\quad \le C\big(\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1+s,\frac{N}{2}+s}_{2,r}} + \|u_0\|_{B^{\frac{N}{2}+s}_{2,r}} + \|F\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1+s,\frac{N}{2}+s}_{2,r})} + \|G\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1+s}_{2,r})}\big).
\end{aligned}$$

Proof. It suffices to follow exactly the same lines as the proof of Proposition 3.6 except that we have to consider the following k_l:

$$k_l^2 = \|u_l\|^2_{L^2} + c\|\nabla q_l\|^2_{L^2} + d\|q_l\|^2_{L^2} + 2\alpha \int_{\mathbb{R}^N} \nabla q_l \cdot u_l dx.$$

Now choosing α suitably small, it turns out that:

$$\frac{1}{2}k_l^2 \le \|u_l\|^2_{L^2} + c\|\nabla q_l\|^2_{L^2} + d\|q_l\|^2_{L^2} \le \frac{3}{2}k_l^2. \tag{3.43}$$

By combining energy estimates in frequencies space, we show as in [17] that there exist $c, C > 0$ such that:

$$\frac{1}{2}\frac{d}{dt}k_l^2 + c2^{2l}k_l^2 \le Ck_l(\|G_l\|_{L^2} + \|(\nabla F_l, F_l)\|_{L^2}).$$

As in the proof of Proposition 3.6 routine computations yield Proposition 3.7. □

We are now interested in studying the following system with $\mu > 0$ and $\kappa > 0$:

$$\begin{cases}
\partial_t h_2 + \mathrm{div} u_2 - \mu\nabla \ln\rho_1 \cdot \nabla h_2 + u_2 \cdot \nabla \ln \rho_1 = F, \\
\partial_t u_2 - \mu\Delta u_2 - \mu\nabla \mathrm{div} u_2 - \kappa\nabla\Delta h_2 + K\nabla h_2 - 2\mu\nabla\ln\rho_1 \cdot Du_2 - 2\mu\nabla h_2 \cdot Du_1 \\
\qquad + u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 - \mu^2\nabla(\nabla\ln\rho_1 \cdot \nabla h_2) = G, \\
(h_2(0,\cdot), u_2(0,\cdot)) = (h_0^2, u_0^2).
\end{cases} \tag{3.44}$$

Here (h_2, u_2) are the unknowns, ρ_1 and u_1 corresponds to some functions defined in suitable Chemin Lerner Besov space $\widetilde{L}^\rho(B^s_{p,r})$ that we will precise below in the Proposition 3.8. F, G are source term (we will also precise their regularity). In order to prove the Theorem 1.3 we will need precise estimates on the solution (h_2, u_2) of (3.44) in terms of Chemin Lerner Besov spaces; to do this we are going to prove the following proposition.

Proposition 3.8. *Let $(h_0^2, u_0^2) \in \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1} \times B^{\frac{N}{2}-1}_{2,1}$ and we assume that $\ln\rho_1$ belongs to $\widetilde{L}^\infty(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}) \cap \widetilde{L}^1(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})$ and u_1 is in $\widetilde{L}^\infty(\mathbb{R}^+, B^{\frac{N}{2}-1}_{2,1}) \cap \widetilde{L}^1(\mathbb{R}^+, B^{\frac{N}{2}+1}_{2,1})$.*

Furthermore (F, G) are in $\widetilde{L}^1(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}) \times \widetilde{L}^1(\mathbb{R}^+, B^{\frac{N}{2}-1}_{2,1})$.

Let (h_2, u_2) be the solution of the linear system (3.44), then there exists $C > 0$ such that (h_2, u_2) verify for any $T > 0$:

$$\begin{aligned}&\|h_2\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})}+\|h_2\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}+\|u_2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,1})}+\|u_2\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,1})}\\
&\leq C\big(\|h_0^2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}}+\|u_0^2\|_{B^{\frac{N}{2}}_{2,1}}+\|F\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}+\|G\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\big)\\
&\times\exp\Big(C\int_0^T\big(\|u_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}}+\|u_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\big)(s)ds\Big).\end{aligned}\tag{3.45}$$

Remark 19. We would like to emphasize on the fact that in the Proposition 3.8 the main estimates (3.45) involves a control of $\nabla\ln\rho_1$ in $L^{\frac{4}{3}}(\mathbb{R}^+,B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})\cap L^4(\mathbb{R}^+,B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})$ and of u_1 in $L^{\frac{4}{3}}(\mathbb{R}^+,B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})\cap L^4(\mathbb{R}^+,B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})$; but by Minkowski inequality (see (2.25)) we know that:

$$\|\nabla\ln\rho_1\|_{L^{\frac{4}{3}}(\mathbb{R}^+,B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})\cap L^4(\mathbb{R}^+,B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\leq\|\nabla\ln\rho_1\|_{\widetilde{L}^{\frac{4}{3}}(\mathbb{R}^+,B^{\frac{N}{2}+\frac{1}{2}}_{2,\frac{4}{3}})\cap\widetilde{L}^4(\mathbb{R}^+,B^{\frac{N}{2}-\frac{1}{2}}_{2,\frac{4}{3}})}.\tag{3.46}$$

Since $\ln\rho_1$ belongs to $\widetilde{L}^\infty(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})\cap\widetilde{L}^1(\mathbb{R}^+,\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})$, it implies by interpolation that $\nabla\ln\rho_1$ is in $\widetilde{L}^{\frac{4}{3}}(\mathbb{R}^+,B^{\frac{N}{2}+\frac{1}{2}}_{2,\frac{4}{3}})\cap\widetilde{L}^4(\mathbb{R}^+,B^{\frac{N}{2}-\frac{1}{2}}_{2,\frac{4}{3}})$. In particular it proves that with such assumptions we control the right hand side of the inequality (3.45).

Let us mention that since in the Theorem 1.3 we will have $u_1=-\mu\nabla\ln\rho_1$ with ρ_1 verifying a heat equation, we may improve the condition on the initial data $\rho_1^0=h_1^0+1$ of Theorem 1.3 by assuming only $h_0^1\in B^{\frac{N}{2}-2}_{2,1}\cap B^{\frac{N}{2}}_{2,\frac{4}{3}}\cap L^\infty$ and $\rho_0^1\geq c>0$. Indeed with such condition $\nabla\ln\rho_1$ would verify exactly the quantity on the right hand side of (3.45). In fact a more accurate proof of the Proposition 3.8 by using critical interpolation estimate would show that $h_0^1\in B^{\frac{N}{2}-2}_{2,1}\cap B^{\frac{N}{2}}_{2,2-\varepsilon}\cap L^\infty$ (with $\varepsilon>0$) and $\rho_0^1\geq c>0$ is sufficient for the Theorem 1.3.

Proof. We observe that (h_2,u_2) are solution of the following system:

$$\begin{cases}\partial_t h_2+\mathrm{div}u_2=F(h_2,u_2),\\ \partial_t u_2-\mu\Delta u_2-\mu\nabla\mathrm{div}u_2-\kappa\nabla\Delta h_2+K\nabla h_2=G(h_2,u_2),\\ (h_2(0,\cdot),u_2(0,\cdot))=(h_0^2,u_0^2),\end{cases}\tag{3.47}$$

with:

$$\begin{aligned}&F(h_2,u_2)=F+\mu\nabla\ln\rho_1\cdot\nabla h_2-u_2\cdot\nabla\ln\rho_1,\\
&G(h_2,u_2)=G+2\mu\nabla\ln\rho_1\cdot Du_2+2\mu\nabla h_2\cdot Du_1-u_1\cdot\nabla u_2-u_2\cdot\nabla u_1+\mu^2\nabla(\nabla\ln\rho_1\cdot\nabla h_2).\end{aligned}$$

By applying the Proposition 3.7, we have for any $T>0$:

$$\begin{aligned}&\|h_2\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})}+\|h_2\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}+\|u_2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,1})}+\|u_2\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,1})}\\
&\leq C\big(\|h_0^2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}}+\|u_0^2\|_{B^{\frac{N}{2}}_{2,1}}+\|F(h_2,u_2)\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}+\|G(h_2,u_2)\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\big).\end{aligned}\tag{3.48}$$

We have only to deal with the right hand side of (3.48), we have in particular:

$$\|\nabla\ln\rho_1\cdot\nabla h_2\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}=\int_0^T\|\nabla\ln\rho_1\cdot\nabla h_2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}}(s)ds$$

$$\lesssim \int_0^T \big(\|\nabla h_2\|_{\widetilde{B}_{2,1}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}} \|\nabla \ln \rho_1\|_{B_{2,\infty}^{\frac{N}{2}-\frac{1}{2}}} + \|\nabla h_2\|_{\widetilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}} \|\nabla \ln \rho_1\|_{B_{2,\infty}^{\frac{N}{2}+\frac{1}{2}}}\big)(s)ds. \tag{3.49}$$

By interpolation (see the Proposition 2.1) we get:

$$\begin{aligned}
&\|\nabla h_2\|_{\widetilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}} \lesssim \|\nabla h_2\|^{\frac{3}{4}}_{\widetilde{B}_{2,1}^{\frac{N}{2}-2,\frac{N}{2}-1}} \|\nabla h_2\|^{\frac{1}{4}}_{\widetilde{B}_{2,1}^{\frac{N}{2},\frac{N}{2}+1}},\\
&\|\nabla h_2\|_{\widetilde{B}_{2,1}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}} \le \|\nabla h_2\|^{\frac{1}{4}}_{\widetilde{B}_{2,1}^{\frac{N}{2}-2,\frac{N}{2}-1}} \|\nabla h_2\|^{\frac{3}{4}}_{\widetilde{B}_{2,1}^{\frac{N}{2},\frac{N}{2}+1}}.
\end{aligned} \tag{3.50}$$

By combining (3.50), (3.49) and Young inequality we have for any $\varepsilon > 0$:

$$\begin{aligned}
\|\nabla \ln \rho_1 \cdot \nabla h_2\|_{\widetilde{L}^1_T(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})} &\lesssim \int_0^T \big(\|\nabla h_2\|^{\frac{1}{4}}_{\widetilde{B}_{2,1}^{\frac{N}{2}-2,\frac{N}{2}-1}} \|\nabla h_2\|^{\frac{3}{4}}_{\widetilde{B}_{2,1}^{\frac{N}{2},\frac{N}{2}+1}} \|\nabla \ln \rho_1\|_{B_{2,\infty}^{\frac{N}{2}-\frac{1}{2}}}\\
&\quad + \|\nabla h_2\|^{\frac{3}{4}}_{\widetilde{B}_{2,1}^{\frac{N}{2}-2,\frac{N}{2}-1}} \|\nabla h_2\|^{\frac{1}{4}}_{\widetilde{B}_{2,1}^{\frac{N}{2},\frac{N}{2}+1}} \|\nabla \ln \rho_1\|_{B_{2,\infty}^{\frac{N}{2}+\frac{1}{2}}}\big)(s)ds\\
&\lesssim \int_0^T \big(2\varepsilon \|\nabla h_2\|_{\widetilde{B}_{2,1}^{\frac{N}{2},\frac{N}{2}+1}} + C_\varepsilon \|\nabla \ln \rho_1\|^4_{B_{2,\infty}^{\frac{N}{2}-\frac{1}{2}}} \|\nabla h_2\|_{\widetilde{B}_{2,1}^{\frac{N}{2}-2,\frac{N}{2}-1}}\\
&\quad + C_\varepsilon \|\nabla h_2\|_{\widetilde{B}_{2,\infty}^{\frac{N}{2}-2,\frac{N}{2}-1}} \|\nabla \ln \rho_1\|^{\frac{4}{3}}_{B_{2,\infty}^{\frac{N}{2}+\frac{1}{2}}}\big)(s)ds.
\end{aligned} \tag{3.51}$$

In a similar way by interpolation we obtain:

$$\begin{aligned}
\|u_2 \cdot \nabla \ln \rho_1\|_{\widetilde{L}^1_T(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})} &= \int_{\mathbb{R}^+} \|u_2 \cdot \nabla \ln \rho_1\|_{\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}}}(s)ds\\
&\lesssim \int_0^T \big(\|\nabla \ln \rho_1\|_{\widetilde{B}_{2,\infty}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}} \|u_2\|_{B_{2,1}^{\frac{N}{2}-\frac{1}{2}}} + \|\nabla \ln \rho_1\|_{\widetilde{B}_{2,\infty}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}} \|u_2\|_{B_{2,1}^{\frac{N}{2}+\frac{1}{2}}}\big)(s)ds\\
&\lesssim \int_0^T \big(\|\nabla \ln \rho_1\|_{\widetilde{B}_{2,\infty}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}} \|u_2\|^{\frac{3}{4}}_{B_{2,1}^{\frac{N}{2}-1}} \|u_2\|^{\frac{1}{4}}_{B_{2,1}^{\frac{N}{2}+1}}\\
&\quad + \|\nabla \ln \rho_1\|_{\widetilde{B}_{2,\infty}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}} \|u_2\|^{\frac{1}{4}}_{B_{2,1}^{\frac{N}{2}-1}} \|u_2\|^{\frac{3}{4}}_{B_{2,1}^{\frac{N}{2}+1}}\big)(s)ds.
\end{aligned} \tag{3.52}$$

Applying Young inequality it yields:

$$\begin{aligned}
\|u_2 \cdot \nabla \ln \rho_1\|_{\widetilde{L}^1_T(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})} &\lesssim \int_0^T \big(2\varepsilon \|u_2\|_{B_{2,1}^{\frac{N}{2}+1}} + C_\varepsilon \|\nabla \ln \rho_1\|^4_{\widetilde{B}_{2,\infty}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}} \|u_2\|_{B_{2,1}^{\frac{N}{2}-1}}\\
&\quad + C_\varepsilon \|u_2\|_{B_{2,\infty}^{\frac{N}{2}-1}} \|\nabla \ln \rho_1\|^{\frac{4}{3}}_{\widetilde{B}_{2,\infty}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}}\big)(s)ds.
\end{aligned} \tag{3.53}$$

Let us proceed in a similar way for $\|G(h_2,u_2)\|_{\widetilde{L}^1_T(B_{2,1}^{\frac{N}{2}-1})}$, we are going only to treat two terms (the other one will be left to the reader). As previously, we get:

$$
\begin{aligned}
\|\nabla\ln\rho_1\cdot Du_2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})} &= \int_0^T \|\nabla\ln\rho_1\cdot Du_2\|_{B^{\frac{N}{2}-1}_{2,1}}(s)ds \\
&\lesssim \int_0^T \big(\|\nabla\ln\rho_1\|_{B^{\frac{N}{2}-\frac12}_{2,\infty}}\|Du_2\|_{B^{\frac{N}{2}-\frac12}_{2,1}} + \|\nabla\ln\rho_1\|_{B^{\frac{N}{2}+\frac12}_{2,\infty}}\|Du_2\|_{B^{\frac{N}{2}-\frac32}_{2,1}}\big)(s)ds \\
&\lesssim \int_0^T \big(2\varepsilon\|u_2\|_{B^{\frac{N}{2}+1}_{2,1}} + C_\varepsilon\|\nabla\ln\rho_1\|^4_{\widetilde{B}^{\frac{N}{2}-\frac32,\frac{N}{2}-\frac12}_{2,\infty}}\|u_2\|_{B^{\frac{N}{2}-1}_{2,1}} \\
&\quad + C_\varepsilon\|u_2\|_{B^{\frac{N}{2}-1}_{2,\infty}}\|\nabla\ln\rho_1\|^{\frac43}_{\widetilde{B}^{\frac{N}{2}-\frac12,\frac{N}{2}+\frac12}_{2,\infty}}\big)(s)ds,
\end{aligned} \tag{3.54}
$$

and:

$$
\begin{aligned}
\|u_1\cdot\nabla u_2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})} &= \int_0^T \|u_1\cdot\nabla u_2\|_{B^{\frac{N}{2}-1}_{2,1}}(s)ds \\
&\lesssim \int_0^T \big(2\varepsilon\|u_2\|_{B^{\frac{N}{2}+1}_{2,1}} + C_\varepsilon\|u_1\|^4_{B^{\frac{N}{2}-\frac12}_{2,\infty}}\|u_2\|_{B^{\frac{N}{2}-1}_{2,1}} + C_\varepsilon\|u_2\|_{B^{\frac{N}{2}-1}_{2,1}}\|u_1\|^{\frac43}_{B^{\frac{N}{2}+\frac12}_{2,\infty}}\big)(s)ds.
\end{aligned} \tag{3.55}
$$

Finally combining (3.48), (3.51), (3.53), (3.54) and (3.55) we have for $C>0$ and $\varepsilon>0$ small enough such that $C\varepsilon\le\frac12$:

$$
\begin{aligned}
&\|h_2\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})} + \|h_2\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})} + \|u_2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,1})} + \|u_2\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,1})} \\
&\le C\Bigg(\|h_0^2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}} + \|u_0^2\|_{B^{\frac{N}{2}}_{2,1}} + \|F\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})} + \|G\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})} \\
&\quad + \int_0^T\Big(\varepsilon\big(\|u_2\|_{B^{\frac{N}{2}+1}_{2,1}} + \|h_2\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})}\big) + C_\varepsilon\big(\|u_2\|_{B^{\frac{N}{2}-1}_{2,1}} + \|h_2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}}\big) \\
&\quad \times\big(\|u_1\|^4_{B^{\frac{N}{2}-\frac12}_{2,\infty}} + \|u_1\|^{\frac43}_{B^{\frac{N}{2}+\frac12}_{2,\infty}} + \|\nabla\ln\rho_1\|^{\frac43}_{B^{\frac{N}{2}+\frac12}_{2,\infty}} + \|\nabla\ln\rho_1\|^4_{B^{\frac{N}{2}-\frac12}_{2,\infty}}\big)\Big)(s)ds\Bigg).
\end{aligned} \tag{3.56}
$$

By a bootstrap argument and the Gronwall lemma where we use the fact that:

$$
\|u_2(s)\|_{B^{\frac{N}{2}-1}_{2,1}} + \|h_2(s)\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}} \lesssim \|u_2\|_{\widetilde{L}^\infty_s(B^{\frac{N}{2}-1}_{2,1})} + \|h_2\|_{\widetilde{L}^\infty_s(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}
$$

we get for $C>0$ large enough:

$$
\begin{aligned}
&\|h_2\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})} + \|h_2\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})} + \|u_2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,1})} + \|u_2\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,1})} \\
&\le C\big(\|h_0^2\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1}} + \|u_0^2\|_{B^{\frac{N}{2}}_{2,1}} + \|F\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})} + \|G\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\big) \\
&\times\exp\Big(C\int_0^T\big(\|u_1\|^4_{B^{\frac{N}{2}-\frac12}_{2,\infty}} + \|u_1\|^{\frac43}_{B^{\frac{N}{2}+\frac12}_{2,\infty}} + \|\nabla\ln\rho_1\|^{\frac43}_{B^{\frac{N}{2}+\frac12}_{2,\infty}} + \|\nabla\ln\rho_1\|^4_{B^{\frac{N}{2}-\frac12}_{2,\infty}}\big)(s)ds\Big).
\end{aligned} \tag{3.57}
$$

It concludes the proof of the Proposition 3.8. □

3.2. Existence of local solutions for system (1.6)

We now are going to prove the existence of strong solutions in finite time with large initial data verifying the hypothesis of Theorem 1.1 for the system (1.6). More precisely we assume that (q_0, u_0) belong to $B^{\frac{N}{2}}_{2,\infty} \times B^{\frac{N}{2}-1}_{2,\infty}$.

Existence of solutions

The existence part of the theorem is proved by an iterative method. We define a sequence (q^n, u^n) as follows:

$$q^n = q_L + \bar{q}^n,\ u^n = u_L + \bar{u}^n,$$

where (q_L, u_L) stands for the solution of:

$$\begin{cases} \partial_t q_L + \mathrm{div} u_L = 0, \\ \partial_t u_L - \mathcal{A} u_L - \kappa \nabla(\Delta q_L) = 0, \end{cases} \tag{3.58}$$

supplemented with initial data:

$$q_L(0) = q_0\ ,\ u_L(0) = u_0.$$

Here $\mathcal{A}$ define the Lamé operator $\mathcal{A}u = \mu\Delta u + (\lambda + \mu)\nabla \mathrm{div} u$. Using the Proposition 3.6, we obtain the following estimates on (q_L, u_L) for all $T > 0$:

$$q_L \in \widetilde{C}([0,T], B^{\frac{N}{2}}_{2,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{2}+2}_{2,\infty}) \text{ and } u_L \in \widetilde{C}([0,T], B^{\frac{N}{2}-1}_{2,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty}).$$

Setting $(\bar{q}^0, \bar{u}^0) = (0,0)$ we now define $(\bar{q}_n, \bar{u}_n)$ as the solution of the following system:

$$\begin{cases} \partial_t \bar{q}^n + \mathrm{div}(\bar{u}^n) = F_{n-1}, \\ \partial_t \bar{u}_n - \mathcal{A}\bar{u}_n - \kappa\nabla(\Delta \bar{q}^n) = G_{n-1}, \\ (\bar{q}_n, \bar{u}_n)_{t=0} = (0,0), \end{cases} \tag{N_1}$$

where:

$$\begin{aligned} F_{n-1} =& -u^{n-1} \cdot \nabla q^{n-1}, \\ G_{n-1} =& -(u^{n-1})^*.\nabla u^{n-1} + 2\mu\nabla q^{n-1} \cdot Du^{n-1} + \lambda\nabla q^{n-1}\,\mathrm{div} u^{n-1} + \frac{\kappa}{2}\nabla(|\nabla q^{n-1}|^2) - K\nabla q^{n-1}. \end{aligned}$$

1) First step, uniform bound

Let ε be a small parameter and choose T small enough such that according to the Proposition 3.6 we have (this is possible since q_0 and u_0 are respectively in $X_0^{\frac{N}{2}}$ and in $X_0^{\frac{N}{2}-1}$):

$$\begin{aligned} &\|u_L\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty})} + \|q_L\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+2}_{2,\infty})} \le \varepsilon, \\ &\|u_L\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,\infty})} + \|q_L\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}}_{2,\infty})} \le CA_0, \end{aligned} \tag{$\mathcal{H}_\varepsilon$}$$

with $A_0 = \|q_0\|_{B^{\frac{N}{2}}_{2,\infty}} + \|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}}$. We are going to show by induction that:

$$\|(\bar{q}^n, \bar{u}^n)\|_{F_T} \le \sqrt{\varepsilon}, \tag{$\mathcal{P}_n$}$$

for ε small enough with:

$$F_T = \big(\widetilde{C}([0,T], B^{\frac{N}{2}}_{2,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{2}+2}_{2,\infty})\big) \times \big(\widetilde{C}([0,T], B^{\frac{N}{2}-1}_{2,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty})\big)^N.$$

As $(\bar{q}^0, \bar{u}^0) = (0,0)$ the result is true for $n = 0$. We now suppose $(\mathcal{P}_{n-1})$ (with $n \geq 1$) true and we are going to show $(\mathcal{P}_n)$. Applying Proposition 3.6 we have:

$$\|(\bar{q}^n, \bar{u}^n)\|_{F_T} \leq C\|(\nabla F_{n-1}, G_{n-1})\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,\infty})}. \tag{3.59}$$

Bounding the right-hand side of (3.59) may be done by applying Proposition 2.2, Lemma 2.3 and Corollary 2. We first treat the case of $\|F_{n-1}\|_{\widetilde{L}^1_T(B^{N/2}_{2,\infty})}$, let us recall that:

$$F_{n-1} = -u_L \cdot \nabla q_L - \bar{u}^{n-1} \cdot \nabla q_L - u^L \cdot \nabla \bar{q}^{n-1} - \bar{u}^{n-1} \cdot \nabla \bar{q}^{n-1}. \tag{3.60}$$

We are going to bound each term of (3.60) we have then:

$$\|u_L \cdot \nabla q_L\|_{\widetilde{L}^1_T(B^{N/2}_{2,\infty})} \leq \|u_L\|_{\widetilde{L}^1_T(B^{N/2+1}_{2,\infty})}\|q_L\|_{\widetilde{L}^\infty_T(B^{N/2}_{2,\infty})} + \|q_L\|_{\widetilde{L}^1_T(B^{N/2+2}_{2,\infty})}\|u_L\|_{\widetilde{L}^\infty_T(B^{N/2-1}_{2,\infty})}. \tag{3.61}$$

Similarly we obtain:

$$\begin{aligned}\|u_L \cdot \nabla \bar{q}^{n-1}\|_{\widetilde{L}^1_T(B^{N/2}_{2,\infty})} &\leq \|u_L\|_{\widetilde{L}^1_T(B^{N/2+1}_{2,\infty})}\|\bar{q}^{n-1}\|_{\widetilde{L}^\infty_T(B^{N/2}_{2,\infty})} \\ &\quad + \|\nabla \bar{q}^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|u_L\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})},\end{aligned} \tag{3.62}$$

$$\begin{aligned}\|\bar{u}^{n-1} \cdot \nabla q_L\|_{\widetilde{L}^1_T(B^{N/2}_{2,\infty})} &\leq \|\bar{u}^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\nabla q_L\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})} \\ &\quad + \|q_L\|_{\widetilde{L}^1_T(B^{N/2+2}_{2,\infty})}\|\bar{u}^{n-1}\|_{\widetilde{L}^\infty_T(B^{N/2-1}_{2,\infty})},\end{aligned} \tag{3.63}$$

and:

$$\begin{aligned}\|\bar{u}^{n-1} \cdot \nabla \bar{q}^{n-1}\|_{\widetilde{L}^1_T(B^{N/2}_{2,\infty})} &\leq \|\bar{u}^{n-1}\|_{\widetilde{L}^1_T(B^{N/2+1}_{2,\infty})}\|\bar{q}^{n-1}\|_{\widetilde{L}^\infty_T(B^{N/2}_{2,\infty})} \\ &\quad + \|\bar{q}^{n-1}\|_{\widetilde{L}^1_T(B^{N/2+2}_{2,\infty})}\|\bar{u}^{n-1}\|_{\widetilde{L}^\infty_T(B^{N/2-1}_{2,\infty})}.\end{aligned} \tag{3.64}$$

By using the previous inequalities (3.61), (3.62), (3.63), (3.64), $(\mathcal{P}_{n-1})$ and by interpolation, we get that for $C > 0$ large enough:

$$\|F_n\|_{\widetilde{L}^1_T(B^{N/2}_{2,\infty})} \leq C\sqrt{\varepsilon}(A_0^{\frac{3}{4}}\varepsilon^{\frac{1}{4}} + \sqrt{\varepsilon}(1 + A_0) + \varepsilon). \tag{3.65}$$

Next we want to control $\|G_n\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})}$. According to Propositions 2.2, 3.6, and Corollary 2, it yields:

$$\begin{aligned}\|(u^{n-1})^*.\nabla u^{n-1}\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,\infty})} &\lesssim \|u^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|u^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}, \\ \|\nabla(|\nabla q^{n-1}|^2)\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,\infty})} &\lesssim \||\nabla q^{n-1}|^2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}}_{2,\infty})} \\ &\lesssim \|\nabla q^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\nabla q^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})} \\ &\lesssim \|q^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{3}{2}}_{2,\infty})}\|q^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})},\end{aligned}$$

$$\begin{aligned}\|\nabla q^{n-1}\cdot Du^{n-1}\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,\infty})} &\lesssim \|\nabla q^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|u^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})} \\ &\quad + \|u^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\nabla q^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}, \\ \|\nabla q^{n-1}\mathrm{div}u^{n-1}\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,\infty})} &\lesssim \|\nabla q^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|u^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})} \\ &\quad + \|u^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\nabla q^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}, \\ \|\nabla q^{n-1}\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,\infty})} &\leq T\|q^{n-1}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}}_{2,\infty})}. \end{aligned} \tag{3.66}$$

Let us recall that we have by interpolation and the condition $(\mathcal{P}_{n-1})$:

$$\begin{aligned}\|u^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})} &\leq \|u^{n-1}\|^{\frac{3}{4}}_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,\infty})}\|u^{n-1}\|^{\frac{1}{4}}_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty})} \leq A_0^{\frac{3}{4}}\varepsilon^{\frac{1}{4}}+\sqrt{\varepsilon}, \\ \|u^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})} &\leq \|u^{n-1}\|^{\frac{1}{4}}_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,\infty})}\|u^{n-1}\|^{\frac{3}{4}}_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty})} \leq A_0^{\frac{1}{4}}\varepsilon^{\frac{3}{4}}+\sqrt{\varepsilon}. \end{aligned} \tag{3.67}$$

Using (3.59), (3.65), (3.66) and (3.67) we have for $C > 0$ large enough:

$$\begin{aligned}\|(\bar{q}^n,\bar{u}^n)\|_{F_T} &\leq C\sqrt{\varepsilon}(A_0^{\frac{3}{4}}\varepsilon^{\frac{1}{4}}+\sqrt{\varepsilon}(1+A_0)+\varepsilon)+C(A_0^{\frac{3}{4}}\varepsilon^{\frac{1}{4}}+\sqrt{\varepsilon})(A_0^{\frac{1}{4}}\varepsilon^{\frac{3}{4}}+\sqrt{\varepsilon})+T(A_0+\sqrt{\varepsilon}) \\ &\leq C\sqrt{\varepsilon}(A_0^{\frac{3}{4}}\varepsilon^{\frac{1}{4}}+2\sqrt{\varepsilon}(1+A_0)+A_0^{\frac{3}{4}}\varepsilon^{\frac{1}{4}}+A_0^{\frac{1}{4}}\varepsilon^{\frac{3}{4}}+\varepsilon)+T(A_0+\sqrt{\varepsilon}).\end{aligned}$$

By choosing T and ε small enough the property $(\mathcal{P}_n)$ is verified, so we have shown by induction that (q^n, u^n) is bounded in F_T.

Second step, convergence of the sequence

We will show that (q^n, u^n) is a Cauchy sequence in the Banach space F_T, hence converges to some $(q, u) \in F_T$. Let:

$$\delta q^n = q^{n+1} - q^n,\ \delta u^n = u^{n+1} - u^n.$$

The system verified by $(\delta q^n, \delta u^n)$ reads:

$$\begin{cases}\partial_t\delta q^n + \mathrm{div}\delta u^n = F_n - F_{n-1}, \\ \partial_t\delta u^n - \mu\Delta\delta u^n - (\lambda+\mu)\nabla\mathrm{div}\delta u^n - \kappa\nabla\Delta\delta q^n = G_n - G_{n-1}, \\ \delta q^n(0) = 0\ ,\ \delta u^n(0) = 0.\end{cases}$$

Applying Propositions 3.6 gives:

$$\|(\delta q^n,\delta u^n)\|_{F_T} \leq C(\|F_n - F_{n-1}\|_{\widetilde{L}^1_T(B^{N/2}_{2,\infty})} + \|G_n - G_{n-1}\|_{\widetilde{L}^1_T(B^{N/2-1}_{2,\infty})}). \tag{3.68}$$

Tedious calculus ensure that:

$$\begin{aligned}&F_n - F_{n-1} = -\delta u^{n-1}\cdot\nabla q^n - u^{n-1}\cdot\nabla\delta q^{n-1}, \\ &G_n - G_{n-1} = -u^n\cdot\nabla\delta u^{n-1} - \delta u^{n-1}\cdot\nabla u^{n-1} + \mu\nabla q^n\cdot D\delta u^{n-1} + \mu\nabla\delta q^{n-1}\cdot Du^{n-1} \\ &+\lambda\nabla q^n\mathrm{div}\delta u^{n-1} + \lambda\nabla\delta q^{n-1}\mathrm{div}u^{n-1} - K\nabla\delta q^{n-1} + \nabla\big(\nabla q^n\cdot\nabla\delta q^{n-1} + \nabla\delta q^{n-1}\cdot\nabla q^{n-1}\big).\end{aligned}$$

It remains only to estimate the terms on the right hand side of (3.68) by using the same type of estimates than in the previous section and the property $(\mathcal{P}_n)$. More precisely according to the Proposition 2.2 and $(\mathcal{P}_n)$, there exists $C > 0$ such that:

$$\begin{aligned}
\|F_n - F_{n-1}\|_{\widetilde{L}^1_T(B^{N/2}_{2,\infty})} &\leq \|\delta u^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})} \|\nabla q^n\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})} \\
&\quad + \|\delta u^{n-1}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,\infty})} \|\nabla q^n\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty})} + \|u^{n-1}\|_{\widetilde{L}^1_T(B^{\frac{N}{2}+1}_{2,\infty})} \|\nabla \delta q^{n-1}\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}-1}_{2,\infty})} \\
&\quad + \|\nabla \delta q^{n-1}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})} \|u^{n-1}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})} \\
&\leq C(A_0^{\frac{3}{4}}\varepsilon^{\frac{1}{4}} + A_0^{\frac{1}{4}}\varepsilon^{\frac{3}{4}} + \sqrt{\varepsilon} + \varepsilon)\|(\delta q^{n-1}, \delta u^{n-1})\|_{F_T}. \qquad (3.69)
\end{aligned}$$

In a similar way we show that there exists $C > 0$ large enough such that:

$$\|G_n - G_{n-1}\|_{\widetilde{L}^1_T(B^{N/2-1}_{2,\infty})} \leq C(A_0^{\frac{3}{4}}\varepsilon^{\frac{1}{4}} + A_0^{\frac{1}{4}}\varepsilon^{\frac{3}{4}} + \sqrt{\varepsilon} + \varepsilon + T)\|(\delta q^{n-1}, \delta u^{n-1})\|_{F_T}. \qquad (3.70)$$

By combining (3.68), (3.69) and (3.70), we get for $C > 0$ large enough:

$$\|(\delta q^n, \delta u^n)\|_{F_T} \leq C(A_0^{\frac{3}{4}}\varepsilon^{\frac{1}{4}} + A_0^{\frac{1}{4}}\varepsilon^{\frac{3}{4}} + \sqrt{\varepsilon} + \varepsilon + T)\|(\delta q^{n-1}, \delta u^{n-1})\|_{F_T}.$$

It implies that choosing ε and T small enough (q^n, u^n) is a Cauchy sequence in F_T which is a Banach. It provides that (q^n, u^n) converges to (q, u) in F_T. The verification that the limit (q, u) is solution of (1.6) in the sense of distributions is a straightforward application of Proposition 2.2.

Third step, uniqueness

Now, we are going to prove the uniqueness of the solution in F_T. Suppose that (q_1, u_1) and (q_2, u_2) are solutions with the same initial conditions; furthermore they belong to F_T and (q_1, u_1) corresponds to the previous solution. We set:

$$\delta q = q_2 - q_1 \quad \text{and} \quad \delta u = u_2 - u_1.$$

We deduce that $(\delta q, \delta u)$ satisfy the following system:

$$\begin{cases}
\partial_t \delta q + \mathrm{div} \delta u = F_2 - F_1, \\
\partial_t \delta u - \mu \Delta \delta u - (\lambda + \mu)\nabla \mathrm{div} \delta u - \kappa \nabla \Delta \delta q = G_1 - G_2, \\
\delta q(0) = 0\ ,\ \delta u(0) = 0.
\end{cases}$$

We now apply Proposition 3.6 to the previous system, and by using the same type of estimates than in the previous part, we show that:

$$\|(\delta q, \delta u)\|_{\widetilde{F}^{\frac{N}{2}}_{T_1}} \lesssim (\|q_1\|_{\widetilde{L}^2 T_1(B^{\frac{N}{2}+1}_{2,\infty})} + \|q_2\|_{\widetilde{L} T_1{}^2(B^{\frac{N}{2}+1}_{2,\infty})} + \|u_1\|_{\widetilde{L}^2 T_1(B^{\frac{N}{2}+1}_{2,\infty})} + \|u_2\|_{\widetilde{L} T_1{}^2(B^{\frac{N}{2}}_{2,\infty})})\|(\delta q, \delta u)\|_{\widetilde{F}^{\frac{N}{2}}_{T_1}}.$$

We have then for T_1 small enough: $(\delta q, \delta u) = (0, 0)$ on $[0, T_1]$ and by connexity we finally conclude that:

$$q_1 = q_2,\ u_1 = u_2 \ \text{ on } \ [0, T]. \qquad \square$$

3.3. Global strong solution near equilibrium for system (1.6)

We are now interested in proving the existence of global strong solution with small initial data for the system (1.6). The main difference with the previous proof consists essentially in taking into account the behavior of the density in low frequencies, to do this we will use the Proposition 3.6. More precisely we are going to use a contracting mapping argument for the function ψ defined as follows:

$$\psi(q,u)=W(t,\cdot)*\begin{pmatrix}q_0\\u_0\end{pmatrix}+\int_0^t W(t-s)\begin{pmatrix}F(q,u)\\G(q,u)\end{pmatrix}ds, \tag{3.71}$$

where W is the semi-group associated with the linear system $(N1)$ with $a=\mu$, $b=\lambda+\mu$, $c=\kappa$ and $d=K$. The nonlinear terms F,G are defined as follows:

$$\begin{aligned}&F(q,u)=-\,u\cdot\nabla q,\\&G(q,u)=-\,u.\nabla u+2\mu\nabla q\cdot Du+\lambda\mathrm{div}u\;\;\nabla q+\frac{\kappa}{2}\nabla(|\nabla q|^2).\end{aligned} \tag{3.72}$$

We are going to check that we can apply a fixed point theorem for the function ψ in $E^{\frac{N}{2}}$ defined below, the proof is divided in two step the stability of ψ for a ball $B(0,R)$ in $E^{\frac{N}{2}}$ and the contraction property. We define $E^{\frac{N}{2}}$ by:

$$E^{\frac{N}{2}}=\big(\widetilde{L}^\infty(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})\cap\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty})\big)\times\big(\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,\infty})\cap\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,\infty})\big)^N.$$

1) First step, stability of $B(0,R)$

Let:

$$\eta=\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty}}+\|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}}.$$

We are going to show that ψ maps the ball $B(0,R)$ into itself if R is small enough. According to Proposition 3.7, we have:

$$\|W(t,\cdot)*\begin{pmatrix}q_0\\u_0\end{pmatrix}\|_{E^{\frac{N}{2}}}\le C\eta\,. \tag{3.73}$$

According to the Proposition 3.7 it implies also that:

$$\|\psi(q,u)\|_{E^{\frac{N}{2}}}\le C\big(\eta+\|F(q,u)\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})}+\|G(q,u)\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})}\big). \tag{3.74}$$

The main task consists in using the Proposition 2.2 and Corollary 2 to obtain estimates on:

$$\|F(q,u)\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})},\;\;\|G(q,u)\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})}.$$

Let us first estimate $\|F(q,u)\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})}$. According to Proposition 2.2, we have:

$$\begin{aligned}\|u\cdot\nabla q\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})}\lesssim&\|\nabla q\|_{\widetilde{L}^{\frac{4}{3}}(\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|u\|_{\widetilde{L}^4(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\\&+\|\nabla q\|_{\widetilde{L}^4(\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\|u\|_{\widetilde{L}^{\frac{4}{3}}(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}.\end{aligned} \tag{3.75}$$

Let us now estimate $\|G(q,u)\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})}$. Hence by Proposition 2.2 it yields:

$$\|u\cdot\nabla u\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})}\lesssim\|u\|_{\widetilde{L}^{\frac{4}{3}}(B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|u\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}. \tag{3.76}$$

In the same way, from the Proposition 2.2 and the fact that $\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty}\hookrightarrow B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}$, $\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty}\hookrightarrow B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}$ (see the Remark 17) we deduce that:

$$\begin{aligned}\|\nabla q\cdot Du\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})}&\lesssim\|\nabla q\|_{\widetilde{L}^{\frac{4}{3}}(\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|Du\|_{\widetilde{L}^4(B^{\frac{N}{2}-\frac{3}{2}}_{2,\infty})}\\&\quad+\|\nabla q\|_{\widetilde{L}^4(\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\|Du\|_{\widetilde{L}^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}.\\ \|\nabla q\,\mathrm{div}u\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})}&\lesssim\|\nabla q\|_{\widetilde{L}^{\frac{4}{3}}(\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\mathrm{div}u\|_{\widetilde{L}^4(B^{\frac{N}{2}-\frac{3}{2}}_{2,\infty})}\\&\quad+\|\nabla q\|_{\widetilde{L}^4(\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty})}\|\mathrm{div}u\|_{\widetilde{L}^{\frac{4}{3}}(B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty})}.\end{aligned} \tag{3.77}$$

It now remains only to deal with the capillary terms:

$$\begin{aligned}\|\nabla(|\nabla q|^2)\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})}&\lesssim\||\nabla q|^2\|_{\widetilde{L}^1(B^{\frac{N}{2}}_{2,\infty})}\\&\lesssim\|\nabla q\|_{\widetilde{L}^{\frac{4}{3}}(\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty})}\|\nabla q\|_{\widetilde{L}^4(\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty})}.\end{aligned} \tag{3.78}$$

We have previously used the fact that $\widetilde{B}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}}_{2,\infty}\hookrightarrow B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}$ and $\widetilde{B}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}}_{2,\infty}\hookrightarrow B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}$. We are now going to assume that (q,u) belong to the ball $B(0,R)$ of $E^{\frac{N}{2}}$ with $R>0$. Combining the estimates (3.75), (3.76), (3.77) and (3.78) we get:

$$\|\psi(q,u)\|_{E^{\frac{N}{2}}}\leq C((C+1)\eta+R)^2. \tag{3.79}$$

By choosing R and η small enough we have:

$$C((C+1)\eta+R)^2\leq R. \tag{3.80}$$

It implies that the ball $B(0,R)$ of $E^{\frac{N}{2}}$ is stable under ψ which means:

$$\psi(B(0,R))\subset B(0,R)\,.$$

2) Second step, property of contraction

We consider (q_1,u_1), (q_2,u_2) in $B(0,R)$ and we are interested in verifying that ψ is a contraction. According to the Proposition 3.7 we have:

$$\begin{aligned}\|\psi(q_2,u_2)-\psi(q_1,u_1)\|_{E^{\frac{N}{2}}}\leq C\big(&\|F(q_2,u_2)-F(q_1,u_1)\|_{\widetilde{L}^1(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})}\\&+\|G(q_2,u_2)-G(q_1,u_1)\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,\infty})}\big),\end{aligned} \tag{3.81}$$

with:

$$\begin{aligned}F(q_2,u_2)-F(q_1,u_1)&=-\delta u\cdot\nabla q_2-u_1\cdot\nabla\delta q.\\ G(q_2,u_2)-G(q_1,u_1)&=-u_2\cdot\nabla\delta u-\delta u\cdot\nabla u_1+\mu\nabla q_2\cdot D\delta u+\mu\nabla\delta q\cdot Du_1\\&\quad+\lambda\nabla q_2\mathrm{div}\delta u+\lambda\nabla\delta q\mathrm{div}u_1+\nabla\big(\nabla q_2\cdot\nabla\delta q+\nabla\delta q\cdot\nabla q_1\big).\end{aligned}$$

We now set $\delta q = q_2 - q_1$ and $\delta u = u_2 - u_1$. Let us first estimate $\|F(q_2,u_2) - F(q_1,u_1)\|_{\widetilde{L}^1(\widetilde{B}_{2,\infty}^{\frac{N}{2}-1,\frac{N}{2}})}$. We have by Proposition 2.2 and the Remark 17:

$$\begin{aligned}
&\|F(q_2,u_2) - F(q_1,u_1)\|_{\widetilde{L}^1(\widetilde{B}_{2,\infty}^{\frac{N}{2}-1,\frac{N}{2}})} \lesssim \|\delta u\|_{\widetilde{L}_T^{\frac{4}{3}}(B_{2,\infty}^{\frac{N}{2}+\frac{1}{2}})} \|\nabla q_2\|_{\widetilde{L}_T^4(\widetilde{B}_{2,\infty}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}})} \\
&\quad + \|\delta u\|_{\widetilde{L}_T^\infty(B_{2,\infty}^{\frac{N}{2}-1})} \|\nabla q_2\|_{\widetilde{L}_T^1(\widetilde{B}_{2,\infty}^{\frac{N}{2},\frac{N}{2}+1})} + \|u_1\|_{\widetilde{L}_T^1(B_{2,\infty}^{\frac{N}{2}+1})} \|\nabla \delta q\|_{\widetilde{L}_T^\infty(\widetilde{B}_{2,\infty}^{\frac{N}{2}-2,\frac{N}{2}-1})} \\
&\quad + \|\nabla \delta q\|_{\widetilde{L}_T^{\frac{4}{3}}(\widetilde{B}_{2,\infty}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}})} \|u_1\|_{\widetilde{L}_T^4(B_{2,\infty}^{\frac{N}{2}-\frac{1}{2}})} \\
&\leq C\big(2\|(q_2,u_2)\|_{E^{\frac{N}{2}}} + 2\|(q_1,u_1)\|_{E^{\frac{N}{2}}}\big)\|(\delta q,\delta u)\|_{E^{\frac{N}{2}}}. \qquad (3.82)
\end{aligned}$$

Next, we have to bound $\|G(q_2,u_2) - G(q_1,u_1)\|_{\widetilde{L}^1(B_{2,\infty}^{\frac{N}{2}-1})}$. We treat only one typical term, the others are of the same form.

$$\begin{aligned}
&\|-u_2\cdot\nabla\delta u - \delta u\cdot\nabla u_1\|_{\widetilde{L}^1(B_{2,\infty}^{\frac{N}{2}-1})} \lesssim \|\nabla \delta u\|_{\widetilde{L}_T^{\frac{4}{3}}(B_{2,\infty}^{\frac{N}{2}-\frac{1}{2}})} \|u_2\|_{\widetilde{L}_T^4(B_{2,\infty}^{\frac{N}{2}-\frac{1}{2}})} \\
&\quad + \|\nabla \delta u\|_{\widetilde{L}_T^\infty(B_{2,\infty}^{\frac{N}{2}-2})} \|u_2\|_{\widetilde{L}_T^1(B_{2,\infty}^{\frac{N}{2}+1})} + \|\nabla u_1\|_{\widetilde{L}_T^1(B_{2,\infty}^{\frac{N}{2}})} \|\delta u\|_{\widetilde{L}_T^\infty(B_{2,\infty}^{\frac{N}{2}-1})} \\
&\quad + \|\delta u\|_{\widetilde{L}_T^{\frac{4}{3}}(B_{2,\infty}^{\frac{N}{2}+\frac{1}{2}})} \|\nabla u_1\|_{\widetilde{L}_T^4(B_{2,\infty}^{\frac{N}{2}-\frac{3}{2}})} \\
&\leq C\big(2\|(q_2,u_2)\|_{E^{\frac{N}{2}}} + 2\|(q_1,u_1)\|_{E^{\frac{N}{2}}}\big)\|(\delta q,\delta u)\|_{E^{\frac{N}{2}}}. \qquad (3.83)
\end{aligned}$$

We can bound the other terms of $\|G(q_2,u_2) - G(q_1,u_1)\|_{\widetilde{L}^1(B_{2,\infty}^{\frac{N}{2}-1})}$ in the same manner and this work is left to the reader. Finally by combining (3.81), (3.82) and (3.83) we obtain for $C > 0$ large enough:

$$\|\psi(q_2,u_2) - \psi(q_1,u_1)\|_{E^{\frac{N}{2}}} \leq C\,\|(\delta q,\delta u)\|_{E^{\frac{N}{2}}}\,\big(\|(q_1,u_1)\|_{E^{\frac{N}{2}}} + \|(q_2,u_2)\|_{E^{\frac{N}{2}}}\|_{E^{\frac{N}{2}}}\big).$$

If one chooses R small enough such that $RC \leq \frac{3}{4}$, we end up with using the previous estimate which yields:

$$\|\psi(q_2,u_2) - \Psi(q_1,u_1)\|_{E^{\frac{N}{2}}} \leq \frac{3}{4}\,\|(\delta q,\delta u)\|_{E^{\frac{N}{2}}}.$$

We thus have the property of contraction and so by the fixed point theorem, we have the existence of a global solution (q,u) for the system (1.6). Indeed we can see easily that $E^{\frac{N}{2}}$ is a Banach space.

Concerning the uniqueness of this solution, it suffices to proceed as previously. More precisely if (q,u) is the previous solution and (q_1,u_1) another solution in $E^{\frac{N}{2}}$ then by setting $\delta q = q - q_1$ and $\delta u = u - u_1$ we show that for T small enough:

$$\delta q = 0 \quad \text{and} \quad \delta u = 0 \;\text{ on } [0,T].$$

We conclude by using an argument of connexity in order to get the uniqueness on $\mathbb{R}^+$. □

3.4. Global strong solution near equilibrium for system (1.1)

We are now interested in proving the existence of global strong solution for the original system (1.1), indeed the system (1.1) is a priori not equivalent to the system (1.6) if we do not control $\frac{1}{\rho}$ and ρ in L^∞ norm. When it will be done, it will be possible to propagate on the density ρ the regularity proved in Theorem 1.1 for the unknown $q = \ln\rho$ by using Proposition 2.3. Then it will be easy to verify that (ρ,u) verify the system (1.1) and is a unique solution. In order to apply this program we are going to assume additional hypothesis on the initial data. (q_0, u_0) is now in $\widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}} \times B_{2,2}^{\frac{N}{2}-1}$ with $q_0 = \ln \rho_0 \in L^\infty$ (this last condition implies in particular that ρ_0 and $\frac{1}{\rho_0}$ belong to L^∞).

The first part of the proof consists in getting L^∞ estimates for the solution of the linear system (N), and the second part consists in splitting the solution (q, u) under the following form:

$$(q, u) = (q_L, u_L) + (\bar{q}, \bar{u}),$$

with (q_L, u_L) solution of (N) with $F = G = 0$ and $(q_L(0, \cdot), u_L(0, \cdot)) = (q_0, u_0)$. The key points will be to show that q_L belongs to $L^\infty_T(L^\infty)$ for any $T > 0$ and that $\bar{q}$ is more regular that q_L. More precisely by a regularizing effect on the third index of the Besov space we shall prove that $\bar{q}$ is in $\widetilde{L}^\infty_T(B^{\frac{N}{2}}_{2,1})$ for any $T > 0$ with is embedded in $L^\infty_T(L^\infty)$. It will be then sufficient to deduce a control of $\ln \rho$ in $L^\infty_T(L^\infty)$ and to show that q and ρ have the same regularity.

3.4.1. Result of maximum principle type for the linear system (N)

Let us start by studying the following system:

$$\begin{cases} \partial_t q + \mathrm{div} u = 0, \\ \partial_t u - \mu\Delta u - (\lambda + \mu)\nabla \mathrm{div} u - \kappa\nabla\Delta q = 0, \\ (q(0, \cdot), u(0, \cdot)) = (q_0, u_0), \end{cases} \tag{3.84}$$

with $\mu > 0$ and $\lambda + 2\mu > 0$. We are now interested in characterizing the Besov spaces in term of the semi-group $B(t)$ associated with the system (3.84), it will be useful in order to obtain L^∞ estimates for q. More precisely we have the following proposition.

Proposition 3.9. *Let s be a positive real number and $(p, r) \in [1, +\infty]^2$. Let (q, u) be the solution of (3.84) with $(q, u)(t) = e^{B(t)}(q_0, u_0)$ and with the following notation:*

$$(\nabla q, u)(t) = e^{B(t)}(\nabla q_0, u_0).$$

Then there exists a constant $C > 0$ which satisfies:

$$\| \|t^s e^{B(t)}(\nabla q_0, u_0)\|_{L^p} \|_{L^r(\mathbb{R}^+, \frac{dt}{t})} \leq C\|(\nabla q_0, u_0)\|_{B^{-2s}_{p,r}} \quad \forall (\nabla q_0, u_0) \in B^{-2s}_{p,r}. \tag{3.85}$$

Proof. Apply operator Δ to the first equation of (3.84) and operators div and curl to the second one; we obtain the following system with $\nu = 2\mu + \lambda$:

$$\begin{cases} \partial_t \Delta q + \Delta \mathrm{div} u = 0, \\ \partial_t \mathrm{div} u - \nu\Delta \mathrm{div} u - \kappa\Delta^2 q = 0, \\ \partial_t \mathrm{curl} u - \mu\Delta \mathrm{curl} u = 0, \\ (q(0, \cdot), u(0, \cdot)) = (q_0, u_0). \end{cases} \tag{3.86}$$

We observe that the third equation is a heat equation and we know via lemma 2.4, p. 54 in [5] that there exist $C, c > 0$ such that for all $l \in \mathbb{Z}$:

$$\|e^{\mu t\Delta} \Delta_l \mathrm{curl} u_0\|_{L^p} \leq C e^{-c2^{2l}\mu} \|\Delta_l \mathrm{curl} u_0\|_{L^p} \quad \forall p \in [1, +\infty]. \tag{3.87}$$

Let us study now the following system:

$$\begin{cases} \partial_t c + \Delta v = 0, \\ \partial_t v - \nu\Delta v - \kappa\Delta c = 0, \\ (c(0,\cdot), v(0,\cdot)) = (c_0, v_0), \end{cases} \tag{3.88}$$

where $c = \Delta q$ and $v = \mathrm{div} u$. Denoting by $U(t)$ the semi-group associated with (3.88), we deduce from Duhamel's formula that:

$$\partial_t \begin{pmatrix} \hat{q}(t,\xi) \\ \hat{d}(t,\xi) \end{pmatrix} = U(t) \begin{pmatrix} q_0 \\ u_0 \end{pmatrix}.$$

We are now interested in proving estimate of the same form than (3.87) for $e^{U(t)}(\Delta_l c, \Delta_l v)$, and to do this we are going to prove the following lemma which is a direct consequence of the lemma 3 of [17]. For the sake of completeness we are going to recall its proof.

Lemma 1. *For any $p \in [1, +\infty]$ and any $t > 0$ we have for all $q \in \mathbb{Z}$:*

$$\|e^{U(t)}(\Delta_q c_0, \Delta_q v_0)\|_{L^p} \leq C e^{-c\min(1, \frac{4\kappa}{\nu^2})2^{2q}\nu t}(\|\Delta_q c_0\|_{L^p} + \|\Delta_q v_0\|_{L^p}). \tag{3.89}$$

Proof. Simple calculus show that $U(t) = e^{-tA(D)}$ with:

$$A(\xi) = \begin{pmatrix} 0 & -|\xi|^2 \\ \kappa|\xi|^2 & \mu|\xi|^2 \end{pmatrix}.$$

Following [17] we show that:

$$e^{-tA(\xi)} = e^{-\frac{t\nu|\xi|^2}{2}} \begin{pmatrix} h_1(t,\xi) + \frac{\nu}{2}h_2(t,\xi) & h_2(t,\xi) \\ -\kappa h_2(t,\xi) & h_1(t,\xi) - \frac{\nu}{2}h_2(t,\xi) \end{pmatrix},$$

with:

$$\begin{aligned} &h_1(t,\xi) = \cos(\nu'|\xi|^2 t), \;\; h_2(t,\xi) = \frac{\sin(\nu'|\xi|^2 t)}{\nu'}, \;\; \text{if } \mu^2 < 4\kappa, \\ &h_1(t,\xi) = 1, \;\; h_2(t,\xi) = t|\xi|^2, \;\; \text{if } \nu^2 = 4\kappa, \\ &h_1(t,\xi) = \cos(\nu'|\xi|^2 t), \;\; h_2(t,\xi) = \frac{\sin(\nu'|\xi|^2 t)}{\mu'}, \;\; \text{if } \nu^2 > 4\kappa, \end{aligned}$$

and $\nu' = \sqrt{|\kappa - \frac{\nu^2}{4}|}$. Let φ defined as in the definition of Littlewood–Paley theory, we denote by $a_{ij}(t,\xi)$ the coefficients of the matrix $e^{-tA(\xi)}$ and:

$$\Delta_q b_{ij}(t,x) = \mathcal{F}^{-1}\Delta_q\big(a_{ij})(t,x)\big) = (2\pi)^{-N}\int_{\mathbb{R}^N} e^{ix\cdot\xi} a_{ij}(t,\xi)\varphi(2^{-q}\xi)d\xi.$$

Let us show now that:

$$\|\Delta_q(b_{ij})(t,\cdot)\|_{L^1} \leq C e^{-c\min(1,\frac{4\kappa}{\nu^2})2^{2q}\nu t}, \tag{3.90}$$

where c depends only on ν, κ and c is a universal constant. We first remark that $\|\Delta_q b_{ij}\|_{L^1} = \|h_{ijq}\|_{L^1}$ with:

$$h_{ijq}(t,y) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{iy\cdot\eta} a_{ij}(t, 2^q\eta)\varphi(\eta)d\eta.$$

Let us observe that the functions h_{ijq} can be rewritten under the form:

$$h_q(t,x) = \int_{\mathbb{R}^N} e^{iy\cdot\xi} f(2^{2q}|\xi|^2 t)\varphi(\xi)d\xi, \tag{3.91}$$

with $f \in C^\infty(\mathbb{R}^+)$. By integration by parts and Leibniz' formula, we obtain for all $\alpha \in \mathbb{N}^N$,

$$(-ix)^\alpha h_q(x) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^N} e^{ix\cdot\xi} \partial^\beta f(2^{2q}|\xi|^2 t)\partial^{\alpha-\beta}\varphi(\xi)d\xi. \tag{3.92}$$

Next using Faà-di-Bruno's formula, it gives:

$$\partial^\beta f(2^{2q}|\xi|^2 t) = \sum_{\gamma_1+\cdots+\gamma_m=\beta, |\gamma_i|\geq 1} f^{(m)}(2^{2q}|\xi|^2 t)(2^{2q}t)^m (\Pi_{j=1}^m \partial^{\gamma_j}(|\xi|^2)). \tag{3.93}$$

Let us start with the case $\nu^2 > 4\kappa$. Then, it suffices to show that:

$$\|h_q\|_{L^1} \leq Ce^{-c2^{2q}\nu t}, \tag{3.94}$$

for $f(u) = e^{i\nu' u}e^{-\nu u/2}$. We have then:

$$|f^{(m)}(u)| \leq (\nu' + \frac{\nu}{2})^m e^{-\nu u/2}.$$

Using (3.92), (3.93), we prove the existence of constant $C_{\alpha,\beta,m}$ such that:

$$|x^\alpha h_q(x)| \leq \sum_{\beta\leq\alpha}\sum_{m=1}^{|\beta|} C_{\alpha,\beta,m}(2^{2q}t)^m e^{-\nu t 2^{2q}/8}.$$

For any constant $c < 1$ and $m \in \mathbb{N}$, there exists C_m such that $u^m e^{-u} \leq C_m e^{-cu}$ which implies (3.94).

When $\nu^2 = 4\kappa$, it suffices to verify (3.94) for $f(u) = ue^{-\nu u/2}$ and $f(u) = e^{-\nu u/2}$. When $\nu^2 > 4\kappa$, we have to check (3.94) for:

$$f(u) = \exp(- - \frac{\nu}{2}(1 + \sqrt{1 - \frac{4\kappa}{\nu^2}})u) \;\text{ and }\; f(u) = \exp(- - \frac{\nu}{2}(1 - \sqrt{1 - \frac{4\kappa}{\nu^2}})u).$$

Using again (3.93) we have:

$$|x^\alpha h_q(x)| \leq C\max(e^{-c\nu t 2^{2q}(1+\sqrt{1-\frac{4\kappa}{\nu^2}})}, e^{-c\nu t 2^{2q}(1-\sqrt{1-\frac{4\kappa}{\nu^2}})}) \leq Ce^{-c(\frac{\kappa}{\nu})2^{2q}t}$$

and we conclude to (3.90).

We obtain finally by using (3.90) and the Young inequality for the convolution:

$$\|e^{U(t)}(\Delta_q c_0, \Delta_q v_0)\|_{L^p} \leq Ce^{-c\min(1,\frac{4\kappa}{\nu^2})2^{2q}\mu t}(\|\Delta_q c_0\|_{L^p} + \|\Delta_q v_0\|_{L^p}).$$

It proves the Lemma 1. $\square$

Via the Bernstein lemma, the Lemma 1, (3.87) we obtain that $\forall l \in \mathbb{Z}$:

$$\|e^{B(t)}(\Delta_l \nabla q_0, \Delta_l u_0)\|_{L^p} \leq Ce^{-c\min(1,\frac{4\kappa}{\nu^2})2^{2l}\nu t}(\|\Delta_l \nabla q_0\|_{L^p} + \|\Delta_l u_0\|_{L^p}). \tag{3.95}$$

According the estimate (3.95) we have by setting $V = (\nabla q_0, u_0)$ for $c, C > 0$:

$$\|t^s \Delta_l e^{B(t)} V\|_{L^p} \leq Ct^s 2^{2ls} e^{-ct2^{2l}} 2^{-2ls} \|\Delta_l V\|_{L^p}.$$

We are now going to define some Besov space in terms of the semi-group $B(t)$ (this is an adaptation of a classical criterion for the heat equation, see theorem 2.34 in [5]).

Since V belongs to $\mathcal{S}'_h$ and the definition of the homogeneous Besov semi-norm we have:

$$\begin{aligned}\|t^s e^{tB(t)} V\|_{L^p} &\leq \sum_{l\in\mathbb{Z}} \|t^s \Delta_l e^{B(t)} V\|_{L^p} \\ &\leq C\|V\|_{B^{-2s}_{p,r}} \sum_{l\in\mathbb{Z}} t^s 2^{2ls} e^{-ct2^{2l}} c_{r,l} \end{aligned} \tag{3.96}$$

where $(c_{rl})_{l\in\mathbb{Z}}$ is an element of the unit sphere of $l^r(\mathbb{Z})$. If $r = +\infty$, we easily show (3.85) by using the following lemma (we left to the reader the proof of this last one).

Lemma 1. *For any s, we have:*

$$\sup_{t>0} \sum_{l\in\mathbb{Z}} t^s 2^{2ls} e^{-ct2^{2l}} < +\infty.$$

Let us deal now with the case $r < +\infty$, combining Hölder's inequality with the weight $2^{2ls}e^{-ct2^{2l}}$ and (3.96), we obtain:

$$\begin{aligned}\int_0^{+\infty} t^{rs} \|e^{B(t)} V\|^r_{L^p} \frac{dt}{t} &\leq C\|V\|^r_{B^{-2s}_{p,r}} \int_0^{+\infty} (\sum_{l\in\mathbb{Z}} t^s 2^{2ls} e^{-ct2^{2l}} c_{rl})^r \frac{dt}{t} \\ &\leq C\|V\|^r_{B^{-2s}_{p,r}} \int_0^{+\infty} (\sum_{l\in\mathbb{Z}} t^s 2^{2ls} e^{-ct2^{2l}})^{r-1} (\sum_{l\in\mathbb{Z}} t^s 2^{2ls} e^{-ct2^{2l}} c^r_{rl}) \frac{dt}{t} \\ &\leq C\|V\|^r_{B^{-2s}_{p,r}} \int_0^{+\infty} (\sum_{l\in\mathbb{Z}} t^s 2^{2ls} e^{-ct2^{2l}} c^r_{rl}) \frac{dt}{t}.\end{aligned}$$

Using Fubini's theorem and the change of variable $u = ct2^j$, we have:

$$\begin{aligned}\int_0^{+\infty} t^{rs} \|e^{B(t)} V\|^r_{L^p} \frac{dt}{t} &\leq C\|V\|^r_{B^{-2s}_{p,r}} \sum_{l\in\mathbb{Z}} c^r_{rl} \int_0^{+\infty} t^s 2^{2ls} e^{-ct2^{2l}} \frac{dt}{t} \\ &\leq C\Gamma(s)\|V\|^r_{B^{-2s}_{p,r}},\end{aligned}$$

with $\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt$. The proof of the Proposition 3.9 is now achieved. $\square$

Let us now prove L^∞ estimate for the density solution q of the system (3.84). We recall that in the sequel q will correspond roughly speaking to $\ln\rho$, in particular L^∞ estimate on q will provide L^∞ estimate on ρ and $\frac{1}{\rho}$.

Proposition 3.10. *Let $q_0 \in B^{\frac{N}{2}}_{2,2}$, $u_0 \in B^{N-1}_{2,2}$ and $q_0 \in L^\infty$. Let (q,u) be the solution of the system 3.84, then there exists $C>0$ such that for all $T>0$ we have:*

$$\sup_{t\in[0,T]} (\sqrt{t}\|(\nabla q,u)(t,\cdot)\|_{L^\infty}) \leq C\|(\nabla q_0,u_0)\|_{B^{\frac{N}{2}-1}_{2,2}}.$$

$$\|q\|_{L^\infty_T(L^\infty)} \leq \|q_0\|_{L^\infty} + C\|(\nabla q_0,u_0)\|_{B^{\frac{N}{2}-1}_{2,2}}. \tag{3.97}$$

Proof. The first estimate in (3.97) is a direct application of the Proposition 3.9 applied to $p=+\infty$, $r=+\infty$, $s=\frac{1}{2}$ and using the fact that $B^{\frac{N}{2}-1}_{2,2}$ is embedded in $B^{-1}_{\infty,\infty}$.

Let $\mathcal{E}$ be the fundamental solution of the Laplacian operator, and we define the operator $(\Delta)^{-1}$ by the convolution operator $(\Delta)^{-1}f=\mathcal{E}*f$ with $f\in\mathcal{S}'(\mathbb{R}^N)$. By applying the operator $(\Delta)^{-1}\mathrm{div}$ to the second equation of (3.84) and using the fact that $\Delta c=\mathrm{div}u$, we obtain the following system with $c=(\Delta)^{-1}\mathrm{div}u$:

$$\begin{cases}\partial_t q - \dfrac{\kappa}{\mu}\Delta q = -\dfrac{1}{\mu}\partial_t c,\\ \partial_t c - \mu\Delta c - \kappa\Delta q = 0.\end{cases}$$

Let us prove that q belongs to $L^\infty_T(L^\infty)$ for any $T>0$; from Duhamel formula we have:

$$q(t,x) = e^{\frac{\kappa}{\mu}t\Delta}q_0 - \frac{1}{\mu}\int_0^t e^{\frac{\kappa}{\mu}(t-s)\Delta}\partial_t c(s)ds. \tag{3.98}$$

According to the maximum principle for the heat equation, we deduce that:

$$\|e^{\frac{\kappa}{\mu}t\Delta}q_0\|_{L^\infty(L^\infty)} \leq \|q_0\|_{L^\infty(\mathbb{R}^N)}. \tag{3.99}$$

Next we are going to consider $\int_0^t e^{\frac{\kappa}{\mu}(t-s)\Delta}\partial_s c(s)ds$, we recall that:

$$\begin{aligned}
e^{\frac{\kappa}{\mu}(t-s)\Delta}\partial_s c(s) &= K(\frac{\cdot}{\sqrt{t-s}}) *_x \partial_s(\Delta)^{-1}\mathrm{div}u(s,\cdot)\\
&= K(\frac{\cdot}{\sqrt{t-s}}) *_x \big(\partial_s[(\mathcal{E} *_x \mathrm{div}u(s,\cdot)]\big)\\
&= K(\frac{x}{\sqrt{t-s}}) *_x \big([\sum_i \partial_i\mathcal{E} *_x \partial_s u_i(s,\cdot)]\big)\\
&= \sum_i \big(K(\frac{x}{\sqrt{t-s}}) *_x \partial_i\mathcal{E}\big) *_x \partial_s u_i(s,\cdot)]\big)\\
&= \sum_i \big(\partial_i[K(\frac{\cdot}{\sqrt{t-s}})] *_x \mathcal{E}\big) *_x \partial_s u_i(s,\cdot)\\
&= \sum_i \partial_i[K(\frac{\cdot}{\sqrt{t-s}})] *_x \big(\mathcal{E} *_x \partial_s u_i(s,\cdot)\big)\\
&= \sum_i \partial_i[K(\frac{\cdot}{\sqrt{t-s}})] *_x (\Delta)^{-1}\partial_s u_i(s,\cdot),
\end{aligned} \tag{3.100}$$

with $*_x$ the convolution in space and setting $\bar{\mu}=\frac{\kappa}{\mu}$:

$$K(\frac{x}{\sqrt{t}}) = \frac{1}{(4\pi\bar{\mu}t)^{\frac{N}{2}}}e^{-\frac{|x|^2}{4\bar{\mu}t}}.$$

We deduce that:

$$\partial_i[K(\frac{\cdot}{\sqrt{t-s}})](x) = \frac{-2x_i}{\pi^{\frac{N}{2}}}\frac{1}{(4\bar{\mu}(t-s))^{\frac{N}{2}+1}}e^{-\frac{|x|^2}{4\bar{\mu}(t-s)}}.$$

We easily check by a change of variable $u = \frac{x}{\sqrt{4\bar{\mu}(t-s)}}$ that:

$$\|\partial_i[K(\frac{\cdot}{\sqrt{t-s}})]\|_{L^1} \leq \frac{C}{\sqrt{t-s}}.$$

Then by Young's inequality we have for $0 < s < t$:

$$\|\partial_i[K(\frac{\cdot}{\sqrt{t-s}})] *_x (\Delta)^{-1}\partial_s u_i(s,\cdot)\|_{L^\infty_x} \leq \frac{C}{\sqrt{t-s}}\frac{1}{\sqrt{s}}\sup_{0<s<t}\sqrt{s}\|(\Delta)^{-1}\partial_s u(s)\|_{L^\infty} \tag{3.101}$$

with $C > 0$. Applying the operator $(\Delta)^{-1}$ to the second equation of (3.84) we have:

$$\partial_t(\Delta)^{-1}u = \mu u + \kappa\nabla q. \tag{3.102}$$

Using (3.100), (3.101) and (3.102), we observe that there exist $C, C_1 > 0$ such that:

$$\begin{aligned}\|\int_0^t e^{\frac{\kappa}{\mu}(t-s)\Delta}\partial_s c(s)ds\|_{L^\infty(L^\infty)} &\leq (\sup_{0<s<t}\sqrt{s}\|\Delta^{-1}\partial_s u(s)\|_{L^\infty})\int_0^t \frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s}}ds \\ &\leq C\sup_{0<s<t}\sqrt{s}\|\mu u(s,\cdot) + \kappa\nabla q(s,\cdot)\|_{L^\infty} \\ &\leq C_1\sup_{0<s<t}\sqrt{s}\|(u(s,\cdot),\nabla q(s,\cdot))\|_{L^\infty}.\end{aligned} \tag{3.103}$$

Combining the first estimate in (3.97) and (3.103), (3.99) we obtain that for any $T > 0$ we have:

$$\|q\|_{L^\infty_T(L^\infty)} \leq \|q_0\|_{L^\infty} + C\|(\nabla q_0, u_0)\|_{B^{\frac{N}{2}-1}_{2,2}},$$

with $C > 0$. It achieves the proof of the Proposition 3.10. □

3.4.2. The unique solution of (1.6) verifies $\rho \in L^\infty_T(L^\infty)$

Regularizing effect on the third index of the Besov spaces

Let us start by recalling that we are concerned now with initial data such that (q_0, u_0) belong to $(B^{\frac{N}{2}}_{2,2} \cap L^\infty) \times B^{\frac{N}{2}-1}_{2,2}$. This additional regularity assumptions on the initial data will be crucial in order to prove L^∞ estimates on $\ln\rho$. Let us start by proving additional regularity assumption on the solution (q, u) of the system (1.6) which verify $(q, u) \in E$ with E defined as follows:

$$E = \big(\widetilde{L}^\infty(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty}) \cap \widetilde{L}^1(\mathbb{R}^+, \widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty}\big) \times \big(\widetilde{L}^\infty(\mathbb{R}^+, B^{\frac{N}{2}-1}_{2,\infty}) \cap \widetilde{L}^1(\mathbb{R}^+, B^{\frac{N}{2}+1}_{2,\infty})\big)^N.$$

Indeed we are interested in splitting the unique solution (q, u) constructed in the subsection 3.3 as the following sum:

$$(q, u) = (q_L, u_L) + (\bar{q}, \bar{u}),$$

with (q_L, u_L) the solution of the system $(N1)$ with $F = G = 0$ $a = \mu$, $b = \lambda + \mu$, $c = \kappa$, $d = K$ and with initial data $(\ln \rho_0, u_0)$. Compared with the subsection 3.3, we are going to show that the remainder term $(\bar{q}, \bar{u})$ is more regular than (q_L, u_L) and is in the space F defined as follows:

$$F = \big(\widetilde{L}^\infty(\mathbb{R}^+, \widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}}) \cap \widetilde{L}^1(\mathbb{R}^+, \widetilde{B}_{2,1}^{\frac{N}{2}+1,\frac{N}{2}+2})\big) \times \big(\widetilde{L}^\infty(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}-1} \cap \widetilde{L}^1(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}+1}))\big)^N.$$

This type of regularizing effect on the remainder $(\bar{q}, \bar{u})$ in term of the third index on the Besov spaces has been observed for the first time by M. Cannone and F. Planchon in [9] for the incompressible Navier–Stokes equations. We are going to use a similar type of ideas in your case, to do this let us use the Proposition 3.7 which ensures:

$$\begin{aligned}
&\|q_L\|_{\widetilde{L}^\infty(\mathbb{R}^+, B_{2,2}^{\frac{N}{2}-1}\cap B_{2,2}^{\frac{N}{2}})} + \|q_L\|_{\widetilde{L}^1(\mathbb{R}^+, B_{2,2}^{\frac{N}{2}+1}\cap B_{2,2}^{\frac{N}{2}+2})} + \|u_L\|_{\widetilde{L}^\infty(\mathbb{R}^+, B_{2,2}^{\frac{N}{2}-1})} \\
&\quad + \|u_L\|_{\widetilde{L}^1(\mathbb{R}^+, B_{2,2}^{\frac{N}{2}+1})} \lesssim \|q_0\|_{\widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}}} + \|u_0\|_{B_{2,2}^{\frac{N}{2}-1}}.
\end{aligned} \tag{3.104}$$

As in the subsection 3.3, we know that $(\bar{q}, \bar{u})$ verify the following system:

$$\begin{cases}
\partial_t \bar{q} + \mathrm{div} \bar{u} = F(\bar{q}, \bar{u}), \\
\partial_t \bar{u} - \mu\Delta\bar{u} - (\lambda+\mu)\nabla\mathrm{div}\bar{u} - \kappa\nabla\Delta\bar{q} + K\nabla q = G(\bar{q}, \bar{u}), \\
(\bar{q}, \bar{u})(0, \cdot) = (0, 0),
\end{cases} \tag{3.105}$$

with:

$$\begin{aligned}
&F(\bar{q}, \bar{u}) = -u \cdot \nabla q, \\
&G(\bar{q}, \bar{u}) = -u \cdot \nabla u + 2\mu\nabla q \cdot Du + \lambda \mathrm{div} u \, \nabla q + \frac{\kappa}{2}\nabla(|\nabla q|^2).
\end{aligned}$$

We are going to apply the Proposition 3.7 to $(\bar{q}, \bar{u})$, it implies that:

$$\begin{aligned}
&\|\bar{q}\|_{\widetilde{L}^\infty(\mathbb{R}^+, \widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})} + \|\bar{q}\|_{\widetilde{L}^1(\mathbb{R}^+, \widetilde{B}_{2,1}^{\frac{N}{2}+1,\frac{N}{2}+2})} + \|\bar{u}\|_{\widetilde{L}^\infty(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}-1})} + \|\bar{u}\|_{\widetilde{L}^1(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}+1})} \\
&\quad \lesssim \|F(\bar{q}, \bar{u})\|_{\widetilde{L}^1(\mathbb{R}^+, \widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})} + \|G(\bar{q}, \bar{u})\|_{\widetilde{L}^1(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}-1})}.
\end{aligned} \tag{3.106}$$

It remains only to bound the terms $F(\bar{q}, \bar{u})$ and $G(\bar{q}, \bar{u})$ on the right hand side of (3.106). Let us start with $F(\bar{q}, \bar{u}) = -u \cdot \nabla q$ that we can rewrite as follows:

$$u \cdot \nabla q = \bar{u} \cdot \nabla q + u_L \cdot \nabla \bar{q} + u_L \cdot \nabla q_L.$$

According to Proposition 2.2 we have:

$$\begin{aligned}
&\|\bar{u} \cdot \nabla q\|_{\widetilde{L}^1_T(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})} \lesssim \|\nabla q\|_{\widetilde{L}^{\frac{4}{3}}_T(\widetilde{B}_{2,\infty}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}})} \|\bar{u}\|_{\widetilde{L}^4_T(B_{2,1}^{\frac{N}{2}-\frac{1}{2}})} + \|\nabla q\|_{\widetilde{L}^4_T(\widetilde{B}_{2,\infty}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}})} \|\bar{u}\|_{\widetilde{L}^{\frac{4}{3}}_T(B_{2,1}^{\frac{N}{2}+\frac{1}{2}})}, \\
&\|u_L \cdot \nabla \bar{q}\|_{\widetilde{L}^1_T(B_{2,1}^{\frac{N}{2}})} \lesssim \|\nabla \bar{q}\|_{\widetilde{L}^{\frac{4}{3}}_T(\widetilde{B}_{2,1}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}})} \|u_L\|_{\widetilde{L}^4_T(B_{2,\infty}^{\frac{N}{2}-\frac{1}{2}})} + \|\nabla \bar{q}\|_{\widetilde{L}^4_T(\widetilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}})} \|u_L\|_{\widetilde{L}^{\frac{4}{3}}_T(B_{2,\infty}^{\frac{N}{2}+\frac{1}{2}})}, \\
&\|u_L \cdot \nabla q_L\|_{\widetilde{L}^1_T(B_{2,1}^{\frac{N}{2}})} \lesssim \|\nabla q_L\|_{\widetilde{L}^{\frac{4}{3}}_T(\widetilde{B}_{2,2}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}})} \|u_L\|_{\widetilde{L}^4_T(B_{2,2}^{\frac{N}{2}-\frac{1}{2}})} + \|\nabla q_L\|_{\widetilde{L}^4_T(\widetilde{B}_{2,2}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}})} \|u_L\|_{\widetilde{L}^{\frac{4}{3}}_T(B_{2,2}^{\frac{N}{2}+\frac{1}{2}})}.
\end{aligned} \tag{3.107}$$

Let us deal now with the term $G(\bar{q}, \bar{u})$ and in particular the term $u \cdot \nabla u$, we have then:

$$u \cdot \nabla u = \bar{u} \cdot \nabla u + u_L \cdot \nabla \bar{u} + u_L \cdot \nabla u_L.$$

From the Proposition 2.2, it yields:

$$\begin{aligned}
&\|\bar{u}\cdot\nabla u\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\lesssim\|\bar{u}\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,1})}\|\nabla u\|_{\widetilde{L}^1(B^{\frac{N}{2}}_{2,\infty})}+\|\nabla u\|_{\widetilde{L}^2(B^{\frac{N}{2}-1}_{2,\infty})}\|\bar{u}\|_{\widetilde{L}^2(B^{\frac{N}{2}}_{2,1})},\\
&\|u_L\cdot\nabla\bar{u}\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\lesssim\|u_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,\infty})}\|\nabla\bar{u}\|_{\widetilde{L}^1(B^{\frac{N}{2}}_{2,1})}+\|\nabla\bar{u}\|_{\widetilde{L}^2(B^{\frac{N}{2}-1}_{2,1})}\|u_L\|_{\widetilde{L}^2(B^{\frac{N}{2}}_{2,\infty})},\\
&\|u_L\cdot\nabla u_L\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\lesssim\|u_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,2})}\|\nabla u_L\|_{\widetilde{L}^1(B^{\frac{N}{2}}_{2,2})}+\|\nabla u_L\|_{\widetilde{L}^2(B^{\frac{N}{2}-1}_{2,2})}\|u_L\|_{\widetilde{L}^2(B^{\frac{N}{2}}_{2,2})}.
\end{aligned}\tag{3.108}$$

We can proceed similarly for the terms $\nabla q\cdot Du$ and $\mathrm{div}u\,\nabla q$. Let us treat the last term $\nabla|\nabla q|^2$ we have then:

$$|\nabla q|^2=\nabla q\cdot\nabla\bar{q}+\nabla\bar{q}\cdot\nabla q_L+|\nabla q_L|^2.$$

By Proposition 2.2, we get:

$$\begin{aligned}
&\|\nabla(|\nabla q|^2)\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\lesssim\||\nabla q|^2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}}_{2,1})},\\
&\||\nabla q_L|^2\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\lesssim\|\nabla q_L\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac12}_{2,2})}\|\nabla q_L\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac12}_{2,2})},\\
&\|\nabla q\cdot\nabla\bar{q}\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\lesssim\|\nabla q\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac12}_{2,\infty})}\|\nabla\bar{q}\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac12}_{2,1})}+\|\nabla\bar{q}\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{2}+\frac12}_{2,1})}\|\nabla q\|_{\widetilde{L}^4_T(B^{\frac{N}{2}-\frac12}_{2,\infty})}.
\end{aligned}\tag{3.109}$$

By collecting the estimates (3.107), (3.108) and (3.109) and by interpolation we obtain that there exists $C>0$ such that:

$$\begin{aligned}
&\|\bar{q}\|_{\widetilde{L}^\infty(\mathbb{R}^+,B^{\frac{N}{2}-1}_{2,1}\cap B^{\frac{N}{2}}_{2,1})}+\|\bar{q}\|_{\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}+1}_{2,1}\cap B^{\frac{N}{2}+2}_{2,1})}+\|\bar{u}\|_{\widetilde{L}^\infty(\mathbb{R}^+,B^{\frac{N}{2}-1}_{2,1})}+\|\bar{u}\|_{\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}+1}_{2,1})}\\
&\le C\big(\|u_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,\infty})}+\|q_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})}+\|u_L\|_{\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,\infty})}+\|q_L\|_{\widetilde{L}^1(B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty})}\\
&+\|u\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,\infty})}+\|q\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})}+\|u\|_{\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,\infty})}+\|q\|_{\widetilde{L}^\infty(B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty})}\big)\\
&\times\big(\|\bar{q}\|_{\widetilde{L}^\infty(\mathbb{R}^+,B^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}+\|\bar{q}\|_{\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,1})}+\|\bar{u}\|_{\widetilde{L}^\infty(\mathbb{R}^+,B^{\frac{N}{2}-1}_{2,1})}+\|\bar{u}\|_{\widetilde{L}^1(\mathbb{R}^+,B^{\frac{N}{2}+1}_{2,1})}\big)\\
&+C\big(\|u_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,2})}+\|q_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,2})}+\|u_L\|_{\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,2})}+\|q_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,2})}\big)^2.
\end{aligned}\tag{3.110}$$

Let us recall that via the first part of the Theorem 1.1 and the Proposition 3.7 (see the proof in the subsection 3.3) we have for a $C>0$ and $M>0$ large enough:

$$\begin{aligned}
&\|u_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,\infty})}+\|q_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})}+\|u_L\|_{\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,\infty})}+\|q_L\|_{\widetilde{L}^\infty(B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty})}\le C\varepsilon_0,\\
&\|u\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1}_{2,\infty})}+\|q\|_{\widetilde{L}^\infty(B^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty})}+\|u\|_{\widetilde{L}^1(B^{\frac{N}{2}+1}_{2,\infty})}+\|q\|_{\widetilde{L}^\infty(B^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty})}\le M\varepsilon_0,
\end{aligned}$$

with:

$$\varepsilon_0=\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty}}+\|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}}.\tag{3.111}$$

By choosing ε_0 small enough we can apply a bootstrap argument in (3.110) which implies that there exists $C>0$ such that

$$\begin{aligned}
&\|\bar{q}\|_{\widetilde{L}^{\infty}(\mathbb{R}^+,B_{2,1}^{\frac{N}{2}-1}\cap B_{2,1}^{\frac{N}{2}})}+\|\bar{q}\|_{\widetilde{L}^{1}(\mathbb{R}^+,B_{2,1}^{\frac{N}{2}+1}\cap B_{2,1}^{\frac{N}{2}+2})}+\|\bar{u}\|_{\widetilde{L}^{\infty}(\mathbb{R}^+,B_{2,1}^{\frac{N}{2}-1})}+\|\bar{u}\|_{\widetilde{L}^{1}(\mathbb{R}^+,B_{2,1}^{\frac{N}{2}+1})}\\
&\leq C\big(\|u_L\|_{\widetilde{L}^{\infty}(B_{2,2}^{\frac{N}{2}-1})}+\|q_L\|_{\widetilde{L}^{\infty}(B_{2,2}^{\frac{N}{2}-1,\frac{N}{2}})}+\|u_L\|_{\widetilde{L}^{1}(B_{2,2}^{\frac{N}{2}+1})}+\|q_L\|_{\widetilde{L}^{\infty}(B_{2,2}^{\frac{N}{2}+1,\frac{N}{2}+2})}\big)^2.
\end{aligned}\tag{3.112}$$

It proves finally that $(\bar{q},\bar{u})$ is in F and concludes this subsection.

q is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$ for any $T>0$

In order to show that q is bounded in $L^\infty_T(L^\infty(\mathbb{R}^N))$ for any $T>0$ we shall use the Proposition 3.10. Indeed we have seen in the previous subsection that:

$$(q,u)=(q_L,u_L)+(\bar{q},\bar{u}),$$

with (q_L,u_L) solution of the system $(N1)$ with $F=G=0$ $a=\mu$, $b=\lambda+\mu$, $c=\kappa$, $d=K$ and with initial data $(\ln\rho_0,u_0)$. In particular it implies via the definition of the semi-group $B(t)$ in the Proposition 3.9 and the Duhamel formula that:

$$(q_L(t),u_L(t))=e^{B(t)}(q_0,u_0)+\int_0^t e^{B(t-s)}(0,K\nabla q_L)(s)ds.$$

By using Propositions 3.6, 3.10 and the embedding of $B_{2,1}^{\frac{N}{2}}$ in L^∞ we deduce that for any $T>0$ there exists $C>0$ independent on T such that (here $[\cdot]_1$ defines the first coordinate of the vector field):

$$\begin{aligned}
\|q_L\|_{L^\infty_T(L^\infty)}&\leq\|[e^{B(t)}(q_0,u_0)]_1\|_{L^\infty_T(L^\infty)}+\|[\int_0^t e^{B(t-s)}(0,K\nabla q_L)(s)ds]_1\|_{\widetilde{L}^\infty_T(B_{2,1}^{\frac{N}{2}})}\\
&\leq C(\|\ln\rho_0\|_{L^\infty}+\|(\nabla q_0,u_0)\|_{B_{2,2}^{\frac{N}{2}-1}}+\|\nabla q_L\|_{\widetilde{L}^1_T(B_{2,1}^{\frac{N}{2}-1})}).
\end{aligned}\tag{3.113}$$

From Proposition 3.7, we know that there exists $C>0$ such that:

$$\|q_L\|_{\widetilde{L}^\infty(\widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}})}+\|q_L\|_{\widetilde{L}^1(\widetilde{B}_{2,2}^{\frac{N}{2}+1,\frac{N}{2}+2})}\leq C(\|q_0\|_{\widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}}}+\|u_0\|_{B_{2,2}^{\frac{N}{2}-1}}).\tag{3.114}$$

Let us deal now with the term $\|\nabla q_L\|_{\widetilde{L}^1_T(B_{2,1}^{\frac{N}{2}-1})}$ on the right hand side of (3.113), we have by interpolation for a constant $C>0$:

$$\|q_L(t)\|_{B_{2,1}^{\frac{N}{2}}}\leq C\|q_L(t)\|^{\frac{1}{2}}_{B_{2,2}^{\frac{N}{2}-1}}\|q_L(t)\|^{\frac{1}{2}}_{B_{2,2}^{\frac{N}{2}+1}}.$$

It implies that by Hölder's inequality and (2.25) that there exist $C,C_1>0$ large enough such that:

$$\begin{aligned}
\|q_L(t)\|_{L^1_T(B_{2,1}^{\frac{N}{2}})}&\leq C\|q_L\|^{\frac{1}{2}}_{L^1_T(B_{2,2}^{\frac{N}{2}-1})}\|q_L\|^{\frac{1}{2}}_{L^1_T(B_{2,2}^{\frac{N}{2}+1})}\\
&\leq CT^{\frac{3}{4}}\|q_L\|^{\frac{1}{2}}_{L^\infty_T(B_{2,2}^{\frac{N}{2}-1})}\|q_L\|^{\frac{1}{2}}_{L^2_T(B_{2,2}^{\frac{N}{2}+1})}\\
&\leq C_1T^{\frac{3}{4}}(\|q_0\|_{\widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}}}+\|u_0\|_{B_{2,2}^{\frac{N}{2}-1}}).
\end{aligned}\tag{3.115}$$

Combining (3.113) and (3.115) we get:

$$\|q_L\|_{L^\infty_T(L^\infty)} \leq \|[e^{B(t)}(q_0,u_0)]_1\|_{L^\infty_T(L^\infty)} + \|[\int_0^t e^{B(t-s)}(0,K\nabla q_L)(s)ds]_1\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}}_{2,1})}$$
$$\leq C(\|\ln\rho_0\|_{L^\infty} + \|(\nabla q_0,u_0)\|_{B^{\frac{N}{2}-1}_{2,2}} + C_1 T^{\frac{3}{4}}(\|q_0\|_{\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,2}} + \|u_0\|_{B^{\frac{N}{2}-1}_{2,2}}). \tag{3.116}$$

We have then proved that q_L belongs to $L^\infty_{loc}(L^\infty)$. Since we have seen that $(\bar{q},\bar{u})$ is in F, it implies in particular that $\bar{q}$ belongs to $\widetilde{L}^\infty(B^{\frac{N}{2}}_{2,1})$ which is embedded in $L^\infty(L^\infty)$. We deduce that $q = q_L + \bar{q}$ belongs to $L^\infty_{loc}(L^\infty)$.

It remains only to prove that $(\rho,u) = (\exp(q),u)$ is a global strong solution of (1.1) and we define $h = \exp(q) - 1$ with $\rho = 1 + h$. By Proposition 2.3 and the fact that ρ and $\frac{1}{\rho}$ belong to $L^\infty_{loc}(L^\infty)$ we easily show that h is for any $T > 0$ in:

$$H_T = \big(\widetilde{L}^\infty_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,\infty}) \cap \widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}+1,\frac{N}{2}+2}_{2,\infty}\big).$$

With such regularity on h and u, the verification that (ρ,u) is a solution of (1.1) in the sense of distribution is a straightforward application of Propositions 2.2 and 2.3. The uniqueness follows also the same line than in the proof of subsection 3.2. □

3.5. Proof of the Theorem 1.2

In the Theorem 1.2 we are interested in extending the results of Theorem 1.1 to the case of general pressure P and also to the case of constant viscosity and capillary coefficients. It is worth pointing out that in this case some terms as the pressure $\nabla P(\rho)$ are nonlinear in terms of the density $q = \ln\rho$; that is why it is crucial to control the L^∞ norm of the density in order to estimate this term in Besov space via composition lemma. For this reason it seems delicate to show a result of global strong solution involving only a smallness hypothesis on $\|q_0\|_{B^{\frac{N}{2}}_{2,\infty}}$ and $\|u_0\|_{B^{\frac{N}{2}-1}_{2,\infty}}$ as it is the case in the Theorem 1.1.

We are now going to explain how to adapt the previous arguments of the proof of Theorem 1.1 to this new situation. We are only dealing with the case $\mu(\rho) = \mu$, $\lambda(\rho) = \lambda$, $\kappa(\rho) = \kappa$ and P a regular function such that $P'(1) > 0$ (the viscosity and capillary coefficients verify here the conditions $\mu > 0$, $2\mu + \lambda > 0$ and $\kappa > 0$). Let us mention that the other case can be handled in a similar way. The system 1.1 is equivalent to the following system where we set in this section $\rho = 1 + h$. Let us mention that in this case a straightforward calculus gives:

$$\mathrm{div}K = \kappa\rho\nabla\Delta\rho.$$

We can then rewrite the system (1.1) as follows:

$$\begin{cases} \partial_t h + \mathrm{div}u = F(h,u), \\ \partial_t u - \mu\Delta u - (\mu+\lambda)\nabla\mathrm{div}u + F'(1)\nabla h - \kappa\nabla\Delta h = G(h,u), \\ (h,u)_{\ t=0} = (h_0,u_0), \end{cases} \tag{3.117}$$

with $F'(\rho) = \frac{P'(\rho)}{\rho}$ and:

$$F(h,u) = -u\cdot\nabla h - h\mathrm{div}u,$$
$$G(h,u) = (F'(1) - F'(\rho))\nabla h + (\frac{\mu}{\rho} - \mu)\Delta u + (\frac{\mu+\lambda}{\rho} - \mu - \lambda)\nabla\mathrm{div}u - u\cdot\nabla u. \tag{3.118}$$

Let (h_L, u_L) be the solution of $(N1)$ with $a = \mu$, $b = \lambda + \mu$, $c = \kappa$, $d = P'(1)$, $F = G = 0$ and the initial data (h_0, u_0). In the sequel we will consider solution under the form:

$$(h, u) = (h_L, u_L) + (\bar{h}, \bar{u}).$$

In order to prove Theorem 1.2 we shall use a fixed point theorem, more precisely we are going to consider the following functional:

$$\psi(\bar{h}, \bar{u}) = \int_0^t W(t-s) \begin{pmatrix} F(h,u) \\ G(h,u) \end{pmatrix} ds\,. \tag{3.119}$$

$W(t)$ is the semi-group associated with $(N1)$ with $a = \mu$, $b = \lambda + \mu$, $c = \kappa$, $d = P'(1)$. The proof is divided in two steps the stability of ψ for a ball $B(0, R)$ in $E^{\frac{N}{2}}$ defined below and the contraction property. We consider $E^{\frac{N}{2}}$ defined as follows:

$$E^{\frac{N}{2}} = \big(\widetilde{L}^\infty(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}}) \cap \widetilde{L}^1(\widetilde{B}_{2,1}^{\frac{N}{2}+1,\frac{N}{2}+2})\big) \times \big(\widetilde{L}^\infty(B_{2,1}^{\frac{N}{2}-1}) \cap \widetilde{L}^1(B_{2,1}^{\frac{N}{2}+1})\big)^N.$$

1) First step, stability of $B(0, R)$

By using the Proposition 3.7 we have: for $C > 0$:

$$\begin{aligned} &\|h_L\|_{\widetilde{L}^\infty_T(\widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}})} + \|h_L\|_{\widetilde{L}^1_T(\widetilde{B}_{2,2}^{\frac{N}{2}+1,\frac{N}{2}+2})} + \|u_L\|_{\widetilde{L}^\infty_T(B_{2,2}^{\frac{N}{2}-1})} + \|u_L\|_{\widetilde{L}^1_T(B_{2,2}^{\frac{N}{2}+1})} \\ &\leq C(\|h_0\|_{\widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}}} + \|u_0\|_{B_{2,2}^{\frac{N}{2}-1}}). \end{aligned} \tag{3.120}$$

By combining the Propositions 3.10 and 3.7, we have for any $T > 0$ and $C > 0$ independent on T:

$$\|h_L\|_{L^\infty_T(L^\infty)} \leq C(\|h_0\|_{L^\infty} + \|h_0\|_{B_{2,1}^{\frac{N}{2}-2}} + \|u_0\|_{B_{2,1}^{\frac{N}{2}-2}}). \tag{3.121}$$

Remark 20. Let us mention that the condition $(h_0, u_0) \in B_{2,1}^{\frac{N}{2}-2} \times (B_{2,1}^{\frac{N}{2}-2})^N$ plays only a role in the previous estimate (3.121) in order to bound the term $P'(1)\nabla h_L$ that we consider as a remainder term for the system (N). Indeed we are interested in applying Proposition 3.10 and the Duhamel formula. It would be possible to avoid this additional regularity by extending the Proposition 3.10 to the system $(N1)$.

It remains only to apply the Proposition 3.7 in order to get a priori estimates on $(\bar{h}, \bar{u})$, we have then for $C > 0$:

$$\|\psi(\bar{h}, \bar{u})\|_{E^{\frac{N}{2}}} \leq C(\|F(h,u)\|_{\widetilde{L}^1_T(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})} + \|G(h,u)\|_{\widetilde{L}^1_T(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})}). \tag{3.122}$$

By applying Proposition 2.2 and Lemma 2.3 (indeed we have seen via the estimate (3.120) and the fact that $\bar{h}$ is in $\widetilde{L}^\infty(B_{2,1}^{\frac{N}{2}})$ that h belongs to L^∞ and $\rho = 1 + h \geq c > 0$) we have as in (3.110) for a function $C > 0$:

$$\begin{aligned} &\|\psi(\bar{h}, \bar{u})\|_{E^{\frac{N}{2}}} \leq C(\|h\|_{L^\infty})\big(\|u_L\|_{\widetilde{L}^\infty(B_{2,\infty}^{\frac{N}{2}-1})} + \|h_L\|_{\widetilde{L}^\infty(B_{2,\infty}^{\frac{N}{2}-1,\frac{N}{2}})} + \|h_L\|_{\widetilde{L}^1(B_{2,\infty}^{\frac{N}{2}+1})} \\ &+ \|h_L\|_{\widetilde{L}^1(B_{2,\infty}^{\frac{N}{2}+1,\frac{N}{2}+2})} + \|u\|_{\widetilde{L}^\infty(B_{2,\infty}^{\frac{N}{2}-1})} + \|h\|_{\widetilde{L}^\infty(B_{2,\infty}^{\frac{N}{2}-1,\frac{N}{2}})} + \|u\|_{\widetilde{L}^1(B_{2,\infty}^{\frac{N}{2}+1})} + \|h\|_{\widetilde{L}^\infty(B_{2,\infty}^{\frac{N}{2}+1,\frac{N}{2}+2})}\big) \\ &\times (\|\bar{h}\|_{\widetilde{L}^\infty(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})} + \|\bar{h}\|_{\widetilde{L}^1(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}+1,\frac{N}{2}+2})} + \|\bar{u}\|_{\widetilde{L}^\infty(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}-1})} + \|\bar{u}\|_{\widetilde{L}^1(\mathbb{R}^+, B_{2,1}^{\frac{N}{2}+1})}\big) \\ &+ C(\|h\|_{L^\infty})\big(\|u_L\|_{\widetilde{L}^\infty(B_{2,2}^{\frac{N}{2}-1})} + \|h_L\|_{\widetilde{L}^\infty(B_{2,2}^{\frac{N}{2}-1,\frac{N}{2}})} + \|u_L\|_{\widetilde{L}^1(B_{2,2}^{\frac{N}{2}+1})} + \|h_L\|_{\widetilde{L}^\infty(B_{2,2}^{\frac{N}{2}+1,\frac{N}{2}+2})}\big)^2. \end{aligned} \tag{3.123}$$

We set

$$\varepsilon_0 = \|h_0\|_{\widetilde{B}_{2,2}^{\frac{N}{2}-1,\frac{N}{2}}} + \|u_0\|_{B_{2,2}^{\frac{N}{2}-1}} + \|h_0\|_{L^\infty} + \|h_0\|_{B_{2,1}^{\frac{N}{2}-2}} + \|u_0\|_{B_{2,1}^{\frac{N}{2}-2}}. \tag{3.124}$$

By choosing $R = M\varepsilon_0$ with $M > 2$, it suffices to choose ε_0 small enough such that (3.123) ensures that:

$$\|\psi(\bar{h},\bar{u})\|_{E^{\frac{N}{2}}} \leq R.$$

The proof of the contraction follows the same lines as in the proof of the Theorem 1.1 and is left to the reader. It concludes the proof of the Theorem 1.2. □

4. Proof of the Theorem 1.3

We now wish to investigate the proof of the Theorem 1.3 where we assume that $\kappa(\rho) = \frac{\mu^2}{\rho}$, $\mu(\rho) = \mu\rho$ and $\lambda(\rho) = 0$. We have observed that there exists quasi-solution for the system (1.1), it means a solution of the form $(\rho_1, -\mu\nabla \ln \rho_1)$ with:

$$\begin{cases} \partial_t \rho_1 - \mu\Delta\rho_1 = 0, \\ (\rho_1)_{t=0} = (\rho_1)_0. \end{cases}$$

This quasi-solution $(\rho_1, -\mu\nabla \ln \rho_1)$ verifies the following system:

$$\begin{cases} \dfrac{\partial}{\partial t}\rho + \mathrm{div}(\rho u) = 0, \\ \dfrac{\partial}{\partial t}(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu\rho D(u)) - \mu^2 \rho\nabla\Delta \ln \rho - \dfrac{\mu^2}{2}\nabla(|\nabla \ln \rho|^2) = 0, \\ (\rho, u)_{t=0} = \big((\rho_1)_0, -\mu\nabla \ln(\rho_1)_0\big). \end{cases} \tag{4.125}$$

Let us work now around this quasi-solution, more precisely we search global solution of (1.1) under the form:

$$q = \ln \rho = \ln \rho_1 + h_2 \;\text{ with } \rho_1 = 1 + h_1 \;\text{ and }\; u = -\mu\nabla \ln \rho_1 + u_2.$$

We deduce from (1.11) that (h_2, u_2) verifies the following system:

$$\begin{cases} \partial_t h_2 + \mathrm{div} u_2 - \mu\nabla \ln \rho_1 \cdot \nabla h_2 + u_2 \cdot \nabla \ln \rho_1 = F(h_2, u_2), \\ \partial_t u_2 - \mu\Delta u_2 - \mu\nabla\mathrm{div} u_2 - \kappa\nabla\Delta h_2 + K\nabla h_2 - 2\mu\nabla \ln \rho_1 \cdot Du_2 - 2\mu\nabla h_2 \cdot Du_1 \\ \quad + u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 - \mu^2\nabla(\nabla \ln \rho_1 \cdot \nabla h_2) = G(h_2, u_2), \\ (h_2(0,\cdot), u_2(0,\cdot)) = (h_0^2, u_0^2), \end{cases} \tag{4.126}$$

with:

$$\begin{aligned} &F(h_2, u_2) = -u_2 \cdot \nabla h_2, \\ &G(h_2, u_2) = -u_2 \cdot \nabla u_2 + 2\mu\nabla h_2 \cdot Du_2 - K\nabla \ln \rho_1 + \frac{\mu^2}{2}\nabla(|\nabla h_2|^2). \end{aligned} \tag{4.127}$$

We have now to solve the previous system (4.126) and to do this we are going to apply a fixed point theorem. We start by defining the following map ψ:

$$\psi(h,u) = W_1(t,\cdot) * \begin{pmatrix} h_0^2 \\ u_0^2 \end{pmatrix} + \int_0^t W_1(t-s) \begin{pmatrix} F(h,u) \\ G(h,u) \end{pmatrix} ds\,, \tag{4.128}$$

where W_1 is the semi-group associated with the following linear system (4.129):

$$\begin{cases} \partial_t h_2 + \mathrm{div} u_2 - \mu \nabla \ln \rho_1 \cdot \nabla h_2 + u_2 \cdot \nabla \ln \rho_1 = 0, \\ \partial_t u_2 - \mu \Delta u_2 - \mu \nabla \mathrm{div} u_2 - \kappa \nabla \Delta h_2 + K \nabla h_2 - 2\mu \nabla \ln \rho_1 \cdot Du_2 - 2\mu \nabla h_2 \cdot Du_1 \\ \qquad + u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 - \mu^2 \nabla (\nabla \ln \rho_1 \cdot \nabla h_2) = 0, \\ (h_2(0,\cdot), u_2(0,\cdot)) = (h_0^2, u_0^2). \end{cases} \tag{4.129}$$

The nonlinear terms F, G are defined in (4.127). Let us now check the stability of ψ for a ball $B(0,R)$ in $E^{\frac{N}{2}}$ (the contraction property will follow the same lines). $E^{\frac{N}{2}}$ is defined as follows:

$$E^{\frac{N}{2}} = \big(\widetilde{L}^\infty(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}}) \cap \widetilde{L}^1(\widetilde{B}_{2,1}^{\frac{N}{2}+1,\frac{N}{2}+2})\big) \times \big(\widetilde{L}^\infty(B_{2,1}^{\frac{N}{2}-1}) \cap \widetilde{L}^1(B_{2,1}^{\frac{N}{2}+1})\big)^N.$$

1) First step, stability of $B(0,R)$

Let:

$$\eta = \|h_0^2\|_{\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}}} + \|u_0^2\|_{B_{2,1}^{\frac{N}{2}-1}}\,.$$

We are going to show that ψ maps the ball $B(0,R)$ into itself if R is small enough. By using the Proposition 3.8 we have:

$$\begin{aligned} \|\psi(h,u)\|_{E^{\frac{N}{2}}} \leq\, & C\big(\eta + \|F(h,u)\|_{\widetilde{L}^1_T(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})} + \|G(h,u)\|_{\widetilde{L}^1_T(B_{2,1}^{\frac{N}{2}-1})}\big) \\ & \times \exp\bigg(C \int_{\mathbb{R}^+} \big(\|u_1\|^4_{B_{2,\infty}^{\frac{N}{2}-\frac{1}{2}}} + \|u_1\|^{\frac{4}{3}}_{B_{2,\infty}^{\frac{N}{2}+\frac{1}{2}}} + \|\nabla \ln \rho_1\|^{\frac{4}{3}}_{B_{2,\infty}^{\frac{N}{2}+\frac{1}{2}}} + \|\nabla \ln \rho_1\|^{\frac{4}{3}}_{B_{2,\infty}^{\frac{N}{2}+\frac{1}{2}}}\big)(s)ds\bigg). \end{aligned} \tag{4.130}$$

The main task consists in using the Proposition 2.2 and Corollary 2 to obtain estimates on

$$\|F(h,u)\|_{\widetilde{L}^1(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})},\ \|G(h,u)\|_{\widetilde{L}^1(B_{2,1}^{\frac{N}{2}-1})}.$$

Let us first estimate $\|F(h,u)\|_{\widetilde{L}^1(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})}$. According to Proposition 2.2, we have:

$$\begin{aligned} \|u\cdot\nabla h\|_{\widetilde{L}^1(\widetilde{B}_{2,1}^{\frac{N}{2}-1,\frac{N}{2}})} \lesssim & \|\nabla h\|_{\widetilde{L}^{\frac{4}{3}}(\widetilde{B}_{2,1}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}})} \|u\|_{\widetilde{L}^4(B_{2,1}^{\frac{N}{2}-\frac{1}{2}})} \\ & + \|\nabla h\|_{\widetilde{L}^4(\widetilde{B}_{2,1}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}})} \|u\|_{\widetilde{L}^{\frac{4}{3}}(B_{2,1}^{\frac{N}{2}+\frac{1}{2}})}. \end{aligned} \tag{4.131}$$

Similarly for $\|G(h,u)\|_{\widetilde{L}^1_T(B_{2,1}^{\frac{N}{2}-1})}$ we have:

$$\|u\cdot\nabla u\|_{\widetilde{L}^1(B_{2,1}^{\frac{N}{2}-1})} \lesssim \|u\|_{\widetilde{L}^{\frac{4}{3}}(B_{2,1}^{\frac{N}{2}+\frac{1}{2}})} \|\nabla u\|_{\widetilde{L}^4(B_{2,1}^{\frac{N}{2}-\frac{3}{2}})} + \|u\|_{\widetilde{L}^4(B_{2,1}^{\frac{N}{2}-\frac{1}{2}})} \|\nabla u\|_{\widetilde{L}^{\frac{4}{3}}(B_{2,1}^{\frac{N}{2}-\frac{1}{2}})}. \tag{4.132}$$

The most important term is certainly $K\nabla \ln \rho_1$ which belongs to $\widetilde{L}^1(B_{2,1}^{\frac{N}{2}-1})$ since we have assumed that $\ln \rho_1^0$ belongs to $B_{2,1}^{\frac{N}{2}-2} \cap L^\infty$ and the fact that ρ_1 verifies a heat equation (1.13). Using Propositions 2.3, 2.4 and the maximum principle there exists a regular function g such that:

$$\begin{aligned}\|\nabla \ln \rho_1\|_{\widetilde{L}^1(B^{\frac{N}{2}-1}_{2,1})} &\lesssim \|\ln \rho_1\|_{\widetilde{L}^1(B^{\frac{N}{2}}_{2,1})}\\ &\lesssim g(\|(\rho_1, \frac{1}{\rho_1})\|_{L^\infty})\|h_1\|_{\widetilde{L}^1(B^{\frac{N}{2}}_{2,1})}\\ &\lesssim g(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}}. \end{aligned} \tag{4.133}$$

We deal with the others terms in a similar way. Since (h,u) is in the ball $B(0,R)$ of $E^{\frac{N}{2}}$ and combining (4.130), (4.131), (4.132) and (4.133), we have for $C>0$ large enough:

$$\begin{aligned}&\|\psi(h,u)\|_{E^{\frac{N}{2}}} \leq C\big(\eta + R^2 + g(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}}\big)\\ &\times \exp\Big(C\int_{\mathbb{R}^+} \big(\|u_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}} + \|u_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}} + \|\nabla \ln \rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}} + \|\nabla \ln \rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\big)(s)ds\Big).\end{aligned} \tag{4.134}$$

Let us recall that ρ_1 verifies a heat equation (1.13), we deduce from Propositions 2.4 and the maximum principle that there exists a regular function g_1 such that:

$$\begin{aligned}&\int_{\mathbb{R}^+} \big(\|u_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}} + \|u_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}} + \|\nabla \ln \rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}} + \|\nabla \ln \rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\big)(s)ds\\ &\quad\lesssim g_1(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}.\end{aligned} \tag{4.135}$$

From (4.135) it yields:

$$\begin{aligned}&g(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}} \times \exp\Big(C\int_{\mathbb{R}^+} \big(\|u_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}} + \|u_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\\ &\quad + \|\nabla \ln \rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}} + \|\nabla \ln \rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\big)(s)ds\Big)\\ &\quad\leq g(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}} \exp\big(Cg_1(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}\big).\end{aligned} \tag{4.136}$$

In particular it implies that:

$$\begin{aligned}&C\eta \exp\Big(C\int_{\mathbb{R}^+} \big(\|u_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}} + \|u_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}} + \|\nabla \ln \rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}} + \|\nabla \ln \rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}}\big)(s)ds\Big)\\ &\quad\leq C\eta \exp(Cg_1(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}).\end{aligned} \tag{4.137}$$

Let us prove now the stability of the functional ψ, from label (4.134) let us choose:

$$R = 4C\eta \exp(Cg_1(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}), \tag{4.138}$$

and suppose that:

$$g(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}} \leq C\eta. \tag{4.139}$$

Now combining (4.134), (4.138) and (4.139) we have:

$$\|\psi(h,u)\|_{E^{\frac{N}{2}}} \leq 2C\eta\exp(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}) + CR^2\exp(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}). \tag{4.140}$$

From (4.140) it suffices that η is small enough such that:

$$R^2 \leq 2C\eta\exp(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}). \tag{4.141}$$

Indeed combining (4.141), (4.140) and (4.138) shows that:

$$\|\psi(h,u)\|_{E^{\frac{N}{2}}} \leq R. \tag{4.142}$$

This concludes the proof of the stability except that it remains to choose η and to verify that it implies the condition (1.16) of Theorem 1.3. The condition (4.141) via (4.138) is equivalent to:

$$16C^2\eta^2\exp(2Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}) \leq 2C\eta\exp(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}), \tag{4.143}$$

and:

$$8C\eta\exp(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}) \leq 1. \tag{4.144}$$

From (4.139) let us choose:

$$\eta = \frac{1}{C}g(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}}. \tag{4.145}$$

The condition (4.144) implies by using (4.145) that:

$$8g(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}-2}_{2,1}}\exp(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B^{\frac{N}{2}}_{2,1}}) \leq 1. \tag{4.146}$$

Let us point out that this condition corresponds exactly to the condition (1.16) of the Theorem 1.3. It concludes the proof of the stability of the functional ψ. Let us prove now some contraction properties for the functional ψ.

2) Second step, contraction properties

Consider two element (h,u) and (h',u') in $B(0,R)$, according to Proposition 3.8 we have:

$$\begin{aligned}&\|\psi(h,u)-\psi(h',u')\|_{E^{\frac{N}{2}}} \leq \\ &C\big(\|F(h,u)-F(h',u')\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})} + \|G(h,u)-G(h',u')\|_{\widetilde{L}^1_T(B^{\frac{N}{2}-1}_{2,1})}\big) \\ &\times\exp\Big(C\int_{\mathbb{R}^+}(\|u_1\|^4_{B^{\frac{N}{2}-\frac{1}{2}}_{2,\infty}} + \|u_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}} + \|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}} + \|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B^{\frac{N}{2}+\frac{1}{2}}_{2,\infty}})(s)ds\Big).\end{aligned} \tag{4.147}$$

Let us deal with the term $\|F(h,u)-F(h',u')\|_{\widetilde{L}^1_T(\widetilde{B}^{\frac{N}{2}-1,\frac{N}{2}}_{2,1})}$, we have then by Proposition 2.2 and by denoting $\delta h = h-h'$ and $\delta u = u-u'$:

$$\begin{aligned}
&\|F(h',u')-F(h,u)\|_{\widetilde{L}^1(\widetilde{B}_{2,\infty}^{\frac{N}{2}-1,\frac{N}{2}})}\\
&\lesssim \|\delta u\|_{\widetilde{L}_T^{\frac{4}{3}}(B_{2,\infty}^{\frac{N}{2}+\frac{1}{2}})}\|\nabla h'\|_{\widetilde{L}_T^{4}(\widetilde{B}_{2,\infty}^{\frac{N}{2}-\frac{3}{2},\frac{N}{2}-\frac{1}{2}})}+\|\delta u\|_{\widetilde{L}_T^{\infty}(B_{2,\infty}^{\frac{N}{2}-1})}\|\nabla h'\|_{\widetilde{L}_T^{1}(\widetilde{B}_{2,\infty}^{\frac{N}{2},\frac{N}{2}+1})}\\
&\quad+\|u\|_{\widetilde{L}_T^{1}(B_{2,\infty}^{\frac{N}{2}+1})}\|\nabla\delta h\|_{\widetilde{L}_T^{\infty}(\widetilde{B}_{2,\infty}^{\frac{N}{2}-2,\frac{N}{2}-1})}+\|\nabla\delta h\|_{\widetilde{L}_T^{\frac{4}{3}}(\widetilde{B}_{2,\infty}^{\frac{N}{2}-\frac{1}{2},\frac{N}{2}+\frac{1}{2}})}\|u\|_{\widetilde{L}_T^{4}(B_{2,\infty}^{\frac{N}{2}-\frac{1}{2}})}\\
&\le C\big(2\|(h,u)\|_{E^{\frac{N}{2}}}+2\|(h',u')\|_{E^{\frac{N}{2}}}\big)\|(\delta h,\delta u)\|_{E^{\frac{N}{2}}}.
\end{aligned}\tag{4.148}$$

We proceed similarly for the term $\|G(h,u)-G(h',u')\|_{\widetilde{L}_T^1(B_{2,1}^{\frac{N}{2}-1})}$ (let us mention only that the delicate term $K\nabla\ln\rho_1$ disappears). We get finally for a $C>0$ large enough by using (4.135):

$$\begin{aligned}
&\|\psi(h,u)-\psi(h',u')\|_{E^{\frac{N}{2}}}\le C\big(2\|(h,u)\|_{E^{\frac{N}{2}}}+2\|(h',u')\|_{E^{\frac{N}{2}}}\big)\|(\delta h,\delta u)\|_{E^{\frac{N}{2}}}\\
&\times\exp\Big(C\int_{\mathbb{R}^+}\big(\|u_1\|^4_{B_{2,\infty}^{\frac{N}{2}-\frac{1}{2}}}+\|u_1\|^{\frac{4}{3}}_{B_{2,\infty}^{\frac{N}{2}+\frac{1}{2}}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B_{2,\infty}^{\frac{N}{2}+\frac{1}{2}}}+\|\nabla\ln\rho_1\|^{\frac{4}{3}}_{B_{2,\infty}^{\frac{N}{2}+\frac{1}{2}}}\big)(s)ds\Big)\\
&\le 4CR\exp\big(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B_{2,1}^{\frac{N}{2}}})\big)\|(\delta h,\delta u)\|_{E^{\frac{N}{2}}}.
\end{aligned}\tag{4.149}$$

From (4.138) and (4.145) we have for a C large enough:

$$\begin{aligned}
&4CR\exp\big(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B_{2,1}^{\frac{N}{2}}})\big)=16C\eta\exp\big(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B_{2,1}^{\frac{N}{2}}})\big)\\
&=16Cg(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B_{2,1}^{\frac{N}{2}-2}}\exp\big(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B_{2,1}^{\frac{N}{2}}})\big).
\end{aligned}\tag{4.150}$$

In particular it ensures the contraction property via (4.149) if:

$$4CR\exp\big(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B_{2,1}^{\frac{N}{2}}})\big)\le\frac{1}{2}.\tag{4.151}$$

Using (4.150), the previous condition (4.151) corresponds to:

$$16Cg(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B_{2,1}^{\frac{N}{2}-2}}\exp\big(Cg_1(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B_{2,1}^{\frac{N}{2}}})\big)\le\frac{1}{2}.\tag{4.152}$$

Let us mention that the assumption (4.152) is equivalent to the condition (1.16) of the Theorem 1.3. It achieves the proof of the contraction property of ψ. It concludes in particular the proof of Theorem 1.3.

5. Proof of Corollary 1

It suffices to apply the same proof than for the Theorem 1.3, in particular to apply a fixed point theorem for the function ψ previously defined. The main difference corresponds to the way we are going to handle the remainder term $K\nabla\ln\rho_1$. Following the same arguments than for the estimate (4.133) we have for a regular function g and $C>0$:

$$\begin{aligned}
\|K\nabla\ln\rho_1\|_{\widetilde{L}^1(B_{2,1}^{\frac{N}{2}-1})}&\lesssim\|\ln\rho_1\|_{\widetilde{L}^1(B_{2,1}^{\frac{N}{2}})}\\
&\le CKg(\|(\rho_1,\frac{1}{\rho_1})\|_{L^\infty})\|h_1\|_{\widetilde{L}^1(B_{2,1}^{\frac{N}{2}})}\\
&\le CKg(\|(\rho_1^0,\frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B_{2,1}^{\frac{N}{2}-2}}.
\end{aligned}\tag{5.153}$$

Using exactly the same idea than in the previous proof we need the following smallness hypothesis with $C > 0$ and g, g_1 regular function in order to ensure the stability of the functional ψ for a ball $B(0, R)$ with R defined as previously:

$$16KCg(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B_{2,1}^{\frac{N}{2}-2}} \exp\big(Cg_1(\|(\rho_1^0, \frac{1}{\rho_1^0})\|_{L^\infty})\|h_1^0\|_{B_{2,1}^{\frac{N}{2}}})\big) \leq \frac{1}{2}. \tag{5.154}$$

If we choose K small enough this last condition will be verified, it achieves the proof of Corollary 1.

Appendix A

In this appendix, we just give a technical lemma on the computation of the capillarity tensor.

Lemma 2. *When $\kappa(\rho) = \frac{\kappa}{\rho}$ with $\kappa > 0$ then:*

$$\mathrm{div}K = \kappa\rho(\nabla\Delta\ln\rho + \frac{1}{2}\nabla(|\nabla\ln\rho|^2)),$$

and:

$$\mathrm{div}K = \kappa\mathrm{div}(\rho\nabla\nabla\ln\rho).$$

Proof. We recall that:

$$\mathrm{div}K = \nabla\big(\rho\kappa(\rho)\Delta\rho + \frac{1}{2}(\kappa(\rho) + \rho\kappa'(\rho))|\nabla\rho|^2\big) - \mathrm{div}\big(\kappa(\rho)\nabla\rho\otimes\nabla\rho\big).$$

When $\kappa(\rho) = \frac{\kappa}{\rho}$, we have:

$$\mathrm{div}K = \kappa\nabla\Delta\rho - \kappa\mathrm{div}(\frac{1}{\rho}\nabla\rho\otimes\nabla\rho). \tag{A.155}$$

But since:

$$\Delta\rho = \rho\Delta\ln\rho + \frac{1}{\rho}|\nabla\rho|^2,$$

plugging this expression in (A.155) we get:

$$\mathrm{div}K = \kappa\rho\nabla\Delta\ln\rho + \kappa\nabla\rho\Delta\ln\rho + \kappa\nabla(\frac{1}{\rho}|\nabla\rho|^2) - \kappa\mathrm{div}(\frac{1}{\rho}\nabla\rho\otimes\nabla\rho). \tag{A.156}$$

Furthermore we have:

$$\kappa\mathrm{div}(\frac{1}{\rho}\nabla\rho\otimes\nabla\rho) = \kappa\Delta\ln\rho\nabla\rho + \nabla(\frac{1}{\rho}|\nabla\ln\rho|^2) - \frac{\kappa}{2}\rho\nabla(|\nabla\ln\rho|^2),$$

which concludes the first part of the lemma.

We now want to prove that we can rewrite the capillarity tensor under the form of a viscosity tensor. To see this, we have:

$$\begin{aligned}\mathrm{div}(\rho\nabla(\nabla\ln\rho))_j &= \sum_i \partial_i(\rho\partial_{ij}\ln\rho)\\ &= \sum_i[\partial_i\rho\partial_{ij}\ln\rho + \rho\partial_{iij}\ln\rho]\\ &= \rho(\Delta\nabla\ln\rho)_j + \sum_i \rho\partial_i\ln\rho\partial_j\partial_i\ln\rho)\\ &= \rho(\Delta\nabla\ln\rho)_j + \frac{\rho}{2}(\nabla(|\nabla\ln\rho|^2))_j\\ &= \mathrm{div}K.\end{aligned}$$

It gives then:

$$\mathrm{div}K = \kappa\mathrm{div}(\rho\nabla\nabla\ln\rho) = \kappa\mathrm{div}(\rho D(\nabla\ln\rho)). \qquad \square$$

References

[1] D.M. Anderson, G.B. McFadden, A.A. Wheller, Diffuse-Interface Methods in Fluid Mechanics, Annu. Rev. Fluid Mech., vol. 30, Annual Reviews, Palo Alto, CA, 1998, pp. 139–165.

[2] P. Antonelli, P. Marcati, On the finite energy weak solutions to a system in quantum fluid dynamics, Comm. Math. Phys. 287 (2009) 657–686.

[3] C. Audiard, B. Haspot, Gross–Pitaevskii equation to Euler Korteweg system, existence of global strong solutions with small irrotational initial data, preprint, 2014, hal-01077281.

[4] C. Audiard, B. Haspot, Global well-posedness of the quasi-linear Euler–Korteweg system for small irrotational data, preprint.

[5] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren Math. Wiss., vol. 343, Springer-Verlag, 2011.

[6] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Ann. Sci. Éc. Norm. Supér. 14 (1981) 209–246.

[7] D. Bresch, B. Desjardins, C.-K. Lin, On some compressible fluid models: Korteweg, lubrication and shallow water systems, Comm. Partial Differential Equations 28 (3–4) (2003) 843–868.

[8] J.W. Cahn, J.E. Hilliard, Free energy of a nonuniform system, I. Interfacial free energy, J. Chem. Phys. 28 (1998) 258–267.

[9] M. Cannone, F. Planchon, On the regularity of the bilinear term for solutions to the incompressible Navier–Stokes equations, Rev. Mat. Iberoam. 16 (1) (2000).

[10] F. Charve, Convergence of a low order non-local Navier–Stokes–Korteweg system: the order-parameter model, preprint, hal 2013.

[11] F. Charve, Local in time results for local and non-local capillary Navier–Stokes systems with large data, J. Differential Equations 256 (7) (2014) 2152–2193.

[12] F. Charve, B. Haspot, Convergence of capillary fluid models: from the non-local to the local Korteweg model, Indiana Univ. Math. J. 6 (2011) 2021–2060.

[13] F. Charve, B. Haspot, On a Lagrangian method for the convergence of a non-local to a local Korteweg capillary fluid model, J. Funct. Anal. 265 (2013) 1264–1323.

[14] F. Charve, B. Haspot, Existence of global strong solution and vanishing capillarity-viscosity limit in one dimension for the Korteweg system, SIAM J. Math. Anal. 45 (2) (2013) 469–494.

[15] J.-Y. Chemin, I. Gallagher, On the global wellposedness of the 3-D incompressible Navier–Stokes equations, Ann. Sci. Éc. Norm. Supér. Paris 39 (2006) 679–698.

[16] J.-Y. Chemin, I. Gallagher, Wellposedness and stability results for the Navier–Stokes equations in $\mathbb{R}^3$, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2) (2009) 599–624.

[17] R. Danchin, B. Desjardins, Existence of solutions for compressible fluid models of Korteweg type, Ann. Inst. H. Poincaré Anal. Non Linéaire 18 (2001) 97–133.

[18] J.E. Dunn, J. Serrin, On the thermomechanics of interstitial working, Arch. Ration. Mech. Anal. 88 (2) (1985) 95–133.

[19] P. Gérard, The Cauchy problem for the Gross–Pitaevskii equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 23 (5) (2006) 765–779.

[20] M. Gisclon, I. Lacroix-Violet, About the barotropic compressible quantum Navier–Stokes equations, preprint, arXiv:1412.1332, 2014.

[21] M.E. Gurtin, D. Poligone, J. Vinals, Two-phases binary fluids and immiscible fluids described by an order parameter, Math. Models Methods Appl. Sci. 6 (6) (1996) 815–831.

[22] B. Haspot, Existence of solutions for compressible fluid models of Korteweg type, Ann. Math. Blaise Pascal 16 (2009) 431–481.

[23] B. Haspot, Cauchy problem for viscous shallow water equations with a term of capillarity, Math. Models Methods Appl. Sci. 20 (7) (2010) 1049–1087.

[24] B. Haspot, Existence of global weak solution for compressible fluid models with a capillary tensor for discontinuous interfaces, Differential Integral Equations 23 (9–10) (2010).

[25] B. Haspot, Existence of global strong solutions in critical spaces for barotropic viscous fluids, Arch. Ration. Mech. Anal. 202 (2) (2011) 427–460.
[26] B. Haspot, Global existence of strong solution for the Saint-Venant system with large initial data on the irrotational part, C. R. Math. Acad. Sci. Paris 350 (5–6) (2012) 229–332.
[27] B. Haspot, Existence of global strong solutions for the barotropic Navier–Stokes system with large initial data on the rotational part of the velocity, C. R. Math. Acad. Sci. Paris 350 (2012) 487–492.
[28] B. Haspot, Existence of global strong solution for the compressible Navier–Stokes system and the Korteweg system in two-dimension, Methods Appl. Anal. 20 (2) (June 2013) 141–164.
[29] B. Haspot, New entropy for Korteweg's system, existence of global weak solution and Prodi–Serrin theorem, preprint, 2013, hal-00778811.
[30] B. Haspot, From the highly compressible Navier–Stokes equations to fast diffusion and porous media equations, existence of global weak solution for the quasi-solutions, J. Math. Fluid Mech. (2016), http://dx.doi.org/10.1007/s00021-015-0226-5, in press.
[31] B. Haspot, Global existence of strong solution for shallow water system with large initial data on the irrotational part, preprint, arXiv:1201.5456.
[32] B. Haspot, E. Zatorska, From the highly compressible Navier–Stokes equations to the porous media equation, rate of convergence, Discrete Contin. Dyn. Syst. Ser. A 36 (6) (June 2016), http://dx.doi.org/10.3934/dcds.2016.36.3107, in press.
[33] H. Hattori, D. Li, The existence of global solutions to a fluid dynamic model for materials for Korteweg type, J. Partial Differ. Equ. 9 (4) (1996) 323–342.
[34] H. Hattori, D. Li, Global solutions of a high-dimensional system for Korteweg materials, J. Math. Anal. Appl. 198 (1) (1996) 84–97.
[35] D. Jamet, O. Lebaigue, N. Coutris, J.M. Delhaye, The second gradient method for the direct numerical simulation of liquid-vapor flows with phase change, J. Comput. Phys. 169 (2) (2001) 624–651.
[36] A. Jüngel, Global weak solutions to compressible Navier–Stokes equations for quantum fluids, SIAM J. Math. Anal. 42 (3) (2011) 1025–1045.
[37] D.J. Korteweg, Sur la forme que prennent les équations du mouvement des fluides si l'on tient compte des forces capillaires par des variations de densité, Arch. Néer. Sci. Exactes Sér. II 6 (1901) 1–24.
[38] M. Kotschote, Strong solutions for a compressible fluid model of Korteweg type, Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (4) (July–August 2008) 679–696.
[39] P.-L. Lions, Mathematical Topics in Fluid Mechanics, vol. 2, Compressible Models, Oxford University Press, 1996.
[40] C. Rohde, Approximation of solutions of conservation laws by non-local regularization and discretization, Habilitation thesis, University of Freiburg, 2004.
[41] C. Rohde, On local and non-local Navier–Stokes–Korteweg systems for liquid-vapour phase transitions, ZAMM Z. Angew. Math. Mech. 85 (12) (2005) 839–857.
[42] C. Rohde, A local and low-order Navier–Stokes–Korteweg system, Preprint-Reihe IANS, Universität Stuttgart, 2009.
[43] C. Truedell, W. Noll, The Nonlinear Field Theories of Mechanics, second edition, Springer-Verlag, Berlin, 1992.
[44] J.F. Van der Waals, Thermodynamische Theorie der Kapillarität unter Voraussetzung stetiger Dichteänderung, Phys. Chem. 13 (1894) 657–725.