



Multiple solutions for Schrödinger-Poisson systems with indefinite potential and combined nonlinearity

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Abstract

In this paper, we study the existence of infinitely many solutions for the following nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x, u) + g(x, u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & \lim_{|x| \rightarrow \infty} \phi(x) = 0, \end{cases}$$

where the potential V may be unbounded from below. Under some mild conditions on the nonlinear terms f and g , we obtain infinitely many solutions of this system. Recent results from the literature are generalized and significantly improved.

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1 Introduction and main results

In this paper, we consider the following nonlinear Schrödinger-Poisson system, also known as the coupled nonlinear Schrödinger-Maxwell equations

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x, u) + g(x, u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & \lim_{|x| \rightarrow \infty} \phi(x) = 0. \end{cases} \quad (\text{SP})$$

Such a system and similar ones arise in many mathematical physical context, such as in quantum electrodynamics, to describe the interaction between a charge particle interacting with the electromagnetic field, and also in semiconductor theory, in nonlinear optics and

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in plasma physics (see [4] for more details in the physics aspects). In particular, if we want to seek electrostatic-type solutions, we usually need to solve (SP).

In recent years, with the aid of variational methods, the existence, nonexistence and multiplicity of various solutions for (SP) have been extensively investigated in the literature. According to the conditions imposed on the potential V , these results can be roughly classified into three cases. Many papers deal with the case where V is a positive constant or radially symmetric, see, for example, [2, 3, 8–12, 16, 20, 24, 27, 28, 32, 38, 39, 41, 42] and the references therein. As far as we know, only [1, 33, 37] treat the case where V possesses some kind of periodicity. In addition to these two cases, there are also a great number of papers devoted to the case where V is not necessarily radially symmetric or periodic, see, for instance, [3, 7, 13, 14, 17–19, 23, 30, 31, 34, 35, 37, 40] and the references therein. Here we emphasize that almost in all these mentioned papers the conditions imposed on the potential V always imply that V is bounded from below, which is crucial for the corresponding results.

In the present paper, different from the references mentioned above, we are going to study the existence of infinitely many solutions for (SP) in the case where the potential V may be unbounded from below. Specifically, we first assume V satisfies

$$(S_1) \quad V \in L^q_{loc}(\mathbb{R}^3) \text{ and } V^- := \min\{V, 0\} \in L^\infty(\mathbb{R}^3) + L^q(\mathbb{R}^3) \text{ for some } q \geq 2.$$

This type of assumptions on the potential V has already been introduced in [21] for studying Schrödinger equations (see also [36]), which ensures that the Schrödinger operator $\mathcal{S} := -\Delta + V$, defined as a form sum, is self-adjoint and semibounded on $L^2(\mathbb{R}^3)$ (c.f. Theorem A.2.7 in [29]). We denote by $\sigma(\mathcal{S}) \subset \mathbb{R}$ the spectrum, by $\sigma_{ess}(\mathcal{S})$ the essential spectrum and by $\sigma_{pp}(\mathcal{S})$ the pure point spectrum of \mathcal{S} respectively.

Consider the nondecreasing sequence of min-max values defined by

$$\lambda_k = \inf_{U \in \mathcal{U}_k} \sup_{u \in U \setminus \{0\}} \frac{\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx}{\int_{\mathbb{R}^3} u^2 dx}, \quad \forall k \in \mathbb{N},$$

where \mathcal{U}_k is the set of k -dimensional subspaces of $C_0^\infty(\mathbb{R}^3)$. It is known that $\lambda_\infty := \lim_{k \rightarrow \infty} \lambda_k = \inf \sigma_{ess}(\mathcal{S})$. Moreover, $\lambda_k \in \sigma_{pp}(\mathcal{S})$ whenever $\lambda_k < \lambda_\infty$ (see [25, 26] for details). Then we make the further assumption on V .

$$(S_2) \quad \lambda_\infty > 0.$$

For the nonlinearity, we present the following assumptions.

(S₃) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ is odd in u , and there exist constants $\nu \in (1, 2)$, $\mu \in (\frac{6}{6-\nu}, \frac{2}{2-\nu}]$ and a nonnegative function $\xi \in L^\mu(\mathbb{R}^3)$ such that

$$|f(x, u)| \leq \xi(x)|u|^{\nu-1}, \quad \forall (x, u) \in \mathbb{R}^3 \times \mathbb{R}.$$

(S₄) There exist an $x_0 \in \mathbb{R}^3$ and a constant $r_0 > 0$ such that

$$\liminf_{u \rightarrow 0} \left(\inf_{x \in B_{r_0}(x_0)} u^{-2} F(x, u) \right) > -\infty,$$

and

$$\limsup_{u \rightarrow 0} \left(\inf_{x \in B_{r_0}(x_0)} u^{-2} F(x, u) \right) = +\infty$$

where $B_{r_0}(x_0)$ is the ball in \mathbb{R}^3 centered at x_0 with radius r_0 and

$$F(x, u) := \int_0^u f(x, t) dt.$$

(S₅) $g \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ is odd in u , and there exists $b \in (0, \lambda_\infty)$ such that

$$|g(x, u)| \leq b|u|, \quad \forall (x, u) \in \mathbb{R}^3 \times \mathbb{R}.$$

Our main result reads as follows.

Theorem 1.1. *Suppose that (S₁)–(S₅) are satisfied. Then (SP) possesses a sequence of weak solutions $\{(u_k, \phi_{u_k})\}_{k \in \mathbb{N}}$ in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ with $u_k \rightarrow 0$ in $H^1(\mathbb{R}^3)$ as $k \rightarrow \infty$.*

Remark 1.2. From the proof of Theorem 1.1 in Section 3, we also know that the energies $\{\mathcal{E}_k\}_{k \in \mathbb{N}}$ of these weak solutions $\{(u_k, \phi_{u_k})\}_{k \in \mathbb{N}}$ satisfy either $\mathcal{E}_k \equiv 0$ or $0 > \mathcal{E}_k \rightarrow 0$ as $k \rightarrow \infty$.

Remark 1.3. In Theorem 1.1, the potential V satisfying (S₁) and (S₂) may not be coercive or bounded from below. Moreover, the nonlinear term f satisfying (S₃) and (S₄) may be partially oscillatory near the origin. This is in sharp contrast with the aforementioned references. To the best of our knowledge, there is little literature concerning infinitely many solutions for (SP) in this situation. In fact, it is easy to see that conditions (S₁) and (S₂) are rather weaker than the usual one in the existing literature that the potential $V \in C(\mathbb{R}^3)$ satisfies $\inf_{x \in \mathbb{R}^3} V(x) > 0$ or $\lim_{|x| \rightarrow \infty} V(x) = +\infty$.

Remark 1.4. Theorem 1.1 also essentially improves some related results in the existing literature. Compared to Theorem 1.1 in [30], our conditions (S₁) and (S₂) on the potential V are weaker than (H₁) there, and our conditions (S₃) and (S₄) on the nonlinear term f are much weaker than (H₂) there if we just take $g = 0$ in (SP). In fact, there are many functions V and f which satisfy our conditions (S₁)–(S₄) but do not satisfy the condition (H₁) and (H₂) in [30]. For instance, let

$$V(x) = V_0(x) + \bar{V},$$

where $V_0 \in L^q(\mathbb{R}^3)$ for some $q \geq 2$ is a given non-positive function and unbounded from below. Then it is evident that V satisfies (S₁) and (S₂) if the positive constant \bar{V} is chosen to be large enough. Moreover, V is also unbounded from below. In addition, let

$$F(x, u) = \begin{cases} e^{-|x|^2} |u|^\alpha \sin^2(|u|^{-\epsilon}), & \forall x \in \mathbb{R}^3, 0 < |u| < \pi^{-1/\epsilon}, \\ 0, & \forall x \in \mathbb{R}^3, u = 0 \text{ or } |u| \geq \pi^{-1/\epsilon} \end{cases}$$

be the primitive function of f with respect to u , where $\epsilon > 0$ small enough and $\alpha \in (1 + \epsilon, 2)$. Then it is easy to check that f satisfies conditions (S₃) and (S₄) with $\nu = \alpha - \epsilon$ and $\xi(x) = (\alpha + \epsilon)e^{-|x|^2}$.

2 Notation and preliminaries

Throughout this paper, we always use the following notation:

- $H^1(\mathbb{R}^3)$ is the usual Sobolev space equipped with the standard norm

$$\|u\|_{H^1}^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx.$$

- $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

- $L^p(\Omega)$, $1 \leq p \leq \infty$, $\Omega \subseteq \mathbb{R}^3$, denotes a Lebesgue space, the norm in $L^p(\Omega)$ is denoted by $\|u\|_{p,\Omega}$ when Ω is a proper subset of \mathbb{R}^3 , by $\|u\|_p$ when $\Omega = \mathbb{R}^3$.
- S is the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$, that is

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\|u\|_{D^{1,2}}^2}{\|u\|_6^2}.$$

- For any $R > 0$, B_R denotes the ball in \mathbb{R}^3 centered at 0 with radius R .
- \rightarrow (resp. \rightharpoonup) denotes the strong (resp. weak) convergence.

In what follows it will always be assumed that (S_1) and (S_2) are satisfied. As pointed out in [21], the form domain of the Schrödinger operator \mathcal{S} is

$$E := \left\{ u \in H^1(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx < \infty \right\},$$

which becomes a Hilbert space if it is equipped with the inner product

$$(u, v)_0 := \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv + m_0 uv) dx, \quad \forall u, v \in E$$

with m_0 being a fixed positive constant and satisfying $m_0 > -\inf \sigma(\mathcal{S}) = -\lambda_1$. We denote by $\|\cdot\|_0$ the associated norm in it.

Lemma 2.1. *E is continuously embedded into $H^1(\mathbb{R}^3)$, that is,*

$$\|u\|_{H^1} \leq c_0 \|u\|_0, \quad \forall u \in E$$

for some $c_0 > 0$

Proof. Arguing indirectly, we assume that there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subset E$ such that

$$\|u_n\|_{H^1}^2 = \int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2) dx \equiv 1, \quad \forall n \in \mathbb{N} \quad (2.1)$$

but

$$\|u_n\|_0^2 = \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x)u_n^2 + m_0 u_n^2) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.2)$$

Since $m_0 > -\inf \sigma(\mathcal{S})$, then it holds that

$$\int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x)u_n^2 + m_0 u_n^2) dx \geq c_1 \int_{\mathbb{R}^3} u_n^2 dx \quad (2.3)$$

for some $c_1 > 0$. By (2.2) and (2.3), we get

$$\|u_n\|_2 = \left(\int_{\mathbb{R}^3} u_n^2 dx \right)^{1/2} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.4)$$

Let

$$V^- = V_1^- + V_2^-$$

with $V_1^- \in L^\infty(\mathbb{R}^3)$ and $V_2^- \in L^q(\mathbb{R}^3)$, where V^- and q are given in (S₁). Combining (2.1), (2.4), Hölder's inequality and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^3} V^- u_n^2 dx \right| &= \left| \int_{\mathbb{R}^3} V_1^- u_n^2 dx + \int_{\mathbb{R}^3} V_2^- u_n^2 dx \right| \\ &\leq \int_{\mathbb{R}^3} |V_1^- u_n^2| dx + \int_{\mathbb{R}^3} |V_2^- u_n^2| dx \\ &\leq \|V_1^-\|_\infty \|u_n\|_2^2 + \|V_2^-\|_q \|u_n\|_{2q/(q-1)}^2 \\ &\leq \|V_1^-\|_\infty \|u_n\|_2^2 + c_2 \|V_2^-\|_q \|\nabla u_n\|_2^{3/q} \|u_n\|_2^{(2q-3)/q} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (2.5)$$

where $c_2 > 0$ is a constant depending on q . This together with (2.2) and (2.4) yields

$$\int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which contradicts (2.1). The proof is completed. \square

For late use, we introduce the new inner product in E as follows. Choose $\bar{b} \in (b, \lambda_\infty)$ such that $\bar{b} \neq \lambda_k$ for all $k \in \mathbb{N}$, where b is the constant given in (S₅). Denote by λ_{k_0} the first eigenvalue of the Schrödinger operator \mathcal{S} greater than \bar{b} . Let E^- be the subspace of E spanned by the eigenfunctions with corresponding eigenvalues less than \bar{b} . Note the fact that $\lambda_\infty = \lim_{k \rightarrow \infty} \lambda_k$ and $\lambda_k \in \sigma_{pp}(\mathcal{S})$ whenever $\lambda_k < \lambda_\infty$. Then it is evident that E^- is a finite dimensional subspace of E . If there is no eigenvalue of the Schrödinger operator \mathcal{S} greater than \bar{b} , then we set $\lambda_{k_0} = \lambda_\infty$ and E^- is empty in this case. Let E^+ be the orthogonal complement of E^- in E with respect to the inner product $(\cdot, \cdot)_0$. Then E possesses the orthogonal decomposition $E = E^- \oplus E^+$. By definition, it holds that

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx \geq \lambda_{k_0} \int_{\mathbb{R}^3} u^2 dx, \quad \forall u \in E^+. \quad (2.6)$$

Now we can define the new inner product (\cdot, \cdot) and the induced norm $\|\cdot\|$ in E by

$$\begin{aligned} (u, v) &= \int_{\mathbb{R}^3} (\nabla u^+ \cdot \nabla v^+ + V(x)u^+ v^+ - \bar{b}u^+ v^+) dx \\ &\quad - \int_{\mathbb{R}^3} (\nabla u^- \cdot \nabla v^- + V(x)u^- v^- - \bar{b}u^- v^-) dx, \end{aligned} \quad (2.7)$$

$$\|u\| = \sqrt{(u, u)} \quad (2.8)$$

for all $u = u^- + u^+$, $v = v^- + v^+ \in E$ with $u^\pm, v^\pm \in E^\pm$. Note the fact that E^- and E^+ are also orthogonal with respect to the usual inner product in $L^2(\mathbb{R}^3)$. Then it is evident

that E possesses the same orthogonal decomposition $E = E^- \oplus E^+$ with respect to the new inner product (\cdot, \cdot) . Moreover, we have

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 - \bar{b}u^2) dx = \|u^+\|^2 - \|u^-\|^2 \quad (2.9)$$

for all $u = u^- + u^+ \in E$ with $u^\pm \in E^\pm$.

Lemma 2.2. *The norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent in E .*

Proof. It suffices to show that $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent in E^+ since E^- is finite dimensional. On the one hand, by (2.9), there holds

$$\begin{aligned} \|u\|^2 &= \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 - \bar{b}u^2) dx \\ &\leq \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 + m_0u^2) dx = \|u\|_0^2, \quad \forall u \in E^+. \end{aligned} \quad (2.10)$$

On the other hand, invoking (2.6) and (2.9), we get

$$\begin{aligned} \|u\|^2 &= \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 - \bar{b}u^2) dx \\ &\geq \frac{\lambda_{k_0} - \bar{b}}{\lambda_{k_0}} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx \\ &\geq \frac{\lambda_{k_0} - \bar{b}}{2\lambda_{k_0}} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx + \frac{\lambda_{k_0} - \bar{b}}{2m_0} \int_{\mathbb{R}^3} m_0u^2 dx \\ &\geq c_3 \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 + m_0u^2) dx \\ &= c_3 \|u\|_0^2, \quad \forall u \in E^+, \end{aligned} \quad (2.11)$$

where $c_3 = \min \left\{ \frac{\lambda_{k_0} - \bar{b}}{2\lambda_{k_0}}, \frac{\lambda_{k_0} - \bar{b}}{2m_0} \right\} > 0$ by the choice of \bar{b} and λ_{k_0} . Combining (2.10) and (2.11), we know that $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent in E^+ . The proof is completed. \square

Hereafter, we always use the inner product (\cdot, \cdot) and the induced norm $\|\cdot\|$ in E . Moreover, we write E^* for the dual space of E , and $\langle \cdot, \cdot \rangle : E^* \times E \rightarrow \mathbb{R}$ for the dual pairing. From Lemma 2.1 and Lemma 2.2, we immediately know that E is continuously embedded into $H^1(\mathbb{R}^3)$. Furthermore, using the Sobolev embedding theorem, we also get the following lemma.

Lemma 2.3. *E is continuously embedded into $L^p(\mathbb{R}^3)$ for all $p \in [2, 6]$, and hence there exists $\tau_p > 0$ such that*

$$\|u\|_p \leq \tau_p \|u\|, \quad \forall u \in E \text{ and } p \in [2, 6]. \quad (2.12)$$

Moreover, for any bounded domain $\Omega \subset \mathbb{R}^3$, E is compactly embedded into $L^p(\Omega)$ for all $p \in [1, 6)$.

By Lemma 2.4 of [5] (see also Section 2 of [28]), for every $u \in E \subseteq H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$-\Delta \phi_u = u^2 \quad (2.13)$$

and

$$\int_{\mathbb{R}^3} u^2 v dx = \int_{\mathbb{R}^3} \nabla \phi_u \cdot \nabla v dx, \quad \forall v \in D^{1,2}(\mathbb{R}^3).$$

Moreover, ϕ_u can be expressed by

$$\phi_u = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy. \quad (2.14)$$

For later use, we list some known properties of the function ϕ_u in the following lemma (see [6, 9, 28]).

Lemma 2.4. *For any $u \in E \subseteq H^1(\mathbb{R}^3)$, we have*

- (1) $\phi_u \geq 0$;
- (2) $\phi_{tu} = t^2 \phi_u, \quad \forall t \in \mathbb{R}$;
- (3) $\|\phi_u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^3} \phi_u u^2 dx \leq S^{-1} \|u\|_{12/5}^4 \leq S^{-1} \tau_{12/5}^4 \|u\|^4$, where $\tau_{12/5}$ is given in (2.12).

At the end of this section, we give the following lemma, which will play an important role in the proof of our main result.

Lemma 2.5. *Let (S₁)-(S₃) be satisfied. If $u_n \rightharpoonup u$ in E , then*

$$\int_{\mathbb{R}^3} f(x, u_n) u_n dx \rightarrow \int_{\mathbb{R}^3} f(x, u) u dx \quad \text{as } n \rightarrow \infty. \quad (2.15)$$

Proof. We first claim that if $u_n \rightharpoonup u$ in E , then for any $R > 0$,

$$\int_{B_R} |f(x, u_n) - f(x, u)|^{p_0} dx \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (2.16)$$

where $p_0 := \max \left\{ \frac{6}{5}, \frac{\mu}{\mu(\nu-1)+1} \right\}$ with μ and ν given in (S₃).

Arguing indirectly, by Lemma 2.3, we assume that there exist constants $R_0, \varepsilon_0 > 0$ and a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$ such that

$$u_{n_k} \rightarrow u \text{ in } L^{p_0^*}(B_{R_0}) \text{ and } u_{n_k} \rightarrow u \text{ a.e. in } B_{R_0} \quad \text{as } k \rightarrow \infty \quad (2.17)$$

but

$$\int_{B_{R_0}} |f(x, u_{n_k}) - f(x, u)|^{p_0} dx \geq \varepsilon_0, \quad \forall k \in \mathbb{N}, \quad (2.18)$$

where $p_0^* := \frac{p_0 \mu(\nu-1)}{\mu-p_0} \in [1, 6)$ by (S₃) and the choice of p_0 above. Due to (2.17), passing to a subsequence if necessary, we can assume that

$$\sum_{k=1}^{\infty} \|u_{n_k} - u\|_{p_0^*, B_{R_0}} < +\infty.$$

Let $w(x) = \sum_{k=1}^{\infty} |u_{n_k}(x) - u(x)|$ for all $x \in B_{R_0}$, then $w \in L^{p_0^*}(B_{R_0})$. By virtue of (S₃) and Hölder's inequality, we get

$$\begin{aligned} & |f(x, u_{n_k}) - f(x, u)|^{p_0} \\ & \leq (|f(x, u_{n_k})| + |f(x, u)|)^{p_0} \\ & \leq \xi(x)^{p_0} (|u_{n_k}|^{\nu-1} + |u|^{\nu-1})^{p_0} \\ & \leq 2^{p_0} \xi(x)^{p_0} (|u_{n_k}|^{p_0(\nu-1)} + |u|^{p_0(\nu-1)}) \\ & \leq 2^{p_0\nu+1} \xi(x)^{p_0} (|u_{n_k} - u|^{p_0(\nu-1)} + |u|^{p_0(\nu-1)}) \\ & \leq 2^{p_0\nu+1} \xi(x)^{p_0} (|w|^{p_0(\nu-1)} + |u|^{p_0(\nu-1)}), \quad \forall k \in \mathbb{N} \text{ and } x \in B_{R_0} \end{aligned} \quad (2.19)$$

and

$$\int_{B_{R_0}} \xi(x)^{p_0} (|w|^{p_0(\nu-1)} + |u|^{p_0(\nu-1)}) dx \leq \|\xi\|_{\mu}^{p_0} \left(\|w\|_{p_0^*, B_{R_0}}^{p_0(\nu-1)} + \|u\|_{p_0^*, B_{R_0}}^{p_0(\nu-1)} \right) < +\infty. \quad (2.20)$$

Combining (2.17), (2.19), (2.20) and Lebesgue's dominated convergence theorem, we have

$$\lim_{k \rightarrow \infty} \int_{B_{R_0}} |f(x, u_{n_k}) - f(x, u)|^{p_0} dx = 0,$$

which contradicts (2.18). Thus the claim is true.

For any given $u \in E$, define an associated linear operator $\mathcal{J}(u) : E \rightarrow \mathbb{R}$ as follows.

$$\langle \mathcal{J}(u), v \rangle = \int_{\mathbb{R}^3} f(x, u) v dx, \quad \forall v \in E. \quad (2.21)$$

For notational simplicity, we set

$$\mu^* := \frac{\mu\nu}{\mu-1}. \quad (2.22)$$

Since $\nu \in (1, 2)$ and $\mu \in (\frac{6}{6-\nu}, \frac{2}{2-\nu}]$ in (S_3) , we get $\mu^* \in [2, 6)$. By (S_3) , (2.12) and Hölder's inequality, there holds

$$\begin{aligned} |\langle \mathcal{J}(u), v \rangle| &\leq \int_{\mathbb{R}^3} \xi(x) |u|^{\nu-1} |v| dx \\ &\leq \|\xi\|_\mu \|u\|_{\mu^*}^{\nu-1} \|v\|_{\mu^*} \\ &\leq \tau_{\mu^*}^\nu \|\xi\|_\mu \|u\|^{\nu-1} \|v\|, \quad \forall v \in E, \end{aligned} \quad (2.23)$$

where τ_{μ^*} is the constant given in (2.12). This implies that $\mathcal{J}(u)$ is well defined and bounded.

Now let $u_n \rightharpoonup u$ in E as $n \rightarrow \infty$. Note that

$$\begin{aligned} &\int_{\mathbb{R}^3} (f(x, u_n)u_n - f(x, u)u) dx \\ &= \int_{\mathbb{R}^3} f(x, u)(u_n - u) dx + \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u)) u_n dx \\ &=: I_n^{(1)} + I_n^{(2)}. \end{aligned} \quad (2.24)$$

Then in order to prove (2.15), it suffices to show that $I_n^{(i)} \rightarrow 0$ as $n \rightarrow \infty$ for $i = 1, 2$. On the one hand, by (2.21) and the definition of $I_n^{(1)}$, we immediately get

$$I_n^{(1)} = \langle \mathcal{J}(u), u_n - u \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (2.25)$$

since $u_n \rightharpoonup u$ in E and $\mathcal{J}(u)$ is bounded. On the other hand, it is evident that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in E and hence there exists a constant $D_0 > 0$ such that

$$\|u_n\|^\nu + \|u_n\| \|u\|^{\nu-1} \leq D_0, \quad \forall n \in \mathbb{N}. \quad (2.26)$$

For any $\epsilon > 0$, by (S_3) , there exists $R_\epsilon > 0$ such that

$$\left(\int_{\mathbb{R}^3 \setminus B_{R_\epsilon}} \xi(x)^\mu dx \right)^{1/\mu} < \frac{\epsilon}{2D_0 \tau_{\mu^*}^\nu}. \quad (2.27)$$

Combining (S₃), (2.26), (2.27) and Hölder's inequality, we have

$$\begin{aligned}
 \int_{\mathbb{R}^3 \setminus B_{R_\epsilon}} |f(x, u_n) - f(x, u)| |u_n| dx &\leq \int_{\mathbb{R}^3 \setminus B_{R_\epsilon}} (|f(x, u_n)| + |f(x, u)|) |u_n| dx \\
 &\leq \int_{\mathbb{R}^3 \setminus B_{R_\epsilon}} \xi(x) (|u_n|^{\nu-1} + |u|^{\nu-1}) |u_n| dx \\
 &\leq \left(\int_{\mathbb{R}^3 \setminus B_{R_\epsilon}} \xi(x)^\mu dx \right)^{1/\mu} (\|u_n\|_{\mu^*}^{\nu-1} + \|u\|_{\mu^*}^{\nu-1}) \|u_n\|_{\mu^*} \\
 &\leq \tau_{\mu^*}^\nu \left(\int_{\mathbb{R}^3 \setminus B_{R_\epsilon}} \xi(x)^\mu dx \right)^{1/\mu} (\|u_n\|^\nu + \|u_n\| \|u\|^{\nu-1}) \\
 &< \frac{\epsilon}{2}, \quad \forall n \in \mathbb{N}.
 \end{aligned} \tag{2.28}$$

Let $\bar{p}_0 := p_0/(p_0 - 1)$ be the conjugate index of p_0 , where p_0 is the constant given in (2.16). By (S₃) and the choice of p_0 , we get $\bar{p}_0 \in (1, 6]$. For the R_ϵ given in (2.27), from Lemma 2.3, we know that the sequence $\{\|u_n\|_{\bar{p}_0, R_\epsilon}\}_{n \in \mathbb{N}}$ is bounded since $\{u_n\}_{n \in \mathbb{N}}$ is bounded in E . Then by (2.16) and Hölder's inequality, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\begin{aligned}
 &\int_{B_{R_\epsilon}} |f(x, u_n) - f(x, u)| |u_n| dx \\
 &\leq \left(\int_{B_{R_\epsilon}} |f(x, u_n) - f(x, u)|^{p_0} dx \right)^{1/p_0} \|u_n\|_{\bar{p}_0, R_\epsilon} \\
 &< \frac{\epsilon}{2}, \quad \forall n \geq N_\epsilon,
 \end{aligned} \tag{2.29}$$

Combining (2.28) and (2.29), we have

$$\begin{aligned}
 |I_n^{(2)}| &\leq \int_{\mathbb{R}^3} |f(x, u_n) - f(x, u)| |u_n| dx \\
 &= \int_{\mathbb{R}^3 \setminus B_{R_\epsilon}} |f(x, u_n) - f(x, u)| |u_n| dx + \int_{B_{R_\epsilon}} |f(x, u_n) - f(x, u)| |u_n| dx \\
 &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \forall n \geq N_\epsilon.
 \end{aligned} \tag{2.30}$$

This shows that $I_n^{(2)} \rightarrow 0$ as $n \rightarrow \infty$. The proof is completed. \square

3 Variational setting and proof of the main result

Here, as usual, we say that the pair $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a weak solution for (SP) if

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi dx + \int_{\mathbb{R}^3} V(x)u\varphi dx + \int_{\mathbb{R}^3} \phi_u u \varphi dx = \int_{\mathbb{R}^3} (f(x, u) + g(x, u))\varphi dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3)$$

with $\phi = \phi_u$ defined as in (2.14). Now we define functionals $\Psi_i (i = 1, 2, 3)$ and Φ on E by

$$\Psi_1(u) = \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx, \quad \Psi_2(u) = \int_{\mathbb{R}^3} F(x, u) dx, \quad \Psi_3(u) = \int_{\mathbb{R}^3} \left(\frac{\bar{b}}{2} u^2 - G(x, u) \right) dx$$

and

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} (F(x, u) + G(x, u)) dx \\ &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 - \bar{b}u^2) dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx \\ &\quad - \int_{\mathbb{R}^3} F(x, u) dx + \int_{\mathbb{R}^3} \left(\frac{\bar{b}}{2} u^2 - G(x, u) \right) dx \\ &= \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 + \Psi_1(u) - \Psi_2(u) + \Psi_3(u) \end{aligned} \quad (3.1)$$

for all $u = u^- + u^+ \in E$ with $u^\pm \in E^\pm$. Here \bar{b} is the constant in (2.9) and $G(x, u) := \int_0^u g(x, t) dt$ is the primitive function of $g(x, u)$ with respect to u .

Proposition 3.1. *Assume that (S₁)-(S₃) and (S₅) are satisfied. Then $\Psi_i \in C^1(E, \mathbb{R})$ for $i = 1, 2, 3$, and hence $\Phi \in C^1(E, \mathbb{R})$. Moreover,*

$$\langle \Psi'_1(u), v \rangle = \int_{\mathbb{R}^3} \phi_u u v dx, \quad (3.2)$$

$$\langle \Psi'_2(u), v \rangle = \int_{\mathbb{R}^3} f(x, u) v dx, \quad (3.3)$$

$$\langle \Psi'_3(u), v \rangle = \int_{\mathbb{R}^3} (\bar{b}u - g(x, u)) v dx, \quad (3.4)$$

$$\begin{aligned} \langle \Phi'(u), v \rangle &= (u^+, v^+) - (u^-, v^-) + \langle \Psi'_1(u), v \rangle \\ &\quad - \langle \Psi'_2(u), v \rangle + \langle \Psi'_3(u), v \rangle \\ &= \int_{\mathbb{R}^3} \nabla u \cdot \nabla v dx + \int_{\mathbb{R}^3} V(x)u v dx + \int_{\mathbb{R}^3} \phi_u u v dx \\ &\quad - \int_{\mathbb{R}^3} (f(x, u) + g(x, u)) v dx \end{aligned} \quad (3.5)$$

for all $u = u^- + u^+$, $v = v^- + v^+ \in E$ with $u^\pm, v^\pm \in E^\pm$. In addition, if $u \in E \subseteq H^1(\mathbb{R}^3)$ is a critical point of Φ on E , then the pair $(u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, with ϕ_u defined as in (2.14), is a weak solution of (SP).

Proof. First, we show that $\Psi_1 \in C^1(E, \mathbb{R})$ and (3.2) holds. Define a functional Ψ_0 on $H^1(\mathbb{R}^3)$ by

$$\Psi_0(u) = \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx.$$

It is known that $\Psi_0 \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ (cf. Lemma 2.1 in [28] and Lemma 2.3 in [38]) and

$$\langle \Psi'_0(u), v \rangle = \int_{\mathbb{R}^3} \phi_u uv dx, \quad \forall u, v \in H^1(\mathbb{R}^3). \quad (3.6)$$

Let $\iota : E \rightarrow H^1(\mathbb{R}^3)$ be the continuous embedding. Since $\Psi_1 = \Psi_0 \circ \iota$, we immediately know by (3.6) that $\Psi_1 \in C^1(E, \mathbb{R})$ and (3.2) holds.

Next, we verify (3.3) by definition and prove that $\Psi_2 \in C^1(E, \mathbb{R})$. By (S₃), there holds

$$|F(x, u)| \leq \nu^{-1} \xi(x) |u|^\nu, \quad \forall (x, u) \in \mathbb{R}^3 \times \mathbb{R}. \quad (3.7)$$

Then for any $u \in E$, by (2.12), (3.7) and Hölder's inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^3} |F(x, u)| dx &\leq \int_{\mathbb{R}^3} \nu^{-1} \xi(x) |u|^\nu dx \\ &\leq \nu^{-1} \|\xi\|_\mu \|u\|_{\mu^*}^\nu \\ &\leq \nu^{-1} \tau_{\mu^*}^\nu \|\xi\|_\mu \|u\|^\nu < \infty, \end{aligned} \quad (3.8)$$

where μ^* and τ_{μ^*} are the constants given in (2.22) and (2.12) respectively. Thus Ψ_2 is well defined. By (S₃), it holds

$$|f(x, u + \eta v)v| \leq 2^{\nu-1} \xi(x) (|u|^{\nu-1}|v| + |v|^\nu), \quad \forall x \in \mathbb{R}^3, \eta \in [0, 1] \text{ and } u, v \in \mathbb{R}. \quad (3.9)$$

For any $u, v \in E$, combining (2.23), (3.8), (3.9), the mean value theorem and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\Psi_2(u + tv) - \Psi_2(u)}{t} &= \lim_{t \rightarrow 0} \int_{\mathbb{R}^3} f(x, u + \theta(x)tv) v dx \\ &= \int_{\mathbb{R}^3} f(x, u) v dx \\ &= \langle \mathcal{J}(u), v \rangle, \end{aligned} \quad (3.10)$$

where $\theta(x) \in [0, 1]$ depends on u, v, t , and \mathcal{J} is the bounded linear operator on E defined in (2.21). This shows that Ψ_2 is Gâteaux differentiable on E and the Gâteaux derivative

of Ψ_2 at u is $\mathcal{J}(u)$. In order to prove that $\Psi_2 \in C^1(E, \mathbb{R})$, it suffices to show that $\mathcal{J} : E \rightarrow E^*$ is continuous. Let $u_n \rightharpoonup u$ in E as $n \rightarrow \infty$. Note that

$$\begin{aligned} \|\mathcal{J}(u_n) - \mathcal{J}(u)\|_{E^*} &= \sup_{\|v\|=1} |\langle \mathcal{J}(u_n) - \mathcal{J}(u), v \rangle| \\ &= \sup_{\|v\|=1} \left| \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u)) v dx \right| \\ &\leq \sup_{\|v\|=1} \int_{\mathbb{R}^3} |f(x, u_n) - f(x, u)| |v| dx. \end{aligned} \quad (3.11)$$

Using similar arguments to the last paragraph of the proof of Lemma 2.5, we can obtain

$$\sup_{\|v\|=1} \int_{\mathbb{R}^3} |f(x, u_n) - f(x, u)| |v| dx \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This together with (3.11) shows that $\mathcal{J} : E \rightarrow E^*$ is completely continuous and hence continuous.

Then, taking (S_5) into account and using the standard arguments, one can easily prove that $\Psi_3 \in C^1(E, \mathbb{R})$ and (3.4) holds. For simplicity, we omit the proof here.

Finally, combining (2.7) and (3.1)-(3.4), we immediately know that $\Phi \in C^1(E, \mathbb{R})$ and (3.5) holds. In addition, it is clear that (SP) is the Euler-Lagrange equations of the functional $\mathcal{I} : E \times D^{1,2}(\mathbb{R}^3) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{I}(u, \phi) &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx - \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} \phi u^2 dx \\ &\quad - \int_{\mathbb{R}^3} (F(x, u) + G(x, u)) dx. \end{aligned}$$

Using the reduction method described in [5], one gets

$$\mathcal{I}(u, \phi_u) = \Phi(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} (F(x, u) + G(x, u)) dx.$$

Similar to [4], it can be proved that $(u, \phi) \in E \times D^{1,2}(\mathbb{R}^3)$ is a weak solution of (SP) if and only if $u \in E$ is a critical point of the functional Φ and $\phi = \phi_u$. The proof is completed.

□

We will use the following variant symmetric mountain pass lemma due to [15] (see also [22]) to prove that (SP) possesses a sequence of weak solutions. Before stating this theorem, we first recall the notion of genus.

Let E be a Banach space and A a subset of E . A is said to be symmetric if $u \in A$ implies $-u \in A$. Denote by Γ the family of all closed symmetric subset of E which does not contain 0. For any $A \in \Gamma$, define the genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such a k , define $\gamma(A) = \infty$. Moreover, set $\gamma(\emptyset) = 0$. For each $k \in \mathbb{N}$, let $\Gamma_k = \{A \in \Gamma \mid \gamma(A) \geq k\}$.

Theorem 3.2 ([15, Theorem 1]). *Let E be an infinite dimensional Banach space and $\Phi \in C^1(E, \mathbb{R})$ an even functional with $\Phi(0) = 0$. Suppose that Φ satisfies*

(Φ_1) *Φ is bounded from below and satisfies (PS) condition.*

(Φ_2) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} \Phi(u) < 0$.*

Then either (i) or (ii) below holds.

- (i) *There exists a critical point sequence $\{u_k\}$ such that $\Phi(u_k) < 0$ and $\lim_{k \rightarrow \infty} u_k = 0$.*
- (ii) *There exist two critical point sequences $\{u_k\}$ and $\{v_k\}$ such that $\Phi(u_k) = 0$, $u_k \neq 0$, $\lim_{k \rightarrow \infty} u_k = 0$, $\Phi(v_k) < 0$, $\lim_{k \rightarrow \infty} \Phi(v_k) = 0$, and $\{v_k\}$ converges to a non-zero limit.*

In order to apply Theorem 3.2, we will show in the following lemmas that the functional Φ defined in (3.1) satisfies conditions (Φ_1) and (Φ_2) in Theorem 3.2. The proof of Lemmas 3.3 and 3.4 is motivated by the paper [7].

Lemma 3.3. *Let (S_1) -(S_3) and (S_5) be satisfied. Then Φ is coercive and bounded from below.*

Proof. We first prove that Φ is coercive. Arguing indirectly, we assume that for some sequence $\{u_n\}_{n \in \mathbb{N}} \subset E$ with $\|u_n\| \rightarrow \infty$, there is a constant $M > 0$ such that $\Phi(u_n) \leq M$ for all $n \in \mathbb{N}$. Let $u_n = u_n^- + u_n^+$ with $u_n^\pm \in E^\pm$. If we set $v_n = u_n / \|u_n\|$ for all $n \in \mathbb{N}$, then $\|v_n\| \equiv 1$, and $v_n = v_n^- + v_n^+$ with $v_n^\pm = u_n^\pm / \|u_n\| \in E^\pm$. Note that E^- is finite dimensional. Thus, passing to a subsequence if necessary, we can assume by Lemma 2.3 that

$$v_n \rightharpoonup v, v_n^- \rightarrow v^-, v_n^+ \rightharpoonup v^+ \text{ and } v_n \rightarrow v \text{ a.e. in } \mathbb{R}^3 \quad \text{as } n \rightarrow \infty \quad (3.12)$$

for some $v = v^- + v^+ \in E$ with $v^\pm \in E^\pm$. By (S₅), there hold

$$0 \leq \frac{\bar{b}-b}{2}u^2 \leq \frac{\bar{b}}{2}u^2 - G(x, u) \leq \frac{\bar{b}+b}{2}u^2, \quad \forall (x, u) \in \mathbb{R}^3 \times \mathbb{R} \quad (3.13)$$

and

$$\bar{b}u^2 - g(x, u)u \geq (\bar{b} - b)u^2 \geq 0, \quad \forall (x, u) \in \mathbb{R}^3 \times \mathbb{R} \quad (3.14)$$

since \bar{b} is chosen to be greater than b in Section 2. Combining (3.1), (3.8), (3.13) and Lemma 2.4 (1), we have

$$\begin{aligned} M \geq \Phi(u_n) &\geq \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 - \int_{\mathbb{R}^3} |F(x, u_n)|dx \\ &\geq \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 - \nu^{-1}\tau_{\mu^*}^\nu \|\xi\|_\mu \|u_n\|^\nu, \quad \forall n \in \mathbb{N}. \end{aligned} \quad (3.15)$$

Multiplying both sides of (3.15) by $\|u_n\|^{-2}$, we get

$$\|v_n^+\|^2 \leq \|v_n^-\|^2 + o(1) \quad \text{as } n \rightarrow \infty \quad (3.16)$$

since $\nu < 2$ in (S₃) and $\|u_n\| \rightarrow \infty$.

If $v = 0$, then $v_n^- \rightarrow 0$ and hence $v_n^+ \rightarrow 0$ by (3.16). This implies $v_n \rightarrow 0$, which leads to a contradiction since $\|v_n\| \equiv 1$.

If $v \neq 0$, by (3.12) and Fatou's lemma, there holds

$$\begin{aligned} &\liminf_{n \rightarrow \infty} \left[\|u_n\|^{-4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|} dx dy \right] \\ &= \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{v_n^2(x)v_n^2(y)}{|x-y|} dx dy \\ &\geq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{v^2(x)v^2(y)}{|x-y|} dx dy > 0. \end{aligned} \quad (3.17)$$

Combining (2.14), (3.1), (3.8) and (3.13), we have

$$\begin{aligned} M \geq \Phi(u_n) &\geq \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 + \frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|} dx dy - \int_{\mathbb{R}^3} |F(x, u_n)|dx \\ &\geq \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 + \frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|} dx dy - \nu^{-1}\tau_{\mu^*}^\nu \|\xi\|_\mu \|u_n\|^\nu, \end{aligned}$$

or equivalently,

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|} dx dy \leq 8\pi(\|u_n^-\|^2 - \|u_n^+\|^2) + 16\pi\nu^{-1}\tau_{\mu^*}^\nu \|\xi\|_\mu \|u_n\|^\nu + 16\pi M, \quad (3.18)$$

where μ^* and τ_{μ^*} are the constants given in (2.22) and (2.12) respectively. Multiplying both sides of (3.18) by $\|u_n\|^{-4}$ and letting $n \rightarrow \infty$, we get

$$\liminf_{n \rightarrow \infty} \left[\|u_n\|^{-4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|} dx dy \right] \leq 0,$$

which contradicts (3.17). Therefore, Φ is coercive.

Next, we show that Φ is bounded from below. Combining (2.12), (3.1), (3.8), (3.13) and Lemma 2.4 (3), we have

$$\begin{aligned} |\Phi(u)| &\leq \frac{1}{2}\|u\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx + \int_{\mathbb{R}^3} |F(x, u)| dx + \int_{\mathbb{R}^3} \left(\frac{\bar{b}}{2} u^2 - G(x, u) \right) dx \\ &\leq \frac{1}{2}\|u\|^2 + \frac{\tau_{12/5}^4}{4S} \|u\|^4 + \nu^{-1} \tau_{\mu^*}^\nu \|\xi\|_\mu \|u\|^\nu + \frac{\bar{b}+b}{2} \tau_2^2 \|u\|^2, \end{aligned} \quad (3.19)$$

where τ_2 is the constant given in (2.12). This implies that Φ maps bounded set in E into bounded set in \mathbb{R} . Then it follows from the coercivity that Φ is bounded from below. The proof is completed. \square

Lemma 3.4. *Assume that (S_1) -(S_3) and (S_5) are satisfied. Then Φ satisfies (PS) condition.*

Proof. Let $\{u_n\}_{n \in \mathbb{N}} \subset E$ be a (PS)-sequence, i.e.,

$$|\Phi(u_n)| \leq D_1 \quad \text{and} \quad \Phi'(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (3.20)$$

for some $D_1 > 0$. Note first that Φ is coercive by Lemma 3.3. This together with (3.20) implies that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in E . Thus there exists a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$ such that

$$u_{n_k} \rightharpoonup u_0 \quad \text{as } k \rightarrow \infty \quad (3.21)$$

for some $u_0 \in E$. Passing to a subsequence if necessary, we can further assume by Lemma 2.12 that

$$u_{n_k} \rightarrow u_0 \text{ a.e. in } \mathbb{R}^3 \quad \text{as } k \rightarrow \infty. \quad (3.22)$$

Let

$$u_{n_k} = u_{n_k}^- + u_{n_k}^+ \quad \text{and} \quad u_0 = u_0^- + u_0^+$$

with $u_{n_k}^\pm, u_0^\pm \in E^\pm$. Since E^- is finite dimensional, we get

$$u_{n_k}^- \rightarrow u_0^- \quad \text{and} \quad u_{n_k}^+ \rightharpoonup u_0^+ \quad \text{as } k \rightarrow \infty. \quad (3.23)$$

By (2.14), (3.5) and (3.20), it holds

$$\begin{aligned}
 o(1) &= \langle \Phi'(u_{n_k}), u_{n_k} \rangle \\
 &= \|u_{n_k}^+\|^2 - \|u_{n_k}^-\|^2 + \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_{n_k}^2(x) u_{n_k}^2(y)}{|x-y|} dx dy - \int_{\mathbb{R}^3} f(x, u_{n_k}) u_{n_k} dx \\
 &\quad + \int_{\mathbb{R}^3} (\bar{b} u_{n_k}^2 - g(x, u_{n_k}) u_{n_k}) dx \quad \text{as } k \rightarrow \infty.
 \end{aligned} \tag{3.24}$$

Note that $C_0^\infty(\mathbb{R}^3)$ is dense in E (c.f. Theorem A.2.8 in [29]). Then standard arguments show that u_0 is a critical point of Φ . This together with (2.14) and (3.5) yields

$$\begin{aligned}
 0 &= \langle \Phi'(u_0), u_0 \rangle \\
 &= \|u_0^+\|^2 - \|u_0^-\|^2 + \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_0^2(x) u_0^2(y)}{|x-y|} dx dy - \int_{\mathbb{R}^3} f(x, u_0) u_0 dx \\
 &\quad + \int_{\mathbb{R}^3} (\bar{b} u_0^2 - g(x, u_0) u_0) dx.
 \end{aligned} \tag{3.25}$$

For comparison, we rewrite (3.24) and (3.25) as follows.

$$\begin{aligned}
 \|u_{n_k}^-\|^2 + o(1) - \|u_{n_k}^+\|^2 &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_{n_k}^2(x) u_{n_k}^2(y)}{|x-y|} dx dy + \int_{\mathbb{R}^3} (\bar{b} u_{n_k}^2 - g(x, u_{n_k}) u_{n_k}) dx \\
 &\quad - \int_{\mathbb{R}^3} f(x, u_{n_k}) u_{n_k} dx \quad \text{as } k \rightarrow \infty.
 \end{aligned} \tag{3.26}$$

$$\begin{aligned}
 \|u_0^-\|^2 - \|u_0^+\|^2 &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_0^2(x) u_0^2(y)}{|x-y|} dx dy + \int_{\mathbb{R}^3} (\bar{b} u_0^2 - g(x, u_0) u_0) dx \\
 &\quad - \int_{\mathbb{R}^3} f(x, u_0) u_0 dx.
 \end{aligned} \tag{3.27}$$

Combining (3.14), (3.22), Lemma 2.5 and Fatou's lemma, we have

$$\begin{aligned}
 &\liminf_{k \rightarrow \infty} \left(\frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_{n_k}^2(x) u_{n_k}^2(y)}{|x-y|} dx dy + \int_{\mathbb{R}^3} (\bar{b} u_{n_k}^2 - g(x, u_{n_k}) u_{n_k}) dx - \int_{\mathbb{R}^3} f(x, u_{n_k}) u_{n_k} dx \right) \\
 &\geq \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_0^2(x) u_0^2(y)}{|x-y|} dx dy + \int_{\mathbb{R}^3} (\bar{b} u_0^2 - g(x, u_0) u_0) dx - \int_{\mathbb{R}^3} f(x, u_0) u_0 dx.
 \end{aligned} \tag{3.28}$$

Taking lower limit on both sides of (3.26) and comparing it with (3.27), we obtain by (3.28) that

$$\liminf_{k \rightarrow \infty} (-\|u_{n_k}^+\|^2) \geq -\|u_0^+\|^2, \tag{3.29}$$

or equivalently,

$$\limsup_{k \rightarrow \infty} \|u_{n_k}^+\|^2 \leq \|u_0^+\|^2. \tag{3.30}$$

Note the fact that

$$\liminf_{k \rightarrow \infty} \|u_{n_k}^+\|^2 \geq \|u_0^+\|^2$$

since the norm $\|\cdot\|$ is weakly sequentially lower semi-continuous. This together with (3.30) implies that $\lim_{k \rightarrow \infty} \|u_{n_k}^+\|^2 = \|u_0^+\|^2$. Hence, $u_{n_k}^+ \rightarrow u_0^+$ in E . Then by (3.23), we know that $u_{n_k} \rightarrow u_0$ in E . Therefore, Φ satisfies (PS) condition. The proof is completed. \square

Lemma 3.5. *Let (S₁)-(S₅) be satisfied. Then for each $k \in \mathbb{N}$, there exists an $A_k \subseteq E$ with genus $\gamma(A_k) = k$ such that $\sup_{u \in A_k} \Phi(u) < 0$.*

Proof. We follow the idea of dealing with the elliptic problem in [15]. By coordinate translation, we can assume $x_0 = 0$ in (S₄). Let \mathcal{C} denote the cube

$$\mathcal{C} := \{x = (x_1, x_2, x_3) \mid -r_0/2 \leq x_i \leq r_0/2, i = 1, 2, 3\},$$

where r_0 is the positive constant given in (S₄). Evidently, $\mathcal{C} \subseteq B_{r_0}(0)$. By (S₄), there exist constants $\delta, \varrho > 0$ and two sequences of positive numbers $\delta_n \rightarrow 0, M_n \rightarrow \infty$ as $n \rightarrow \infty$ such that

$$F(x, u) \geq -\varrho u^2, \quad \forall x \in \mathcal{C} \text{ and } |u| \leq \delta \quad (3.31)$$

and

$$F(x, \delta_n)/\delta_n^2 \geq M_n, \quad \forall x \in \mathcal{C} \text{ and } n \in \mathbb{N}. \quad (3.32)$$

For any fixed $k \in \mathbb{N}$, let $m \in \mathbb{N}$ be the smallest positive integer satisfying $m^3 \geq k$. We divide the cube \mathcal{C} equally into m^3 small cubes by planes parallel to each face of \mathcal{C} and denote them by \mathcal{C}_i with $1 \leq i \leq m^3$. Then the edge of each \mathcal{C}_i has the length of $a := r_0/m$. For each $1 \leq i \leq k$, we make a cube \mathcal{D}_i in \mathcal{C}_i such that \mathcal{D}_i has the same center as that of \mathcal{C}_i , the faces of \mathcal{D}_i and \mathcal{C}_i are parallel and the edge of \mathcal{D}_i has the length of $a/2$.

Choose a function $\psi \in C_0^\infty(\mathbb{R}, \mathbb{R})$ such that $\psi(t) \equiv 1$ for $t \in [-a/4, a/4]$, $\psi(t) \equiv 0$ for $t \in \mathbb{R} \setminus [-a/2, a/2]$, and $0 \leq \psi(t) \leq 1$ for all $t \in \mathbb{R}$. Define

$$\varphi(x) := \psi(x_1)\psi(x_2)\psi(x_3), \quad \forall x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

For each $1 \leq i \leq k$, let $y_i \in \mathbb{R}^3$ be the center of both \mathcal{C}_i and \mathcal{D}_i , and define

$$\varphi_i(x) = \varphi(x - y_i), \quad \forall x \in \mathbb{R}^3.$$

Then it is easy to see that

$$\text{supp}\varphi_i \subseteq \mathcal{C}_i, \quad (3.33)$$

and

$$\varphi_i(x) = 1, \quad \forall x \in \mathcal{D}_i, \quad 0 \leq \varphi_i(x) \leq 1, \quad \forall x \in \mathbb{R}^3 \quad (3.34)$$

for all $1 \leq i \leq k$. Set

$$\mathcal{V}_k := \left\{ (t_1, t_2, \dots, t_k) \in \mathbb{R}^k \mid \max_{1 \leq i \leq k} |t_i| = 1 \right\}$$

and

$$\mathcal{W}_k := \left\{ \sum_{i=1}^k t_i \varphi_i \mid (t_1, t_2, \dots, t_k) \in \mathcal{V}_k \right\}.$$

Evidently, \mathcal{V}_k is homeomorphic to the unit sphere in \mathbb{R}^k by an odd mapping. Thus $\gamma(\mathcal{V}_k) = k$. If we define the mapping $\mathcal{H} : \mathcal{V}_k \rightarrow \mathcal{W}_k$ by

$$\mathcal{H}(t_1, t_2, \dots, t_k) = \sum_{i=1}^k t_i \varphi_i, \quad \forall (t_1, t_2, \dots, t_k) \in \mathcal{V}_k,$$

then \mathcal{H} is odd and homeomorphic. Therefore $\gamma(\mathcal{W}_k) = \gamma(\mathcal{V}_k) = k$. Moreover, it is evident that \mathcal{W}_k is compact and hence there is a constant $C_k > 0$ such that

$$\|u\| \leq C_k, \quad \forall u \in \mathcal{W}_k. \quad (3.35)$$

For any $s \in (0, \delta)$ and $u = \sum_{i=1}^k t_i \varphi_i \in \mathcal{W}_k$, combining (2.12), (3.1), (3.13), (3.33), (3.34) and Lemma 2.4 (2), (3), we have

$$\begin{aligned} \Phi(su) &= \frac{1}{2} \|su^+\|^2 - \frac{1}{2} \|su^-\|^2 + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{su}(su)^2 dx \\ &\quad - \int_{\mathbb{R}^3} F(x, s \sum_{i=1}^k t_i \varphi_i) dx + \int_{\mathbb{R}^3} \left(\frac{\bar{b}}{2} (su)^2 - G(x, su) \right) dx \\ &\leq \frac{s^2}{2} (\|u^+\|^2 - \|u^-\|^2) + \frac{s^4}{4} \int_{\mathbb{R}^3} \phi_u u^2 dx \\ &\quad - \sum_{i=1}^k \int_{\mathcal{C}_i} F(x, st_i \varphi_i) dx + \frac{\bar{b} + b}{2} s^2 \int_{\mathbb{R}^3} u^2 dx \\ &\leq \frac{s^2}{2} \|u\|^2 + \frac{s^4 \tau_{12/5}^4}{4S} \|u\|^4 + \frac{\bar{b} + b}{2} \tau_2^2 s^2 \|u\|^2 - \sum_{i=1}^k \int_{\mathcal{C}_i} F(x, st_i \varphi_i) dx, \end{aligned} \quad (3.36)$$

where τ_2 and $\tau_{12/5}$ are the constants given in (2.12). By the definition of \mathcal{V}_k , there exists some integer $1 \leq i_u \leq k$ such that $|t_{i_u}| = 1$. Then it follows that

$$\begin{aligned} & \sum_{i=1}^k \int_{\mathcal{C}_i} F(x, st_i \varphi_i) dx \\ &= \int_{\mathcal{D}_{i_u}} F(x, st_{i_u} \varphi_{i_u}) dx + \int_{\mathcal{C}_{i_u} \setminus \mathcal{D}_{i_u}} F(x, st_{i_u} \varphi_{i_u}) dx \\ &+ \sum_{i \neq i_u} \int_{\mathcal{C}_i} F(x, st_i \varphi_i) dx. \end{aligned} \quad (3.37)$$

By (3.31) and (3.34), there holds

$$\int_{\mathcal{C}_{i_u} \setminus \mathcal{D}_{i_u}} F(x, st_{i_u} \varphi_{i_u}) dx + \sum_{i \neq i_u} \int_{\mathcal{C}_i} F(x, st_i \varphi_i) dx \geq -\varrho r_0^3 s^2, \quad (3.38)$$

Here we use the fact that the volume of cube \mathcal{C} is r_0^3 . For each $\delta_n \in (0, \delta)$ given in (3.32), combining (3.32) and (3.35)–(3.38), we have

$$\begin{aligned} \Phi(\delta_n u) &\leq \frac{C_k^2 \delta_n^2}{2} + \frac{\tau_{12/5}^4 C_k^4 \delta_n^4}{4S} + \frac{(\bar{b} + b) \tau_2^2 C_k^2 \delta_n^2}{2} + \varrho r_0^3 \delta_n^2 - \int_{\mathcal{D}_{i_u}} F(x, \delta_n t_{i_u} \varphi_{i_u}) dx \\ &\leq \delta_n^2 \left(\frac{C_k^2}{2} + \frac{\tau_{12/5}^4 C_k^4 \delta_n^2}{4S} + \frac{(\bar{b} + b) \tau_2^2 C_k^2}{2} + \varrho r_0^3 - \frac{a^3 M_n}{8} \right). \end{aligned} \quad (3.39)$$

Here we use the fact that $|\delta_n t_{i_u} \varphi_{i_u}(x)| \equiv \delta_n$ for all $x \in \mathcal{D}_{i_u}$ and the volume of cube \mathcal{D}_{i_u} is $a^3/8$. Since $\delta_n \rightarrow 0$ and $M_n \rightarrow \infty$ as $n \rightarrow \infty$, we can choose $n_0 \in \mathbb{N}$ large enough such that the right-hand side of (3.39) is negative. Define

$$A_k := \{\delta_{n_0} u \mid u \in \mathcal{W}_k\}. \quad (3.40)$$

Then we have

$$\gamma(A_k) = \gamma(\mathcal{W}_k) = k \quad \text{and} \quad \sup_{u \in A_k} \Phi(u) < 0.$$

The proof is completed. \square

Now we are in a position to give the proof of our main result.

Proof of Theorem 1.1. Evidently, the functional Φ defined in (3.1) is an even functional with $\Phi(0) = 0$. Besides, Proposition 3.1 and Lemmas 3.3–3.5 show that $\Phi \in C^1(E, \mathbb{R})$ and satisfies conditions (Φ_1) and (Φ_2) in Theorem 3.2. Thus, by Theorem 3.2, we get a sequence of nontrivial critical points $\{u_k\}_{k \in \mathbb{N}}$ of Φ satisfying $\Phi(u_k) \leq 0$ for all $k \in \mathbb{N}$ and $u_k \rightarrow 0$ in E as $k \rightarrow \infty$. Taking into account Proposition 3.1 again and the fact that E is continuously embedding into $H^1(\mathbb{R}^3)$, we know that $\{(u_k, \phi_{u_k})\}_{k \in \mathbb{N}}$ is a sequence of weak solutions of (SP) with $u_k \rightarrow 0$ in $H^1(\mathbb{R}^3)$ as $k \rightarrow \infty$. This ends the proof. \square

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