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Martingale weak Orlicz–Karamata–Hardy spaces associated with concave functions ☆

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ABSTRACT

In this paper, we introduce the weak Orlicz–Karamata–Hardy spaces associated with concave functions and establish the atomic decompositions. As applications, we prove several martingale inequalities and obtain a duality theorem.

1. Introduction

The main purpose of this article is to introduce and to investigate the martingale weak Orlicz–Karamata–Hardy spaces defined on a probability space Ω. The family of weak Orlicz–Karamata spaces is a generalization of weak Lebesgue spaces and weak Orlicz spaces. They are defined via the slowly varying functions. The most typical spaces defined via the slowly varying functions are the Lorentz–Karamata spaces. These spaces, generalizing Lorentz spaces and Lorentz–Zygmund spaces, were studied in [21,2,3]. We also refer the reader to [1, Chapter I] for a detailed study of the Karamata theory.

Very recently, there are some works which extend Karamata theory into martingale theory. Atomic decompositions and duality results of martingale Lorentz–Karamata–Hardy spaces were studied by Ho [9]. Jiao et al. [14] introduced the generalized BMO martingale spaces and improved some results of [9]. Inspired by [14], Weisz [25] obtained a duality result of multi-parameter martingale Hardy space. Liu and Zhou [16] proved the dual spaces of the weak Karamata–Hardy spaces.

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It is well-known that the classical weak Hardy spaces in harmonic analysis appear naturally in critical cases of the study on the boundedness of the operators. Indeed, the classical weak Hardy space $wH^1(\mathbb{R}^n)$ was originally introduced by Fefferman and Soria [4] when they tried to find out the biggest space from which the Hilbert transform is bounded to weak Lebesgue space $wL^1(\mathbb{R}^n)$ by establishing the atomic decomposition of $wH^1(\mathbb{R}^n)$. In martingale theory, Weisz [24] first introduced weak martingale Hardy spaces and studied several martingale inequalities and duality theorems. Recently, some new weak-type martingale spaces have attracted a lot of attention. Liu et al. [15] introduced weak Orlicz–Hardy space with convex function Φ. Jiao [11] obtained some embeddings between vector-valued weak Orlicz martingale spaces. Lately, Jiao et al. [12] established the atomic decomposition of weak Orlicz–Hardy space with concave function Φ. As applications, they obtained martingale inequalities and duality theorems. Liu et al. [17] improved some results of [12] and gave an equivalent characterization of $w\mathcal{L}_{r,\phi}$ defined in [12]. Yu [27] investigated the dual space of weak Orlicz–Hardy spaces for Banach space valued martingale. Yang [26] proved the atomic decompositions of weak Musielak–Orlicz martingale spaces and several martingale inequalities for vector-valued martingales.

Our proofs mainly depend on the establishment of atomic decompositions of weak Orlicz–Karamata–Hardy spaces. Recall that atomic decompositions were first introduced by Herz [8], generalized by Weisz [22,23] and developed in [13,9,10,18,20,7] and so on.

The paper is organized as follows. In Section 2, we recall some notation and state some basic properties about weak Orlicz–Karamata spaces. The weak Orlicz–Karamata–Hardy spaces are also defined in this section via slowly varying functions. Section 3 is devoted to establishing the atomic decompositions of weak Orlicz–Karamata–Hardy spaces. Applying the atomic decompositions obtained in Section 3, we prove several martingale inequalities among weak Orlicz–Karamata spaces in Section 4. In the last section, we show a duality result.

Throughout this paper, the sets of integers, non-negative integers and complex numbers are always denoted by $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{C}$, respectively. We use C to denote a positive constant which may vary from line to line, and denote by C_Φ the constant depends only on Φ. The symbol $A \lesssim B$ stands for the inequality $A \le CB$ or $A \le C_\Phi B$. The positive function f is said to be equivalent to the positive function g if $f \lesssim g$ and $g \lesssim f$. The symbol $\subset$ means the continuous embedding.

2. Preliminaries

2.1. Weak Orlicz–Karamata spaces

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We denote by $L_0(\Omega, \mathcal{F}, \mathbb{P})$, or simply $L_0(\Omega)$, the space of all measurable functions on $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.1. ([2]) A Lebesgue measurable function $b : [1, \infty) \to (0, \infty)$ is said to be a slowly varying function, if for any given $\epsilon > 0$, the function $t^\epsilon b(t)$ is equivalent to a non-decreasing function and the function $t^{-\epsilon} b(t)$ is equivalent to a non-increasing function on $[1, \infty)$.

Let b be a slowly varying function on $[1, \infty)$. Define γ_b on $(0, \infty)$ by

$$\gamma_b(t) = b(\max\{t, 1/t\}), \qquad t > 0.$$

Remark 2.2. ([2, Proposition 3.4.33]) (1) If b is a non-decreasing function, by the definition of γ_b, we know that γ_b is non-increasing on $(0, 1]$. For any given $\epsilon > 0$, the function $t^\epsilon \gamma_b(t)$ is equivalent to a non-decreasing function and the function $t^{-\epsilon} \gamma_b(t)$ is equivalent to a non-increasing function on $(0, \infty)$. (2) Let $r > 0$. Then $\gamma_b(rt) \approx \gamma_b(t)$ for all $t > 0$. (3) In the paper, we always assume that b is non-decreasing. We refer the reader to [2, p. 108] for some examples of non-decreasing slowly varying function.

Let $\mathcal{G}$ be the set of all increasing functions $\Phi : [0, \infty) \to [0, \infty)$ satisfying $\Phi(0) = 0$. The Orlicz space L_Φ is defined as follows:

$$L_\Phi = \{f \in L_0(\Omega) : \|f\|_{L_\Phi} < \infty\},$$

where

$$\|f\|_{L_\Phi} = \inf\Big\{t > 0 : \int_\Omega \Phi\Big(\frac{f}{t}\Big) d\mathbb{P} \le 1\Big\}.$$

Now we introduce the weak Orlicz–Karamata spaces.

Definition 2.3. Let $\Phi \in \mathcal{G}$ and b be a slowly varying function. The weak Orlicz–Karamata space consists of those functions $f \in L_0(\Omega)$ such that $\|f\|_{wL_{\Phi,b}} < \infty$, where

$$\|f\|_{wL_{\Phi,b}} = \sup_{t>0} t\|\chi_{\{|f|>t\}}\|_{L_\Phi} \gamma_b(\mathbb{P}(|f| > t)).$$

Note that

$$\|\chi_{\{|f|>t\}}\|_{L_\Phi} = \frac{1}{\Phi^{-1}(1/\mathbb{P}(|f| > t))}.$$

By a simple calculation, one can check that

$$\|f\|_{wL_{\Phi,b}} = \inf\Big\{s > 0 : \sup_{t>0} \Phi\Big(\frac{t\gamma_b(\mathbb{P}(|f| > t))}{s}\Big)\mathbb{P}(|f| > t) \le 1\Big\}.$$

If $b \equiv 1$, then $wL_{\Phi,b} = wL_\Phi$ (see [12] for the definition of wL_Φ). It is also obvious that for any $t > 0$,

$$\Phi\Big(\frac{t\gamma_b(\mathbb{P}(|f| > t))}{\|f\|_{wL_{\Phi,b}}}\Big)\mathbb{P}(|f| > t) \le 1.$$

Let $\Phi \in \mathcal{G}$. The lower index and upper index of Φ are respectively defined by

$$p_\Phi = \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)}, \qquad q_\Phi = \sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)}.$$

It is well known that $1 \le p_\Phi \le q_\Phi \le \infty$ if Φ is convex and $0 < p_\Phi \le q_\Phi \le 1$ if Φ is concave.

Lemma 2.4. ([12, Lemma 1.6]) *Let $\Phi \in \mathcal{G}$ be concave with $q_{\Phi^{-1}} < \infty$. Denote $p = p_{\Phi^{-1}}$, $q = q_{\Phi^{-1}}$. Then $\frac{\Phi^{-1}(t)}{t^p}$, $\frac{\Phi(t)}{t^{1/q}}$ are non-decreasing on $(0, \infty)$ and $\frac{\Phi^{-1}(t)}{t^q}$, $\frac{\Phi(t)}{t^{1/p}}$ are non-increasing on $(0, \infty)$.*

Example 2.5. Let $\Phi(t) = t^p$ for $0 < p \le 1$. Then Φ satisfies the condition in the lemma above.

Proposition 2.6. *Let b be a non-decreasing slowly varying function. For a concave function $\Phi \in \mathcal{G}$ with $q_{\Phi^{-1}} < \infty$, the functional $\|\cdot\|_{wL_{\Phi,b}}$ is a quasi-norm, i.e., it satisfies the following properties:*

(i) $\|f\|_{wL_{\Phi,b}} \ge 0$, *and* $\|f\|_{wL_{\Phi,b}} = 0$ *if and only if* $f = 0$;
(ii) $\|Cf\|_{wL_{\Phi,b}} = |C|\|f\|_{wL_{\Phi,b}}$, $\forall C \in \mathbb{C}$;
(iii) $\|f + g\|_{wL_{\Phi,b}} \le C_\Phi(\|f\|_{wL_{\Phi,b}} + \|g\|_{wL_{\Phi,b}})$.

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Proof. We only show (iii) here. Denote $p = p_{\Phi^{-1}}$, $q = q_{\Phi^{-1}}$. Set $A = \{|f| > t\}$ and $B = \{|g| > t\}$. So $\{|f + g| > 2t\} \subset A \cup B$. Then, by Remark 2.2,

$$\gamma_b(\mathbb{P}(|f+g| > 2t)) \geq \gamma_b(\mathbb{P}(A \cup B)).$$

Without loss of generality, we let $\|f\|_{wL_{\Phi,b}} = \|g\|_{wL_{\Phi,b}} = 1$. Applying Lemma 2.4 for $t > 0$, we have the following estimate

$$\begin{aligned}
&\Phi(t\gamma_b(\mathbb{P}(|f+g| > 2t)))\mathbb{P}(|f+g| > 2t) \\
&\quad \leq \Phi(t\gamma_b(\mathbb{P}(A \cup B)))\gamma_b(\mathbb{P}(A \cup B))^{-\frac{1}{p}} \cdot \gamma_b(\mathbb{P}(|f+g| > 2t))^{\frac{1}{p}}\mathbb{P}(|f+g| > 2t) \\
&\quad \leq C_\Phi \Phi(t\gamma_b(\mathbb{P}(A \cup B)))\gamma_b(\mathbb{P}(A \cup B))^{-\frac{1}{p}} \cdot \gamma_b(\mathbb{P}(A \cup B))^{\frac{1}{p}}\mathbb{P}(A \cup B) \\
&\quad \leq C_\Phi(\Phi(t\gamma_b(\mathbb{P}(A)))\mathbb{P}(A) + \Phi(t\gamma_b(\mathbb{P}(B)))\mathbb{P}(B)) \\
&\quad \leq 2C_\Phi,
\end{aligned}$$

where the second "$\leq$" is because $t^\epsilon \gamma_b(t)$ is equivalent to a non-decreasing function for any $\epsilon > 0$ and the third "$\leq$" is due to Remark 2.2 (1). Using Lemma 2.4 again, we deduce that

$$\begin{aligned}
&\Phi\Big(\frac{2t\gamma_b(\mathbb{P}(|f+g| > 2t))}{2 \cdot (2C_\Phi)^q}\Big)\mathbb{P}(|f+g| > 2t) \\
&\quad \leq \left(\frac{1}{(2C_\Phi)^q}\right)^{\frac{1}{q}} \Phi(t\gamma_b(\mathbb{P}(|f+g| > 2t)))\mathbb{P}(|f+g| > 2t) \leq 1,
\end{aligned}$$

which implies $\|f + g\|_{wL_{\Phi,b}} \leq 2 \cdot (2C_\Phi)^q(\|f\|_{wL_{\Phi,b}} + \|g\|_{wL_{\Phi,b}})$. $\square$

Remark 2.7. Let $\Phi \in \mathcal{G}$ be concave with $q_{\Phi^{-1}} < \infty$ and b be a non-decreasing slowly varying function. By the Aoki–Rolewicz theorem ([6, Exercise 1.4.6]), we find that there exists a positive constant $v \in (0, p_\Phi \wedge 1)$ such that, for all $R \in \mathbb{N}$ and $\{f_j\}_{j=0}^R$,

$$\|\sum_{j=0}^R f_j\|_{wL_{\Phi,b}}^v \leq 4\sum_{j=0}^R \|f_j\|_{wL_{\Phi,b}}^v.$$

Proposition 2.8. *Let $\Phi \in \mathcal{G}$ be concave and b be a non-decreasing slowly varying function. If $0 < r_2 \leq p_\Phi \leq q_\Phi < r_1 \leq \infty$, then*

$$L_{r_1} \subset wL_{r_1} \subset wL_{\Phi,b} \subset wL_{r_2}.$$

Proof. The first inclusion is well-known. Note that $t^\epsilon\gamma_b(t)$ is equivalent to a non-decreasing function and $t^{-\epsilon}\gamma_b(t)$ is equivalent to a non-increasing function for any $\epsilon > 0$. Then, by the Hölder inequality, we have, for any $t > 0$,

$$\begin{aligned}
t\|\chi_{\{|f|>t\}}\|_{L_\Phi}\gamma_b(\mathbb{P}(|f| > t)) &\lesssim t\|\chi_{\{|f|>t\}}\|_{q_\Phi}\gamma_b(\mathbb{P}(|f| > t)) \\
&\leq t\|\chi_{\{|f|>t\}}\|_{r_1}\mathbb{P}(|f| > t)^{\frac{1}{q_\Phi}-\frac{1}{r_1}}\gamma_b(\mathbb{P}(|f| > t)) \\
&\lesssim t\|\chi_{\{|f|>t\}}\|_{r_1}\gamma_b(1),
\end{aligned}$$

which implies that

$$\|f\|_{wL_{\Phi,b}} \lesssim \|f\|_{wL_{r_1}}.$$

Similarly, we have for any $t > 0$,

$$\begin{aligned}
t\|\chi_{\{|f|>t\}}\|_{L_\Phi}\gamma_b(\mathbb{P}(|f|>t)) &\gtrsim t\|\chi_{\{|f|>t\}}\|_{p_\Phi}\gamma_b(\mathbb{P}(|f|>t)) \\
&= t\mathbb{P}(|f|>t)^{\frac{1}{r_2}}\mathbb{P}(|f|>t)^{\frac{1}{p_\Phi}-\frac{1}{r_2}}\gamma_b(\mathbb{P}(|f|>t)) \\
&\gtrsim t\mathbb{P}(|f|>t)^{\frac{1}{r_2}}\gamma_b(1).
\end{aligned}$$

If $r_2 = p_\Phi$, then we may apply Remark 2.2 (1) to show the second "$\gtrsim$" in the above inequalities. Thus, we deduce that

$$\|f\|_{wL_{r_2}} \lesssim \|f\|_{wL_{\Phi,b}}. \quad \square$$

Proposition 2.9. *Let $\Phi \in \mathcal{G}$ be concave and b be a non-decreasing slowly varying function. Then $wL_{\Phi,b}$ is complete.*

Proof. To prove the completeness, we take a Cauchy sequence $(f_l)_{l\geq 1} \subset wL_{\Phi,b}$. We choose a subsequence (which we denote by $(f_l)_{l\geq 1} \subset wL_{\Phi,b}$ again) with

$$\|f_{l+1} - f_l\|_{wL_{\Phi,b}} \leq \frac{1}{2^{2l}}.$$

For notational reason, we set $f_0 = 0$. We consider the function

$$g(w) := \sum_{l=0}^{\infty} |f_{l+1}(w) - f_l(w)|, \quad w \in \Omega.$$

For $\lambda > 0$, we find that

$$\chi_{\{g>\lambda\}} \leq \sum_{l=0}^{\infty} \chi_{\{|f_{l+1}-f_l|>\lambda/2^{l+1}\}}.$$

Let v be the same as in Remark 2.7. Then, by Proposition 2.8, we get$\|f\|_v \lesssim \|f\|_{wL_{\Phi,b}}$ for $f \in wL_{\Phi,b}$ (in fact, $wL_{\Phi,b} \subset wL_{r_2} \subset L_v$ for $v < r_2 < p_\Phi$). Then

$$\begin{aligned}
\|\chi_{\{g>\lambda\}}\|_v^v &\leq \sum_{l=0}^{\infty} \|\chi_{\{|f_{l+1}-f_l|>\lambda/2^{l+1}\}}\|_v^v \leq \sum_{l=0}^{\infty} \frac{2^{(l+1)v}}{\lambda^v}\|f_{l+1} - f_l\|_v^v \\
&\lesssim \sum_{l=0}^{\infty} \frac{2^{(l+1)v}}{\lambda^v}\|f_{l+1} - f_l\|_{wL_{\Phi,b}}^v \leq \sum_{l=0}^{\infty} \frac{2^{(l+1)v}}{\lambda^v} 2^{-2lv} < \infty.
\end{aligned}$$

This implies that $\|\chi_{\{g>\lambda\}}\|_v \to 0$ as $\lambda \to \infty$ and g is finite almost everywhere. Therefore, the series

$$f = \sum_{l=0}^{\infty}(f_{l+1} - f_l) \quad \text{and} \quad \tilde{f} = \sum_{l=1}^{\infty}(f_{l+1} - f_l) = f - f_1$$

converge almost everywhere.

According to Remark 2.7, we have

$$\|\tilde{f}\|_{wL_{\Phi,b}}^v \leq 4\sum_{l=1}^{\infty} \|f_{l+1} - f_l\|_{wL_{\Phi,b}}^v \leq 4\sum_{l=1}^{\infty} 2^{-2lv} < \infty,$$

which deduces that $\tilde{f} \in wL_{\Phi,b}$. Hence, $\|f\|_{wL_{\Phi,b}} \lesssim \|\tilde{f}\|_{wL_{\Phi,b}} + \|f_1\|_{wL_{\Phi,b}}$ and $f \in wL_{\Phi,b}$.

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For $l \in \mathbb{N}$, consider

$$f - f_l = \sum_{m=l}^{\infty}(f_{m+1} - f_m).$$

By a similar argument as above, we obtain

$$\|f - f_l\|_{wL_{\Phi,b}}^v \lesssim \sum_{m=l}^{\infty} 2^{-2lv}.$$

Thus $\|f - f_l\|_{wL_{\Phi,b}} \to 0$ as $l \to \infty$. The proof is complete. □

2.2. Weak martingale Orlicz–Karamata–Hardy spaces

Now we introduce some standard notations from martingale theory. We refer to books [19,23,5] for the classical martingale space theory. Let $(\mathcal{F}_n)_{n\geq 0}$ be a non-decreasing sequence of σ-subalgebras of $\mathcal{F}$ such that $\sigma(\cup_{n\geq 0}\mathcal{F}_n) = \mathcal{F}$. Let $\mathbb{E}$ and $\mathbb{E}_n$ denote the expectation operator and the conditional expectation operator with respect to $\mathcal{F}_n$, respectively. For a martingale $f = (f_n)_{n\geq 0}$ adapted to $\{\mathcal{F}_n\}_{n\geq 0}$, we denote its martingale difference by $df_n = f_n - f_{n-1}$ ($n \geq 0$, with convention $f_{-1} = 0$). Then the maximal function, the square function and the conditional quadratic variation of martingale f are defined respectively by

$$M_n(f) = \sup_{0\leq i\leq n} |f_i|, \qquad M(f) = \sup_{i\geq 0} |f_i|;$$

$$S_n(f) = \left(\sum_{i=0}^{n} |df_i|^2\right)^{1/2}, \quad S(f) = \left(\sum_{i=0}^{\infty} |df_i|^2\right)^{1/2};$$

$$s_n(f) = \Big(\sum_{i=0}^{n} \mathbb{E}_{i-1}|df_i|^2\Big)^{1/2}, \qquad s(f) = \Big(\sum_{i=0}^{\infty} \mathbb{E}_{i-1}|df_i|^2\Big)^{1/2}.$$

The martingale $f = (f_n)_{n\geq 0}$ is $wL_{\Phi,b}$-bounded if $f_n \in wL_{\Phi,b}$ for all $n \geq 0$ and

$$\|f\|_{wL_{\Phi,b}} = \sup_{n\geq 0} \|f_n\|_{wL_{\Phi,b}} < \infty.$$

We now introduce the martingale weak Orlicz–Karamata–Hardy spaces. Denote by Λ the collection of all sequences $(\lambda_n)_{n\geq 0}$ of non-decreasing, non-negative and adapted functions with $\lambda_\infty = \lim_{n\to\infty} \lambda_n$. As usual, our martingale spaces are defined as follows:

$$wH_{\Phi,b} = \{f = (f_n)_{n\geq 0} : \|f\|_{wH_{\Phi,b}} = \|M(f)\|_{wL_{\Phi,b}} < \infty\};$$

$$wH_{\Phi,b}^S = \{f = (f_n)_{n\geq 0} : \|f\|_{wH_{\Phi,b}^S} = \|S(f)\|_{wL_{\Phi,b}} < \infty\};$$

$$wH_{\Phi,b}^s = \{f = (f_n)_{n\geq 0} : \|f\|_{wH_{\Phi,b}^s} = \|s(f)\|_{wL_{\Phi,b}} < \infty\};$$

$$wQ_{\Phi,b} = \{f = (f_n)_{n\geq 0} : \exists(\lambda_n)_{n\geq 0} \in \Lambda, \text{ s.t. } S_n(f) \leq \lambda_{n-1}, \lambda_\infty \in wL_{\Phi,b}\},$$

$$\|f\|_{wQ_{\Phi,b}} = \inf_{(\lambda_n)\in\Lambda} \|\lambda_\infty\|_{wL_{\Phi,b}};$$

$$wP_{\Phi,b} = \{f = (f_n)_{n\geq 0} : \exists(\lambda_n)_{n\geq 0} \in \Lambda, \text{ s.t. } |f_n| \leq \lambda_{n-1}, \lambda_\infty \in wL_{\Phi,b}\},$$

$$\|f\|_{wP_{\Phi,b}} = \inf_{(\lambda_n)\in\Lambda} \|\lambda_\infty\|_{wL_{\Phi,b}}.$$

In the end, we recall the definition of regularity. We refer the reader to [19, Chapter 7] for more details. The stochastic basis $(\mathcal{F}_n)_{n\geq 0}$ is said to be regular, if for $n \geq 0$ and $A \in \mathcal{F}_n$, there exists $B \in \mathcal{F}_{n-1}$ such that $A \subset B$ and $\mathbb{P}(B) \leq R\mathbb{P}(A)$, where R is a positive constant independent of n. A martingale is said to be regular if it is adapted to a regular σ-algebra sequence. This amounts to saying that there exists a constant $R > 0$ such that

$$f_n \leq R f_{n-1} \tag{2.1}$$

for all non-negative martingales $(f_n)_{n\geq 0}$ adapted to the stochastic basis $(\mathcal{F}_n)_{n\geq 0}$.

3. Atomic decompositions

In this section, we establish the atomic decompositions of martingale weak Orlicz–Karamata–Hardy spaces.

Let $\mathcal{T}$ be the set of all stopping times with respect to $(\mathcal{F}_n)_{n\geq 0}$. For a martingale $f = (f_n)_{n\geq 0}$ and $\tau \in \mathcal{T}$, we denote the stopped martingale by $f^\nu = (f_n^\nu)_{n\geq 0} = (f_{n\wedge\nu})_{n\geq 0}$, where $a \wedge b = \min(a, b)$. We recall the definition of an atom.

Definition 3.1. Let $\Phi \in \mathcal{G}$ be concave and $1 < r \leq \infty$. A measurable function a is called a $(1, L_\Phi, r)$-atom (or $(2, L_\Phi, r)$-atom, $(3, L_\Phi, r)$-atom, respectively) if there exists a stopping time $\nu \in \mathcal{T}$ such that

(1) $\mathbb{E}_n(a) = 0, \ \forall\, n \leq \nu$,
(2) $\|s(a)\|_r$(or $\|S(a)\|_r$, $\|M(a)\|_r$, respectively) $\leq \frac{\|\chi_{\{\nu<\infty\}}\|_r}{\|\chi_{\{\nu<\infty\}}\|_{L_\Phi}}$.

Theorem 3.2. *Let $\Phi \in \mathcal{G}$ be concave, $1 < r \leq \infty$ and b be a slowly varying function. If $f \in wH^s_{\Phi,b}$, then there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of $(1, L_\Phi, r)$-atoms associating with stopping times ν_k and a sequence $(\mu_k)_{k\in\mathbb{Z}}$ of non-negative real numbers satisfying $\mu_k = 3 \cdot 2^k \|\chi_{\{\nu_k<\infty\}}\|_{L_\Phi}$ such that for all $n \geq 0$,*

$$f_n = \sum_{k\in\mathbb{Z}} \mathbb{E}_n a^k, \quad a.e. \tag{3.1}$$

and

$$\inf\left\{t > 0 : \sup_{k\in\mathbb{Z}} \Phi\Big(\frac{2^k \gamma_b(\mathbb{P}(\nu_k < \infty))}{t}\Big)\mathbb{P}(\nu_k < \infty) \leq 1\right\} \lesssim \|f\|_{wH^s_{\Phi,b}}.$$

Proof. It suffices to show the result for $(1, L_\Phi, \infty)$-atoms. Let us consider the following stopping times for all $k \in \mathbb{Z}$,

$$\nu_k = \inf\{n \in \mathbb{N} : s_{n+1}(f) > 2^k\}.$$

The sequence of these stopping times is obviously non-decreasing. For each stopping time ν, denote $f_n^\nu = f_{n\wedge\nu}$. It is easy to see that

$$f_n = \sum_{k\in\mathbb{Z}} (f_n^{\nu_{k+1}} - f_n^{\nu_k}).$$

For all $k \in \mathbb{Z}, n \in \mathbb{N}$, let

$$\mu_k = 3 \cdot 2^k \left\|\chi_{\{\nu_k<\infty\}}\right\|_{L_\Phi} \quad \text{and} \quad a_n^k = \frac{f_n^{\nu_{k+1}} - f_n^{\nu_k}}{\mu_k}.$$

If $\mu_k = 0$, then set $a_n^k = 0$. Then $(a_n^k)_{n\geq 0}$ is a martingale for each fixed $k \in \mathbb{Z}$. Since $s(f^{\nu_k}) = s_{\nu_k}(f) \leq 2^k$, by the sublinearity of the operator s, we get

$$s\left((a_n^k)_{n\geq 0}\right) \leq \frac{s(f^{\nu_{k+1}}) + s(f^{\nu_k})}{\mu_k} \leq \left\|\chi_{\{\nu_k<\infty\}}\right\|_{L_\Phi}^{-1}.$$

Hence $(a_n^k)_{n\geq 0}$ is a bounded L_2-martingale. Consequently, there exists an element $a^k \in L_2$ such that $\mathbb{E}_n a^k = a_n^k$. If $n \leq \nu_k$, then $a_n^k = 0$. Thus we conclude that a^k is really a $(1, L_\Phi, \infty)$-atom. Since $\{\nu_k < \infty\} = \{s(f) > 2^k\}$ for any $k \in \mathbb{Z}$, we have

$$\Phi\left(\frac{2^k\gamma_b(\mathbb{P}(\nu_k < \infty))}{\|f\|_{wH^s_{\Phi,b}}}\right)\mathbb{P}(\nu_k < \infty) = \Phi\left(\frac{2^k\gamma_b(\mathbb{P}(s(f) > 2^k))}{\|f\|_{wH^s_{\Phi,b}}}\right)\mathbb{P}(s(f) > 2^k) \leq 1.$$

Thus

$$\inf\left\{t > 0 : \sup_{k\in\mathbb{Z}} \Phi\Big(\frac{2^k\gamma_b(\mathbb{P}(\nu_k < \infty))}{t}\Big)\mathbb{P}(\nu_k < \infty) \leq 1\right\} \leq \|f\|_{wH^s_{\Phi,b}} < \infty. \quad \square$$

Theorem 3.3. *Let $\Phi \in \mathcal{G}$ be concave with $q_{\Phi^{-1}} < \infty$, $1 < r \leq \infty$ and b be a non-decreasing slowly varying function. Assume that martingale f has a decomposition (3.1) with a sequence $(a^k)_{k\in\mathbb{Z}}$ of $(1, L_\Phi, r)$-atoms and a sequence $(\mu_k)_{k\in\mathbb{Z}}$ of nonnegative real numbers satisfying $\mu_k = 3 \cdot 2^k\|\chi_{\{\nu_k<\infty\}}\|_{L_\Phi}$, where ν_k is the stopping time associated with a^k. Then $f \in wH^s_{\Phi,b}$ and*

$$\|f\|_{wH^s_{\Phi,b}} \lesssim \inf\left\{t > 0 : \sup_{k\in\mathbb{Z}} \Phi\Big(\frac{2^k\gamma_b(\mathbb{P}(\nu_k < \infty))}{t}\Big)\mathbb{P}(\nu_k < \infty) \leq 1\right\}.$$

In order to prove Theorem 3.3, we firstly present several lemmas. Set

$$B := \inf\left\{t > 0 : \sup_{k\in\mathbb{Z}} \Phi\Big(\frac{2^k\gamma_b(\mathbb{P}(\nu_k < \infty))}{t}\Big)\mathbb{P}(\nu_k < \infty) \leq 1\right\} < \infty.$$

For an arbitrary integer k_0, set

$$f = \sum_{k\in\mathbb{Z}} \mu_k a^k := F_1 + F_2,$$

where

$$F_1 = \sum_{k=-\infty}^{k_0-1} \mu_k a^k \quad \text{and} \quad F_2 = \sum_{k=k_0}^{\infty} \mu_k a^k.$$

Note that

$$s(F_1) \leq \sum_{k=-\infty}^{k_0-1} \mu_k s(a^k), \quad s(F_2) \leq \sum_{k=k_0}^{\infty} \mu_k s(a^k).$$

The symbols B, k_0, F_1 and F_2 will be used in the following lemmas.

Lemma 3.4. *Let $\Phi \in \mathcal{G}$ be concave with $q_{\Phi^{-1}} < \infty$ and b be a non-decreasing slowly varying function. Then*

$$\Phi\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(s(F_2) > 3\cdot 2^{k_0}))}{B}\Big)\mathbb{P}(s(F_2) > 3\cdot 2^{k_0}) \leq \tag{3.2}$$
$$C_\Phi \sum_{k=k_0}^{\infty} \Phi\Big(\frac{2^k\gamma_b(\mathbb{P}(\nu_k < \infty)}{B}\Big)\mathbb{P}(\nu_k < \infty).$$

Proof. Since $s(a^k) = 0$ on the set $\{\tau_k = \infty\}$, we have $\{s(a^k) > 0\} \subset \{\nu_k < \infty\}$. Then,

$$\{s(F_2) > 3\cdot 2^{k_0}\} \subset \{s(F_2) > 0\} \subset \bigcup_{k=k_0}^{\infty}\{s(a^k) > 0\} \subset \bigcup_{k=k_0}^{\infty}\{\nu_k < \infty\}. \tag{3.3}$$

We get that

$$\mathbb{P}(s(F_2) > 3\cdot 2^{k_0}) \leq \mathbb{P}(\bigcup_{k=k_0}^{\infty}\{\nu_k < \infty\})$$

and

$$\gamma_b(\mathbb{P}(s(F_2) > 3\cdot 2^{k_0})) \geq \gamma_b(\mathbb{P}(\bigcup_{k=k_0}^{\infty}\{\nu_k < \infty\}))$$

since $\gamma_b(\cdot)$ is decreasing on $(0, 1]$ (see Remark 2.2). Denote $p = p_{\Phi^{-1}}$. According to Lemma 2.4, $\frac{\Phi(t)}{t^{1/p}}$ is decreasing on $(0, \infty)$. Thus we have the following estimate

$$\begin{aligned}
&\Phi\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(s(F_2) > 3\cdot 2^{k_0}))}{B}\Big)\mathbb{P}(s(F_2) > 3\cdot 2^{k_0})\\
&\leq \Phi\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(\bigcup_{k=k_0}^{\infty}\{\nu_k < \infty\}))}{B}\Big)\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(\bigcup_{k=k_0}^{\infty}\{\nu_k < \infty\}))}{B}\Big)^{-\frac{1}{p}}\\
&\quad\cdot\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(s(F_2) > 3\cdot 2^{k_0}))}{B}\Big)^{\frac{1}{p}}\mathbb{P}(s(F_2) > 3\cdot 2^{k_0})\\
&\leq C_\Phi\Phi\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(\bigcup_{k=k_0}^{\infty}\{\nu_k < \infty\}))}{B}\Big)\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(\bigcup_{k=k_0}^{\infty}\{\nu_k < \infty\}))}{B}\Big)^{-\frac{1}{p}}\\
&\quad\cdot\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(\bigcup_{k=k_0}^{\infty}\{\nu_k < \infty\}))}{B}\Big)^{\frac{1}{p}}\mathbb{P}(\bigcup_{k=k_0}^{\infty}\{\nu_k < \infty\})\\
&\leq C_\Phi\sum_{k=k_0}^{\infty}\Phi\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(\nu_k < \infty))}{B}\Big)\mathbb{P}(\nu_k < \infty),
\end{aligned}$$

where the second "$\leq$" is because $t^\epsilon\gamma_b(t)$ is equivalent to a non-decreasing function for any $\epsilon > 0$. $\square$

Lemma 3.5. *Let $\Phi \in \mathcal{G}$ be concave with $q_{\Phi^{-1}} < \infty$ and b be a non-decreasing slowly varying function. Then $s(F_2) \in wL_{\Phi,b}$ and*

$$\|s(F_2)\|_{wL_{\Phi,b}} \lesssim B.$$

Proof. By the definition of B, $\mathbb{P}(\nu_k < \infty) \leq 1/\Phi\big(2^k\gamma_b(\mathbb{P}(\nu_k < \infty))/B\big)$ for each $k \in \mathbb{Z}$. Denote $q = q_{\Phi^{-1}}$. Then, by Lemma 2.4, $\frac{\Phi(t)}{t^{1/q}}$ is increasing on $(0, \infty)$. Then we obtain

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} \Phi\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(\nu_k<\infty))}{B}\Big)\mathbb{P}(\nu_k<\infty) \\
&\quad\le \sum_{k=k_0}^{\infty} \Phi\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(\nu_k<\infty))}{B}\Big)\frac{1}{\Phi\big(2^k\gamma_b(\mathbb{P}(\nu_k<\infty))/B\big)} \\
&\quad\le \sum_{k=k_0}^{\infty} \Phi\Big(\frac{2^{k}\gamma_b(\mathbb{P}(\nu_k<\infty))}{B}\Big)\Big(\frac{2^{k}\gamma_b(\mathbb{P}(\nu_k<\infty))}{B}\Big)^{-\frac{1}{q}} \\
&\qquad\cdot\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(\nu_k<\infty))}{B}\Big)^{\frac{1}{q}}\frac{1}{\Phi\big(2^k\gamma_b(\mathbb{P}(\nu_k<\infty))/B\big)} \\
&\quad= \sum_{k=k_0}^{\infty} 2^{(k_0-k)\frac{1}{q}} = \frac{1}{1-2^{-1/q}} := C_1.
\end{aligned}$$

Applying Lemmas 2.4 and 3.4, we have

$$\begin{aligned}
&\Phi\Big(\frac{3\cdot 2^{k_0}\gamma_b(\mathbb{P}(s(F_2)>3\cdot 2^{k_0}))}{3(C_\Phi C_1)^q B}\Big)\mathbb{P}(s(F_2)>3\cdot 2^{k_0}) \\
&\quad\le \Big(\frac{1}{(C_\Phi C_1)^q}\Big)^{1/q}\Phi\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(s(F_2)>3\cdot 2^{k_0}))}{B}\Big)\mathbb{P}(s(F_2)>3\cdot 2^{k_0}) \le 1,
\end{aligned}$$

which implies that

$$\|s(F_2)\|_{wL_{\Phi,b}} \le 3(C_\Phi C_1)^q B. \quad \square$$

Lemma 3.6. *Let $\Phi\in\mathcal{G}$ be concave with $q_{\Phi^{-1}}<\infty$, $1<r\le\infty$ and b be a non-decreasing slowly varying function. Then*

$$\begin{aligned}
&\Phi\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(s(F_1)>3\cdot 2^{k_0}))}{B}\Big)\mathbb{P}(s(F_1)>3\cdot 2^{k_0}) \le \qquad (3.4)\\
&C_\Phi\Big(\sum_{k=-\infty}^{k_0-1}\Phi\Big(\frac{2^{k_0}\gamma_b(2^{(k-k_0)r}\mathbb{P}(\nu_k<\infty))}{B}\Big)^{\frac{1}{r}}2^{k-k_0}\mathbb{P}(\nu_k<\infty)^{\frac{1}{r}}\Big)^{r}.
\end{aligned}$$

Proof. Assume that a^k is a $(1,L_\Phi,r)$-atom for each $k\in\mathbb{Z}$. By Chebychev's inequality, we have

$$\begin{aligned}
\mathbb{P}(s(F_1)>3\cdot 2^{k_0}) &\le \Big\|\frac{s(F_1)}{3\cdot 2^{k_0}}\Big\|_r^r \le \Big(\frac{1}{3\cdot 2^{k_0}}\sum_{k=-\infty}^{k_0-1}\mu_k\|s(a^k)\|_r\Big)^r \\
&\le \Big(\frac{1}{2^{k_0}}\sum_{k=-\infty}^{k_0-1}2^k\mathbb{P}(\nu_k<\infty)^{\frac{1}{r}}\Big)^r \le \Big(\sum_{k=-\infty}^{k_0-1}2^{k-k_0}\Big)^r \le 1.
\end{aligned}$$

Denote

$$I := \Big(\frac{1}{2^{k_0}}\sum_{k=-\infty}^{k_0-1}2^k\mathbb{P}(\nu_k<\infty)^{\frac{1}{r}}\Big)^r.$$

Since $\gamma_b(\cdot)$ is decreasing on $(0,1]$, we find that $\gamma_b(\mathbb{P}(s(F_1)>3\cdot 2^{k_0}))\ge\gamma_b(I)$. It follows from Lemma 2.4 that $\frac{\Phi(t)}{t^{1/p}}$ is decreasing on $(0,\infty)$ with $p=p_{\Phi^{-1}}$. Taking similar argument as Lemma 3.4, we get

$$\begin{aligned}
&\Phi\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(s(F_1)>3\cdot 2^{k_0}))}{B}\Big)\mathbb{P}(s(F_2)>3\cdot 2^{k_0})\\
&\quad\le \Phi\Big(\frac{2^{k_0}\gamma_b(I)}{B}\Big)\Big(\frac{2^{k_0}\gamma_b(I)}{B}\Big)^{-\frac{1}{p}}\\
&\qquad\cdot\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(s(F_1)>3\cdot 2^{k_0}))}{B}\Big)^{\frac{1}{p}}\mathbb{P}(s(F_2)>3\cdot 2^{k_0})\\
&\quad\le C_\Phi\Phi\Big(\frac{2^{k_0}\gamma_b(I)}{B}\Big)I\\
&\quad\le C_\Phi\Big(\sum_{k=-\infty}^{k_0-1}\Phi\Big(\frac{2^{k_0}\gamma_b(2^{(k-k_0)r}\mathbb{P}(\nu_k<\infty))}{B}\Big)^{\frac{1}{r}}2^{k-k_0}\mathbb{P}(\nu_k<\infty)^{\frac{1}{r}}\Big)^r,
\end{aligned}$$

where the second "$\le$" is because $t^\epsilon\gamma_b(t)$ is equivalent to a non-decreasing function and the third one is due to Remark 2.2. The proof is complete. $\square$

Lemma 3.7. *Let $\Phi\in\mathcal{G}$ be concave with $q_{\Phi^{-1}}<\infty$ and b be a non-decreasing slowly varying function. Then $s(F_1)\in wL_{\Phi,b}$ and*

$$\|s(F_1)\|_{wL_{\Phi,b}}\lesssim B.$$

Proof. The proof is similar to Lemma 3.5. We give some main calculation. For $1<r<\infty$, set $J:=2^{(k-k_0)r}\mathbb{P}(\nu_k<\infty)$. Then, for $k<k_0$,

$$J\le\mathbb{P}(\nu_k<\infty),\quad \gamma_b(J)\ge\gamma_b(\mathbb{P}(\nu_k<\infty)).$$

Take $0<\epsilon<\frac{1}{r}(1-\frac{1}{rp})$. Using Lemma 2.4 for the decreasing function $\frac{\Phi(t)}{t^{1/p}}$, we obtain

$$\begin{aligned}
&\sum_{k=-\infty}^{k_0-1}\Phi\Big(\frac{2^{k_0}\gamma_b(J)}{B}\Big)^{\frac{1}{r}}2^{k-k_0}\mathbb{P}(\nu_k<\infty)^{\frac{1}{r}}\\
&\quad\le\sum_{k=-\infty}^{k_0-1}\Phi\Big(\frac{2^{k}\gamma_b(\mathbb{P}(\nu_k<\infty))}{B}\Big)^{\frac{1}{r}}\Big(\frac{2^{k}\gamma_b(\mathbb{P}(\nu_k<\infty))}{B}\Big)^{-\frac{1}{rp}}\\
&\qquad\cdot\Big(\frac{2^{k_0}\gamma_b(J)}{B}\Big)^{\frac{1}{rp}}2^{k-k_0}\mathbb{P}(\nu_k<\infty)^{\frac{1}{r}}\\
&\quad=\sum_{k=-\infty}^{k_0-1}2^{(k-k_0)(1-\frac{1}{rp})}\Phi\Big(\frac{2^{k}\gamma_b(\mathbb{P}(\nu_k<\infty))}{B}\Big)^{\frac{1}{r}}(\gamma_b(\mathbb{P}(\nu_k<\infty)))^{-\frac{1}{rp}}\\
&\qquad\cdot(\gamma_b(J))^{\frac{1}{rp}}J^\epsilon\cdot J^{-\epsilon}\mathbb{P}(\nu_k<\infty)^{\frac{1}{r}}\\
&\quad:=K.
\end{aligned}$$

Since $t^\epsilon\gamma_b(t)$ is equivalent to a non-decreasing function for any $\epsilon>0$, we have

$$(\gamma_b(J))^{\frac{1}{rp}}J^\epsilon\le C(\gamma_b(\mathbb{P}(\nu_k<\infty)))^{\frac{1}{rp}}\mathbb{P}(\nu_k<\infty)^\epsilon.$$

So, for K, we deduce that

$$K\le C\sum_{k=-\infty}^{k_0-1}2^{(k-k_0)(1-\frac{1}{rp}-\epsilon r)}\Phi\Big(\frac{2^{k}\gamma_b(\mathbb{P}(\nu_k<\infty))}{B}\Big)^{\frac{1}{r}}\mathbb{P}(\nu_k<\infty)^{\frac{1}{r}}.$$

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Note that $r > 1$ and $p \geq 1$. We find that $1 - \frac{1}{rp} - \epsilon r > 0$. Then, it follows from $\Phi\big(2^k \gamma_b(\mathbb{P}(\nu_k < \infty))/B\big)\mathbb{P}(\nu_k < \infty) \leq 1$ that

$$K \leq C \sum_{k=-\infty}^{k_0-1} 2^{(k-k_0)(1-\frac{1}{rp}-\epsilon r)} := C_2 < \infty.$$

Applying Lemmas 2.4 and 3.6, we have

$$\begin{aligned}
&\Phi\Big(\frac{3 \cdot 2^{k_0}\gamma_b(\mathbb{P}(s(F_1) > 3 \cdot 2^{k_0}))}{3(C_\Phi C_2^r)^q B}\Big)\mathbb{P}(s(F_1) > 3 \cdot 2^{k_0}) \\
&\leq \Big(\frac{1}{(C_\Phi C_2^r)^q}\Big)^{1/q}\Phi\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(s(F_1) > 3 \cdot 2^{k_0}))}{B}\Big)\mathbb{P}(s(F_1) > 3 \cdot 2^{k_0}) \leq 1,
\end{aligned}$$

which implies that

$$\|s(F_1)\|_{wL_{\Phi,b}} \leq 3(C_\Phi C_2^r)^q B. \quad \square$$

Remark 3.8. Assume that $f = f_{k_0}^1 + f_{k_0}^2$. Combining Lemmas 3.5 and 3.7, we also have shown that if $f_{k_0}^1$ satisfies (3.2) and $f_{k_0}^2$ satisfies (3.4) (replacing $s(F_1)$, $s(F_2)$ by $f_{k_0}^1$, $f_{k_0}^2$), then $f \in wL_{\Phi,b}$ and $\|f\|_{wL_{\Phi,b}} \lesssim B$. This observation will be used to prove martingale inequalities between various weak Orlicz–Karamata–Hardy spaces.

Proof of Theorem 3.3. Assume that a martingale f has the decomposition as (3.1). We use the symbols mentioned before Lemma 3.4. Let us firstly deal with $r = \infty$. Since a^k is a $(1, L_\Phi), \infty)$-atom for every $k \in \mathbb{Z}$, we find that

$$\begin{aligned}
\|s(F_1)\|_\infty &\leq \sum_{k=-\infty}^{k_0-1} \mu_k \|s(a^k)\|_\infty \\
&\leq \sum_{k=-\infty}^{k_0-1} \mu_k \|\chi_{\{\tau_k<\infty\}}\|_{L_\Phi}^{-1} \leq 3 \cdot 2^{k_0}.
\end{aligned}$$

Thus we can deduce that

$$\{s(f) > 6 \cdot 2^{k_0}\} \subset \{s(F_2) > 3 \cdot 2^{k_0}\}$$

and

$$\|f\|_{wH^s_{\Phi,b}} \lesssim \|s(F_2)\|_{wL_{\Phi,b}}.$$

Then, by Lemma 3.5, we prove the conclusions of theorem.

For the case $r < \infty$, note that

$$\|f\|_{wH^s_{\Phi,b}} \lesssim \|s(F_1)\|_{wL_{\Phi,b}} + \|s(F_2)\|_{wL_{\Phi,b}}.$$

We apply Lemmas 3.5 and 3.7 to complete the proof. $\square$

Theorem 3.9. *In Theorems 3.2 and 3.3, if we replace $wH^s_{\Phi,b}$ and the $(1, L_\Phi, r)$-atoms by $wQ_{\Phi,b}$ and $(2, L_\Phi, \infty)$-atoms (or by $wP_{\Phi,b}$ and $(3, L_\Phi, \infty)$-atoms, respectively), then the conclusions still hold.*

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Proof. The proof is similar to Theorems 3.2 and 3.3, so we only give it in sketch. Let $f = (f_n)_{n\geq 0} \in wQ_{\Phi,b}$ (or $wP_{\Phi,b}$). The stopping times ν_k's are defined by

$$\nu_k = \inf\{n \in \mathbb{N} : \lambda_n > 2^k\}, \quad (\inf \emptyset = \infty),$$

where $(\lambda_n)_{n\geq 0} \in \Lambda$. Let a_n^k and μ_k $(k \in \mathbb{Z})$ be the same as in the proof of Theorem 3.2. Then we get (3.1), where $(a^k)_{k\in\mathbb{Z}}$ is a sequence of $(2, L_\Phi, \infty)$-atoms (or $(3, L_\Phi, \infty)$-atoms). Moreover,

$$\inf\left\{t > 0 : \sup_{k\in\mathbb{Z}} \Phi\Big(\frac{2^k \gamma_b(\mathbb{P}(\nu_k < \infty))}{t}\Big)\mathbb{P}(\nu_k < \infty) \leq 1\right\} \leq \|f\|_{wQ_{\Phi,b}} \text{ (or } \ \|f\|_{wP_{\Phi,b}})$$

still holds.

To prove the converse part, let

$$\lambda_n = \sum_{k\in\mathbb{Z}} \mu_k \chi_{\{\nu_k \leq n\}} \|S(a^k)\|_\infty \text{ (or } \lambda_n = \sum_{k\in\mathbb{Z}} \mu_k \chi_{\{\nu_k \leq n\}} \|M(a^k)\|_\infty).$$

Then $(\lambda_n)_{n\geq 0}$ is a non-decreasing, non-negative and adapted sequence with $S_{n+1}(f) \leq \lambda_n$ (or $|f_{n+1}| \leq \lambda_n$) for any $n \geq 0$. For any given integer k_0, let

$$\lambda_\infty = \lambda_\infty^{(1)} + \lambda_\infty^{(2)},$$

where

$$\lambda_\infty^{(1)} = \sum_{k=-\infty}^{k_0-1} \mu_k \chi_{\{\nu_k < \infty\}} \|S(a^k)\|_\infty \text{ (or } \lambda_\infty^{(1)} = \sum_{k=-\infty}^{k_0-1} \mu_k \chi_{\{\nu_k < \infty\}} \|M(a^k)\|_\infty),$$

and

$$\lambda_\infty^{(2)} = \sum_{k=k_0}^{\infty} \mu_k \chi_{\{\nu_k < \infty\}} \|S(a^k)\|_\infty \text{ (or } \lambda_\infty^{(2)} = \sum_{k=k_0}^{\infty} \mu_k \chi_{\{\nu_k < \infty\}} \|M(a^k)\|_\infty).$$

By replacing $s(F_1)$ and $s(F_2)$ in Lemmas 3.5 and 3.7 with $\lambda_\infty^{(1)}$ and $\lambda_\infty^{(2)}$, we obtain $f \in wQ_{\Phi,b}$ (or $f \in wP_{\Phi,b}$). $\square$

Proposition 3.10. *Let $\Phi \in \mathcal{G}$ be concave with $q_{\Phi^{-1}} < \infty$ and b be a non-decreasing slowly varying function. If $(\mathcal{F}_n)_{n\geq 0}$ is regular, then*

$$wH^S_{\Phi,b} \subset wQ_{\Phi,b}, \quad wH_{\Phi,b} \subset wP_{\Phi,b}.$$

Proof. We only give the proof for $wH_{\Phi,b} \subset wP_{\Phi,b}$. The other one can be shown in a similar way. Take $f \in wH_{\Phi,b}$. It follows from the regularity of $(\mathcal{F}_n)_{n\geq 0}$ that there exists a sequence of stopping times ν_k such that

$$\{M(f) > 2^k\} \subset \{\nu_k < \infty\}, \quad M_{\nu_k}(f) \leq 2^k, \quad \mathbb{P}(\nu_k < \infty) \leq R\mathbb{P}(M(f) > 2^k)$$

and $\nu_k \leq \nu_{k+1}$, $\nu_k \uparrow \infty$ according to [19, Definition 7.1.1]. Then we have the following decomposition

$$f_n = \sum_{k\in\mathbb{Z}} (f_n^{\nu_{k+1}} - f_n^{\nu_k}).$$

Also, define

$$\mu_k = 3 \cdot 2^k \left\|\chi_{\{\nu_k<\infty\}}\right\|_{L_\Phi}, \quad \text{and} \quad a_n^k = \frac{f_n^{\nu_{k+1}} - f_n^{\nu_k}}{\mu_k}.$$

Then $a^k = (a_n^k)_{n\geq 0}$ is a $(3, L_\Phi, \infty)$-atom for each $k \in \mathbb{Z}$.

Now we know that f has a decomposition of (3.1) with a sequence $(\mu_k)_{k\in\mathbb{Z}}$ of non-negative real numbers satisfying $\mu_k = 3 \cdot 2^k \|\chi_{\{\nu_k<\infty\}}\|_{L_\Phi}$. Applying Theorem 3.9, we get that

$$f \in wP_{\Phi,b}.$$

The proof is complete. □

4. Martingale inequalities

As an application of the atomic decompositions, we obtain a sufficient condition for a σ-sublinear operator to be bounded from the martingale Hardy spaces to $wL_{\Phi,b}$.

An operator $T : X \to Y$ is called a σ-sublinear operator if for any $\alpha \in \mathbb{C}$ it satisfies

$$\left|T\left(\sum_{k=1}^{\infty} f_k\right)\right| \leq \sum_{k=1}^{\infty} |T(f_k)| \quad \text{and} \quad |T(\alpha f)| = |\alpha||T(f)|,$$

where X is a martingale space and Y is a measurable function space. For $0 < r < \infty$, the martingale Hardy space H_r^s is defined as $H_r^s = \{f : \|f\|_{H_r^s} = \|s(f)\|_r < \infty\}$. Similarly, we may define H_r^S and H_r with respect to the operators S and M.

Theorem 4.1. *Let $\Phi \in \mathcal{G}$ be concave with $q_{\Phi^{-1}} < \infty$, $1 < r \leq \infty$ and b be a non-decreasing slowly varying function. If $T : H_r^s \to L_r$ is a bounded σ-sublinear operator and*

$$\{|Ta| > 0\} \subset \{\nu < \infty\} \tag{4.1}$$

for all $(1, L_\Phi, \infty)$-atoms a, where ν is the stopping time associated with a, then

$$\|Tf\|_{wL_{\Phi,b}} \lesssim \|f\|_{wH^s_{\Phi,b}}, \quad f \in wH^s_{\Phi,b}.$$

Proof. The proof is similar to Lemmas 3.4–3.7. So we only give it in sketch. Let a martingale $f \in wH^s_{\Phi,b}$. By Theorem 3.2 we know that f has the decomposition as (3.1) such that a^k is a $(1, L_\Phi, \infty)$-atom and $\mu_k = 3 \cdot 2^k \|\chi_{\{\tau_k<\infty\}}\|_{L_\Phi}$. For an arbitrary integer k_0, we set again

$$f = \sum_k \mu_k a^k := F_1 + F_2,$$

where

$$F_1 = \sum_{k=-\infty}^{k_0-1} \mu_k a^k \quad \text{and} \quad F_2 = \sum_{k=k_0}^{\infty} \mu_k a^k.$$

Note that, by the σ-sublinearity of the operator T, we have

$$|T(F_1)| \leq \sum_{k=-\infty}^{k_0-1} \mu_k |T(a^k)|, \quad |T(F_2)| \leq \sum_{k=k_0}^{\infty} \mu_k |T(a^k)|. \tag{4.2}$$

We need to estimate $\|T(F_1)\|_{wL_{\Phi,b}}$ and $\|T(F_2)\|_{wL_{\Phi,b}}$, separately.

We first estimate $\|T(F_1)\|_{wL_{\Phi,b}}$. Since a^k is a $(1, L_\Phi, \infty)$-atom for each $k \in \mathbb{Z}$, we have $\|s(a^k)\|_\infty \leq \|\chi_{\{\nu_k<\infty\}}\|_{L_\Phi}^{-1}$. It follows from the boundedness of T that $\|Tf\|_r \leq C_r\|f\|_{H_r^s}$. Then, by Chebychev's inequality, we have

$$\begin{aligned}\mathbb{P}(T(F_1) > 3\cdot 2^{k_0}) &\leq \Big\|\frac{T(F_1)}{3\cdot 2^{k_0}}\Big\|_r^r \leq \Big(\frac{1}{3\cdot 2^{k_0}}\sum_{k=-\infty}^{k_0-1}\mu_k\|T(a^k)\|_r\Big)^r \\ &\leq C_r\Big(\frac{1}{3\cdot 2^{k_0}}\sum_{k=-\infty}^{k_0-1}\mu_k\|s(a^k)\|_r\Big)^r \\ &\leq C_r\Big(\frac{1}{3\cdot 2^{k_0}}\sum_{k=-\infty}^{k_0-1}\mu_k\|s(a^k)\|_\infty\|\chi_{\{\nu_k<\infty\}}\|_r\Big)^r \\ &\leq C_r\Big(\frac{1}{2^{k_0}}\sum_{k=-\infty}^{k_0-1}2^k\mathbb{P}(\nu_k<\infty)^{\frac{1}{r}}\Big)^r \leq C_r\Big(\sum_{k=-\infty}^{k_0-1}2^{k-k_0}\Big)^r \leq C_r.\end{aligned}$$

By Remark 2.2(2), we have

$$\gamma_b(\mathbb{P}(T(F_1) > 3\cdot 2^{k_0})) \leq C_3\gamma_b(\frac{1}{C_r}\mathbb{P}(T(F_1) > 3\cdot 2^{k_0})).$$

Then, by using the same argument of Lemma 3.6, we obtain

$$\Phi\Big(\frac{2^{k_0}\gamma_b(\mathbb{P}(T(F_1) > 3\cdot 2^{k_0}))}{B}\Big)\mathbb{P}(T(F_1) > 3\cdot 2^{k_0}) \leq C_\Phi C_3\Phi\Big(\frac{2^{k_0}C_3\gamma_b(I)}{B}\Big)I,$$

where I is same to the one appeared in Lemma 3.6. Following the line of Lemma 3.7, we get

$$\|T(F_1)\|_{wL_{\Phi,b}} \lesssim B.$$

Now we start to estimate $\|T(F_2)\|_{wL_{\Phi,b}}$. According to condition (4.1),

$$\{T(F_2) > 2^{k_0}\} \subset \{T(F_2) > 0\} \subset \bigcup_{k=k_0}^{\infty}\{T(a^k) > 0\} \subset \bigcup_{k=k_0}^{\infty}\{\nu_k < \infty\}.$$

So, repeating the calculation of Lemmas 3.4 and 3.5, we easily obtain

$$\|T(F_2)\|_{wL_{\Phi,b}} \lesssim \|f\|_{wH_{\Phi,b}^s}.$$

The proof is complete now. □

Remark 4.2. Compared with [12, Theorem 3.1], we do not need the condition $\frac{1}{p_{\Phi^{-1}}} < r$ any more.

Similar to Theorem 4.1, we obtain the following theorem by applying Theorem 3.9.

Theorem 4.3. *Let $\Phi \in \mathcal{G}$ be concave with $q_{\Phi^{-1}} < \infty$, $1 < r \leq \infty$ and b be a non-decreasing slowly varying function. If $T : H_r^S \to L_r$ (or $H_r \to L_r$) is bounded σ-sublinear operator and (4.1) holds for all $(2, L_\Phi, \infty)$-atoms (or $(3, L_\Phi, \infty)$-atoms), then*

$$\begin{aligned}&\|Tf\|_{wL_{\Phi,b}} \lesssim \|f\|_{wQ_{\Phi,b}}, \quad f \in wQ_{\Phi,b}, \\ (or \quad &\|Tf\|_{wL_{\Phi,b}} \lesssim \|f\|_{wP_{\Phi,b}}, \quad f \in wP_{\Phi,b}).\end{aligned}$$

Theorem 4.4. *Let $\Phi \in \mathcal{G}$ be concave with $q_{\Phi^{-1}} < \infty$ and b be a non-decreasing slowly varying function. Then the following inequalities hold:*

$$\|f\|_{wH_{\Phi,b}} \lesssim \|f\|_{wH^s_{\Phi,b}}, \quad \|f\|_{wH^S_{\Phi,b}} \lesssim \|f\|_{wH^s_{\Phi,b}}; \tag{4.3}$$

$$\|f\|_{wH_{\Phi,b}} \leq \|f\|_{wP_{\Phi,b}}, \quad \|f\|_{wH^S_{\Phi,b}} \leq \|f\|_{wQ_{\Phi,b}}; \tag{4.4}$$

$$\|f\|_{H^S_{\Phi,b}} \lesssim \|f\|_{wP_{\Phi,b}}, \quad \|f\|_{wH_{\Phi,b}} \lesssim \|f\|_{wQ_{\Phi,b}}; \tag{4.5}$$

$$\|f\|_{wH^s_{\Phi,b}} \lesssim \|f\|_{wP_{\Phi,b}}, \quad \|f\|_{wH^s_{\Phi,b}} \lesssim \|f\|_{wQ_{\Phi,b}}; \tag{4.6}$$

$$\|f\|_{wP_{\Phi,b}} \lesssim \|f\|_{wQ_{\Phi,b}} \lesssim \|f\|_{wP_{\Phi,b}}. \tag{4.7}$$

Moreover, if $\{\mathcal{F}_n\}_{n\geq 0}$ is regular, then

$$wH^S_{\Phi,b} = wQ_{\Phi,b} = wP_{\Phi,b} = wH_{\Phi,b} = wH^s_{\Phi,b}$$

with equivalent quasi-norms.

Proof. First we show (4.3). Let $f \in wH^s_{\Phi,b}$. The maximal operator $T = M$ is σ-sublinear and $\|Mf\|_2 \lesssim \|s(f)\|_2$ (see [23, Theorem 2.11(i)]). If a is a $(1, L_\Phi, \infty)$-atom and τ is the stopping time associated with a, then $\{|Ta| > 0\} = \{M(a) > 0\} \subset \{\nu < \infty\}$ and hence (4.1) holds. Thus it follows from Theorem 4.1 that

$$\|f\|_{wH_{\Phi,b}} = \|Tf\|_{wL_{\Phi,b}} \lesssim \|f\|_{wH^s_{\Phi,b}}.$$

Similarly, considering $Tf = Sf$, we get the second inequality of (4.3) by Theorem 4.1.

(4.4) comes easily from the definitions of these martingale spaces.

Next we show (4.5). Consider $Tf = Mf$ or Sf. Then (4.5) follows from the combination of the Burkholder–Gundy (see [23, Theorem 2.12]), Doob's maximal inequalities

$$\|S(f)\|_r \approx \|Mf\|_r \approx \|f\|_r \qquad (1 < r < \infty)$$

and Theorem 4.3.

(4.6) can be deduced by applying the inequalities (see [23, Theorem 2.11(ii)])

$$\|s(f)\|_r \lesssim \|Mf\|_r, \quad \|s(f)\|_r \lesssim \|S(f)\|_r, \quad 2 < r < \infty,$$

and Theorem 4.3.

To prove (4.7), we use (4.5). Assume that $f = (f_n)_{n\geq 0} \in wQ_{\Phi,b}$, then there exists an optimal control $(\lambda_n^{(1)})_{n\geq 0}$ such that $S_n(f) \leq \lambda_{n-1}^{(1)}$ with $\lambda_\infty^{(1)} \in wL_{\Phi,b}$. Since

$$|f_n| \leq M_{n-1}(f) + \lambda_{n-1}^{(1)},$$

by the second inequality of (4.5) we have

$$\|f\|_{wP_{\Phi,b}} \leq C\big(\|f\|_{wH_{\Phi,b}} + \|\lambda_\infty^{(1)}\|_{wL_{\Phi,b}}\big) \lesssim \|f\|_{wQ_{\Phi,b}}.$$

On the other hand, if $f = (f_n)_{n\geq 0} \in wP_{\Phi,b}$, then there exists an optimal control $(\lambda_n^{(2)})_{n\geq 0}$ such that $|f_n| \leq \lambda_{n-1}^{(2)}$ with $\lambda_\infty^{(2)} \in wL_{\Phi,b}$. Notice that

$$S_n(f) \leq S_{n-1}(f) + 2\lambda_{n-1}^{(2)}.$$

Using the first inequality of (4.5), we get the rest of (4.7).

Further, assume that $\{\mathcal{F}_n\}_{n\geq 0}$ is regular. Then according to [23, p. 33], we have

$$S_n(f) \leq R^{1/2} s_n(f) \quad \text{and} \quad \|f\|_{wH^S_{\Phi,b}} \lesssim \|f\|_{wH^s_{\Phi,b}}.$$

Since $s_n(f) \in \mathcal{F}_{n-1}$, by the definition of $wQ_{\Phi,b}$ we have

$$\|f\|_{wQ_{\Phi,b}} \lesssim \|s(f)\|_{wL_{\Phi,b}} = \|f\|_{wH^s_{\Phi,b}}$$

Hence, by (4.6) we obtain

$$wQ_{\Phi,b} = wH^s_{\Phi,b}.$$

Combining Proposition 3.10, the inequalities (4.5) and (4.7), we get

$$wH^S_{\Phi,b} = wQ_{\Phi,b} = wP_{\Phi,b} = wH_{\Phi,b}.$$

Consequently, we conclude that these five kinds of weak Orlicz–Karamata–Hardy martingale spaces are equivalent. □

5. Duality result

In order to deal with the duality, we define, strongly inspired by [13, Definition 1.1] (see also [14]), the following generalized weak BMO martingale spaces associated with slowly varying functions.

Definition 5.1. Let b be a slowly varying function. For a function $\phi : (0, \infty) \to (0, \infty)$, the generalized weak BMO martingale space for $1 \leq r < \infty$ is defined by

$$wBMO_{r,\phi,b} = \{f \in L_r : \|f\|_{wBMO_{r,\phi,b}} < \infty\},$$

where

$$\|f\|_{wBMO_{r,\phi,b}} = \sup \frac{\sum\limits_{k\in\mathbb{Z}} 2^k \mathbb{P}(\nu_k < \infty)^{1-\frac{1}{r}} \|f - f^{\nu_k}\|_r}{\sup\limits_{k\in\mathbb{Z}} 2^k \gamma_b(\mathbb{P}(\nu_k < \infty))\mathbb{P}(\nu_k < \infty)\phi(\mathbb{P}(\nu_k < \infty))},$$

where the supremum is taken over all stopping time sequences $\{\nu_k\}_{k\in\mathbb{Z}} \subset \mathcal{T}$ such that $\{2^k \gamma_b(\mathbb{P}(\nu_k < \infty))\mathbb{P}(\nu_k < \infty)\phi(\mathbb{P}(\nu_k < \infty))\}_{k\in\mathbb{Z}} \in \ell_\infty$.

Definition 5.2. *([12, Definition 1.4])* Let $\Phi \in \mathcal{G}$ be concave with $q_{\Phi^{-1}} < \infty$ and b be a non-decreasing slowly varying function. Denote by $w\mathcal{L}_{\Phi,b}$ the set of all $f \in wL_{\Phi,b}$ having the absolute continuous quasi-norm defined by

$$w\mathcal{L}_{\Phi,b} = \Big\{f \in wL_{\Phi,b} : \lim_{\mathbb{P}(A)\to 0} \|f\chi_A\|_{wL_{\Phi,b}} = 0\Big\}.$$

Note that $w\mathcal{L}_{\Phi,b}$ is a linear closed subspace of $wL_{\Phi,b}$. Moreover, $L_2 \subset w\mathcal{L}_{\Phi,b}$ for concave $\Phi \in \mathcal{G}$. Now we define a closed subspace of $wH^s_{\Phi,b}$ as follows

$$w\mathcal{H}^s_{\Phi,b} = \{f = (f_n)_{n\geq 0} : s(f) \in w\mathcal{L}_{\Phi,b}\}.$$

Similarly, we can define $w\mathcal{H}_{\Phi,b}$ which is also a linear closed subspace of $wH_{\Phi,b}$.

Remark 5.3. Using similar argument as in [12, Remark 2.2], one can check that the space $H_2^s = L_2$ is dense in $w\mathcal{H}^s_{\Phi,b}$. Moreover, if the stochastic basis $\{\mathcal{F}_n\}_{n\geq 0}$ is regular, then L_∞ is dense in $w\mathcal{H}^s_{\Phi,b}$.

The following duality theorem is the main result of the section.

Theorem 5.4. *Let $\Phi \in \mathcal{G}$ be concave with $q_{\Phi^{-1}} < \infty$ and b be a non-decreasing slowly varying function. Let $\phi(s) = 1/(s\Phi^{-1}(1/s))$ for $s \in (0,\infty)$. Then*

$$(w\mathcal{H}^s_{\Phi,b})^* = wBMO_{2,\phi,b}.$$

Proof. Let $g \in wBMO_{2,\phi,b} \subset L_2$. Define

$$l_g(f) = \mathbb{E}\Big(\sum_{k\in\mathbb{Z}} df_k dg_k\Big), \quad f \in L_2.$$

By Remark 5.3, $L_2 \subset w\mathcal{H}^s_{\Phi,b}$. Then according to the Theorem 3.2, there exists a sequence $(a^k)_{k\in\mathbb{Z}}$ of $(1, L_\Phi, \infty)$-atoms and a sequence $(\mu_k)_{k\in\mathbb{Z}}$ of non-negative real numbers satisfying $\mu_k = 3\cdot 2^k \|\chi_{\{\nu_k<\infty\}}\|_{L_\Phi}$ such that

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_n a^k \qquad \text{a.e.} \quad \forall n \in \mathbb{N}.$$

By the definition of $\|\cdot\|_{wBMO_{2,\phi}}$, we have

$$\begin{aligned}
|l_g(f)| = |\mathbb{E}(fg)| &\leq \sum_k \mu_k |\mathbb{E}(a^k(g - g^{\nu_k}))| \\
&\leq \sum_k \mu_k \|a^k\|_2 \|g - g^{\nu_k}\|_2 \\
&\leq 3\|g\|_{wBMO_{2,\phi,b}} \sup_k 2^k \gamma_b(\mathbb{P}(\nu_k<\infty))\mathbb{P}(\nu_k<\infty)\phi(\mathbb{P}(\nu_k<\infty)) \\
&\leq 3\|g\|_{wBMO_{2,\phi,b}} \|f\|_{wH^s_{\Phi,b}}.
\end{aligned}$$

Since L_2 is dense in $w\mathcal{H}^s_{\Phi,b}$ (see Remark 5.3), l can be uniquely extended to a continuous linear functional on $w\mathcal{H}^s_{\Phi,b}$.

Conversely, let $l \in (w\mathcal{H}^s_{\Phi,b})^*$. Since L_2 can be embedded continuously in $w\mathcal{H}^s_{\Phi,b}$, there exists $g \in L_2$ such that

$$l(f) = \mathbb{E}(fg) \qquad f \in L_2.$$

Let $\{\nu_k\}_{k\in\mathbb{Z}}$ be a sequence of stopping times satisfying

$$\{2^k \gamma_b(\mathbb{P}(\nu_k<\infty))\mathbb{P}(\nu_k<\infty)\phi(\mathbb{P}(\nu_k<\infty))\}_{k\in\mathbb{Z}} \in \ell_\infty.$$

For $k \in \mathbb{Z}$ and arbitrary non-negative N, we define

$$h_k = \frac{|g - g^{\nu_k}|\operatorname{sign}(g - g^{\nu_k})}{\|g - g^{\nu_k}\|_2}, \quad f^N = \sum_{k=-N}^{N} 2^k (h_k - h_k^{\nu_k})\mathbb{P}(\nu_k<\infty)^{\frac{1}{2}}.$$

Set

$$a^k = (h_k - h_k^{\nu_k})\mathbb{P}(\nu_k<\infty)^{\frac{1}{2}} \|\chi_{\{\nu_k<\infty\}}\|^{-1}_{L_\Phi}.$$

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It is obvious that a^k is a $(1, L_\Phi, 2)$-atom for each $k \in \mathbb{Z}$. Then

$$f^N = \frac{1}{3} \sum_{k=-N}^{N} \mu_k a^k,$$

where $\mu_k = 3 \cdot 2^k \|\chi_{\{\nu_k<\infty\}}\|_{L_\Phi}$. Thus, applying Theorem 3.3, we obtain that $f^N \in wH^s_{\Phi,b}$ and

$$\|f^N\|_{wH^s_{\Phi,b}} \le C \sup_k 2^k \gamma_b(\mathbb{P}(\nu_k < \infty))\mathbb{P}(\nu_k < \infty)\phi(\mathbb{P}(\nu_k < \infty)).$$

So we obtain

$$\begin{aligned}
&\sum_{k=-N}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{1-\frac{1}{2}} \|g - g^{\nu_k}\|_2 \\
&= \sum_{k=-N}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{2}} \mathbb{E}(h_k(g - g^{\nu_k})) \\
&= \sum_{k=-N}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{2}} \mathbb{E}((h_k - h_k^{\nu_k})g) \\
&= \mathbb{E}(f^N g) = l(f^N) \le \|f^N\|_{wH^s_{\Phi,b}} \|l\| \\
&\le C \sup_k 2^k \gamma_b(\mathbb{P}(\nu_k < \infty))\mathbb{P}(\nu_k < \infty)\phi(\mathbb{P}(\nu_k < \infty))\|l\|.
\end{aligned}$$

By the definition of $\|\cdot\|_{wBMO_{2,\phi,b}}$, we get $\|g\|_{wBMO_{2,\phi,b}} \le C\|l\|$. $\square$

Remark 5.5. Let $\Phi \in \mathcal{G}$ be concave with $q_{\Phi^{-1}} < \infty$ and b be a non-decreasing slowly varying function. Let $\phi(s) = 1/(s\Phi^{-1}(1/s))$ for all $s \in (0, \infty)$. Moreover, assume that the stochastic basis $\{\mathcal{F}_n\}_{n\ge0}$ is regular. Taking similar argument as above, we may prove that for $1 < r < \infty$,

$$(w\mathcal{H}_{\Phi,b})^* = wBMO_{1,\phi,b} \quad \text{and} \quad (w\mathcal{H}_{\Phi,b})^* = wBMO_{r,\phi,b}.$$

Hence we have the following John–Nirenberg theorem.

Corollary 5.6. *Let $\Phi \in \mathcal{G}$ be concave with $q_{\Phi^{-1}} < \infty$ and b be a non-decreasing slowly varying function. Let $\phi(s) = 1/(s\Phi^{-1}(1/s))$ for all $s \in (0, \infty)$. If the stochastic basis $\{\mathcal{F}_n\}_{n\ge0}$ is regular and $1 \le r < \infty$, then*

$$wBMO_{1,\phi,b} = wBMO_{r,\phi,b}$$

with equivalent norms.

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