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The question on characteristic endpoints for iterative roots of PM functions [☆]

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A B S T R A C T

Iterative roots of mappings are of special interests because it defines fractional iteration, displays middle procedure of evolution and proposes a weak version of the embedding flow problem. For PM functions of height 1, the class of 1-dimensional mappings having the simplest nonmonotonicity, the existence of continuous iterative roots of any order was obtained under the characteristic endpoints condition and the condition was proved to be necessary for those orders greater than the number of forts plus 1. This suggests an open question about iterative roots without that condition, called the question on characteristic endpoints. In this paper, the question is answered completely in the case that the number of forts is equal to the order. Although results of nonexistence are also obtained for the case that the number of forts is greater than the order, a full description is still open.

1. Introduction

In the theory of dynamical systems, it has been concerned for a long time whether a mapping can be embedded into a flow ([1,4,5,11,13,14]). A weak version of the embedding flow problem is the problem of iterative roots. Given a nontrivial topological space X and an integer $n > 0$, the n-th iterate of a continuous self-mapping $F : X \to X$, denoted by F^n, is defined inductively by $F^n(x) := F(F^{n-1}(x))$ and $F^0(x) := x$

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for all $x \in X$. An iterative root of order n of a given continuous self-mapping $F : X \to X$ is a continuous self-mapping $f : X \to X$ such that

$$f^n(x) = F(x), \quad \forall x \in X. \tag{1.1}$$

Iterative root, which defines a fractional iterate $F^{1/n}$, attracts extensive interests ([6,7,12]).

The iterative root theory was set up well for 1-dimensional monotone mappings (see e.g. [6,7,12]), but there are less results on iterative roots of nonmonotonic mappings. Difficulties with nonmonotonic mappings mainly come from those points which break the monotonicity and their orbits. Those points are the so-called *nonmonotonic points* or *forts* ([15,16]), each of which is an interior point c in $I := [a, b]$ such that the continuous function $F : I \to I$ is not monotone on any neighborhood of c. A nonmonotonic continuous mapping $F : I \to I$ having a finite set $S(F)$ of forts is referred to a *PM function*, an abbreviation of *strictly piecewise monotone continuous function*, which plays an important role in the study of dynamical systems (see e.g. [2,15,16]). Let $PM(I, I)$ denote the set of all PM functions mappings from I into itself. For each $F \in PM(I, I)$, one can easily prove that the number $N(F)$ of forts satisfies the ascending relation

$$0 = N(F^0) \le N(F) \le N(F^2) \le ... \le N(F^k) \le N(F^{k+1}) \le ...$$

and that if $N(F^k) = N(F^{k+1})$ then $N(F^k) = N(F^{k+i})$ for all $i \ge 1$. Thus, the least number k such that $N(F^k) = N(F^{k+1})$, called the *nonmonotonicity height* or *height* simply and denoted by $H(F)$ in ([9]), is an important index to describe the complexity of F under iteration. Clearly, F is strictly monotone if $H(F) = 0$. In this paper we focus on functions of $H(F) = 1$.

Each PM function F admits a partition: The interval I is divided by its forts into finitely many subintervals, called laps. One can prove that exact one of those laps, called the *characteristic interval* of F, covers the range of F if and only if $H(F) = 1$. The characteristic interval is useful in finding iterative roots. In 1983 Jingzhong Zhang and Lu Yang ([16]) gave the following in Chinese:

Theorem A. (Theorem 4 in [15]). *Let $F \in PM(I, I)$ be of height 1. Suppose that* $(\mathbf{K}^+)$ *F is strictly increasing on its characteristic interval $K(F) = [a', b']$ and, additionally,* $(\mathbf{K}_0^+)$ *F on I cannot reach a' and b' unless $F(a') = a'$ or $F(b') = b'$. Then, for any integer $n > 1$, F has continuous iterative roots of order n. Conversely, conditions* $(\mathbf{K}^+)$ *and* $(\mathbf{K}_0^+)$ *are necessary for $n > N(F) + 1$.*

The converse part of this theorem suggests an open question (see [16] or [15]): *Does a function $F \in PM(I, I)$ with $H(F) = 1$ have an iterative root of an order $n \le N(F) + 1$ if condition* $(\mathbf{K}_0^+)$*, called the 'characteristic endpoints condition', is not satisfied?* For convenience, we simply call it the 'question on characteristic endpoints'.

This question can be discussed in the three cases:

(CEQ1) $a' = m < M = b'$ and either $F(a') > a'$ or $F(b') < b'$,
(CEQ2) $a' = m < M < b'$ and $F(a') > a'$, and
(CEQ3) $a' < m < M = b'$ and $F(b') < b'$,

where m denotes $\min F := \min\{F(x) : x \in I\}$ and M denotes $\max F := \max\{F(x) : x \in I\}$. In 2008 Li, Yang and Zhang ([8, Theorem 3]) investigated **(CEQ1)** and gave a negative answer: F has no continuous iterative roots f of order $n = N(F)$ such that $H(f) = n$. Clearly, most of those cases are unsolved yet. Although Liu and Zhang ([10]) gave $\min\{n, N(F)\}$ modes of extension, which extend a monotone iterative root from the characteristic interval to the whole I, and proved that all continuous iterative roots of F of order n can be given with those modes, those results obtained in ([10]) cannot be applied immediately to answer the question in the above cases.

In this paper we study with the question on characteristic endpoints. For $n = N(F)$, we discuss the above cases **(CEQ2)** and **(CEQ3)** and, including the negative answer of ([8]) to case **(CEQ1)**, we answer this question completely. For $n < N(F)$ or $= N(F) + 1$, we only need to consider n-th order iterative roots f of height $H(f) < n$ or $= n$ because none of n-th order iterative roots is of $H(f) > n$ by Lemma 2 of [8] (or the following Lemma 2.3). For $n < N(F)$, we prove that no roots f of $H(f) < n$ exist in all the three cases but still leave the existence of roots f of $H(f) = n$ open. For $n = N(F) + 1$, we prove in all the three cases that F has neither a continuous iterative root which is increasing on $[a', b']$ nor a continuous iterative root f which is decreasing on $[a', b']$ and of height $H(f) < n - 1$, but leave the question of existence of continuous iterative roots f which are decreasing on $[a', b']$ and of height $H(f) = n - 1$ open.

2. PM functions of height 1

When $H(F) = 1$, as mentioned in ([15,16]) and the Introduction, the mapping F has a characteristic interval of F, denoted by $K(F)$. This interval is bounded by two consecutive forts (or an endpoint) and covers the range of F. Therefore, $K(F)$ is a positively invariant interval and F restricted to $K(F)$ is monotone. The idea of finding iterative roots of such PM functions used in ([10,15,16]) is to find an iterative root on $K(F)$ for a monotone function and then extend it to the whole interval I. Therefore, the following result on iterative roots of monotone functions is useful and can be found in ([3,17]).

Lemma 2.1 *(Bödewadt Theorem). Let $CI(I, I) := \{f : I \to I|\ f$ is continuous and strictly increasing, $f(a) = a, f(b) = b\}$. Then for any integer $n \geq 2$, every $F \in CI(I, I)$ has iterative roots f of order n in $CI(I, I)$.*

This result remains true if an endpoint is not a fixed point.

Proposition 2.1. *Let $F : I \to I$ be continuous and strictly increasing, $n > 1$ be an arbitrarily given integer, and A, B be real constant such that either $a < A < B < b$ and $F(a) \neq a, F(b) \neq b$, or $F(a) = a$ and $a = A < B < b$, or $F(b) = b$ and $a < A < B = b$. Then equation (1.1) has strictly increasing continuous solutions f on I satisfying*

$$F(a) \leq f(A) < f(B) \leq F(b). \tag{2.1}$$

Additionally, if $f, F \in CI(I, I)$ satisfy $f^n(x) = F(x)$ for all $x \in I$, then there is $h \in CI(I, I)$ such that $h(f(x)) = F(h(x))$ for all $x \in I$.

Proof. Without loss of generality, we assume that F has no fixed points in (a, b). Since the case that $F(a) = a$ and $F(b) = b$ is done in Lemma 2.1, it suffices to consider either a or b is not a fixed point of F.

Firstly, suppose that $F(a) = a$ and $F(b) < b$. Choose $c > b$ and an arbitrary strictly increasing and continuous function $\hat{F} : [b, c] \to [F(b), c]$ such that $\hat{F}(b) = F(b)$, $\hat{F}(c) = c$. Let

$$F_1(x) := \begin{cases} F(x), & \forall x \in [a, b), \\ \hat{F}(x), & \forall x \in [b, c]. \end{cases}$$

Clearly, $F_1 \in CI(I_1, I_1)$ where $I_1 := [a, c]$. By Lemma 2.1, there is an $f_1 \in CI(I_1, I_1)$ such that $f_1^n(x) = F_1(x)$ for any $x \in I_1$. Then the restriction $f := f_1|I$ is a continuous iterative root of F of order n.

In order to prove inequality (2.1), it suffices to verify $f_1(B) \leq F(b)$ because $f_1(A) \geq a = F(a)$. In fact, we infer from $F_1(c) = c$ that $F_1(x) < x$ for all $x \in (a, c)$. In particular, since f_1 is strictly increasing on I_1, it follows that $f_1(x) < x$ for all $x \in (a, c)$. Hence, (2.1) holds if $B \leq F(b)$. If $B > F(b)$, the result can be established by using the property that the initial function in Bödewadt's Theorem is not unique. More precisely, for every h_0 defined on $[F_1(x_0), x_0]$, $\forall x_0 \in (a, b]$, such that h_0 is continuous, strictly increasing, and

$$h_0(x_0) = F_1(x_0), \qquad h_0(F_1(x_0)) = F_1^{n+1}(x_0), \tag{2.2}$$

it has an extension h to I satisfying $F(x) = h^{-1}(F_1^n(h(x)))$ for all $x \in I$. Thus, $f(x) = h^{-1}(F_1(h(x)))$ is an iterative root of F_1 of order n. Now we choose a suitable h_0 as follows. Let $x_0 = b$ and h_0 be any function on $[F_1(b), b]$ such that

$$h_0(B) = F_1^n(b) - \epsilon, \quad 0 \le \epsilon < (F_1^n(b) - F_1^{n+1}(b))/2. \tag{2.3}$$

Note that the above definition is fine. Indeed, it follows from (2.2)–(2.3) that

$$h_0(F_1(b)) = F_1^{n+1}(b) < F_1^n(b) - \epsilon = h_0(B) < F_1(b) = h_0(b).$$

Furthermore, since F_1 is strictly increasing, we have

$$F_1(h(B)) = F_1(F_1^n(b) - \epsilon) \le F_1^{n+1}(b) = F_1^n(h(b)) = h(F_1(b)),$$

which implies that $f_1(B) = h^{-1}(F_1(h(B))) \le F_1(b) = F(b)$, and thus inequality (2.1) holds.

Similar arguments can be applied to prove the other cases that $F(a) > a$, $F(b) = b$ and $F(a) > a$, $F(b) < b$. This completes the proof. □

In latter sections we discuss PM functions and consider the number of forts for their iterates. The following lemmas are useful.

Lemma 2.2. (Lemma 2.3 in [9]) *Let $F : [a, b] \to \mathbb{R}$ and $G : [\alpha, \beta] \to \mathbb{R}$ be continuous functions such that $F([a, b]) \subset [\alpha, \beta]$. Then*

$$S(G \circ F) = S(F) \cup \{c \in (a, b) : F(c) \in S(G)\}.$$

Lemma 2.3. (Lemma 2 in [8]) *Let $F \in PM(I, I)$ be of height 1. Then F has no continuous iterative roots f of order $n > 1$ such that $H(f) > n$.*

Lemma 2.4. (Theorem 3 in [8]) *Let $F \in PM(I, I)$ be of height 1. Suppose that F is strictly increasing on its characteristic interval $[a', b']$, $N(F) = n > 1$, and that $\min F = a'$ and $\max F = b'$. Then F has no continuous iterative roots f of order n such that $H(f) = n$.*

Note that, for each $F \in PM(I, I)$ with $H(F) = 1$, Lemma 2.3 asserts that we only need to verify the existence of its iterative roots f with $H(f) \le n$ and Lemma 2.4 gives corresponding results when $H(f) = N(F) = n$ under some assumptions.

The following result is Theorem 3 of [15], which gives a method of extension and will be used in the proofs of our Theorems.

Lemma 2.5. (Theorem 3 in [15]) *Suppose $F \in PM(I, I)$ with height 1. Let $[a', b']$ be the characteristic interval, let m and M the minimum and maximum of F on $[a, b]$, and m' and M' those on $[a', b']$. If, restricted to $[a', b']$, equation (1.1) has a continuous solution f_1 which maps $[a', b']$ into itself and maps $[m, M]$ into $[m', M']$, then there exists a continuous function f from I into I such that (i) $f(x) = f_1(x)$ for all $x \in [a', b']$, and (ii) f satisfies equation (1.1) on the whole interval I.*

Having these knowledges, in what follows, we always assume that $F \in PM(I, I)$ with $H(F) = 1$ such that F is strictly increasing on its characteristic interval $[a', b']$, as considered in Theorem A in the Introduction. As known from the theory of monotone iterative roots ([6,7]), F has two classes of continuous iterative roots: roots increasing on $[a', b']$ and roots decreasing on $[a', b']$. We separately discuss the two classes in the following two sections.

3. Roots increasing on characteristic interval

In this section we discuss the first class, i.e., iterative roots which are increasing on the characteristic interval $[a', b']$.

Theorem 3.1. *Suppose that $F \in PM(I, I)$ with $H(F) = 1$ such that $(\mathbf{K}^+)$ holds but condition $(\mathbf{K}_0^+)$ is not true. Then for any integer $n > 1$ mapping F has no continuous iterative roots f of order n such that f is strictly increasing on $[a', b']$ and $H(f) < n$. In particular, for $n = N(F)$ mapping F has no continuous iterative roots f of order n such that f is strictly increasing on $[a', b']$ and $H(f) \le n$.*

Remind that the case $H(f) > n$ does not exist by Lemma 2.3, as mentioned in the Introduction. Besides, this theorem covers more than [8] even in case **(CEQ1)** because [8] only deals with roots f of height $H(f) = n$ but this theorem works for $H(f) \le n$.

Proof. Let F satisfy all conditions given in the theorem. As mentioned in the Introduction, we need to consider the three cases (**CEQ1**)–(**CEQ3**). For an indirect proof to the result, suppose that F has a continuous iterative root f of order $n = N(F) > 1$ such that f is strictly increasing on $[a', b']$ and satisfies $H(f) \le n$. By Lemma 1 of [10], we know that f maps $[a', b']$ into itself.

First, we consider case (**CEQ1**). Since roots f of height $H(f) = n = N(F)$ was discussed in Theorem 3 of [8] as indicated in Lemma 2.4, it suffices to consider those roots f of $H(f) < n$. We divide the case into the following three subcases:

(CEQ1A): $a' = m < M = b'$, $F(a') = a'$ and $F(b') < b'$;
(CEQ1B): $a' = m < M = b'$, $F(a') > a'$ and $F(b') = b'$; and
(CEQ1C): $a' = m < M = b'$, $F(a') > a'$ and $F(b') < b'$.

In subcase (CEQ1A), we have $f(b') < b'$. The inequality $H(f) < n$ yields that $N(f^{n-1}) = N(f^n)$, implying that f^{n-1} and f^n being equal to F, have a common fort. Moreover, the interval $[a', b']$ is also the characteristic interval of f^{n-1} because $[\min f^n, \max f^n] \subseteq [\min f^{n-1}, \max f^{n-1}]$. Let $x_0 \in I$ satisfy $F(x_0) = b'$, being equal to M. Since $f^{n-1}(x_0) \in [a', b']$, we have

$$b' = F(x_0) = f \circ f^{n-1}(x_0) \le f(b') < b', \tag{3.1}$$

which is a contradiction. Thus F has no continuous iterative roots f of order $n > 1$ such that $H(f) < n$. In subcase (CEQ1B), we have $f(a') > a'$. The same arguments as given in subcase (CEQ1A) show that f^{n-1} and f^n have both a common fort and the same characteristic interval $[a', b']$. Let $y_0 \in I$ satisfy $F(y_0) = a'$, being equal to m. Since $f^{n-1}(y_0) \in [a', b']$, we obtain

$$a' = F(y_0) = f \circ f^{n-1}(y_0) \ge f(a') > a', \tag{3.2}$$

a contradiction. Thus F has no continuous iterative roots f of order $n > 1$ such that $H(f) < n$. In subcase (CEQ1C), we have $F(b') < b'$ and $F(a') > a'$, which lead to contradictions (3.1) and (3.2) respectively, as done in subcases (CEQ1A) and (CEQ1B). Hence, it is proved in case (**CEQ1**) that mapping F has no continuous iterative roots f of order n which are strictly increasing on $[a', b']$.

Next, we consider case (**CEQ2**). If $H(f) < n$, the result can be given as in subcase (CEQ1B). In what follows, we consider the situation that $H(f) = n = N(F)$, in which

$$N(f^k) = k, \quad 0 \le k \le n. \tag{3.3}$$

By Lemma 2.2, f is not monotone on $[\min f^{n-1}, \max f^{n-1}]$. Let $\xi \in (\min f^{n-1}, \max f^{n-1})$ be a fort of f. By (3.3), ξ is the unique fort of f on I. On the other hand, let $x_0 \in S(f^n)\backslash S(f^{n-1})$ since the cardinality $\#S(f^n)\backslash S(f^{n-1}) = 1$ by (3.3). Then $f^{n-1}(x_0) \in S(f)$. It implies that $f^{n-1}(x_0) = \xi$ and therefore

$$f(\xi) = F(x_0) \in [a', b']. \tag{3.4}$$

Since ξ is the unique fort of f on I, $f(\xi)$ is equal to either $\min f$, the minimum of f on I, or $\max f$, the maximum of f on I. We first assume that

$$f(\xi) = \min f. \tag{3.5}$$

Then, the relation (3.4) associated with the inequality $a' = \min f^n \ge \min f = f(\xi)$ implies that $f(\xi) = a'$. Noting that f is strictly decreasing on $[a, \xi]$ (or strictly increasing on $[\xi, b]$) and $f(a') > a'$, we have $\xi < a'$ because f is assumed to be strictly increasing on $[a', b']$. Furthermore, $a' = \min f \le \min f^n = a'$, implying that $\min f^{n-1} = a'$. This contradicts to the fact $\xi \in (\min f^{n-1}, \max f^{n-1}) = (a', \max f^{n-1})$. Next, opposite to (3.5), we assume that

$$f(\xi) = \max f.$$

We similarly obtain that $\xi \ge b'$. By (3.4), it leads to a contradiction to the fact that $\xi \in (\min f^{n-1}, \max f^{n-1}) \subseteq (\min f^{n-1}, \max f) = (\min f^{n-1}, f(\xi)) \subseteq (\min f^{n-1}, b')$.

Finally, we consider case (**CEQ3**). If $H(f) < n$, the result can be given as in subcase (CEQ1A). If $H(f) = N(F) = n$, in the case that $f(\xi) = \max f$ the discussion is similar to case (**CEQ2**); in the case that $f(\xi) = \min f$ we get $\xi \le a'$ because f is assumed to be strictly increasing on the characteristic interval $[a', b']$ and ξ is the unique fort. This contradicts to the fact that $\xi \in (\min f^{n-1}, \max f^{n-1}) \subseteq (\min f, \max f^{n-1}) = (f(\xi), \max f^{n-1}) \subseteq (a', \max f^{n-1})$.

With the obtained contradictions in cases (**CEQ1**)–(**CEQ3**), the proof is completed. □

Corollary 3.1. *Let F be the same as supposed in Theorem 3.1. Then F does not have a continuous iterative root of order $n = N(F) + 1$ or $n = N(F)$ which is strictly increasing on $[a', b']$.*

Proof. For an indirect proof, assume that F has a continuous iterative root f of order $n = N(F)$ or $N(F)+1$ such that f is strictly increasing on $[a', b']$. By Lemma 2.3, we see that $H(f) \le n$. If $n = N(F)$, it follows from Theorem 3.1 that F has no such iterative roots f satisfying $H(f) \le n$, which gives a contradiction. On the other hand, for the case $n = N(F) + 1$ we claim that $H(f) < n$. Otherwise, $1 \le N(f) < ... < N(f^n) = N(F) \le N(f^{n+1})$, implying $n \le N(F)$, a contradiction to the assumption of $n = N(F) + 1$. Furthermore, the results about $H(f) < n$ given in Theorem 3.1 yield the contradiction again. □

Aiming to the question on characteristic endpoints, Theorem 3.1 denies the existence of continuous iterative roots of order $n \le N(F) + 1$ which are of the height $H(f) < n$ and strictly increasing on $[a', b']$. Corollary 3.1 further denies the existence of continuous iterative roots of order $n = N(F) + 1$ or $N(F)$ without the restriction $H(f) < n$. The following question is not answered yet:

(P): *Does a function $F \in PM(I, I)$ given in Theorem 3.1 have an iterative root f of order $n < N(F)$ such that f is strictly increasing on $[a', b']$ and $H(f) = n$?*

4. Roots decreasing on characteristic interval

This section is devoted to the second class, i.e., roots decreasing on the characteristic interval $[a', b']$.

Theorem 4.1. *Let F be the same as supposed in Theorem 3.1. Then for any $n > 2$ mapping F has no continuous iterative roots f of order n such that f is strictly decreasing on $[a', b']$ and $H(f) < n - 1$.*

Proof. For an indirect proof, assume that F has a continuous iterative root f of order $n > 2$ such that f is strictly decreasing on $[a', b']$ and $H(f) < n - 1$. The monotonicity of f on $[a', b']$ implies that n must be even, that is, $n = 2m$ for an integer $m > 1$. Let $g := f^2$. Then g is a continuous iterative root of F of order m which is strictly increasing on $[a', b']$. Furthermore, we infer from the assumption $H(f) < n - 1$ that

$$N(f) \le N(f^2) \le \ldots \le N(f^{n-2}) = N(f^{n-1}) = N(f^n) = \ldots,$$

implying that $H(g) < m$, a contradiction to the conclusion of Theorem 3.1. $\square$

By Lemma 2.3, we have $H(f) \le n$. Therefore, opposite to Theorem 4.1, we ask: Does mapping F supposed in Theorem 3.1 have a continuous iterative root f of order n $(2 < n \le N(F) + 1)$ such that f is strictly decreasing on $[a', b']$ and $n - 1 \le H(f) \le n$? What happens with $n = 2$? In what follows, we only work in the case that $H(f) = n$ for $2 \le n \le N(F) + 1$ and the case that $H(f) = 1$ for $n = 2$ because the case that $H(f) = n - 1$ for $2 < n \le N(F) + 1$ can be reduced to the open question **(P)** stated in the end of last section. In fact, as shown in the proof of Theorem 4.1, the monotonicity of f on $[a', b']$ implies that $n = 2m$ for an integer $m > 1$ and the function $g := f^2$ is a continuous iterative root of F of order m such that g is strictly increasing on $[a', b']$ and $H(g) = m$. However, the existence of such g is still unknown as indicated in the question **(P)**.

We first discuss for $n = 2$ in the three cases (**CEQ1**)–(**CEQ3**) because F does not satisfy condition ($\mathbf{K}_0^+$) if and only if F lies in one of cases (**CEQ1**)–(**CEQ3**) as mentioned in the Introduction.

Theorem 4.2. *Let F be the same as supposed in Theorem 3.1. If F satisfies (**CEQ1**), then F has no continuous iterative roots f of order 2 such that f is strictly decreasing on $[a', b']$ and $H(f) = 1$. If F satisfies (**CEQ2**) (or (**CEQ3**)), then F has a continuous iterative root f of order 2 such that f is strictly decreasing on $[a', b']$ and $H(f) = 1$ if and only if the restriction $F|_{[a',b']}$ is a reversing correspondence and $F(b') = \max F$ (or $F(a') = \min F$).*

The phrase "reversing correspondence" comes from [6,9]. A strictly increasing function ϕ mapping a compact interval I into itself is said to be a *reversing correspondence* if there are a $\xi \in \mathrm{Fix}\phi$, i.e., a fixed point of ϕ, and a strictly decreasing function ω mapping $\mathrm{Fix}\phi$ onto itself such that $\omega(\xi) = \xi$ and the difference $\phi(x) - x$ on the interval (ξ_1, ξ_2) has an opposite sign to that on the interval $(\omega(\xi_2), \omega(\xi_1))$ for every $\xi_1, \xi_2 \in \mathrm{Fix}\phi$ satisfying that $\xi_1 < \xi_2$ and $(\xi_1, \xi_2) \cap \mathrm{Fix}\phi = \emptyset$.

Proof of Theorem 4.2. If F satisfies (**CEQ1**), we generally claim that for any $n > 1$ mapping F has no continuous iterative roots f of order n such that f is strictly decreasing on $[a', b']$ and $H(f) < n$. For an indirect proof, assume that F has a continuous iterative root f of order n such that f is strictly decreasing on $[a', b']$ and $H(f) < n$. According to the conditions given in (**CEQ1**), there are $x', y' \in I \backslash [a', b']$ such that $F(x') = a'$ and $F(y') = b'$. Then

$$a' = F(x') = f \circ f^{n-1}(x'), \qquad b' = F(y') = f \circ f^{n-1}(y'). \tag{4.1}$$

Since $H(f) < n$, the interval $[a', b']$ is also the characteristic interval of f^{n-1}, i.e., $f^{n-1}(I) \subset [a', b']$. In view of (4.1), we obtain $f^{n-1}(x') = b'$ and $f^{n-1}(y') = a'$ because f is a strictly decreasing self-mapping on $[a', b']$. Hence, $f(b') = F(x') = a'$ and $f(a') = F(y') = b'$, implying that $F(b') = b'$ and $F(a') = a'$, a contradiction to the fact of (**CEQ1**). Clearly, the first result is a corollary of the general claim.

The discussion for cases (**CEQ2**)–(**CEQ3**) is quite different from the above for case (**CEQ1**). We only give a proof to case (**CEQ2**). Case (**CEQ3**) can be proved similarly.

In order to prove the necessity, assume that F has a continuous square iterative root f which is strictly decreasing on $[a', b']$ and $H(f) = 1$. Then [6, Theorem 15.10] implies that $F|_{[a',b']}$ is a reversing correspondence. By [10, Theorem 1], we see that f is an extension from a square iterative root f_0 of $F|_{[a',b']}$, fulfilling

$$f(x) = f_0^{-1} \circ F(x), \quad \forall x \in I. \tag{4.2}$$

From the conditions given in (**CEQ2**), there exists $x' \in I \backslash [a', b']$ such that $F(x') = a'$. Then the equality $f(f(x')) = a'$ implies $f(x') = b'$ since f is strictly decreasing on $[a', b']$. We further obtain

$$f(b') = F(x') = a' \quad \text{and} \quad f(a') = F(b'). \tag{4.3}$$

Therefore, f_0 maps $[a', b']$ onto $[a', F(b')]$. By virtue of (4.2), f is well defined if $F(b') = \max F$.

In order to prove the sufficiency, we need to find a strictly decreasing continuous iterative root f_0 of $F|_{[a',b']}$ of order 2 and then extend it continuously to an iterative root well defined on the whole interval $[a, b]$. To ensure the existence of such an extension, it suffices to choose a root f_0 satisfying condition (4.3). Without loss of generality, assume F has a unique fixed point ξ on $[a', b']$; otherwise, we discuss between two consecutive fixed points separately. Since $F|_{[a',b']}$ is a reversing correspondence, it follows from [6, Theorem 3.1] that there is a continuous strictly decreasing function $\alpha : [a', \xi] \to [\xi, F(b')]$ fulfilling

$$\alpha(a') = F(b'), \qquad \alpha(\xi) = \xi \tag{4.4}$$

such that

$$\alpha(F(x)) = F(\alpha(x)), \quad x \in [a', \xi]. \tag{4.5}$$

Hence, define $f_0 : [a', b'] \to [a', b']$ by

$$f_0(x) := \begin{cases} \alpha(x), & \forall x \in [a', \xi], \\ \alpha^{-1}(F(x)), & \forall x \in (\xi, b']. \end{cases} \tag{4.6}$$

Clearly, f_0 is a continuous and strictly decreasing square iterative root of $F|_{[a',b']}$ by (4.5)–(4.6). Moreover, we infer from (4.4) that f_0 satisfies condition (4.3). Therefore, function f defined in (4.2) is a continuous iterative root we want. This completes the proof. $\square$

The above proof actually gives a general result for case (**CEQ1**), i.e., if $F \in PM(I, I)$ with $H(F) = 1$ is strictly increasing on its characteristic interval $[a', b']$ and satisfies (**CEQ1**), then for any $n > 1$ mapping F has no continuous iterative roots f of order n such that f is strictly decreasing on $[a', b']$ and $H(f) < n$.

Finally, we consider iterative roots f with $H(f) = n$ for $2 \leq n \leq N(F) + 1$. In case (**CEQ1**), i.e., $\min F = a'$ and $\max F = b'$, Lemma 2.4 guarantees that if $n = N(F)$ then F has no continuous iterative

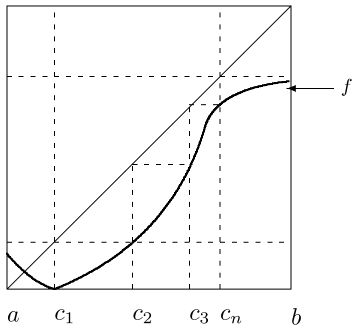

Fig. 1. $S(f) = \{c_1\}$.

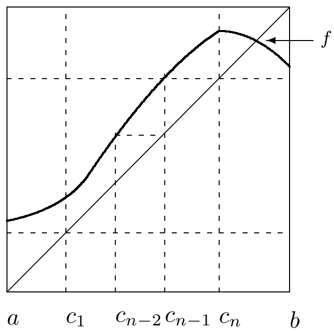

Fig. 2. $S(f) = \{c_n\}$.

roots f of order n such that f is strictly decreasing on $[a', b']$ and $H(f) = n$. Thus, it is unknown yet in the cases (**CEQ2**) and (**CEQ3**) whether there are iterative roots f of order n such that f are strictly decreasing on $[a', b']$ and $H(f) = n$. The following lemmas give necessary conditions for such continuous iterative roots. Let $S(F) = \{c_1, c_2, \dots, c_n\}$, where $a = c_0 < c_1 < c_2 < \dots < c_n < c_{n+1} = b$.

Lemma 4.1. *Suppose that $F \in PM(I, I)$ with $H(F) = 1$ satisfies ($\mathbf{K}^+$). If $f \in PM(I, I)$ is an iterative root of F of order $n > 1$ such that $H(f) = N(F) = n$, then $N(F) = 1$ and there is a permutation $(\ell_1, \dots, \ell_n)$ of $\{1, \dots, n\}$ such that*

$$f(c_{\ell_k}) = c_{\ell_{k-1}}, \qquad k = 2, \dots, n, \tag{4.7}$$

and $S(f) = \{c_{\ell_1}\}$. Moreover, one of the following assertions holds:

$\mathcal{K}_1$: *either $S(f) = \{c_1\}$, the function f reaches the minimum value at c_1 and $f(c_1) < c_1$, or $S(f) = \{c_n\}$, the function f reaches the maximum value at c_n and $f(c_n) > c_n$;*

$\mathcal{K}_2$: *either $S(f) = \{c_1\}$, the function f reaches the maximum value at c_1 and $f(c_1) > c_1$, or $S(f) = \{c_n\}$, the function f reaches the minimum value at c_n and $f(c_n) < c_n$.*

The shape of f in $\mathcal{K}_1$ (resp. $\mathcal{K}_2$) looks almost the same as the type $\mathcal{T}_1$ (resp. the type $\mathcal{T}_2$) iterative roots given in [9] in the case that $H(F) > 1$, but in our paper we consider $H(F) = 1$. One can see, from $\mathcal{K}_1$ for example, that the difference between the two versions comes from the values of $f(b)$ (Fig. 1) and $f(a)$ (Fig. 2). In [9] we have $f(b) > c_n$ (resp. $f(a) < c_1$) but in our paper $f(b) \le c_n$ (resp. $f(a) \ge c_1$), which will be indicated below Lemma 4.2.

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Proof of Lemma 4.1. It follows from $H(f) = n$ that $N(f) < N(f^2) < ... < N(f^n) = N(F) = n$, that is

$$N(f^k) = k, \quad k = 1, ..., n. \tag{4.8}$$

In particular, $N(f) = 1$.

Let $S_1 = S(f)$ and $S_k := \{x \in (a, b) : f(x) \in S(f^{k-1})\} \backslash S(f^{k-1})$, $k = 2, ..., n$. According to the proof of Lemma 3.1 in [9], there is a permutation $(\ell_1, ..., \ell_n)$ of $\{1, ..., n\}$ such that $S_k = \{c_{\ell_k}\}$, $k = 1, ..., n$. This proves (4.7). Furthermore, the proof shows that either $S(f) = \{c_1\}$ or $S(f) = \{c_n\}$. Finally, we consider the case that f takes the maximum value at c_ℓ, where $\ell \in \{1, n\}$. It suffices to prove that $f(c_\ell) > c_\ell$. Suppose that $f(c_\ell) \le c_\ell$. Then $f([a, b]) \subset [a, f(c_\ell)] \subset [a, c_\ell]$. It follows that $H(f) = 1$, a contradiction to the assumption that $H(f) = n > 1$. The other case can be proved similarly. This completes the proof. □

By Lemma 4.1, one can easily see from the proofs of Lemmas 3.3–3.4 of [9] the following lemma about behaviors of iterative roots at forts of function F.

Lemma 4.2. *Suppose that $F \in PM(I, I)$ with $H(F) = 1$ satisfies $(\mathbf{K}^+)$. Let $f \in PM(I, I)$ be an iterative root of F of order $n > 1$ such that $H(f) = N(F) = n$. Then either $f(c_i) = c_{i-1}$ for every $i \in \{2, \ldots, n\}$ (resp. $f(c_i) = c_{i+1}$ for every $i \in \{1, \ldots, n-1\}$) if f is of type $\mathcal{K}_1$ and $S(f) = \{c_1\}$ (resp. $S(f) = \{c_n\}$), or $f(c_i) = c_{n+2-i}$ for every $i \in \{2, \ldots, [\frac{n+1}{2}]\}$ and $f(c_i) = c_{n+1-i}$ for every $i \in \{[\frac{n+1}{2}] + 1, \ldots, n\}$ (resp. $f(c_{n-i}) = c_i$ for every $i \in \{1, \ldots, [\frac{n-1}{2}]\}$ and $f(c_{n-i}) = c_{i+1}$ for every $i \in \{[\frac{n-1}{2}] + 1, \ldots, n-1\}$) if f is of type $\mathcal{K}_2$ and $S(f) = \{c_1\}$ (resp. $S(f) = \{c_n\}$).*

Note that our aim is to find under the assumption of (**CEQ2**) or (**CEQ3**) the iterative roots f of order $n > 1$ which are strictly decreasing on $[a', b']$. Obviously, n is even. As mentioned in Lemma 1 of [10], f is a self-mapping on $[a', b']$, implying that $\mathcal{K}_2$ is impossible. In fact, assume that f is of type $\mathcal{K}_2$. If $S(f) = \{c_1\}$ then, by Lemmas 4.1–4.2, $f([a, c_1]) \not\subset [a, c_1]$ since $f(c_1) > c_1$, and $f([c_i, c_{i+1}]) \not\subset [c_i, c_{i+1}]$ for $i = 1, ..., n-1$. If $S(f) = \{c_n\}$, then $f([c_n, b]) \not\subset [c_n, b]$ since $f(c_n) < c_n$, and $f([c_i, c_{i+1}]) \subset [c_i, c_{i+1}]$ if and only if $i = \frac{n}{2}$, i.e., $f(c_{\frac{n}{2}}) = c_{\frac{n}{2}+1}$, $f(c_{\frac{n}{2}+1}) = c_{\frac{n}{2}}$ by Lemma 4.2. However, the above two equalities about f imply that $[c_{\frac{n}{2}}, c_{\frac{n}{2}+1}]$ is the characteristic interval of F and $F(c_{\frac{n}{2}}) = c_{\frac{n}{2}}$, $F(c_{\frac{n}{2}+1}) = c_{\frac{n}{2}+1}$, which contradicts to the conditions of (**CEQ2**)–(**CEQ3**). Therefore, under the hypothesis (**CEQ2**) or (**CEQ3**), such an iterative root f (if exists) is of type $\mathcal{K}_1$. By Lemma 4.2, $[a, c_1]$ is the characteristic interval of F when $S(f) = \{c_1\}$ (see Fig. 1), and $[c_n, b]$ is the characteristic interval of F when $S(f) = \{c_n\}$ (see Fig. 2). Moreover, $f(b) \le c_n$ (resp. $f(a) \ge c_1$) by the fact $H(F) = 1$.

The main result of this section reads as follows.

Theorem 4.3. *Let F be the same as supposed in Theorem 3.1. Then for $n = N(F)$ mapping F has a continuous iterative root f of order n such that f is strictly decreasing on $[a', b']$ and $H(f) = n$ if and only if n is even and one of the following conditions is fulfilled:*

(i) *$F|_{[a,c_1]}$ is a reversing correspondence and*

$$\begin{aligned} &F(a) \ge F(c_2) > \ldots > F(c_{n-2}) > F(c_n) \ge a, \\ &F(c_1) < F(c_3) < \ldots < F(c_{n-1}) < c_1, \end{aligned} \tag{4.9}$$

where either $F(a) > F(c_2)$ and $F(c_n) = a$ or $F(a) = F(c_2)$ and $F(c_n) > a$ (see Figs. 3 and 4).

(ii) *$F|_{[c_n,b]}$ is a reversing correspondence and*

$$\begin{aligned} &F(b) \le F(c_{n-1}) < F(c_{n-3}) < \ldots < F(c_1) \le b, \\ &F(c_n) > F(c_{n-2}) > \ldots > F(c_2) > c_n, \end{aligned} \tag{4.10}$$

where either $F(b) < F(c_{n-1})$ and $F(c_1) = b$ or $F(b) = F(c_{n-1})$ and $F(c_1) < b$ (see Figs. 5 and 6).

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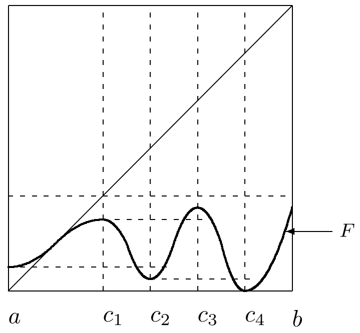

Fig. 3. $F(a) > F(c_2)$ and $F(c_4) = a$.

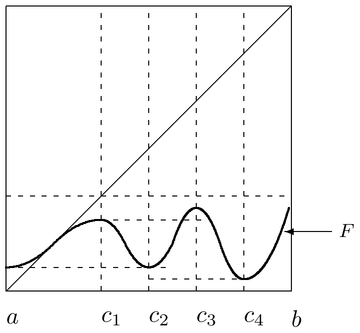

Fig. 4. $F(a) = F(c_2)$ and $F(c_4) > a$.

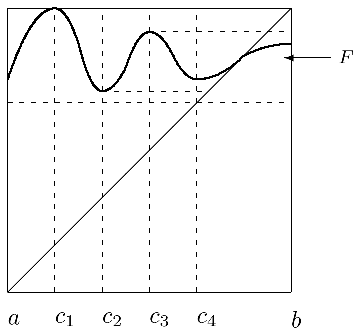

Fig. 5. $F(b) < F(c_3)$ and $F(c_1) = b$.

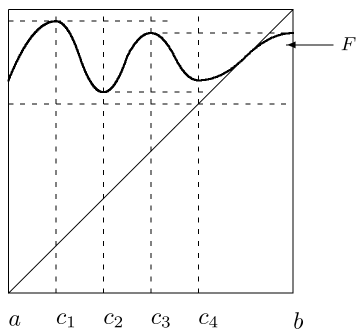

Fig. 6. $F(b) = F(c_3)$ and $F(c_1) < b$.

Proof of Theorem 4.3. First, we prove the necessity. Assume F has a continuous iterative root f of order $n = N(F)$ such that f is strictly decreasing on $[a', b']$ and $H(f) = n$. Then n is even, and we infer from [6, Theorem 15.10] that $F|_{[a',b']}$ is a reversing correspondence. From Lemma 4.1, we need to discuss in the two cases: $S(f) = \{c_1\}$ and $S(f) = \{c_n\}$.

In the case that $S(f) = \{c_1\}$, by Lemma 4.2 we have $f(c_i) = c_{i-1}$ for every $i \in \{2, \dots, n\}$. Then

$$F(c_i) = f^{n-i+1}(c_1), \quad i = 1, ..., n. \tag{4.11}$$

It suffices to consider cases (**CEQ2**) and (**CEQ3**) since the situation of (**CEQ1**) was done in Lemma 2.4. Assume that F satisfies (**CEQ2**) (resp. F satisfies (**CEQ3**)). Then there is $x' \in I \backslash [a, c_1]$ such that $F(x') = a$ (resp. there is $y' \in I \backslash [a, c_1]$ such that $F(y') = c_1$). We only discuss case (**CEQ2**) because the other one can be proved similarly. Since $f|_{[a,c_1']}$ is a self-mapping and strictly decreasing, it follows that $f(c_1) = a$ and $f(a) < c_1$. Note that f^{n-1} is also decreasing in $[a, c_1]$, we see from (4.11) that $F(a) = f^{n-1}(f(a)) > f^{n-1}(c_1) = F(c_2)$. Furthermore, for even $i \in \{1, ..., n-2\}$ we have

$$F(c_i) = f^{n-i+1}(c_1) = f^{n-i-1}(f^2(c_1)) > f^{n-i-1}(c_1) = F(c_{i+2}).$$

Similarly, for odd $i \in \{1, ..., n-2\}$ we have $F(c_i) < F(c_{i+2})$. Moreover, using (4.11) again, we get $F(c_n) = f(c_1) = a$ and $F(c_{n-1}) = f^2(c_1) < c_1$. Thus, both the strict inequalities and equality given in **(i)** are proved.

In the case that $S(f) = \{c_n\}$, by Lemma 4.2 we have $f(c_i) = c_{i+1}$ for every $i \in \{1, \dots, n-1\}$. Then

$$F(c_i) = f^i(c_n), \quad i = 1, ..., n. \tag{4.12}$$

Assume that F satisfies (**CEQ2**) (resp. F satisfies (**CEQ3**)). Then there is $x' \in I \backslash [c_n, b]$ such that $F(x') = c_n$ (resp. there is $y' \in I \backslash [c_n, b]$ such that $F(y') = b$). It suffices to consider (**CEQ2**) because the other one can be discussed similarly. Thus, $f(b) = c_n$ and $f(c_n) < b$. Furthermore, it follows from (4.12) that $F(b) = f^{n-1}(f(b)) = f^{n-1}(c_n) = F(c_{n-1})$ and

$$F(c_i) = f^i(c_n) = f^{i-2}(f^2(c_n)) > f^{i-2}(c_n) = F(c_{i-2})$$

for even $i \in \{3, ..., n\}$. Similarly, for odd $i \in \{3, ..., n\}$ we obtain $F(c_i) < F(c_{i-2})$. Using (4.12) again, we get $F(c_1) = f(c_n) < b$ and $F(c_2) = f^2(c_n) > c_n$. Thus, both the strict inequalities and equality given in **(ii)** are proved, and this completes the proof of necessity.

Next, we prove the sufficiency. We confine ourselves to the case **(i)** because case **(ii)** can be proved similarly. Our strategy is to find a strictly decreasing continuous root f_0 of $F|_{[a,c_1]}$ at first and then extend f_0 to a continuous root on the whole interval $[a, b]$. Actually, under the conditions of (4.9), the proof of Theorem 4.1 in [9] (pp. 296–297) shows that $F|_{[a,c_1]}$, no matter whether $F(a) \geq F(c_2)$ or $F(c_n) \geq a$, has a strictly decreasing continuous iterative root $f_0 : [a, c_1] \to [a, c_1]$ of even order n fulfilling condition (4.11), i.e.,

$$f_0(F(c_i)) = F(c_{i-1}), \quad i = 2, \dots, n \quad \text{and} \quad f_0(c_1) = F(c_n), \tag{4.13}$$

which will be used in the extension. In order to extend f_0 from $[a, c_1]$ to $[a, b]$, let

$$f_1(x) := f_0^{-n+1} \circ F(x), \quad \forall x \in [c_1, c_2].$$

By (4.13), f_0 maps $[F(c_2), F(c_1)]$ onto $[f_0^{n+1}(c_1), F(c_1)]$ and $F^{-1}|_{[a,c_1]}$ maps $[f_0^{n+1}(c_1), F(c_1)]$ onto $[f_0(c_1), c_1]$. These imply that $f_1 : [c_1, c_2] \to [f_0(c_1), c_1]$ is well defined, continuous and strictly increasing on $[c_1, c_2]$ since n is even. Further, let

$$f_2(x) := f_1^{-1} \circ f_0^{-n+2} \circ F(x), \quad \forall x \in [c_2, c_3].$$

By (4.13), f_2 maps $[c_2, c_3]$ onto $[c_1, c_2]$, which is continuous and strictly increasing. Then we generally define

$$f_i(x) := f_{i-1}^{-1} \circ ... \circ f_1^{-1} \circ f_0^{-n+i} \circ F(x), \quad \forall x \in [c_i, c_{i+1}] \tag{4.14}$$

for $i = 3, ..., n$. In view of (4.13)–(4.14) we obtain

$$f_i(c_i) = f_{i-1}^{-1} \circ ... \circ f_1^{-1} \circ f_0^{-n+i} \circ F(c_i) = f_{i-1}^{-1} \circ ... \circ f_1^{-1}(f_0(c_1)) = c_{i-1},$$
$$f_i(c_{i+1}) = f_{i-1}^{-1} \circ ... \circ f_1^{-1} \circ f_0^{-n+i} \circ F(c_{i+1}) = f_{i-1}^{-1} \circ ... \circ f_1^{-1}(c_1) = c_i$$

for $i = 3, ..., n-1$. Hence, $f_i : [c_i, c_{i+1}] \to [c_{i-1}, c_i]$ for $i = 1, 2, ..., n$ is an orientation-preserving homeomorphism. Let

$$f(x) := \begin{cases} f_0(x), & \forall x \in [a, c_1], \\ f_i(x), & \forall x \in (c_i, c_{i+1}],\ i = 1, 2, ..., n, \end{cases} \tag{4.15}$$

where f_is are defined in (4.14). Clearly, f is continuous on I because

$$f_{i-1}(c_i) = f_{i-2}^{-1} \circ ... \circ f_1^{-1} \circ f_0^{-n+i-1} \circ F(c_i) = f_{i-2}^{-1} \circ ... \circ f_1^{-1}(c_1) = c_{i-1}$$

by (4.14). Moreover, for each $x \in I \backslash [a, c_1]$ there exists $i = 1, ..., n$ such that $x \in [c_i, c_{i+1}]$. One can check that

$$\begin{aligned} f^n(x) &= f_0^{n-i} \circ f_1 \circ ... \circ f_i(x) \\ &= f_0^{n-i} \circ f_1 \circ ... \circ f_{i-1} \circ f_{i-1}^{-1} \circ ... \circ f_1^{-1} \circ f_0^{-n+i} \circ F(x) \\ &= F(x) \end{aligned}$$

by (4.14) again. Therefore, f defined in (4.15) is a continuous iterative root of F of order n. This completes the proof. □

Concerning the question on characteristic endpoints, studied in this paper and mentioned in the Introduction, Theorems 4.2–4.3 give a complete answer to the existence of continuous iterative roots f of order $n = N(F) \geq 2$ which are of the height $H(f) = n$ and strictly decreasing on $[a', b']$. For $2 < n \leq N(F) + 1$, Theorem 4.1 denies the existence of continuous iterative roots f of order n which are of the height $H(f) < n - 1$ and strictly decreasing on $[a', b']$, but the existence of such roots f decreasing on $[a', b']$ of order n with the height $H(f) = n$ is still open in the case $n = N(F) + 1$ and the case that $2 < n < N(F)$.

5. Some remarks

First, we demonstrate our theorems with the following examples.

Example 5.1. Consider $F_1 : [0, 1] \to [0, 1]$ (see Fig. 7), defined by

$$F_1(x) := \begin{cases} \frac{1}{2}x + \frac{1}{4}, & \forall\ x \in [0, \frac{1}{2}], \\ -2x + \frac{3}{2}, & \forall\ x \in (\frac{1}{2}, \frac{3}{4}), \\ x - \frac{3}{4}, & \forall\ x \in [\frac{3}{4}, 1]. \end{cases}$$

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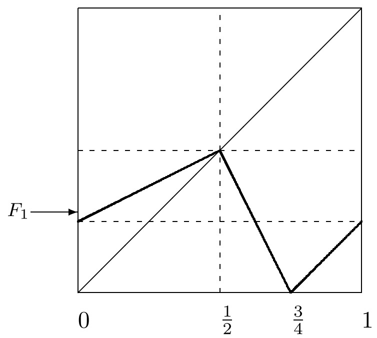

Fig. 7. F_1 with $N(F_1) = 2$.

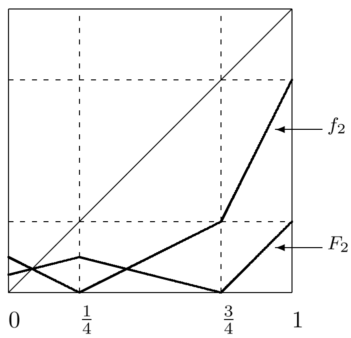

Fig. 8. F_2 and f_2.

Obviously, F_1 does not satisfy condition $(\mathbf{K}_0^+)$ because $K(F_1) = [0, \frac{1}{2}]$ and F_1 reaches 0 at the unique point $\frac{3}{4} \notin [0, \frac{1}{2}]$. By Theorem 3.1, F_1 does not have a continuous iterative root f of order $n > 1$ which is strictly increasing on $K(F_1)$ and satisfies either $H(f) \neq n$ or $H(f) = N(F_1) = 2$. On the other hand, by Theorems 4.1 and 4.2, F_1 does not have a continuous iterative root f of order $n > 1$ which is strictly decreasing on $K(F_1)$ and satisfies $H(f) < n - 1$. Therefore, F_1 does not have a continuous iterative root f of any order $n > 1$ which satisfies $H(f) < n - 1$.

Example 5.2. Consider $F_2 : [0, 1] \to [0, 1]$ (see Fig. 8), defined by

$$F_2(x) := \begin{cases} \frac{1}{4}x + \frac{1}{16}, & \forall\ x \in [0, \frac{1}{4}], \\ -\frac{1}{4}x + \frac{3}{16}, & \forall\ x \in (\frac{1}{4}, \frac{3}{4}), \\ x - \frac{3}{4}, & \forall\ x \in [\frac{3}{4}, 1]. \end{cases}$$

Clearly, F_2 does not satisfy condition $(\mathbf{K}_0^+)$ because $K(F_2) = [0, \frac{1}{4}]$ and F_2 reaches 0 at the unique point $\frac{3}{4} \notin [0, \frac{1}{4}]$. Moreover, the assumption (i) of Theorem 4.3 is satisfied with $n = N(F_2) = 2$, $c_1 = \frac{1}{4}$ and $c_2 = \frac{3}{4}$. It is easy to verify that the mapping $f_2 : [0, 1] \to [0, 1]$ (see Fig. 8), defined by

$$f_2(x) := \begin{cases} -\frac{1}{2}x + \frac{1}{8}, & \forall\ x \in [0, \frac{1}{4}], \\ \frac{1}{2}x - \frac{1}{8}, & \forall\ x \in (\frac{1}{4}, \frac{3}{4}), \\ 2x - \frac{5}{4}, & \forall\ x \in [\frac{3}{4}, 1], \end{cases}$$

is a continuous iterative root of F_2 of order 2 such that $H(f_2) = 2$.

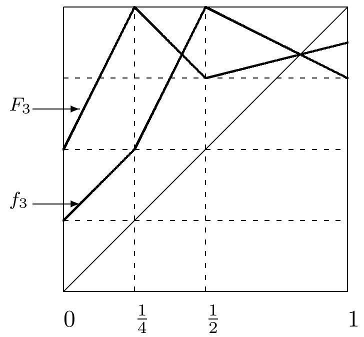

Fig. 9. F_3 and f_3.

Example 5.3. Consider $F_3 : [0, 1] \to [0, 1]$ (see Fig. 9), defined by

$$F_3(x) := \begin{cases} 2x + \frac{1}{2}, & \forall\, x \in [0, \frac{1}{4}], \\ -x + \frac{5}{4}, & \forall\, x \in (\frac{1}{4}, \frac{1}{2}), \\ \frac{1}{4}x + \frac{5}{8}, & \forall\, x \in [\frac{1}{2}, 1]. \end{cases}$$

Clearly, F_3 does not satisfy condition $(\mathbf{K}_0^+)$ because $K(F_3) = [\frac{1}{2}, 1]$ and F_3 reaches $\frac{1}{2}$ at the unique point $0 \notin [\frac{1}{2}, 1]$. Moreover, the assumption (ii) of Theorem 4.3 is satisfied with $n = N(F_3) = 2$, $c_1 = \frac{1}{4}$ and $c_2 = \frac{1}{2}$. One can check that the mapping $f_3 : [0, 1] \to [0, 1]$ (see Fig. 9), defined by

$$f_3(x) := \begin{cases} x + \frac{1}{4}, & \forall\, x \in [0, \frac{1}{4}], \\ 2x, & \forall\, x \in (\frac{1}{4}, \frac{1}{2}), \\ -\frac{1}{2}x + \frac{5}{4}, & \forall\, x \in [\frac{1}{2}, 1], \end{cases}$$

is a continuous iterative root of F_3 of order 2 such that $H(f_3) = 2$.

Our this paper is focused on the question on characteristic endpoints mentioned in the Introduction (also seen in [15]). Assuming that $F \in PM(I, I)$ with $H(F) = 1$ such that $(\mathbf{K}^+)$ holds but condition $(\mathbf{K}_0^+)$ is not true, we discuss for $n = N(F)$, $n = N(F) + 1$ and $2 \le n \le N(F) - 1$ separately:

- For $n = N(F)$, F has neither a continuous iterative root which is strictly increasing on $[a', b']$ (Corollary 3.1) nor a continuous iterative root of height $< n - 1$ which is strictly decreasing on $[a', b']$ (Theorem 4.1). A necessary and sufficient condition is given in Theorem 4.3 for the existence of iterative roots f of height $H(f) = n$ which are strictly decreasing on $[a', b']$. The existence of continuous iterative roots f of height $H(f) = n - 1$ which are decreasing on $[a', b']$ is still unknown, which can be reduced to the open problem **(P)** as indicated in the end of section 3.
- For $n = N(F) + 1$, F has neither a continuous iterative root which is strictly increasing on $[a', b']$ (Corollary 3.1) nor an iterative root f of height $H(f) < n - 1$ which is strictly decreasing on $[a', b']$ (Theorem 4.1), but the existence of such roots f decreasing on $[a, b]$ is still unknown.
- For $2 < n < N(F)$, F has neither a continuous iterative root f of height $H(f) < n$ which is strictly increasing on $[a', b']$ (Theorem 3.1) nor an iterative root f of height $H(f) < n - 1$ which is strictly decreasing on $[a', b']$ (Theorem 4.1), but the existence of such a root f of height $H(f) = n$ which is increasing on $[a', b']$ is still unknown, which is proposed as the open problem **(P)**.
- For $n = 2$, we obtain a necessary and sufficient condition for the existence of iterative roots f of height $H(f) = 1$ which are strictly decreasing on $[a', b']$ (Theorem 4.2).

Remark that the question on characteristic endpoints, which was raised in [16], does not concern about those F decreasing on the characteristic interval $[a', b']$. However, for those F decreasing on the characteristic interval $[a', b']$, the following result was also obtained in Theorem 10 in [16] (also seen in Theorem 5 in [15]), which is in contrast to Theorem A stated in the Introduction.

Theorem B. *Let $F \in PM(I, I)$ be of height 1. Suppose that $(\mathbf{K}^-)$ F is strictly decreasing on its characteristic interval $K(F) = [a', b']$ and, additionally, $(\mathbf{K}_0^-)$ either $F(a') = b'$ and $F(b') = a'$ or $a' < F(x) < b'$ on I. Then, for any odd $n > 1$, F has continuous iterative roots of order n.*

Theorem 2 of [15] shows that condition $(\mathbf{K}_0^-)$ is necessary for odd order $n > N(F) + 1$ if $(\mathbf{K}^-)$ holds. Hence, it is also interesting to discuss a similar question: *What happens if condition $(\mathbf{K}_0^-)$, also called the 'characteristic endpoints condition' for decreasing case, is not satisfied?*

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