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On the positivity, monotonicity, and stability of a semi-adaptive LOD method for solving three-dimensional degenerate Kawarada equations

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ABSTRACT

This paper concerns the numerical solution of three-dimensional degenerate Kawarada equations. These partial differential equations possess highly nonlinear source terms, and exhibit strong quenching singularities which pose severe challenges to the design and analysis of highly reliable schemes. Arbitrary fixed nonuniform spatial grids, which are not necessarily symmetric, are considered throughout this study. The numerical solution is advanced through a semi-adaptive Local One-Dimensional (LOD) integrator. The temporal adaptation is achieved via a suitable arc-length monitoring mechanism. Criteria for preserving the positivity and monotonicity are investigated and acquired. The numerical stability of the splitting method is proven in the von Neumann sense under the spectral norm. Extended stability expectations are proposed and investigated.

1. Introduction

Let $\mathcal{D} = (0,a)\times(0,b)\times(0,c) \subset \mathbb{R}^3$, where $a, b, c > 0$, and $\partial\mathcal{D}$ be its boundary. Denote $\Omega = \mathcal{D}\times(t_0,\ T)$, $\mathcal{S} = \partial\mathcal{D} \times (t_0,\ T)$ for given $0 \le t_0 < T < \infty$. We consider the following degenerate Kawarada problem,

$$s(x,y,z)u_t = u_{xx} + u_{yy} + u_{zz} + f(u), \quad (x,y,z,t) \in \Omega, \tag{1.1}$$

$$u(x,y,z,t) = 0, \quad (x,y,z,t) \in \mathcal{S}, \tag{1.2}$$

$$u(x,y,z,t_0) = u_0(x,y,z), \quad (x,y,z) \in \mathcal{D}, \tag{1.3}$$

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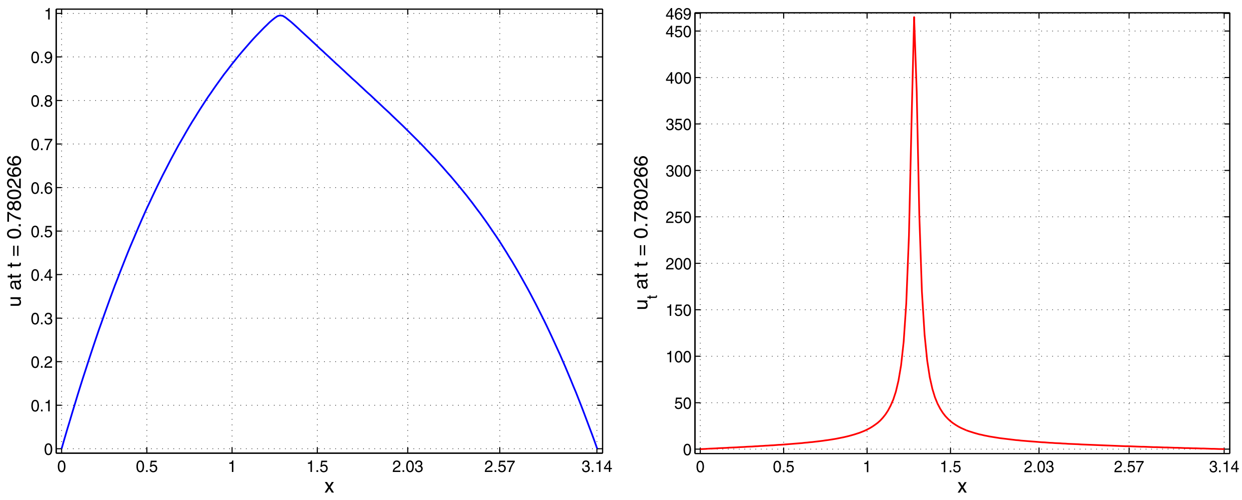

Fig. 1. Numerical solution (left) and its temporal derivative (right) immediately before quenching. It is observed that as $\max_x u(x) \to 1^-$, we have $\max_x u_t \gg 600$. The computed quenching time is $T \approx 0.780265747310047$.

where $s(x,y,z) = \left(x^2+y^2+z^2\right)^{q/2}$, $q \in [0,2]$. The nonlinear source function, $f(u)$, is strictly increasing for $0 \le u < 1$ with

$$f(0) = f_0 > 0, \quad \lim_{u \to 1^-} f(u) = \infty.$$

In idealized thermal combustion applications [1,4,17], u represents the temperature in the combustion channel, and the x-, y-, and z-coordinates coincide with the channel walls. The initial temperature $0 \le u_0 \ll 1$ is typically chosen to be small. The function $s(x,y,z)$ represents certain singularities in the temperature transportation speed within the channel, which causes the degeneracy in the differential equation (1.1) [3,14,18,20]. The solution u of (1.1)–(1.3) is said to *quench* if there exists a finite time $T > 0$ such that

$$\sup\left\{|u_t(x,y,z,t)| : (x,y,z) \in \mathcal{D}\right\} \to \infty \text{ as } t \to T^-. \tag{1.4}$$

The value T is then defined as the *quenching time* [2,1,13]. It has been shown that a necessary condition for quenching to occur is

$$\max\left\{|u(x,y,z,t)| : (x,y,z) \in \bar{\mathcal{D}}\right\} \to 1^- \text{ as } t \to T^-. \tag{1.5}$$

Further, such a T exists only when certain spatial references, such as the size and shape of $\mathcal{D}$, reach their critical limits. A domain $\mathcal{D}^*$ is called the *critical domain* if the solution of (1.1)–(1.3) exists for all time when $\mathcal{D} \subseteq \mathcal{D}^*$, and (1.5) occurs when $\mathcal{D}^* \subseteq \mathcal{D}$ for a finite T [13].

Systematic mathematical investigations of quenching phenomena can be traced back to Kawarada's original work involving the one-dimensional model equation [11]. It was observed that for any spatial domain $[0,a]$, there exists a unique value $a^* > 0$ such that for $a < a^*$, the solution of the equation exists globally; and for $a \ge a^*$, there exists a finite time $T(a)$, such that $\lim_{t \to T(a)} \max_{0 \le x \le a} u(x,t) = 1$. In the latter case, u stops existing in finite time and this phenomenon is referred to as *quenching* [11,13,19].

As an illustration, in Fig. 1, we show the numerical solution and its temporal derivative of a typical one-dimensional Kawarada problem over the interval $[0,\pi]$. The initial function $u_0(x) = 0.001\sin(x)$, $f(u) = 1/(1-u)$, $s(x) = x^p(\pi - x)^{1-p}$ with $p = (\sqrt{5}-1)/2$, and homogeneous Dirichlet boundary condition are utilized. When considering the numerical solution v, it is evident that v_t changes dramatically when compared with v. This suggests that properly chosen nonuniform steps can be vital in computations. There have been considerable developments in the study of Kawarada equations, although discussions of multidimensional problems were extremely limited until recently.

Numerous computational procedures, including moving mesh adaptive methods, have been constructed for solving blow-up and Kawarada problems in the past decades (interested readers are referred to [2,6,7,18,20] and references therein). Though in the former case, adaptations are frequently achieved via monitoring functions on the arc-length of the function u; in the latter situation, adaptations are more likely to be built upon the arc-length of u_t, since it is directly proportional to $f(u)$, which blows up as u quenches [5,13,16].

As reported in several recent investigations, when quenching locations can be predetermined, it is preferable to use nonuniform spatial grids throughout the computations [3,12,20]. To that end, this paper develops a temporally adaptive splitting scheme utilizing predetermined nonuniform spatial grids. In this case, key quenching characteristics such as the quenching time, critical domain, and important numerical properties of underlying algorithms can be more precisely studied. Our discussions will be organized as follows. In the next section, the semi-adaptive LOD scheme for solving (1.1)–(1.3) will be constructed and discussed. Then, in Section 3, criteria to guarantee the positivity of the numerical scheme will be determined. In Section 4, appropriate criteria for guaranteeing the monotonicity will be obtained. These two sections together serve as the platform for carrying out investigations of stability. Section 5 is devoted to the stability analysis of the semi-adaptive LOD scheme. The analysis will first be carried out for a fully linearized scheme, and then a more realistic stability analysis is proposed without freezing the source term. Finally, concluding remarks and proposed future work will be given in Section 6.

2. Semi-adaptive LOD scheme

Utilizing the transformations $\tilde{x} = x/a$, $\tilde{y} = y/b$, $\tilde{z} = z/c$, and reusing the original variables for simplicity, we may reformulate (1.1)–(1.3) as

$$u_t = \frac{1}{a^2\phi}u_{xx} + \frac{1}{b^2\phi}u_{yy} + \frac{1}{c^2\phi}u_{zz} + g(u), \quad (x,y,z,t) \in \Omega, \tag{2.1}$$

$$u(x,y,z,t) = 0, \quad (x,y,z) \in \mathcal{S}, \tag{2.2}$$

$$u(x,y,z,t_0) = u_0, \quad (x,y,z) \in \mathcal{D}, \tag{2.3}$$

where $g(u) = \dfrac{f(u)}{\phi}$, $\phi = \phi(x,y,z) = \left(a^2x^2 + b^2y^2 + c^2z^2\right)^{q/2}$, and $\mathcal{D} = (0,1)\times(0,1)\times(0,1) \subset \mathbb{R}^3$.

Let $N_1, N_2, N_3 \gg 1$. We inscribe over $\bar{\mathcal{D}}$ the following variable grid: $\mathcal{D}_h = \{(x_i, y_j, z_k) \mid i = 0,\dots,N_1+1;\ j = 0,\dots,N_2+1;\ k = 0,\dots,N_3+1;\ x_0 = y_0 = z_0 = 0,\ x_{N_1+1} = y_{N_2+1} = z_{N_3+1} = 1\}$. Denote $h_{1,i} = x_{i+1} - x_i > 0$, $h_{2,j} = y_{j+1} - y_j > 0$, and $h_{3,k} = z_{k+1} - z_k > 0$ for $1 \le i \le N_1$, $1 \le j \le N_2$, $1 \le k \le N_3$. Let $u_{i,j,k}(t)$ be an approximation of the solution of (2.1)–(2.3) at (x_i, y_j, z_k, t) and consider the following first-order finite differences [20],

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j,k} \approx \frac{2u_{i-1,j,k}}{h_{1,i-1}(h_{1,i-1}+h_{1,i})} - \frac{2u_{i,j,k}}{h_{1,i-1}h_{1,i}} + \frac{2u_{i+1,j,k}}{h_{1,i}(h_{1,i-1}+h_{1,i})},$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{i,j,k} \approx \frac{2u_{i,j-1,k}}{h_{2,j-1}(h_{2,j-1}+h_{2,j})} - \frac{2u_{i,j,k}}{h_{2,j-1}h_{2,j}} + \frac{2u_{i,j+1,k}}{h_{2,j}(h_{2,j-1}+h_{2,j})},$$

$$\left.\frac{\partial^2 u}{\partial z^2}\right|_{i,j,k} \approx \frac{2u_{i,j,k-1}}{h_{3,k-1}(h_{3,k-1}+h_{3,k})} - \frac{2u_{i,j,k}}{h_{3,k-1}h_{3,k}} + \frac{2u_{i,j,k+1}}{h_{3,k}(h_{3,k-1}+h_{3,k})}.$$

Further, denote $v(t) = (u_{1,1,1}, u_{2,1,1}, \dots, u_{N_1,1,1}, u_{1,2,1}, u_{2,2,1}, \dots, u_{N_1,2,1}, \dots, u_{1,N_2,1}, u_{2,N_2,1}, \dots, u_{N_1,N_2,1}, \dots, u_{1,N_2,N_3}, u_{2,N_2,N_3}, \dots, u_{N_1,N_2,N_3})^\intercal \in \mathbb{R}^{N_1N_2N_3}$ and let $g(v)$ be a discretization of the nonhomogeneous term of (2.1). We obtain from (2.1)–(2.3) the following semi-discretized system

$$v'(t) = \sum_{\sigma=1}^{3} M_\sigma v(t) + g(v(t)), \quad t_0 < t < T, \tag{2.4}$$

$$v(t_0) = v_0, \tag{2.5}$$

where $M_1 = \dfrac{1}{a^2}B(I_{N_3} \otimes I_{N_2} \otimes T_1)$, $M_2 = \dfrac{1}{b^2}B(I_{N_3} \otimes T_2 \otimes I_{N_1})$, $M_3 = \dfrac{1}{c^2}B(T_3 \otimes I_{N_2} \otimes I_{N_1})$, $\otimes$ stands for the Kronecker product, $I_{N_\sigma} \in \mathbb{R}^{N_\sigma \times N_\sigma}$, $\sigma = 1,2,3$, are identity matrices, and

$$B = \mathrm{diag}\left(\phi_{1,1,1}^{-1}, \phi_{2,1,1}^{-1}, \ldots, \phi_{N_1,1,1}^{-1}, \phi_{1,2,1}^{-1}, \ldots, \phi_{N_1,N_2,N_3}^{-1}\right) \in \mathbb{R}^{N_1N_2N_3 \times N_1N_2N_3},$$

$$\phi_{i,j,k} = \left[a^2\left(\sum_{\ell=0}^{i-1} h_{1,\ell}\right)^2 + b^2\left(\sum_{\ell=0}^{j-1} h_{2,\ell}\right)^2 + c^2\left(\sum_{\ell=0}^{k-1} h_{3,\ell}\right)^2\right]^{q/2},$$

$$T_\sigma = \begin{pmatrix} m_{\sigma,1} & n_{\sigma,1} & & & \\ l_{\sigma,1} & m_{\sigma,2} & n_{\sigma,2} & & \\ & \cdots & \cdots & \cdots & \\ & & l_{\sigma,N_\sigma-2} & m_{\sigma,N_\sigma-1} & n_{\sigma,N_\sigma-1} \\ & & & l_{\sigma,N_\sigma-1} & m_{\sigma,N_\sigma} \end{pmatrix} \in \mathbb{R}^{N_\sigma \times N_\sigma}, \quad \sigma = 1,2,3,$$

and for the above

$$l_{\sigma,j} = \frac{2}{h_{\sigma,j}(h_{\sigma,j} + h_{\sigma,j+1})}, \; n_{\sigma,j} = \frac{2}{h_{\sigma,j}(h_{\sigma,j-1} + h_{\sigma,j})}, \quad j = 1, \ldots, N_\sigma - 1,$$

$$m_{\sigma,j} = -\frac{2}{h_{\sigma,j-1}h_{\sigma,j}}, \quad j = 1, \ldots, N_\sigma; \; \sigma = 1,2,3.$$

The formal solution of (2.4), (2.5) can thus be written as

$$v(t) = E(tC)v_0 + \int_{t_0}^{t} E((t-\tau)C)g(v(\tau))d\tau, \quad t_0 < t < T, \tag{2.6}$$

where $E(\cdot) = \exp(\cdot)$ is the matrix exponential and $C = \sum_{\sigma=1}^{3} M_\sigma$ [17].

In principle, different approximation techniques can be used to yield different splitting methods based on (2.6) [9,17,16]. Yet, we are particularly interested in approximating (2.6) via a trapezoidal rule and a [1/1] Padé approximation, $E(tC) = p(t) + \mathcal{O}\left(t^2\right)$, where

$$p(t) = \prod_{\sigma=1}^{3}\left(I - \frac{t}{2}M_\sigma\right)^{-1}\left(I + \frac{t}{2}M_\sigma\right), \quad t_0 < t < T.$$

The above leads to

$$v(t) = p(t)\left[v_0 + \frac{t}{2}g(v_0)\right] + \frac{t}{2}g(v(t)) + \mathcal{O}\left((t-t_0)^2\right), \quad t \to t_0. \tag{2.7}$$

The above LOD algorithm provides a highly efficient way to compute numerical solutions of multidimensional problems such as (2.1)–(2.3) [9,15,18,20]. Based on (2.7), we obtain the following first-order in space and time semi-adaptive LOD scheme:

$$v_{\ell+1} = \left[\prod_{\sigma=1}^{3}\left(I - \frac{\tau_\ell}{2}M_\sigma\right)^{-1}\left(I + \frac{\tau_\ell}{2}M_\sigma\right)\right]\left(v_\ell + \frac{\tau_\ell}{2}g(v_\ell)\right) + \frac{\tau_\ell}{2}g(v_{\ell+1}), \tag{2.8}$$

where v_ℓ and $v_{\ell+1}$ are approximations of $v(t_\ell)$ and $v(t_{\ell+1})$, respectively, v_0 is the initial vector, $t_\ell = t_0 + \sum_{k=0}^{\ell-1}\tau_k$, $\ell = 0, 1, 2, \ldots$, and $\{\tau_\ell\}_{\ell\geq 0}$ is a set of variable temporal steps determined by an adaptive procedure. In order to avoid a fully implicit scheme, $g(v_{\ell+1})$ may be approximated by $g(w_\ell)$, where w_ℓ is an approximation to $v_{\ell+1}$, such as

$$w_\ell = v_\ell + \tau_\ell(Cv_\ell + g(v_\ell)), \quad 0 < \tau_\ell \ll 1, \tag{2.9}$$

in practical computations.

Recalling (1.4) and (1.5), we consider the following arc-length monitoring function on v_t,

$$m\left(\frac{\partial v}{\partial t}, t\right) = \sqrt{1 + \left(\frac{\partial^2 v}{\partial t^2}\right)^2}, \quad t_0 < t < T.$$

Setting the two maximal arc-lengths in neighboring intervals $[t_{\ell-2},\ t_{\ell-1}]$ and $[t_{\ell-1},\ t_\ell]$ equal [8,12,19,20], we acquire the following quadratic equations from the above,

$$\tau_\ell^2 = \tau_{\ell-1}^2 + \left(\frac{\partial v_{\ell-1}}{\partial t} - \frac{\partial v_{\ell-2}}{\partial t}\right)^2 - \left(\frac{\partial v_\ell}{\partial t} - \frac{\partial v_{\ell-1}}{\partial t}\right)^2, \quad \ell = 1, 2, 3, \ldots,$$

with τ_0 given.

In the above temporal adaptation procedures, we may consider a minimal temporal step size controller $\tilde\tau_0$, $0 < \tilde\tau_0 \ll \tau_0$, to avoid sudden changes in grid movements or unnecessarily large numbers of computations. Further, in all subsequent discussions, inequalities involving vectors and matrices are considered as component-wise.

3. Positivity

The positivity property is one of the most profound characteristics of the solution of the Kawarada problem (1.1)–(1.3) or (2.1)–(2.3) [2,1,5,13]. Since positive computational solutions preserve the correct physical features of quenching phenomena, it is crucial that our numerical solution also possesses this feature.

Lemma 3.1. $\|T_\sigma\|_2 \leq \max_{j=1,\ldots,N_\sigma}\left\{4/h_{\sigma,j}^2\right\}$, $\sigma = 1, 2, 3$.

Proof. Due to the similarity in structure, we only consider T_1 since the other two cases follow by similar arguments. Note that, in general, T_1 is not symmetric. However, $\|T_1\|_2^2 = \rho(T_1^\intercal T_1)$ and $T_1^\intercal T_1$ is symmetric with a bandwidth of five. Thus,

$$T_1^\intercal T_1 = \begin{pmatrix} \tilde m_{1,1} & \tilde n_{1,1} & \tilde N_{1,1} & & & & \\ \tilde l_{1,1} & \tilde m_{1,2} & \tilde n_{1,2} & \tilde N_{1,2} & & & \\ \tilde L_{1,1} & \tilde l_{1,2} & \tilde m_{1,3} & \tilde n_{1,3} & \tilde N_{1,3} & & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & \tilde L_{1,N_1-4} & \tilde l_{1,N_1-3} & \tilde m_{1,N_1-2} & \tilde n_{1,N_1-2} & \tilde N_{1,N_1-2} \\ & & & \tilde L_{1,N_1-3} & \tilde l_{1,N_1-2} & \tilde m_{1,N_1-1} & \tilde n_{1,N_1-1} \\ & & & & \tilde L_{1,N_1-2} & \tilde l_{1,N_1-1} & \tilde m_{1,N_1} \end{pmatrix} \in \mathbb{R}^{N_1\times N_1},$$

where

$$\tilde{N}_{1,j} = \tilde{L}_{1,j} = l_{1,j} n_{1,j+1}, \quad j = 1, \ldots, N_1 - 2,$$
$$\tilde{n}_{1,j} = \tilde{l}_{1,j} = m_{1,j} n_{1,j} + m_{1,j+1} l_{1,j}, \quad j = 1, \ldots, N_1 - 1,$$
$$\tilde{m}_{1,j} = \begin{cases} m_{1,1}^2 + l_{1,1}^2, & j = 1, \\ n_{1,j-1}^2 + m_{1,j}^2 + l_{1,j}^2, & j = 2, \ldots, N_1 - 1, \\ n_{1,N_1-1}^2 + m_{1,N_1}^2, & j = N_1. \end{cases}$$

We may determine a bound on the spectral radius of $T_1^{\intercal} T_1$ by using Geršchgorin's circle theorem. In fact, only rows containing five nontrivial elements, *i.e.*, $j = 3, \ldots, N_1 - 2$, need to be considered. To this end, $|\lambda_{1,j} - \tilde{m}_{1,j}| \le |\tilde{L}_{1,j-2}| + |\tilde{l}_{1,j-1}| + |\tilde{n}_{1,j}| + |\tilde{N}_{1,j}|$, $j = 3, \ldots, N_1 - 2$, which gives

$$\begin{aligned} &-|m_{1,j-1} n_{1,j-1} + m_{1,j} l_{1,j-1}| - |m_{1,j} n_{1,j} + m_{1,j+1} l_{1,j}| \\ &\quad - |l_{1,j-2} n_{1,j-1}| - |l_{1,j} n_{1,j+1}| + n_{1,j-1}^2 + m_{1,j}^2 + l_{1,j}^2 \le \lambda_{1,j} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \lambda_{1,j} &\le |m_{1,j-1} n_{1,j-1} + m_{1,j} l_{1,j-1}| + |m_{1,j} n_{1,j} + m_{1,j+1} l_{1,j}| \\ &\quad + |l_{1,j-2} n_{1,j-1}| + |l_{1,j} n_{1,j+1}| + n_{1,j-1}^2 + m_{1,j}^2 + l_{1,j}^2. \end{aligned} \tag{3.2}$$

Let $h_1 \equiv \min_{j=1,\ldots,N_1} \{h_{1,j}\}$. From (3.2) we acquire that

$$\begin{aligned} \lambda_{1,j} &\le \frac{2}{h_1^2} \cdot \frac{2}{2h_1^2} + \frac{2}{h_1^2} \cdot \frac{2}{2h_1^2} + \frac{2}{h_1^2} \cdot \frac{2}{2h_1^2} + \frac{2}{h_1^2} \cdot \frac{2}{2h_1^4} + \frac{4}{4h_1^4} + \frac{4}{4h_1^2} \\ &\quad + \left(\frac{2}{2h_1^2}\right)^2 + \left(\frac{2}{h_1^2}\right)^2 + \left(\frac{2}{2h_1^2}\right)^2 = \frac{16}{h_1^4}. \end{aligned}$$

Now, reverse (3.1) and by the same token,

$$\begin{aligned} -\lambda_{1,j} &\le |m_{1,j-1} n_{1,j-1} + m_{1,j} l_{1,j-1}| + |m_{1,j} n_{1,j} + m_{1,j+1} l_{1,j}| + |l_{1,j-2} n_{1,j-1}| \\ &\quad + |l_{1,i} n_{1,j+1}| + n_{1,j-1}^2 + m_{1,j}^2 + l_{1,j}^2 \le \frac{16}{h_1^4}\ , \end{aligned}$$

where, once again, $h_1 \equiv \min_{j=1,\ldots,N_1} \{h_{1,j}\}$. Thus, combining the bounds we have $\|T_1\|_2 \le \max_{i=1,\ldots,N_1} \{4/h_{1,j}^2\}$. The other bounds follow similarly. $\square$

Lemma 3.2. *Let*

$$\beta_{\min} = \frac{h^2}{2\|B\|_2},\ h = \min_{j=1,\ldots,N_\sigma;\ \sigma=1,2,3} \{h_{\sigma,j}\},$$
$$\frac{1}{\|B\|_2} = \min_{i,j,k} \phi_{i,j,k} = \left[a^2 h_{1,0}^2 + b^2 h_{2,0}^2 + c^2 h_{3,0}^2\right]^{q/2}.$$

If

$$\frac{\tau_\ell}{\beta_{\min}} < \min\left\{a^2, b^2, c^2\right\}, \tag{3.3}$$

then the matrices $I - \frac{\tau_\ell}{2}M_\sigma$, $I + \frac{\tau_\ell}{2}M_\sigma$, $\sigma = 1,2,3$, *are nonsingular. Further,* $I - \frac{\tau_\ell}{2}M_\sigma$, $\sigma = 1,2,3$, *are monotone and inverse-positive, and* $I + \frac{\tau_\ell}{2}M_\sigma$, $\sigma = 1,2,3$, *are nonnegative.*

Proof. First, note that

$$\begin{aligned}\left\|\frac{\tau_\ell}{2}M_1\right\|_2 &= \frac{\tau_\ell}{2a^2}\|B(I_{N_3}\otimes I_{N_2}\otimes T_1)\|_2\\ &\le \frac{\tau_\ell}{2a^2}\|B\|_2\|I_{N_3}\otimes I_{N_2}\otimes T_1\|_2 = \frac{\tau_\ell}{2a^2}\|B\|_2\|T_1\|_2\\ &\le \frac{\tau_\ell}{a^2}\|B\|_2 \max_{j=1,\dots,N_1}\left\{\frac{2}{h_{1,j}^2}\right\} < 1.\end{aligned}$$

Hence, $I + \frac{\tau_\ell}{2}M_1$ is nonsingular, and also nonnegative. Similar arguments give that $I + \frac{\tau_\ell}{2}M_2$ and $I + \frac{\tau_\ell}{2}M_3$ are nonsingular and nonnegative.

Now, consider $A = I - \frac{\tau_\ell}{2}M_1$. As $A_{ij} \le 0$ for $i \ne j$ and the weak row sum criterion is satisfied, A is monotone, and hence an inverse exists and is nonnegative. So, A must be inverse-positive [10]. Similar arguments can be given for $I - \frac{\tau_\ell}{2}M_2$ and $I - \frac{\tau_\ell}{2}M_3$. This ensures the proof. □

We also need the following lemma as a result of the definitions used.

Lemma 3.3. *Let* $A \in \mathbb{R}^{n\times n}$ *be nonsingular and nonnegative and* $\beta \in \mathbb{R}^n$ *be positive. Then* $A\beta > 0$.

4. Monotonicity

Another key characteristic which distinguishes a solution to a quenching problem from a solution to most blow-up problems is its monotonicity with respect to time $t \ge t_0$ [2,1,5,13,18]. Thus, it is necessary to guarantee that our numerical solution preserves this property strictly while solving the Kawarada equation (1.1)–(1.3) or (2.1)–(2.3).

Lemma 4.1. *If* (3.3) *holds for all* $\ell \ge k \ge 0$, *and*

(a) $Cv_0 + \frac{1}{2}g(v_0) > 0$;
(b) $\left(I - \frac{\tau_0}{2}g_v(\xi_0)\right)^{-1} > 0$

hold, then $v_{\ell+1} \ge v_\ell$ *for all* $\ell \ge 0$. *That is, the sequence* $\{v_\ell\}_{\ell=0}^\infty$ *is monotonically increasing.*

Proof. By (3.3) we have $\left\|\frac{\tau_k}{2}M_\sigma\right\| < 1$, and thus, $\left(I - \frac{\tau_k}{2}M_\sigma\right)^{-1} = I + \frac{\tau_k}{2}M_\sigma + \mathcal{O}\left(\tau_k^2\right)$, $\sigma = 1,2,3$. From (2.8) and the above, we have

$$\begin{aligned}v_{k+1} - v_k &= \left[\prod_{\sigma=1}^{3}\left(I - \frac{\tau_k}{2}M_\sigma\right)^{-1}\left(I + \frac{\tau_k}{2}M_\sigma\right)\right]\left(v_k + \frac{\tau_k}{2}g(v_k)\right) + \frac{\tau_k}{2}g(v_{k+1}) - v_k\\ &= \left[\prod_{\sigma=1}^{3}(I + \tau_k M_\sigma) + \mathcal{O}\left(\tau_k^2\right)\right]\left(v_k + \frac{\tau_k}{2}g(v_k)\right) + \frac{\tau_k}{2}g(v_{k+1}) - v_k\\ &= \frac{\tau_k}{2}g(v_k) + \tau_k C v_k + \frac{\tau_k}{2}g(v_{k+1}) + \mathcal{O}\left(\tau_k^2\right) \qquad (4.1)\end{aligned}$$

as $\tau_k \to 0$. Note that $g(v_{k+1}) = g(v_k) + g_v(\xi_k)(v_{k+1} - v_k)$ for some $\xi_k \in \mathcal{L}(v_{k+1}; v_k)$, where $\mathcal{L}(v_{k+1}; v_k)$ is the line segment connecting v_{k+1} to v_k in $\mathbb{R}^{N_1N_2N_3}$. Using this fact, we derive from (4.1) that

$$\left(I - \frac{\tau_k}{2} g_v(\xi_k)\right)(v_{k+1} - v_k) = \tau_k \left(Cv_k + \frac{1}{2} g(v_k)\right) + \mathcal{O}\left(\tau_k^2\right),$$

and thus, $v_{k+1} - v_k = \tau_k \left(I - \frac{\tau_k}{2} g_v(\xi_k)\right)^{-1} \left(Cv_k + \frac{1}{2} g(v_k)\right) + \mathcal{O}\left(\tau_k^2\right)$.

We now proceed by induction. Letting $k = 0$, we have

$$v_1 - v_0 = \tau_0 \left(I - \frac{\tau_0}{2} g_v(\xi_0)\right)^{-1} \left(Cv_0 + \frac{1}{2} g(v_0)\right) + \mathcal{O}\left(\tau_0^2\right).$$

Thus, if τ_0 is sufficiently small, we have $v_1 - v_0 > 0$ by our assumption and then Lemma 3.3. For the sake of induction, assume that the monotonicity holds for $k = \ell - 1$. Then we have

$$\begin{aligned} v_{\ell+1} - v_\ell = &\left[\prod_{\sigma=1}^{3} \left(I - \frac{\tau_\ell}{2} M_\sigma\right)^{-1} \left(I + \frac{\tau_\ell}{2} M_\sigma\right)\right] \\ &\times \left(v_\ell - v_{\ell-1} + \frac{\tau_\ell}{2}(g(v_\ell) - g(v_{\ell-1}))\right) + \frac{\tau_\ell}{2}(g(v_{\ell+1}) - g(v_\ell)). \end{aligned}$$

Note that $g(v)$ is strictly increasing since $f(v)$ is strictly increasing. Utilizing Lemmas 3.2–3.3 we find that $v_{\ell+1} - v_\ell > 0$ if $v_\ell - v_{\ell-1} > 0$, which completes the induction. $\square$

It is not uncommon to set $v_0 \equiv 0$ in practical combustion simulations. The following corollary shows that in this case conditions in Lemma 4.1 are satisfied for $\ell = 0$.

Corollary 4.1. *If $v_0 \equiv 0$ and $\tau_0 < \min\left\{\beta_{\min} \min\{a^2, b^2, c^2\}, \min_{i,j,k} \frac{2\phi_{i,j,k}}{f_v(\xi_0(x_i, y_j, z_k))}\right\}$, then conditions* (a), (b) *are true.*

Proof. We first consider (a). Clearly, $Cv_0 + \frac{1}{2} g(v_0) = \frac{1}{2} g(0) > 0$, since $f(0) = f_0 > 0$.

We now consider (b), and under these circumstances we need to show $\left(I - \frac{\tau_0}{2} g_v(\xi_0)\right)^{-1} > 0$. First, we note that $g_v(\xi_0)$ is diagonal by definition, since

$$g(v) = (g_{1,1,1}, \ldots, g_{N_1,N_2,N_3})^{\mathsf{T}} = \left(\frac{f(v_{1,1,1})}{\phi_{1,1,1}}, \ldots, \frac{f(v_{N_1,N_2,N_3})}{\phi_{N_1,N_2,N_3}}\right)^{\mathsf{T}}$$

and

$$g_v(v) = \begin{pmatrix} \frac{\partial g_{1,1,1}}{\partial v_{1,1,1}} & \cdots & \frac{\partial g_{1,1,1}}{\partial v_{N_1,N_2,N_3}} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{N_1,N_2,N_3}}{\partial v_{1,1,1}} & \cdots & \frac{\partial g_{N_1,N_2,N_3}}{\partial v_{N_1,N_2,N_3}} \end{pmatrix} = \operatorname{diag}\left(\frac{f_v(v_{1,1,1})}{\phi_{1,1,1}}, \ldots, \frac{f_v(v_{N_1,N_2,N_3})}{\phi_{N_1,N_2,N_3}}\right).$$

Let us denote

$$g_v(\xi_0) = \operatorname{diag}\left(\frac{f_v((\xi_0)_{1,1,1})}{\phi_{1,1,1}}, \ldots, \frac{f_v((\xi_0)_{N_1,N_2,N_3})}{\phi_{N_1,N_2,N_3}}\right) = \operatorname{diag}\left(d_{1,1,1}^{(0)}, \ldots, d_{N_1,N_2,N_3}^{(0)}\right).$$

It follows readily that $\left(I - \frac{\tau_0}{2} g_v(\xi_0)\right)^{-1} = \operatorname{diag}\left(\frac{2}{2 - \tau_0 d_{1,1,1}^{(0)}}, \ldots, \frac{2}{2 - \tau_0 d_{N_1,N_2,N_3}^{(0)}}\right)$, and (b) holds if $\tau_0 d_{i,j,k}^{(0)} < 2$, $1 \le i \le N_1$, $1 \le j \le N_2$, $1 \le k \le N_3$. Denote $d^{(0)} \equiv \max_{i,j,k}\left\{d_{i,j,k}^{(0)}\right\}$, then $\tau_0 d^{(0)} < 2$, which leads to (b). $\square$

Lemma 4.2. *For any $\tau_\ell > 0$ we have*

$$\left(I - \frac{\tau_\ell}{2} M_\sigma\right) x \geq x, \quad \sigma = 1, 2, 3,$$

where $x = (1, 1, \ldots, 1)^\intercal$.

Proof. We only need to show the case with $w = \left(I - \frac{\tau_\ell}{2} M_1\right) x = (w_{1,1,1}, \ldots, w_{i,j,k}, \ldots, w_{N_1,N_2,N_3})^\intercal$. First, we observe that

$$\begin{aligned} w_{1,1,1} &= \left(1 - \frac{\tau_\ell}{2} \cdot \frac{-2}{a^2 \phi_{1,1,1} h_{1,0} h_{1,1}}\right) - \frac{\tau_\ell}{2} \cdot \frac{2}{a^2 \phi_{1,1,1} h_{1,1}(h_{1,0} + h_{1,1})} \\ &= 1 + \frac{\tau_\ell}{a^2 \phi_{1,1,1}} \left(\frac{1}{h_{1,0} h_{1,1}} - \frac{1}{h_{1,1}(h_{1,0} + h_{1,1})}\right) > 1. \end{aligned}$$

Second, for $i = 2, \ldots, N_1 - 1$ we have

$$\begin{aligned} w_{i,1,1} &= -\frac{\tau_\ell}{2} \cdot \frac{2}{a^2 \phi_{i,1,1} h_{1,i-1}(h_{1,i-1} + h_{1,i})} + \left(1 - \frac{\tau_\ell}{2} \cdot \frac{-2}{a^2 \phi_{i,1,1} h_{1,i-1} h_{1,i}}\right) \\ &\quad - \frac{\tau_\ell}{2} \cdot \frac{2}{a^2 \phi_{i,1,1} h_{1,i}(h_{1,i-1} + h_{1,i})} \\ &= 1 + \frac{\tau_\ell}{a^2 \phi_{i,1,1}} \left[\frac{-h_{1,i} + (h_{1,i-1} + h_{1,i}) - h_{1,i-1}}{h_{1,i-1} h_{1,i}(h_{1,i-1} + h_{1,i})}\right] = 1. \end{aligned}$$

Third, we have

$$\begin{aligned} w_{N_1,1,1} &= -\frac{\tau_\ell}{2} \cdot \frac{2}{a^2 \phi_{N_1,1,1} h_{1,N_1-1}(h_{1,N_1-1} + h_{1,N_1})} + \left(1 - \frac{\tau_\ell}{2} \cdot \frac{-2}{a^2 \phi_{N_1,1,1} h_{1,N_1-1} h_{1,N_1}}\right) \\ &= 1 + \frac{\tau_\ell}{a^2 \phi_{N_1,1,1}} \left[\frac{1}{h_{1,N_1}(h_{1,N_1-1} + h_{1,N_1})}\right] > 1. \end{aligned}$$

Hence, we conclude that $w_{i,1,1} \geq 1$, $i = 1, \ldots, N_1$. Similar arguments may show that all remaining elements of w are also bounded below by 1. Therefore we have $w \geq x$. Similar discussions may be utilized for the cases involving M_2 or M_3. □

In the next lemma we show that numerical quenching, *i.e.*, one or more components of v_ℓ reaching or exceeding unity, cannot occur immediately after the first time step under appropriate constraints. To this end, we denote $h = \max_{j=1,\ldots,N_\sigma,\ \sigma=1,2,3} \{h_{\sigma,j}\}$.

Lemma 4.3. *If (3.3) holds and $h^2 < \frac{1}{2\min\{a^2, b^2, c^2\}} \min\left\{\frac{1}{f_0}, \frac{4}{f(\tau_0 f_0/\phi_{\min})}\right\}$, then for given $v_0 \equiv 0$, we have that all components of $v_1 < 1$.*

Proof. If $v_0 \equiv 0$, then from (2.8) we have $v_1 = \left[\prod_{\sigma=1}^{3} \left(I - \frac{\tau_0}{2} M_\sigma\right)^{-1} \left(I + \frac{\tau_0}{2} M_\sigma\right)\right] \frac{\tau_0}{2} g(0) + \frac{\tau_0}{2} g(v_1)$. Using $g(v_1) \approx g(w_0) = g(v_0 + \tau_0(Cv_0 + g(v_0))) = g(\tau_0 f_0 \xi)$, where $\xi = \left(\phi_{1,1,1}^{-1}, \ldots, \phi_{N_1,N_2,N_3}^{-1}\right)^\intercal \in \mathbb{R}^{N_1 N_2 N_3}$, we have following decomposed connections

$$\left(I - \frac{\tau_0}{2} M_1\right) \tilde{v}_0 = \left(I + \frac{\tau_0}{2} M_1\right) \frac{\tau_0}{2} f_0 \xi, \tag{4.2}$$

$$\left(I - \frac{\tau_0}{2}M_2\right)\bar{v}_0 = \left(I + \frac{\tau_0}{2}M_2\right)\tilde{v}_0, \tag{4.3}$$

$$\left(I - \frac{\tau_0}{2}M_3\right)\left(v_1 - \frac{\tau_0}{2}g(\tau_0 f_0 \xi)\right) = \left(I + \frac{\tau_0}{2}M_3\right)\bar{v}_0. \tag{4.4}$$

From (4.2) we observe that

$$\tilde{v}_0 - \frac{1}{4}x = \left(I - \frac{\tau_0}{2}M_1\right)^{-1}\left[\left(I + \frac{\tau_0}{2}M_1\right)\frac{\tau_0}{2}f_0\xi - \frac{1}{4}\left(I - \frac{\tau_0}{2}M_1\right)x\right] = \left(I - \frac{\tau_0}{2}M_1\right)^{-1}\left(s_1^+ + s_1^-\right)$$

for which

$$\begin{aligned}|s_1^+| &= \left|\left(I + \frac{\tau_0}{2}M_1\right)\frac{\tau_0 f_0}{2}\xi\right| \le \frac{\tau_0 f_0}{2}\phi_{\min}^{-1}\left\|I + \frac{\tau_0}{2}M_1\right\|_2 \\ &< \tau_0 f_0 \phi_{\min}^{-1} < \frac{h^2 f_0}{2\|B\|_2}\min\left\{a^2, b^2, c^2\right\}\phi_{\min}^{-1} \le \frac{h^2 f_0}{2}\min\left\{a^2, b^2, c^2\right\}.\end{aligned}$$

The above indicates that $s_1^+ \le \dfrac{h^2 f_0}{2}\min\left\{a^2, b^2, c^2\right\}x$. On the other hand, according to Lemma 4.3 we have $s_1^- \le -\dfrac{1}{4}x$, and thus, $s_1^+ + s_1^- \le \left(\dfrac{h^2 f_0}{2}\min\left\{a^2, b^2, c^2\right\} - \dfrac{1}{4}\right)x$. Since we have $\left(I - \dfrac{\tau_0}{2}M_1\right)^{-1}$ is positive by (3.3), we wish each component of $s_1^+ + s_1^-$ to be negative. Thus, we require

$$\frac{h^2 f_0}{2}\min\left\{a^2, b^2, c^2\right\} - \frac{1}{4} < 0, \quad \text{or} \quad h < \frac{1}{\sqrt{2f_0 \min\left\{a^2, b^2, c^2\right\}}}. \tag{4.5}$$

Now, recall (4.3). It follows that

$$\bar{v}_0 - \frac{1}{2}x = \left(I - \frac{\tau_0}{2}M_2\right)^{-1}\left[\left(I + \frac{\tau_0}{2}M_2\right)\tilde{v}_0 - \frac{1}{2}x\right] = \left(I - \frac{\tau_0}{2}M_2\right)^{-1}\left(s_2^+ + s_2^-\right).$$

Note that $\left|s_2^+\right| = \left|\left(I + \frac{\tau_0}{2}M_2\right)\tilde{v}_0\right| < \dfrac{1}{4}\left\|I + \frac{\tau_0}{2}M_2\right\|_2 \le \frac{1}{2}$, which implies that $s_2^+ < \dfrac{1}{2}x$. Therefore we arrive at $s_2^+ + s_2^- < \dfrac{1}{2}x - \dfrac{1}{2}x = 0$, and the result follows from the fact that $\left(I - \dfrac{\tau_0}{2}M_2\right)^{-1}$ is positive by (3.3). By the same token, based on (4.4) we observe that

$$\begin{aligned}v_1 - x &= \left(I - \frac{\tau_0}{2}M_3\right)^{-1}\left[\left(I + \frac{\tau_0}{2}M_3\right)\bar{v}_0 + \left(I - \frac{\tau_0}{2}M_3\right)\left(\frac{\tau_0}{2}g(\tau_0 f_0\xi) - x\right)\right] \\ &= \left(I - \frac{\tau_0}{2}M_3\right)^{-1}\left(s_3^+ + s_3^-\right).\end{aligned}$$

It can be seen that

$$\begin{aligned}|s_3^+| &= \left|\left(I + \frac{\tau_0}{2}M_3\right)\bar{v}_0 + \left(I - \frac{\tau_0}{2}M_3\right)\frac{\tau_0}{2}g(\tau_0 f_0\xi)\right| \\ &\le \max\left\{|\bar{v}_0|, \left|\frac{\tau_0}{2}g(\tau_0 f_0\xi)\right|\right\}\left\|\left(I + \frac{\tau_0}{2}M_3\right) + \left(I - \frac{\tau_0}{2}M_3\right)\right\|_2 \\ &= \max\left\{1, \frac{h^2 f(\tau_0 f_0 \phi_{\min}^{-1})}{2}\min\left\{a^2, b^2, c^2\right\}\right\},\end{aligned}$$

and the above indicates that $s_3^+ \le \max\left\{1, \dfrac{h^2 f(\tau_0 f_0 \phi_{\min}^{-1})}{2}\min\left\{a^2, b^2, c^2\right\}\right\}x$. By Lemma 4.2 we conclude that $s_3^- \le -x$, and therefore,

$$
\begin{aligned}
s_3^+ + s_3^- &\le \max\left\{1, \frac{h^2 f(\tau_0 f_0 \phi_{\min}^{-1})}{2} \min\left\{a^2, b^2, c^2\right\}\right\} x - x \\
&= \max\left\{0, \frac{h^2 f(\tau_0 f_0 \phi_{\min}^{-1})}{2} \min\left\{a^2, b^2, c^2\right\} - 1\right\} x.
\end{aligned}
$$

Since we again wish each component of the above vector to be negative, we need

$$
\frac{h^2 f(\tau_0 f_0 \phi_{\min}^{-1})}{2} \min\left\{a^2, b^2, c^2\right\} - 1 < 0, \quad \text{or} \quad h^2 < \frac{2}{f(\tau_0 f_0/\phi_{\min}) \min\left\{a^2, b^2, c^2\right\}}.
$$

Hence, since $\left(I - \frac{\tau_0}{2} M_3\right)^{-1}$ is positive, $v_1 - x \le 0$ follows immediately from (4.5) and the above. $\square$

Combining above results we obtain the following theorem.

Theorem 4.1. *For any beginning step $\ell_0 \ge 0$ if τ_ℓ is sufficiently small for $\ell \ge \ell_0$ and*

(i) (3.3) *holds for all $\ell \ge \ell_0$,*
(ii) $h^2 < \frac{1}{2\min\{a^2, b^2, c^2\}} \min\left\{\frac{1}{f_0}, \frac{4}{f(\tau_0 f_0/\phi_{\min})}\right\}$, *where* $h = \max_{j=1,\ldots,N_\sigma,\ \sigma=1,2,3} \{h_{\sigma,j}\}$,
(iii) $C v_{\ell_0} + \frac{1}{2} g(v_{\ell_0}) > 0$ *and* $\left(I - \frac{\tau_{\ell_0}}{2} g_v(\xi_{\ell_0})\right)^{-1} > 0$,

then the sequence $\{v_\ell\}_{\ell \ge \ell_0}$ produced by the semi-adaptive LOD scheme (2.8) increases monotonically until unity is reached or exceeded by one or more components of the solution vector, i.e., until quenching occurs.

5. Stability

When the numerical solution varies relatively slowly, that is, before reaching a certain neighborhood of quenching, instability may be detected through a linear stability analysis of the nonlinear scheme utilized [6,12,21]. Although the application of such an analysis to nonlinear problems cannot be rigorously justified, it has been found to be remarkably informative in practical computations. In the following study, we will first carry out a linearized stability analysis in the von Neumann sense for (2.8) with its nonlinear source term frozen. The analysis will then be extended to circumstances where the nonlinear term is not frozen. In the later case, the boundedness of the Jacobian of the source term, $\|g_v(v)\|_2$, which is equivalent to assuming that we are some neighborhood away from quenching, is assumed.

In the following, let $A \in \mathbb{C}^{n\times n}$ and again denote $E(\cdot) = \exp(\cdot)$ for $n > 1$.

Definition 5.1. Let $\|\cdot\|$ be an induced matrix norm. Then the associated logarithmic norm $\mu : \mathbb{C}^{n\times n} \to \mathbb{R}$ of A is defined as

$$
\mu(A) = \lim_{h\to 0^+} \frac{\|I_n + hA\| - 1}{h},
$$

where $I_n \in \mathbb{C}^{n\times n}$ is the identity matrix.

Remark 5.1. If the matrix norm being considered is the spectral norm, then $\mu(A) = \max\{\lambda : \lambda$ is an eigenvalue of $(A + A^*)/2\} = \frac{1}{2}\lambda_{\max}(A + A^*)$.

Lemma 5.1. *(See [10].) For $\alpha \in \mathbb{C}$ we have*

$$
\|E(\alpha A)\| \le E(\alpha \mu(A)).
$$

For the semi-adaptive LOD method (2.8) with its nonlinear source term frozen, regularity conditions need to be imposed upon the nonuniform spatial grids for a linear stability analysis. For this purpose, let us denote $h_\sigma = \min_{j=1,\ldots,N_\sigma} \{h_{\sigma,j}\}$, $\sigma = 1,2,3$.

Lemma 5.2. *If*

$$\frac{1}{h_1^2\phi_{i-1,j,k}} - \frac{1}{h_{1,i-1}h_{1,i}\phi_{i,j,k}} \le \frac{K}{2}, \tag{5.1}$$

$$\frac{1}{h_2^2\phi_{i,j-1,k}} - \frac{1}{h_{2,j-1}h_{2,j}\phi_{i,j,k}} \le \frac{K}{2}, \tag{5.2}$$

$$\frac{1}{h_3^2\phi_{i,j,k-1}} - \frac{1}{h_{3,k-1}h_{3,k}\phi_{i,j,k}} \le \frac{K}{2}, \tag{5.3}$$

where the constant $K > 0$ is independent of $h_{\sigma,j}$, $j = 1,\ldots,N_\sigma$, $\sigma = 1,2,3$, then

$$\mu(M_\sigma) \le K, \ \ \sigma = 1,2,3.$$

Proof. We only need to consider the case involving M_1 since the other cases are similar. Note that $\mu(M_1) = \frac{1}{2}\lambda_{\max}\left(M_1 + M_1^\intercal\right)$ and

$$\frac{1}{2}\left(M_1 + M_1^\intercal\right) = \operatorname{diag}(X_{1,1},\ldots,X_{N_2,1},X_{1,2},\ldots,X_{N_2,N_3}) \in \mathbb{R}^{N_1N_2N_3\times N_1N_2N_3},$$

where

$$(X_{j,k})_{n,p} = \begin{cases} m_{1,n}/\phi_{n,j,k}, & \text{if } n = p, \\ n_{1,n-1}/2\phi_{n-1,j,k} + l_{1,n-1}/2\phi_{n,j,k}, & \text{if } n-p = 1, \\ n_{1,n}/2\phi_{n,j,k} + l_{1,n}/2\phi_{n+1,j,k}, & \text{if } p-n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We apply Geršchgorin's circle theorem to an arbitrary $X_{j,k}$, $j = 1,\ldots,N_2$, $k = 1,\ldots,N_3$. Further, notice that we only need to consider circumstances where the bandwidth of $M_1 + M_1^\intercal$ is three. Thus,

$$\left|\lambda_{1,i} - \frac{m_{1,i}}{\phi_{i,j,k}}\right| \le \left|\frac{n_{1,i-1}}{2\phi_{i-1,j,k}} + \frac{l_{1,i-1}}{2\phi_{i,j,k}}\right| + \left|\frac{n_{1,i}}{2\phi_{i,j,k}} + \frac{l_{1,i}}{2\phi_{i+1,j,k}}\right| \le \frac{2}{h_1^2\phi_{i-1,j,k}},$$
$$i = 2,\ldots,N_1-1, \ j = 1,\ldots,N_2, \ k = 1,\ldots,N_3.$$

We then see that (5.1) follows immediately from the above and the fact that

$$\frac{2}{h_1^2\phi_{i-1,j,k}} - \frac{2}{h_{1,i-1}h_{1,i}\phi_{i,j,k}} \le K, \quad i = 2,\ldots,N_1-1, \ j = 1,\ldots,N_2, \ k = 1,\ldots,N_3. \qquad \square$$

Lemma 5.3. *If (5.1)–(5.3) hold then*

$$\left\|\left(I - \frac{\tau_\ell}{2}M_\sigma\right)^{-1}\left(I + \frac{\tau_\ell}{2}M_\sigma\right)\right\|_2 \le 1 + \tau_\ell K + \mathcal{O}\left(\tau_\ell^2\right), \quad \ell \ge 0, \ \sigma = 1,2,3, \tag{5.4}$$

for sufficiently small $\tau_\ell > 0$.

Proof. Recalling the [1/1] Padé approximation utilized in Section 2, we have

$$\left(I-\frac{\tau_\ell}{2}M_\sigma\right)^{-1}\left(I+\frac{\tau_\ell}{2}M_\sigma\right)=E(\tau_\ell M_\sigma)+\mathcal{O}\left(\tau_\ell^3\right),\quad \sigma=1,2,3.$$

Now, based on Lemmas 5.1 and 5.2,

$$\begin{aligned}\left\|\left(I-\frac{\tau_\ell}{2}M_\sigma\right)^{-1}\left(I+\frac{\tau_\ell}{2}M_\sigma\right)\right\|_2 &\le E(\tau_\ell\mu(M_\sigma))+\mathcal{O}\left(\tau_\ell^3\right)\\ &\le \left[1+\tau_\ell K+\mathcal{O}\left(\tau_\ell^2\right)\right]+\mathcal{O}\left(\tau_\ell^3\right),\end{aligned}$$

which yields the desired bound. □

Combining the above results gives the following theorem.

Theorem 5.1. *If (3.3) and (5.1)–(5.3) hold, then the semi-adaptive LOD method (2.8) with the source term frozen is unconditionally stable in the von Neumann sense under the spectral norm, that is,*

$$\|z_{\ell+1}\|_2\le c\|z_0\|_2,\quad \ell\ge 0,$$

where $z_0=v_0-\tilde{v}_0$ is an initial error, $z_{\ell+1}=v_{\ell+1}-\tilde{v}_{\ell+1}$ is the $(\ell+1)$th perturbed error vector, and $c>0$ is a constant independent of ℓ and τ_ℓ.

Proof. When the nonlinear source term is frozen, $z_{\ell+1}$ takes the form of

$$z_{\ell+1}=\prod_{\sigma=1}^{3}\left(I-\frac{\tau_\ell}{2}M_\sigma\right)^{-1}\left(I+\frac{\tau_\ell}{2}M_\sigma\right)z_\ell,\quad \ell\ge 0. \tag{5.5}$$

Recall that $\sum_{k=0}^{\ell}\tau_k\le T,\ \ell>0$. It follows by taking the norm on both sides of (5.5) that

$$\begin{aligned}\|z_{\ell+1}\|_2 &\le \prod_{\sigma=1}^{3}\left\|\left(I-\frac{\tau_\ell}{2}M_\sigma\right)^{-1}\left(I+\frac{\tau_\ell}{2}M_\sigma\right)\right\|_2\|z_\ell\|_2\\ &\le \left(1+3\tau_\ell K+c_2\tau_\ell^2\right)\|z_\ell\|_2\le \prod_{k=0}^{\ell}\left(1+3\tau_k K+c_3\tau_k^2\right)\|z_0\|_2\\ &\le \left(1+3KT+c_4\sum_{k=0}^{\ell}\tau_k^2\right)\|z_0\|_2\le c\|z_0\|_2,\end{aligned}$$

where c_1,c_2,c_3,c_4 and c are positive constants independent of $\ell,\ \tau_k,\ 0\le k\le\ell$. Therefore the theorem is clear. □

We now consider the case without freezing the nonlinear source term in (2.8). In this situation, restrictions upon the Jacobian matrix $g_v(v)$ become necessary.

Theorem 5.2. *Let $\tau_k,\ 0\le k\le\ell$, be sufficiently small and (3.3), (5.1)–(5.3) hold. If there exists a constant $G<\infty$ such that*

$$\|g_v(\xi)\|_2\le G,\quad \xi\in\mathbb{R}^{N_1N_2N_3}, \tag{5.6}$$

then the semi-adaptive LOD method (2.8) is unconditionally stable in the von Neumann sense, that is,

$$\|z_{\ell+1}\|_2 \le \tilde{c}\,\|z_0\|_2, \quad \ell > 0,$$

where $z_0 = v_0 - \tilde{v}_0$ is an initial error, $z_{\ell+1} = v_{\ell+1} - \tilde{v}_{\ell+1}$ is the $(\ell+1)$th perturbed error vector, and $\tilde{c} > 0$ is a constant independent of ℓ and τ_ℓ.

Proof. By definition we have

$$\begin{aligned} v_{\ell+1} &= \prod_{\sigma=1}^{3}\left(I - \frac{\tau_\ell}{2}M_\sigma\right)^{-1}\left(I + \frac{\tau_\ell}{2}M_\sigma\right)\left(v_\ell + \frac{\tau_\ell}{2}g(v_\ell)\right) + \frac{\tau_\ell}{2}g(v_{\ell+1}) \\ &= \Phi_\ell\left(v_\ell + \frac{\tau_\ell}{2}g(v_\ell)\right) + \frac{\tau_\ell}{2}g(v_{\ell+1}), \end{aligned}$$

where $\Phi_\ell = \prod_{\sigma=1}^{3}\left(I - \frac{\tau_\ell}{2}M_\sigma\right)^{-1}\left(I + \frac{\tau_\ell}{2}M_\sigma\right)$. It follows that

$$\begin{aligned} z_{\ell+1} &= \Phi_\ell z_\ell + \frac{\tau_\ell}{2}\Phi_\ell(g(v_\ell) - g(\tilde{v}_\ell)) + \frac{\tau_\ell}{2}(g(v_{\ell+1}) - g(\tilde{v}_{\ell+1})) \\ &= \Phi_\ell z_\ell + \frac{\tau_\ell}{2}\Phi_\ell g_v(\xi_\ell) z_\ell + \frac{\tau_\ell}{2}g_v(\xi_{\ell+1}) z_{\ell+1}, \end{aligned}$$

where $\xi_k \in \mathcal{L}(v_k, \tilde{v}_k),\ k = \ell, \ell+1$. Rearranging the above equality, we have

$$\left(I - \frac{\tau_\ell}{2}g_v(\xi_{\ell+1})\right) z_{\ell+1} = \Phi_\ell\left(I + \frac{\tau_\ell}{2}g_v(\xi_\ell)\right) z_\ell.$$

Further, recall (5.6). When τ_k is sufficiently small we may claim that

$$\left(I - \frac{\tau_k}{2}g_v(\xi)\right)^{-1},\ I + \frac{\tau_k}{2}g_v(\xi) = E\left(\frac{\tau_k}{2}g_v(\xi)\right) + \mathcal{O}\left(\tau_k^2\right).$$

Thus,

$$\begin{aligned} z_{\ell+1} &= \left(I - \frac{\tau_\ell}{2}g_v(\xi_{\ell+1})\right)^{-1}\Phi_\ell\left(I + \frac{\tau_\ell}{2}g_v(\xi_\ell)\right) z_\ell \\ &= \left\{\prod_{k=0}^{\ell}\left[E\left(\frac{\tau_k}{2}g_v(\xi_{k+1})\right) + \mathcal{O}\left(\tau_k^2\right)\right]\Phi_k\left[E\left(\frac{\tau_k}{2}g_v(\xi_k)\right) + \mathcal{O}\left(\tau_k^2\right)\right]\right\} z_0. \end{aligned}$$

It follows therefore

$$\begin{aligned} \|z_{\ell+1}\|_2 &\le \left\{\prod_{k=0}^{\ell}\|\Phi_k\|_2\left\|E\left(\frac{\tau_k}{2}g_v(\xi_{k+1})\right)\right\|_2\left\|E\left(\frac{\tau_k}{2}g_v(\xi_k)\right)\right\|_2 + c_{1,k}\tau_k^2\right\}\|z_0\|_2 \\ &\le \left(1 + 3KT + c\sum_{k=0}^{\ell}\tau_k^2\right)\left(e^{GT} + c_1\sum_{k=0}^{\ell}\tau_k^2\right)\|z_0\|_2 \le \tilde{c}\,\|z_0\|_2, \end{aligned}$$

where $c_{1,k},\ k = 1, 2, \ldots, \ell$, are positive constants and $c, c_1, \tilde{c}$ are positive constants independent of ℓ and $\tau_\ell,\ \ell > 0$. Thus giving the desired stability. $\square$

The above theorem provides further insight as to why the standard linear analysis can be useful in estimating the nonlinear stability. The extra cost paid, however, is assuming the boundedness of $\|g_v(\xi)\|_2$.

Nevertheless, this is an improvement upon the traditional methodology of having the nonlinear source term frozen. In fact, the aforementioned bound is well-maintained in numerical experiments until certain neighborhoods of quenching are reached. This serves as an indication that the new analysis is valid and effective.

6. Conclusions

A semi-adaptive LOD scheme is developed for solving degenerate Kawarada equations possessing a strong quenching nonlinearity and singularity. While a temporal adaptation is performed via an arc-length monitoring mechanism of the temporal derivative of the solution, fixed nonuniform spatial grids are adopted. The novel splitting method is implicit and the impact of the degeneracy is found to be limited. Rigorous analysis is given for key computational features, including the positivity, monotonicity, and stability, of the numerical solution. Important criteria to guarantee these properties, which depend upon the variable steps and degenerate function, are obtained.

Under much weaker requirements (see the latest results in [3]), the temporal step restriction for guaranteeing monotone numerical solutions of our LOD scheme has been reduced to only one-half of those in uniform spatial mesh cases [18]. Furthermore, a realistic method of targeting the realization of nonlinear stability analysis is proposed and shown to be successful. Though this new strategy needs the boundedness of $\|g_v(\xi)\|_2$, the requirement is well-justified before quenching is reached. This improved methodology not only provides further insight into the stability, but also offers explanations as to why the linear stability analysis must be valid before quenching. On the other hand, simulations of real three-dimensional solutions still remain as one of the most challenging tasks. In anticipated future work we plan to utilize the latest *High Performance Computing* tools with large data computations for this purpose. More rigorous and generalized analysis, as well as non-exponential splitting based higher order splitting methods [17,16] will also be investigated, studied, and experimented with.

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