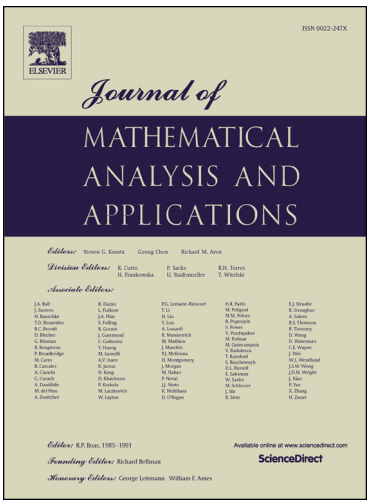
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A Variation On Uncertainty Principles For The Generalized q-Bessel Fourier Transform

Manel Hleili * & Bochra Nefzi † & Anis Bsaissa ‡

Abstract

The aim of this paper is to prove new uncertainty principles for the generalized q-Fourier Bessel transform early studied in [7]. To do so we prove a Nash-type inequality and a Carlson-type inequality for this transformation. From this we deduce a variation on Heisenberg's uncertainty inequality and Faris's local uncertainty principle. We also prove a variation on Donoho-Stark's uncertainty principle. Our results can be applied to the q-Bessel Fourier transform [9].

Keywords : Heisenberg uncertainty inequality, Generalized q-Bessel function, Generalized q-Bessel Fourier transform.

AMS 2010 Subject Classification: 33B15, 33D05, 44A20, 42A38, 42B10.

1 Introduction

A Fourier uncertainty principle is an inequality or uniqueness theorem concerning the joint localization of a function and its Fourier transform. Every discussion of the uncertainty principle must necessarily begin with the classical uncertainty principle which also called the Heisenberg-Pauli-Weil uncertainty inequality [14] in which concentration is measured in terms of dispersions. This leads to this quantitative formulation in the form of a lower bound of the product of the dispersions of a nonzero signal f and its Fourier transform $\mathcal{F}(f)$:

$$\||x|f\|_{L^2(\mathbb{R})}\ \||\xi|\mathcal{F}(f)\|_{L^2(\mathbb{R})} \geq\ \frac{1}{2}\ \|f\|^2_{L^2(\mathbb{R})}, \tag{1}$$

where for $\xi \in \mathbb{R}$

$$\mathcal{F}(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx.$$

Generalizations of this result in both classical and quantum analysis have been treated and many versions of Heisenberg-Pauli-Weil type uncertainty inequalities have been obtained for several generalized Fourier transforms (see [2, 12, 13, 24]). Therefore, one may obtain new uncertainty principles involving weighted L^{p_1} and L^{p_2}-norms ($1 < p_1 \leq p_2 < \infty$) by replacing the weight $|x|$ with other weight functions u and $\tilde{u}$ and search for inequalities of the form (see [3, 14])

$$\|uf\|_{L^{p_1}(\mathbb{R})}\ \|\tilde{u}\mathcal{F}(f)\|_{L^{p_2}(\mathbb{R})} \geq\ c\ \|f\|^2_{L^2(\mathbb{R})};\quad f \in L^2(\mathbb{R}).$$

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Moreover in [18, 20] the authors considered another variation of the uncertainty inequality (1) involving L^1-norms of the form

$$\||x|^2 f\|_{L^1(\mathbb{R})} \, \||\xi|\mathcal{F}(f)\|^2_{L^2(\mathbb{R})} \geq \; c \, \|f\|_{L^1(\mathbb{R})}\|f\|^2_{L^2(\mathbb{R})}. \tag{2}$$

Uncertainty principles when concentration is measured in terms of fast decay for the generalized q-Bessel Fourier transform have been stated in [7]. We will here concentrate on uncertainty principles in which concentration is measured in terms of (generalized) dispersions like (2) in Heisenberg's uncertainty. To do so we prove a Nash-type inequality [4, 21] a Carlson-type inequality [5, 19] for this transformation.

This paper is motivated by recent results stated in [7] which will be useful to our study. To describe these results in more detail, we need some notations.
Let us now be more precise. For $\alpha \in \mathbb{R}, \; n \in \mathbb{N}$ we set

$$\nu = (\alpha, -n) \qquad , |\nu| = \alpha - n > -1.$$

In all the sequel, we assume $0 < q < 1$ and we adapt the same notations as in [15, 16].
Let $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a,q)_0 = 1; \quad (a,q)_j = \prod_{k=0}^{j-1}(1-aq^k); \quad \text{and} \quad (a,q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

We recall that

$$[a]_q = \frac{1-q^a}{1-q}.$$

The q-derivative of a function f is given by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad \text{if} \quad x \neq 0,$$

and $D_q f(0) = f'(0)$ provided $f'(0)$ exists. Note that when f is differentiable, at x, then $D_q f(x)$ tends to $f'(x)$ as q tends to 1^-.
The q-Jackson integrals from 0 to a and from 0 to ∞ are defined by (see [16])

$$\int_0^a f(x) d_q x = (1-q)a \sum_{k=0}^{\infty} f(aq^k) q^k,$$

$$\int_0^\infty f(x) d_q x = (1-q) \sum_{k=-\infty}^{\infty} f(q^k) q^k,$$

provided the sums converge absolutely.
Note that

$$\int_a^b D_q f(x) d_q x = f(b) - f(a), \quad \forall a, b \in \mathbb{R}_q^+.$$

The normalized form of the q-Bessel function is defined by (see [8, 9])

$$j_\alpha(x,q) = \sum_{k=0}^{\infty} (-1)^k \frac{q^{\frac{k(k+1)}{2}}}{(q,q)_k (q^{\alpha+1},q)_k} x^{2k}.$$

We need the following spaces and norms:

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• $\mathbb{R}_q^+ = \{q^k,\ k \in \mathbb{Z}\}$.

•$B_{r,q} = \{x \in \mathbb{R}_q^+,\ |x| \leq r\}$ be the ball of radius r and centre 0. We note $\chi_r = \chi_{B_{r,q}}$ and $\tilde{\chi_r} = 1 - \chi_r$, where χ_r is the characteristic function of $[0, r]$.

• $\mathcal{C}_{q,0}$: the set of all functions defined on $\mathbb{R}_q^+$, continuous in 0 and vanishing at infinity, equipped with the induced topology of uniform convergence such that

$$\|f\|_{q,\infty} = \sup_{x \in \mathbb{R}_q^+} |f(x)| < \infty.$$

• The space $\mathcal{L}_{q,p,\nu}$, $1 \leq p < \infty$ denotes the set of functions on $\mathbb{R}_q^+$ such that

$$\|f\|_{q,p,\nu} = \left[\int_0^\infty |f(x)|^p x^{2|\nu|+1} d_q x\right]^{1/p} < \infty.$$

The generalized q-Bessel function is defined by (see [7])

$$\widetilde{j}_{q,\nu}(x, q^2) = x^{2n} j_{\alpha+n}(q^n x, q^2), \quad n \in \mathbb{N},$$

where $j_{\alpha+n}(., q^2)$ is the normalized Hahn-Exton function of index $\alpha + n$.
For the particular case $\nu = (\alpha, 0),\ \alpha > -1,\ \widetilde{j}_{q,\nu}(., q^2) = j_\alpha(., q^2)$.

In [7], the authors proved the following results

$$|\widetilde{j}_{q,\nu}(q^k, q^2)| \leq q^{2kn} \frac{(-q^2, q^2)_\infty (-q^{2\alpha+2}, q^2)_\infty (q^{2\alpha+2}, q^2)_n}{(-q^{2\alpha+2}, q^2)_n (q^{2\alpha+2}, q^2)_\infty} \times \begin{cases} q^{2nk} & \text{if} \quad n+k \geq 0 \\ q^{(n+k)^2-(n+k)(2\alpha+1)-2n^2} & \text{if} \quad n+k < 0 \end{cases}.$$

The function $x \mapsto \widetilde{j}_{q,\nu}(ax, q^2)$ is a solution of the following q-differential equation

$$\left[1 - \frac{\widetilde{\Delta}_{q,\nu}}{a^2}\right] f(x) = 0,$$

where $\widetilde{\Delta}_{q,\nu}$ is the generalized q-Bessel operator

$$\widetilde{\Delta}_{q,\nu} f(x) = \frac{f(q^{-1}x) - (q^{2\alpha} + q^{-2n}) f(x) + q^{2\alpha-2n} f(qx)}{x^2},$$

which can be factorized as follows

$$\widetilde{\Delta}_{q,\nu} = \partial^*_{q,\alpha} \partial_{q,n},$$

where

$$\partial_{q,n} f(x) = \frac{f(q^{-1}x) - q^{-2n} f(x)}{x}$$

$$\partial^*_{q,\alpha} f(x) = \frac{f(x) - q^{2\alpha+1} f(qx)}{x}.$$

The generalized q-Bessel Fourier transform $\mathcal{F}_{q,\nu}$ is defined as follows (see [7])

$$\mathcal{F}_{q,\nu} f(x) = c_{q,\nu} \int_0^\infty f(t)\ \widetilde{j}_{q,\nu}(tx, q^2)\ t^{2|\nu|+1} d_q t, \tag{3}$$

where

$$c_{q,\nu} = \frac{q^{n(\alpha+n)}}{(1-q)} \frac{(q^{2\alpha+2}, q^2)_\infty}{(q^2, q^2)_\infty (q^{2\alpha+2}, q^2)_n}.$$

Moreover, it was shown in [7], that the generalized q-Bessel Fourier transform verifies the following properties.

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Proposition 1 *The generalized q-Bessel Fourier transform*

$$\mathcal{F}_{q,\nu} : \mathcal{L}_{q,1;\nu} \to \mathcal{C}_{q,0},$$

satisfies

$$\|\mathcal{F}_{q,\nu} f\|_{q,\infty} \leq B_{q,\nu} \|f\|_{\nu,1,q}, \tag{4}$$

where

$$B_{q,\nu} = \frac{q^{n(\alpha+n+2k)}}{(1-q)} \frac{(-q^2,q^2)_\infty (-q^{2\alpha+2},q^2)_\infty}{(q^2,q^2)_\infty (-q^{2\alpha+2},q^2)_n}. \tag{5}$$

Theorem 1 *1. Let f be a function in the $\mathcal{L}_{\nu,p,q}$ space where $p \geq 1$ then*

$$\mathcal{F}_{q,\nu}^2 f = f.$$

2. Let f be a function in the $\mathcal{L}_{q,2,\nu}$ then

$$\|\mathcal{F}_{q,\nu} f\|_{q,2,\nu} = \|f\|_{q,2,\nu}.$$

3. Let f be a function in the $\mathcal{L}_{q,1,\nu} \cap \mathcal{L}_{q,p,\nu}, p \geq 2$ then

$$f \in \mathcal{L}_{q,2,\nu} \quad \textit{and} \quad \|\mathcal{F}_{q,\nu} f\|_{q,2,\nu} = \|f\|_{q,2,\nu}. \tag{6}$$

Throughout this paper, if S is q-measurable subset in $\mathbb{R}_q^+$, we will write:

$$|S| = \int_0^\infty \chi_S \; x^{2|\nu|+1} d_q x, \tag{7}$$

where χ_S is the characteristic function of S.

Our first result will be the following variation Heisenberg's uncertainty inequality for the generalized q-Bessel Fourier transform $\mathcal{F}_{q,\nu}$:

THEOREM A: Let $a > 0$. Then there exists a constant $k_q(\nu, a)$ such that for all $f \in \mathcal{L}_{q,1,\nu} \cap \mathcal{L}_{q,p,\nu},\ p \geq 2$,

$$\left\| |x|^{2a} f \right\|_{q,1,\nu} \left\| |\lambda|^a \mathcal{F}_{q,\nu} f \right\|_{q,2,\nu}^2 \geq k_q(\nu, a) \left\| f \right\|_{q,1,\nu} \left\| f \right\|_{q,2,\nu}^2 .$$

This theorem implies that if f is highly localized in the neighborhood of $x = 0$ i.e $\left\| |x|^{2a} f \right\|_{q,1,\nu}$ is small enough, then $\mathcal{F}_{q,\nu}$ cannot be concentrated in the neighborhood of $\lambda = 0$.

We show below that a more general local principle holds for $\mathcal{F}_{q,\nu}$:

THEOREM B: Let $a > 0$. Then there exists a constant $C_q(\nu, a)$ such that for all $f \in \mathcal{L}_{q,1,\nu} \cap \mathcal{L}_{q,p,\nu},\ p \geq 2$ and all q-mesurable subset E of $\mathbb{R}_q^+$:

$$\left\| \mathcal{F}_{q,\nu}(f) \chi_r \right\|_{q,2,\nu} \quad \leq \quad C_q(\nu, a) |E|^{1/2} \left\| f \right\|_{q,2,\nu}^{\frac{2a}{2a+|\nu|+1}} \left\| |x|^{2a} f \right\|_{q,1,\nu}^{\frac{|\nu|+1}{2a+|\nu|+1}} .$$

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Another uncertainty principle which is of particular interest is an analogue of Donoho-Stark's for $\mathcal{F}_{q,\nu}$:

THEOREM C: Let $0 < \epsilon_1 < 1$ and $0 < \epsilon_2 < 1$. Let S and E be a pair of q-measurable subsets of $\mathbb{R}_q^+$ and $f \in \mathcal{L}_{q,1,\nu} \cap \mathcal{L}_{q,p,\nu},\ p \geq 2$. If f is ϵ_1-timelimited on S (see 16) and ϵ_2-bandlimited on E (see 17), then

$$|S||E| \geq B_{q,\nu}^{-2}(1-\epsilon_1^2)(1-\epsilon_2^2),$$

where $B_{q,\nu}$ is given by (5).

The main body of the paper is organized as follows. In section 2, we prove a variation on Heisenberg's uncertainty inequality for the generalized q-Bessel Fourier transform. Then, we state Faris's local uncertainty inequality and Donoho-Stark's uncertainty principle for this transformation. Finally in section 3, we apply our results for the q-Bessel Fourier transform early studied in [9].

2 A variation on Heisenberg's uncertainty principles for $\mathcal{F}_{q,\nu}$

Variations for the Fourier transform have appeared in the literature in [12, 14, 18] to obtain an upper bound for the so-called Laue constant, and in the survey paper [20] for an extensive discussion.
Now we will show an analogue of inequality (2) for the transformation $\mathcal{F}_{q,\nu}f$. To do so we start by proving a Nash-type and a Carlson-type inequalities for $\mathcal{F}_{q,\nu}f$:

1. Nash-type inequality:

The classical Nash inequality in $\mathbb{R}^d$ may be stated as

$$\left(\int_{\mathbb{R}^d} |u(x)|^2 dx\right)^{1+\frac{2}{d}} \leq A_d \left(\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx\right) \left(\int_{\mathbb{R}^d} |u(x)| dx\right)^{\frac{4}{d}}, \tag{8}$$

for all functions $u \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Such an inequality first appeared in the celebrated paper of Nash [21], where he discussed the Hölder regularity of solutions of divergence form in uniformly elliptic equations and the optimal constant A_d has been computed more recently in [4].

Proposition 2 *(Nash-type inequality for $\mathcal{F}_{q,\nu}$)*
Let $a > 0$. Then there exists a constant $k_{1,q}(\nu, a)$ such that for all $f \in \mathcal{L}_{q,1,\nu} \cap \mathcal{L}_{q,p,\nu},\ p \geq 2$

$$\|f\|_{q,2,\nu}^2 \leq k_{1,q}(\nu,a)\, \|f\|_{q,1,\nu}^{\frac{2a}{a+|\nu|+1}}\, \||x|^a \mathcal{F}_{q,\nu} f\|_{q,2,\nu}^{\frac{2(|\nu|+1)}{a+|\nu|+1}}, \tag{9}$$

where $k_{1,q}(\nu, a)$ is given by

$$k_{1,q}(\nu,a) = \left(1+\frac{a}{|\nu|+1}\right)\left[\frac{1}{[2(|\nu|+1)]_q} B_{q,\nu}^2 \left(\frac{|\nu|+1}{a}\right)\right]^{\frac{a}{a+|\nu|+1}}.$$

Proof. For all $r > 0$ and $f \in \mathcal{L}_{q,1,\nu} \cap \mathcal{L}_{q,p,\nu},\ p \geq 2$, we have

$$\begin{aligned}
\int_0^\infty |\mathcal{F}_{q,\nu} f(x)|^2 x^{2|\nu|+1} d_q x &= \|\mathcal{F}_{q,\nu}(f)\|_{q,2,\nu}^2 \\
&= \|f\|_{q,2,\nu}^2 \\
&\leq \|\mathcal{F}_{q,\nu}(f)\chi_r\|_{q,2,\nu}^2 + \|\mathcal{F}_{q,\nu}(f)\tilde{\chi_r}\|_{q,2,\nu}^2.
\end{aligned}$$

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Hence using the Hölder inequality and if k is the integer such that $q^k \leq r < q^{k-1}$, one gets

$$\begin{aligned} \|\mathcal{F}_{q,\nu}(f)\chi_r\|_{q,2,\nu}^2 &\leq \int_0^{q^k} x^{2|\nu|+1} d_q x \; \|\mathcal{F}_{q,\nu}(f)\|_{q,\infty}^2 \\ &\leq B_{q,\nu}^2 \frac{r^{2(|\nu|+1)}}{[2(|\nu|+1)]_q} \; \|f\|_{q,1,\nu}^2. \end{aligned} \tag{10}$$

However, since $f \in \mathcal{L}_{q,1,\nu} \cap \mathcal{L}_{q,p,\nu},\ p \geq 2$ and $a > 0$, we obtain

$$\begin{aligned} \|\mathcal{F}_{q,\nu}(f)\tilde{\chi_r})\|_{q,2,\nu}^2 &= \||x|^a \mathcal{F}_{q,\nu}(f)|x|^{-a}\tilde{\chi_r})\|_{q,2,\nu}^2 \\ &\leq \||x|^{-a}\tilde{\chi_r}\|_{q,\infty}^2 \||x|^a \mathcal{F}_{q,\nu}(f)\|_{q,2,\nu}^2 \leq r^{-2a}\||x|^a \mathcal{F}_{q,\nu}(f)\|_{q,2,\nu}^2. \end{aligned} \tag{11}$$

Then by (10) and (11) we have

$$\|f\|_{q,2,\nu}^2 \leq B_{q,\nu}^2 \frac{r^{2(|\nu|+1)}}{[2(|\nu|+1)]_q} \; \|f\|_{q,1,\nu}^2 + r^{-2a}\||x|^a \mathcal{F}_{q,\nu}(f)\|_{q,2,\nu}^2.$$

By minimization of the right-hand side of the previous inequality over $r > 0$, we obtain the desired result (9). ■

2. Carlson-type inequality:

In 1934, Carlson [5] proved that the following inequality

$$\left(\int_0^\infty |u(x)|dx\right)^4 \leq \pi^2 \left(\int_0^\infty |u(x)|^2 dx\right)\left(\int_0^\infty x^2|u(x)|^2 dx\right), \tag{12}$$

holds and π^2 is the best constant. Several generalizations and applications in different branches of mathematics have been given during the years. For a complete survey of the results and applications concerning the above inequalities and also interesting historical remarks see the book [17].

Proposition 3 *(Carlson-type inequality for $\mathcal{F}_{q,\nu}$)*
Let $a > 0$. Then there exists a constant $k_{2,q}(\nu, a)$ such that for all $f \in \mathcal{L}_{q,1,\nu} \cap \mathcal{L}_{q,p,\nu},\ p \geq 2$

$$\|f\|_{q,1,\nu}^{2a+|\nu|+1} \leq k_{2,q}(\nu, a) \, \|f\|_{q,2,\nu}^{2a} \left\||x|^{2a} f\right\|_{q,1,\nu}^{|\nu|+1}, \tag{13}$$

where $k_{2,q}(\nu, a)$ is given by

$$k_{2,q}(\nu, a) = \left(1 + \tfrac{2a}{|\nu|+1}\right)^{2a+|\nu|+1} \left(\tfrac{|\nu|+1}{2a\sqrt{[2(|\nu|+1)]_q}}\right)^{2a}.$$

Proof. For all $r > 0$, we may write

$$\begin{aligned} \|f\|_{q,1,\nu} &= \|f\chi_r\|_{q,1,\nu} + \|f\tilde{\chi}_r\|_{q,1,\nu} \\ &\leq \|f\chi_r\|_{q,1,\nu} + r^{-2a}\||x|^{2a} f\|_{q,1,\nu}. \end{aligned}$$

Hence it follows by the Cauchy-Schwarz inequality, that

$$\begin{aligned} \|f\|_{q,1,\nu} &\leq \left(\int_0^{q^k} x^{2|\nu|+1} d_q x\right)^{1/2} \|f\|_{q,2,\nu} + r^{-2a}\||x|^{2a} f\|_{q,1,\nu} \\ &\leq \frac{r^{(|\nu|+1)}}{\sqrt{[2(|\nu|+1)]_q}} \|f\|_{q,2,\nu} + r^{-2a}\||x|^{2a} f\|_{q,1,\nu}. \end{aligned}$$

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By minimizing the right-hand side of that inequality over $r > 0$, we obtain the desired result (13). ■

Multiplying Inequality (13) by $\|f\|^2_{q,2,\nu}$ and substituting (9) to the right-hand side give the following variation on Heisenberg's uncertainty inequality for $\mathcal{F}_{q,\nu}$:

Theorem 2 *(q-analogue uncertainty inequality type for $\mathcal{F}_{q,\nu}$:)*
Let $a > 0$. Then there exists a constant $k_q(\nu, a)$ such that for all $f \in \mathcal{L}_{q,1,\nu} \cap \mathcal{L}_{q,p,\nu},\ p \geq 2$

$$\left\||x|^{2a} f\right\|_{q,1,\nu} \left\||\lambda|^a \mathcal{F}_{q,\nu} f\right\|^2_{q,2,\nu} \geq k_q(\nu, a) \|f\|_{q,1,\nu} \|f\|^2_{q,2,\nu}, \tag{14}$$

where

$$k_q(\nu, a) = (k_{1,q}(\nu, a))^{-\frac{a+|\nu|+1}{|\nu|+1}} (k_{2,q}(\nu, a))^{-\frac{1}{|\nu|+1}}.$$

The uncertainty inequality (14) states that the more a function $f \in \mathcal{L}_{q,1,\nu} \cap \mathcal{L}_{q,p,\nu},\ p \geq 2$ is concentrated around $x = 0$, the more its generalized q-Bessel Fourier transform $\mathcal{F}_{q,\nu}$ must be spread and vice versa. However, they do not preclude that this spreading may be achieved by two or more peaks far apart. In fact, the latter phenomenon cannot occur either, and it is the object of local uncertainty inequalities to make this precise. The first such inequalities for the Fourier transform were obtained by Faris [11], and they were subsequently sharpened and generalized by Price [22, 23].

In the following theorem, we give another local uncertainty inequality which compares $\|\mathcal{F}_{q,\nu}(f)\chi_r\|_{q,2,\nu}$ with the dispersion $\left\||x|^{2a} f\right\|_{q,1,\nu}$ and this result is based on the Carlson-type inequality (13).

Theorem 3 *(q-analogue local uncertainty inequality type for $\mathcal{F}_{q,\nu}$:)*
Let $a > 0$. Then there exists a constant $C_q(\nu, a)$ such that for all $f \in \mathcal{L}_{q,1,\nu} \cap \mathcal{L}_{q,p,\nu},\ p \geq 2$ and all q-mesurable subset E of $\mathbb{R}_q^+$:

$$\|\mathcal{F}_{q,\nu}(f)\chi_r\|_{q,2,\nu} \leq C_q(\nu, a)|E|^{1/2} \|f\|^{\frac{2a}{2a+|\nu|+1}}_{q,2,\nu} \left\||x|^{2a} f\right\|^{\frac{|\nu|+1}{2a+|\nu|+1}}_{q,1,\nu}, \tag{15}$$

where

$$C_q(\nu, a) = B_{q,\nu} \left(1 + \frac{2a}{|\nu|+1}\right) \left(\frac{|\nu|+1}{2a\sqrt{[2(|\nu|+1)]_q}}\right)^{\frac{2a}{2a+|\nu|+1}}.$$

Proof. We have

$$\begin{aligned} \|\mathcal{F}_{q,\nu}(f)\chi_r\|_{q,2,\nu} &\leq \sqrt{|E|}\, \|\mathcal{F}_{q,\nu}(f)\|_{q,\infty} \\ &\leq B_{q,\nu}\sqrt{|E|}\, \|f\|_{q,1,\nu}. \end{aligned}$$

The desired result (15) follows from the Carlson Inequality (13). ■

Uncertainty principles for integral operators with bounded kernel have been first considered by de Jeu [6] to prove a generalization of Donoho-Stark's uncertainty principle [10].
Now we will investigate the case where f and $\mathcal{F}_{q,\nu}$ are close to zero outside q-measurable subsets. To do so we start with the following definition:

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Definition 1 *Let $0 < \epsilon < 1$ and S, E be a pair of q-measurable subsets on $\mathbb{R}_q^+$. For $f \in \mathcal{L}_{q,1,\nu} \cap \mathcal{L}_{q,p,\nu},\ p \geq 2$:*

1. *We say that f is ϵ-timelimited on S if*

$$\|\mathcal{F}_{q,\nu}(f)\chi_{S^c}\|_{q,1,\nu} \leq \epsilon \|f\|_{q,1,\nu}. \tag{16}$$

2. *We say that f is ϵ-bandlimited on E if*

$$\|\mathcal{F}_{q,\nu}(f)\chi_{E^c}\|_{q,2,\nu} \leq \epsilon \|f\|_{q,2,\nu}. \tag{17}$$

Theorem 4 *(Donoho-Stark's uncertainty principle for $\mathcal{F}_{q,\nu}$)*
Let $0 < \epsilon_1 < 1$ and $0 < \epsilon_2 < 1$. Let S and E be a pair of q-measurable subsets of $\mathbb{R}_q^+$ and $f \in \mathcal{L}_{q,1,\nu} \cap \mathcal{L}_{q,p,\nu},\ p \geq 2$. If f is ϵ_1-timelimited on S and ϵ_2-bandlimited on E, then

$$|S||E| \geq B_{q,\nu}^{-2}(1-\epsilon_1^2)(1-\epsilon_2^2). \tag{18}$$

Proof. We have

$$\|f\chi_s\|_{q,1,\nu} = \|f\|_{q,1,\nu} - \|f\chi_{s^c}\|_{q,1,\nu},$$

it follows then by (16) that

$$\|f\chi_s\|_{q,1,\nu} \geq (1-\epsilon_1)\|f\|_{q,1,\nu}.$$

Moreover by the Cauchy-Schwarz inequality, we have

$$\|f\chi_s\|_{q,1,\nu}^2 \leq |S|\,\|f\|_{q,2,\nu}^2.$$

Hence

$$|S|\,\|f\|_{q,2,\nu}^2 \geq (1-\epsilon_1^2)\|f\|_{q,1,\nu}^2. \tag{19}$$

On the other hand, by (17) and Plancherel formula we have

$$\begin{aligned}
\|\mathcal{F}_{q,\nu}(f)\chi_E\|_{q,2,\nu}^2 &= \|\mathcal{F}_{q,\nu}(f)\|_{q,2,\nu}^2 - \|\mathcal{F}_{q,\nu}(f)\chi_{E^c}\|_{q,2,\nu}^2 \\
&\leq \|\mathcal{F}_{q,\nu}(f)\|_{q,2,\nu}^2 - \epsilon_2^2\|f\|_{q,2,\nu}^2 \\
&\leq (1-\epsilon_2^2)\|f\|_{q,2,\nu}^2.
\end{aligned}$$

Moreover, we have

$$\|\mathcal{F}_{q,\nu}(f)\chi_E\|_{q,2,\nu}^2 \leq |E|\,\|\mathcal{F}_{q,\nu}(f)\|_{q,\infty}^2 \leq |E|B_{q,\nu}^2\|f\|_{q,1,\nu}^2.$$

So

$$|E|B_{q,\nu}^2\|f\|_{q,1,\nu}^2 \geq (1-\epsilon_2^2)\|f\|_{q,2,\nu}^2. \tag{20}$$

Inequalities (19) and (20) give the desired result (18). ■

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3 Application

In this section, we give different kind of uncertainty principles for the q-Bessel Fourier transform early studied in [8, 9]. Note that other versions of the Heisenberg uncertainty principle for the q-Fourier transform have recently appeared in the literature [1, 2, 9, 12]. There are some differences of the results cited above and our result.

For the particular case $\nu = (\alpha, 0),\ \alpha > -1$, the q-Bessel Fourier transform $\mathfrak{F}_{q,\alpha}$ is defined as follows

$$\mathfrak{F}_{q,\alpha} f(x) = c_{q,\alpha} \int_0^\infty f(t)\ j_\alpha(tx, q^2)\ t^{2\alpha+1} d_q t, \tag{21}$$

where

$$c_{q,\alpha} = \frac{1}{(1-q)} \frac{(q^{2\alpha+2}, q^2)_\infty}{(q^2, q^2)_\infty}.$$

It was shown in [9] that we have the following results:

1. For $f \in \mathcal{L}_{q,1,\alpha}$, one has $\mathfrak{F}_{q,\alpha} \in \mathcal{C}_{q,0}$ and

$$\|\mathfrak{F}_{q,\alpha} f\|_{q,\infty} \leq B_{q,\alpha} \|f\|_{q,1,\alpha}, \tag{22}$$

   where

$$B_{q,\alpha} = \frac{1}{(1-q)} \frac{(-q^2, q^2)_\infty (-q^{2\alpha+2}, q^2)_\infty}{(q^2, q^2)_\infty}. \tag{23}$$

2. For all $f \in \mathcal{L}_{q,2,\alpha} \cap \mathcal{L}_{q,p,\alpha},\ p \geq 2$, we have

$$f \in \mathcal{L}_{q,2,\alpha} \quad \text{and} \quad \|\mathfrak{F}_{q,\alpha} f\|_{q,2,\alpha} = \|f\|_{q,2,\alpha}.$$

So by application of Theorems 2, 3 and 4 , we deduce the following results.

Theorem 5 *Let $s > 0$. Let S and E be a pair of q-measurable subsets of $\mathbb{R}_q^+$. Then we have*

1. A q-analogue uncertainty inequality type for $\mathfrak{F}_{q,\alpha}$:

There exists a constant $A_{1,q}(\alpha, s)$, such that for all $f \in \mathcal{L}_{q,2,\alpha} \cap \mathcal{L}_{q,p,\alpha},\ p \geq 2$,

$$\left\| |x|^{2s} f \right\|_{q,1,\alpha} \left\| |\lambda|^s \mathfrak{F}_{q,\alpha} f \right\|_{q,2,\alpha}^2 \geq A_{1,q}(\alpha, s) \|f\|_{q,1,\alpha} \|f\|_{q,2,\alpha}^2 .$$

2. A q-analogue local uncertainty inequality type for $\mathfrak{F}_{q,\alpha}$:

There exists a constant $A_{2,q}(\alpha, s)$, such that for all $f \in \mathcal{L}_{q,2,\alpha} \cap \mathcal{L}_{q,p,\alpha},\ p \geq 2$,

$$\|\mathfrak{F}_{q,\alpha}(f)\chi_r\|_{q,2,\alpha} \ \leq \ A_{2,q}(\alpha, s) |E|^{1/2} \|f\|_{q,2,\alpha}^{\frac{2s}{2s+\alpha+1}} \left\| |x|^{2s} f \right\|_{q,1,\alpha}^{\frac{\alpha+1}{2s+\alpha+1}} .$$

3. A Donoho-Stark uncertainty inequality type for $\mathfrak{F}_{q,\alpha}$:

If $f \in \mathcal{L}_{q,2,\alpha} \cap \mathcal{L}_{q,p,\alpha},\ p \geq 2$ is ϵ_1-timelimited on S and ϵ_2-bandlimited on E, then

$$|S||E| \geq B_{q,\alpha}^{-2} (1 - \epsilon_1^2)(1 - \epsilon_2^2),$$

where $B_{q,\alpha}$ is given by (23).

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