J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Strong law of large numbers and Chover's law of the iterated logarithm under sub-linear expectations ☆

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ARTICLE INFO

Article history:
Received 5 March 2017
Available online xxxx
Submitted by G. Chen

Keywords:
Sub-linear expectation
Strong laws of large numbers
Chover's law of the iterated logarithm
Extended independence random variables

ABSTRACT

Limit theorems for sub-linear expectations are challenging field which have raised a large number of issues of interest recently. The aim of this paper is to establish general strong law of large numbers and the Chover's law of the iterated logarithm for a sequence of random variables under a sub-linear expectation space. As applications, several results on strong laws of large numbers which contain Marcinkiewicz strong law of large numbers and the Chover's law of the iterated logarithm for extended independence and identically distributed random variables have been generalized to the sub-linear expectation space context. Our results of strong laws of large numbers are more general than some related results previously reported obtained by Zhang (2016) [23], Cheng (2016) [5], Liu et al. (2015) [10] and Chen (2016) [3]. There is no report on the Chover's law of the iterated logarithm under sub-linear expectation, and we provide a method to study this subject.

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1. Introduction

Limit theorems are important research topics in probability and statistics. They were widely used in finance and other fields. Classical limit theorems only hold in the case of model certainty. However, in practice, such model certainty assumption is not realistic in many areas of applications because the uncertainty phenomena cannot be modeled using model certainty. Motivated by modeling uncertainty in practice, Peng (2006 [15]) introduced a new notion of sub-linear expectation. As an alternative to the traditional probability/expectation, capacity/sub-linear expectation has been studied in many fields such as statistics, finance, economics, and measures of risk (see Denis and Martini (2006 [8]), Gilboa (1987 [9]), Marinacci (1999 [11]),

☆ Supported by the National Natural Science Foundation of China (11661029, 11361019), and the Support Program of the Guangxi China Science Foundation (2015GXNSFAA139008).

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https://doi.org/10.1016/j.jmaa.2017.11.053
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Please cite this article in press as: Q. Wu, Y. Jiang, Strong law of large numbers and Chover's law of the iterated logarithm under sub-linear expectations, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2017.11.053

Peng (1997 [13], 1999 [14], 2006 [15], 2008 [16]) etc.). The general framework of the sub-linear expectation in a general function space was introduced by Peng (2006 [15], 2008 [16], 2008 [17]), and sub-linear expectation is a natural extension of the classical linear expectation.

Because the sub-linear expectation provides a very flexible framework to model sub-linear probability problems, the limit theorems of the sub-linear expectation have received more and more attention and research recently. A series of useful results have been established. Peng (2006 [15], 2008 [16], 2008 [17]) constructed the basic framework, basic properties and a new central limit theorem under sublinear expectations, Zhang (2016 [23], 2016 [24], 2016 [25]) established the exponential inequalities, Rosenthal's inequalities, Kolmogorov's strong law of larger numbers and Hartman–Wintner's law of iterated logarithm, Cheng (2016 [5]) and Chen (2016 [3]) also studied Kolmogorov's strong law of larger numbers, and so on. In general, extending the limit properties of conventional probability space to the cases of sub-linear expectation is highly desirable and of considerably significance in the theory and application. Because sub-linear expectation and capacity are not additive, many powerful tools and common methods for linear expectations and probabilities are no longer valid, so that the study of the limit theorems under sub-linear expectation becomes much more complex and difficult.

As it is well known, the strong law of large numbers (SLLN) is an essential theory in probability, statistics and the related fields, many of its related results have been obtained in the probabilistic space. In sub-linear expectation, this issue was considered by some authors. For example, under conditions $\hat{\mathbb{E}}|X_1|^{1+\alpha}, \alpha > 0$, Chen (2016 [3]) obtained Kolmogorov's SLLN for a sequence independently and identically distributed random variables. Cheng (2016 [5]) improved Chen's (2016 [3]) moment condition for a sequence of independent random variables. Further, Zhang (2016 [23]) established sufficient and necessary conditions of Kolmogorov's SLLN. However, they only studied the Kolmogorov's SLLN, i.e., SLLN for $p = 1$. One of the main purposes of this paper is to establish the very extensive version of the SLLN which contains the Marcinkiewicz SLLN for extended independence random variables in the general sub-linear expectation space. Our methods are different from their methods, our methods are suitable for studying the general strong limit theorem, and their methods are only suitable for studying the strong limit theorem of S_n/n. Our results of SLLN (Theorem 3.3–3.6) are more general than related results of Zhang (2016 [23]), Cheng (2016 [5]), Liu et al. (2015 [10]) and Chen (2016 [3]).

The classical laws of the iterated logarithm (LIL) as fundamental limit theorems in probability theory play an important role in the development of probability theory and its applications. In probability space, Chover (1966 [6]) established first the classical Chover's LIL for sequence of independent and identically distributed random variables with a stable distribution for an exponent in $(0, 2)$. Under more general assumptions, some results of Chover's LIL obtained by Mikosch (1984 [12]), Vasudeva (1984 [20]), Qi and Cheng (1996 [18]) for sequences of independent random variables. Some papers have been devoted to the study of Chover's LIL for dependent sequence. We refer the reader to Chen (2005 [2]), Cai (2006 [1]) and Wu and Jiang (2010 [21], 2010 [22]). However, the Chover's LIL under sub-linear expectation has not been reported. In this paper, by applying our general SLLN (3.3) for sub-linear expectation space, we studied and extended the Chover's LIL from traditional probability space to the general sub-linear expectation space. We provide a method to study this subject.

This paper is organized as follows. In the next section, we summarize some basic notations and concepts, related properties under the sub-linear expectations and present the preliminary lemmas that are useful to obtain the main results. In Section 3, general strong laws of large numbers for extended independence and identically distributed random variables under sub-linear expectation space are established. The Chover's law of the iterated logarithm under sub-linear expectation is established in Section 4.

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2. Preliminaries

We use the framework and notations of Peng (2008 [17]). Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq c(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_n,$$

for some $c > 0, m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of "random variables". In this case we denote $X \in \mathcal{H}$.

Definition 2.1. A sub-linear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\hat{\mathbb{E}} : \mathcal{H} \to \bar{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geq Y$ then $\hat{\mathbb{E}}X \geq \hat{\mathbb{E}}Y$;
(b) Constant preserving: $\hat{\mathbb{E}}c = c$;
(c) Sub-additivity: $\hat{\mathbb{E}}(X + Y) \leq \hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$ whenever $\hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;
(d) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}X, \lambda \geq 0$.

Here $\bar{\mathbb{R}} = [-\infty, \infty]$. The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-linear expectation space. For $\hat{\mathbb{E}}$, the linear property of expectation is replaced by the sub-additivity and positive homogeneity.

Given a sub-linear expectation $\hat{\mathbb{E}}$, let us denote the conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ by

$$\hat{\varepsilon}X := -\hat{\mathbb{E}}(-X), \quad \forall X \in \mathcal{H}.$$

From the definition, it is easily shown that for all $X, Y \in \mathcal{H}$

$$\hat{\varepsilon}X \leq \hat{\mathbb{E}}X, \quad \hat{\mathbb{E}}(X + c) = \hat{\mathbb{E}}X + c, \quad |\hat{\mathbb{E}}(X - Y)| \leq \hat{\mathbb{E}}|X - Y| \quad \text{and} \quad \hat{\mathbb{E}}(X - Y) \geq \hat{\mathbb{E}}X - \hat{\mathbb{E}}Y$$

If $\hat{\mathbb{E}}Y = \hat{\varepsilon}Y$, then $\hat{\mathbb{E}}(X + aY) = \hat{\mathbb{E}}X + a\hat{\mathbb{E}}Y$ for any $a \in \mathbb{R}$.

Next, we consider the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\emptyset) = 0, \quad V(\Omega) = 1 \quad \text{and} \quad V(A) \leq V(B) \quad \text{for} \quad \forall A \subseteq B, A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A \bigcup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \bigcup B \in \mathcal{G}$. In the sub-linear space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, we denote a pair $(\mathbb{V}, \nu)$ of capacities by

$$\mathbb{V}(A) := \inf\{\hat{\mathbb{E}}\xi; I(A) \leq \xi, \xi \in \mathcal{H}\}, \quad \nu(A) := 1 - \mathbb{V}(A^c), \forall A \in \mathcal{F},$$

where A^c is the complement set of A. By definition of $\mathbb{V}$ and ν, it is obvious that $\mathbb{V}$ is sub-additive, and

$$\begin{aligned} &\nu(A) \leq \mathbb{V}(A), \quad \forall A \in \mathcal{F}; \quad \mathbb{V}(A) = \hat{\mathbb{E}}(I(A)), \quad \nu(A) = \hat{\varepsilon}(I(A)), \quad \text{if} \quad I(A) \in \mathcal{H}, \\ &\hat{\mathbb{E}}f \leq \mathbb{V}(A) \leq \hat{\mathbb{E}}g, \hat{\varepsilon}f \leq \nu(A) \leq \hat{\varepsilon}g, \quad \text{if} \quad f \leq I(A) \leq g, \quad f, g \in \mathcal{H}. \end{aligned} \tag{2.1}$$

This implies Markov inequality: $\forall X \in \mathcal{H}$,

$$\mathbb{V}(|X| \geq x) \leq \hat{\mathbb{E}}(|X|^p)/x^p, \quad \forall \quad x > 0, p > 0$$

from $I(|X| \geq x) \leq |X|^p/x^p \in \mathcal{H}$. From Lemma 4.1 in Zhang (2016 [24]), we have Hölder inequality: $\forall X, Y \in \mathcal{H}, p, q > 1$ satisfying $p^{-1} + q^{-1} = 1$,

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$$\hat{\mathbb{E}}(|XY|) \le \left(\hat{\mathbb{E}}(|X|^p)\right)^{1/p} \left(\hat{\mathbb{E}}(|Y|^q)\right)^{1/q},$$

particularly, Jensen inequality: $\forall X \in \mathcal{H}$,

$$\left(\hat{\mathbb{E}}(|X|^r)\right)^{1/r} \le \left(\hat{\mathbb{E}}(|X|^s)\right)^{1/s} \qquad \text{for} \quad 0 < r \le s.$$

Definition 2.2. (i) A sub-linear expectation $\hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ is called to be countably sub-additive if it satisfies

$$\hat{\mathbb{E}}(X) \le \sum_{n=1}^{\infty} \hat{\mathbb{E}}(X_n), \quad \text{whenever} \quad X \le \sum_{n=1}^{\infty} X_n, \quad X, X_n \in \mathcal{H}, \quad X \ge 0, X_n \ge 0, \quad n \ge 1;$$

It is called to be continuous if it satisfies

$$\hat{\mathbb{E}}(X_n) \uparrow \hat{\mathbb{E}}(X), \quad \text{if} \quad 0 \le X_n \uparrow X, \quad \text{and} \quad \hat{\mathbb{E}}(X_n) \downarrow \hat{\mathbb{E}}(X), \quad \text{if} \quad 0 \le X_n \downarrow X, \quad \text{where} \quad X, X_n \in \mathcal{H}.$$

(ii) A function $V : \mathcal{F} \to [0, 1]$ is called to be countably sub-additive if

$$V\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} V(A_n), \quad \forall A_n \in \mathcal{F};$$

It is called to be continuous if it satisfies

$$V(A_n) \uparrow V(A), \quad \text{if} \quad A_n \uparrow A, \quad \text{and} \quad V(A_n) \downarrow V(A), \quad \text{if} \quad A_n \downarrow A, \quad \text{where} \quad A, A_n \in \mathcal{F}.$$

It is obvious that a continuous sub-additive capacity V (resp. a sub-linear expectation $\hat{\mathbb{E}}$) is countably sub-additive. Also, we define the Choquet integrals/expecations $(C_{\mathbb{V}}, C_{\nu})$ by

$$C_V(X) := \int_0^{\infty} V(X > x)\mathrm{d}x + \int_{-\infty}^{0} (V(X > x) - 1)\mathrm{d}x$$

with V being replaced by $\mathbb{V}$ and ν respectively.

Definition 2.3. (Peng (2006 [14], 2008 [17]), Zhang (2016 [24]))

(i) (Identical distribution) Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two n-dimensional random vectors defined respectively in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. They are called identically distributed if

$$\hat{\mathbb{E}}_1(\varphi(\mathbf{X}_1)) = \hat{\mathbb{E}}_2(\varphi(\mathbf{X}_2)), \quad \forall \varphi \in C_{l,Lip}(\mathbb{R}_n),$$

whenever the sub-expectations are finite. A sequence $\{X_n; n \ge 1\}$ of random variables is said to be identically distributed if for each $i \ge 1$, X_i and X_1 are identically distributed.

(ii) (Independence) In a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent to another random vector $\mathbf{X} = (X_1, \ldots, X_m), X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}$ if for each test function $\varphi \in C_{l,Lip}(\mathbb{R}_m \times \mathbb{R}_n)$ we have $\hat{\mathbb{E}}(\varphi(\mathbf{X}, \mathbf{Y})) = \hat{\mathbb{E}}[\hat{\mathbb{E}}(\varphi(\mathbf{x}, \mathbf{Y}))|_{\mathbf{x}=\mathbf{X}}]$, whenever $\bar{\varphi}(\mathbf{x}) := \hat{\mathbb{E}}(|\varphi(\mathbf{x}, \mathbf{Y})|) < \infty$ for all $\mathbf{x}$ and $\hat{\mathbb{E}}(|\bar{\varphi}(\mathbf{X})|) < \infty$.

(iii) (Negatively dependent) In a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be negatively dependent (ND) to another random vector $\mathbf{X} = (X_1, \ldots, X_m), X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}$ if for each pair of test functions $\varphi_1 \in C_{l,Lip}(\mathbb{R}_m)$ and $\varphi_2 \in C_{l,Lip}(\mathbb{R}_n)$ we have $\hat{\mathbb{E}}(\varphi_1(\mathbf{X})\varphi_2(\mathbf{Y})) \le \hat{\mathbb{E}}(\varphi_1(\mathbf{X})\hat{\mathbb{E}}(\varphi_2(\mathbf{Y}))$, whenever φ_1, φ_2 are coordinatewise nondecreasing or φ_1, φ_2

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are coordinatewise non-increasing with $\varphi_1(\mathbf{X}) \geq 0, \hat{\mathbb{E}}\varphi_2(\mathbf{Y}) \geq 0$, $\hat{\mathbb{E}}|\varphi_1(\mathbf{X})\varphi_2(\mathbf{Y})| < \infty, \hat{\mathbb{E}}|\varphi_1(\mathbf{X})| < \infty$, $\hat{\mathbb{E}}|\varphi_2(\mathbf{Y})| < \infty$.

(iv) (Independent (Negatively dependent) random variables) A sequence of random variables $\{X_n; n \geq 1\}$ is said to be independent (negative dependence), if X_{i+1} is independent (negatively dependent) to $(X_1, \ldots, X_i)$ for each $i \geq 1$.

(v) (Extended independence) A sequence of random variables $\{X_n; n \geq 1\}$ is said to be extended independent, if

$$\hat{\mathbb{E}}\left(\prod_{i=1}^{n} \varphi_i(X_i)\right) = \prod_{i=1}^{n} \hat{\mathbb{E}}(\varphi_i(X_i)), \quad \forall n \geq 2, \quad \forall \varphi_i(x) \in C_{l,Lip}(\mathbb{R}).$$

It can be showed that the independence implies the extended independence, the extended independence implies negatively dependent, and if $\{X_n; n \geq 1\}$ is a sequence of extended independent random variables and $f_1(x), f_2(x), \ldots \in C_{l,Lip}(\mathbb{R})$, then $\{f_n(X_n); n \geq 1\}$ is also a sequence of extended independent random variables. It shall be noted that the extended independence of $\{X_n; n \geq 1\}$ under $\hat{\mathbb{E}}$ does not imply the extended independence under $\hat{\varepsilon}$.

In the following, let $\{X_n; n \geq 1\}$ be a sequence of random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, and $S_n = \sum_{i=1}^{n} X_i$. The symbol c stands for a generic positive constant which may differ from one place to another. Let $a_n \ll b_n$ denote that there exists a constant $c > 0$ such that $a_n \leq cb_n$ for sufficiently large n, and $I(\cdot)$ denotes an indicator function.

To prove our results, we need the following four lemmas.

Lemma 2.4. *(Qi and Cheng 1996 [18]). Suppose that $h(x)$ is a slowly varying function at infinity and $g(x)$ is a positive function with $\lim_{x\to\infty} g(x) = \infty$. Then, for any given $\delta > 0$, there exists an $x_0 > 0$ such that*

$$g^{-\delta}(x) \leq \inf_{x \leq y \leq xg(x)} \frac{h(y)}{h(x)} \leq \sup_{x \leq y \leq xg(x)} \frac{h(y)}{h(x)} \leq g^{\delta}(x) \quad \text{for all} \quad x > x_0.$$

Lemma 2.5. *(Zhang 2016, Lemma 3.9 [24]). Let $\{A_n; n \geq 1\}$ be a sequence of events in $\mathcal{F}$. Suppose that V is a countably sub-additive capacity. If $\sum_{n=1}^{\infty} V(A_n) < \infty$, then $V(A_n; \text{i.o.}) = 0$, where $\{A_n; \text{i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$.*

Lemma 2.6. *(Zhang (2016, Theorem 3.1 [23], 2016, Theorem 2.1 [25])) Suppose that X_k is negatively dependent to $(X_{k+1}, \ldots, X_n)$ for each $k = 1, \ldots, n-1$, and $\hat{\mathbb{E}}X_n \leq 0$. Then*

$$\hat{\mathbb{E}}\left(\left|\max_{k\leq n} S_k\right|^p\right) \leq 2^{2-p} \sum_{k=1}^{n} \hat{\mathbb{E}}X_k^2 \quad \text{for} \quad 1 \leq p \leq 2. \tag{2.2}$$

$$\mathbb{V}(S_n \geq x) \leq c\frac{\sum_{k=1}^{n} \hat{\mathbb{E}}X_k^2}{x^2}, \quad \forall x > 0. \tag{2.3}$$

In particular, Lemma 2.6 holds for a sequence of extended independent random variables.

Lemma 2.7. *Let $\{f_n\}$ be a sequence of positive nondecreasing numbers satisfying*

$$\sum_{n=1}^{\infty} \frac{1}{nf_n} \begin{cases} < \infty, \\ = \infty. \end{cases}$$

Then there exists a nondecreasing sequence of $\{q_n\}$ such that $0 < q_n \leq f_n, \sup_{n\geq 1} \frac{q_{2n}}{q_n} < \infty$ and

$$\sum_{n=1}^{\infty} \frac{1}{nq_n} \begin{cases} < \infty, \\ = \infty. \end{cases}$$

Proof. We can prove the lemma by page 53 of Chen and Qi (2006 [4]).

3. Strong law of large numbers

We define $X_n \to X$ a.s. $\mathbb{V}$ (ν) as follows.

Definition 3.1. A sequence of random variables $\{X_n; n \geq 1\}$ is said to converge to X almost surely V (a.s. V), denoted by $X_n \to X$ a.s. V as $n \to \infty$ if, $V(X_n \nrightarrow X) = 0$.

V can be replaced by $\mathbb{V}$ and ν respectively. By $\nu(A) \leq \mathbb{V}(A)$ and $\nu(A) + \mathbb{V}(A^c) = 1$ for any $A \in \mathcal{F}$, it is obvious that $X_n \to X$ a.s. $\mathbb{V}$ implies $X_n \to X$ a.s. ν, but $X_n \to X$ a.s. ν does not imply $X_n \to X$ a.s. $\mathbb{V}$. Further

$$X_n \to X \quad \text{a.s.} \quad \mathbb{V} \Longleftrightarrow \nu(X_n \to X) = 1 \Longleftrightarrow \mathbb{V}(|X_n - X| \geq \varepsilon, \text{i.o.}) = 0 \quad \text{for} \quad \forall \varepsilon > 0,$$

and

$$X_n \to X \quad \text{a.s.} \quad \nu \Longleftrightarrow \nu(X_n \nrightarrow X) = 0 \Longleftrightarrow \mathbb{V}(X_n \to X) = 1.$$

Remark 3.2. In probability space, it is well known $X_n \to X$ a.s. $\Leftrightarrow P(X_n \to X) = 1 \Leftrightarrow P(X_n \nrightarrow X) = 0$ from $P(A) + P(A^c) = 1$. However, in the sub-linear expectation space, the formula $\mathbb{V}(A) + \mathbb{V}(A^c) = 1$ is no longer valid, which implies $\mathbb{V}(X_n \to X) = 1 \not\Leftrightarrow \mathbb{V}(X_n \nrightarrow X) = 0$. In fact, we have $\mathbb{V}(X_n \nrightarrow X) = 0 \Rightarrow \mathbb{V}(X_n \to X) = 1$, but $\mathbb{V}(X_n \to X) = 1 \not\Rightarrow \mathbb{V}(X_n \nrightarrow X) = 0$. Therefore, we can't define $X_n \to X$ a.s. with $\mathbb{V}(X_n \to X) = 1$.

In sub-linear expectations, due to the uncertainty of expectation and capacity, the strong convergence is essentially different from the ordinary probability space. The study of strong convergence for sub-linear expectations is much more complex and difficult.

Suppose that $\{X_n; n \geq 1\}$ is a sequence of extended independent random variables. It is important to note that the extended independence under $\hat{\mathbb{E}}$ is defined through continuous functions in $C_{l,Lip}$ and the indicator function $I(|x| \leq a)$ is not continuous. Therefore, in order to ensure that the sequence of truncated random variables is also extended independence, we cannot censor X_i using the indicator function. This needs to modify the indicator function by functions in $C_{l,Lip}$.

For $0 < \mu < 1$, let $g(x) \in C_{l,Lip}(\mathbb{R})$ be a non-increasing function such that $0 \leq g(x) \leq 1$ for all x and $g(x) = 1$ if $x \leq \mu$, $g(x) = 0$ if $x > 1$. Then

$$I(|x| \leq \mu) \leq g(|x|) \leq I(|x| \leq 1). \tag{3.1}$$

For any $i \leq n$, let $X_i(a_n) = X_i g\left(\frac{|X_i|}{a_n}\right)$.

Theorem 3.3. *Assume that $\{X_n; n \geq 1\}$ is a sequence of extended independent and identically distributed random variables, $\mathbb{V}$ is continuous, and $\{a_n; n \geq 1\}$ is a sequence of positive numbers with $a_n/n^{1/\beta} \uparrow$ for some $\beta \in (0, 2)$ and $\sup_{n \geq 1} a_{2n}/a_n < \infty$.*

(i) *If*

$$\sum_{n=1}^{\infty} \mathbb{V}(|X_1| > a_n) < \infty, \tag{3.2}$$

then

$$\limsup_{n\to\infty} \frac{S_n - c_n}{a_n} \le 0 \qquad a.s. \quad \mathbb{V}, \tag{3.3}$$

where $c_n = \sum_{k=1}^{n} \hat{\mathbb{E}}(X_k(a_n)) = n\hat{\mathbb{E}}(X_1(a_n)) = n\hat{\mathbb{E}}(X_1 g(|X_1|/a_n))$, *in particular,* $c_n = 0$ *if* $\frac{a_n}{n} \uparrow \infty$ *and* $c_n = n\hat{\mathbb{E}}X_1$ *if* $\frac{n}{a_n} \uparrow, \hat{\mathbb{E}}|X_1| < \infty, \hat{\mathbb{E}}$ *is countably sub-additive. And*

$$\liminf_{n\to\infty} \frac{S_n - \tilde{c}_n}{a_n} \ge 0 \qquad a.s. \quad \mathbb{V}, \tag{3.4}$$

where $\tilde{c}_n = n\hat{\varepsilon}(X_1(a_n))$, *in particular,* $\tilde{c}_n = 0$ *if* $\frac{a_n}{n} \uparrow \infty$ *and* $\tilde{c}_n = n\hat{\varepsilon}X_1$ *if* $\frac{n}{a_n} \uparrow, \hat{\mathbb{E}}|X_1| < \infty, \hat{\mathbb{E}}$ *is countably sub-additive.*

(ii) *If*

$$\sum_{n=1}^{\infty} \mathbb{V}(|X_1| > a_n) = \infty, \tag{3.5}$$

then

$$\limsup_{n\to\infty} \frac{|S_n - d_n|}{a_n} = \infty \qquad a.s. \quad \nu \tag{3.6}$$

for every sequence $\{d_n; n \ge 1\}$ *such that*

$$\limsup_{n\to\infty} \frac{|d_n - d_{n-1}|}{a_n} = c < \infty. \tag{3.7}$$

Remark 3.4. From the proof of Theorem 3.3, if the condition (3.5) is strengthened as $\sum_{n=1}^{\infty} \nu(|X_1| > a_n) = \infty$ and assume that $\{X_n; n \ge 1\}$ are extended independent under $\hat{\varepsilon}$, then the conclusion (3.6) is correspondingly strengthened $\limsup_{n\to\infty} \frac{|S_n - d_n|}{a_n} = \infty$ a.s. $\mathbb{V}$, the inverse part of the strong law of large numbers.

Remark 3.5. Theorem 3.3 is a powerful results, as we can obtain various forms of strong law of large numbers by taking different forms of a_n.

Taking $a_n = n^{1/p}, p \in (0, 2)$, then we have the following Marcinkiewicz strong law of large numbers.

Theorem 3.6. *Assume that* $\{X_n; n \ge 1\}$ *is a sequence of extended independent and identically distributed random variables,* $\mathbb{V}$ *is continuous, and* $p \in (0, 2)$. *For* $p \ge 1$, *assume that* $\hat{\mathbb{E}}$ *is countably sub-additive.*

(i) *If*

$$C_{\mathbb{V}}(|X_1|^p) < \infty, \tag{3.8}$$

then

$$\limsup_{n\to\infty} \frac{S_n - c_n}{n^{1/p}} \le 0 \quad a.s. \quad \mathbb{V}, \quad and \quad \liminf_{n\to\infty} \frac{S_n - \tilde{c}_n}{n^{1/p}} \ge 0 \quad a.s. \quad \mathbb{V},$$

where $c_n = \tilde{c}_n = 0$ *if* $p < 1$ *and* $c_n = n\hat{\mathbb{E}}X_1, \tilde{c}_n = n\hat{\varepsilon}X_1$ *if* $p \ge 1$.

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In particular, if $\hat{\mathbb{E}}X_1 = \hat{\varepsilon}X_1, p \geq 1$, *then (3.8) implies*

$$\lim_{n\to\infty} \frac{S_n - c_n}{n^{1/p}} = 0 \quad a.s. \quad \mathbb{V},$$

where $c_n = 0$ *if* $p < 1$ *and* $c_n = n\hat{\mathbb{E}}X_1$ *if* $p \geq 1$.

(ii) *If*

$$C_{\mathbb{V}}(|X_1|^p) = \infty,$$

then

$$\limsup_{n\to\infty} \frac{|S_n - d_n|}{n^{1/p}} = \infty \qquad a.s. \quad \nu$$

for every sequence $\{d_n; n \geq 1\}$ *such that*

$$\limsup_{n\to\infty} \frac{|d_n - d_{n-1}|}{n^{1/p}} = c < \infty.$$

Remark 3.7. Theorem 3.6 is Marcinkiewicz strong law of large numbers under the sub-linear expectations. Therefore, Theorem 3.3 not only extends the famous Marcinkiewicz SLLN from the probability space to sub-linear expectation space, but also expands the range of a_n. The corresponding results obtained by Zhang (2016 [23]), Cheng (2016 [5]), Liu et al. (2015 [10]) and Chen (2016 [3]) are special cases of Theorem 3.6.

Proof of Theorem 3.3. (i) We just need to prove (3.3). Because of considering $\{-X_n; n \geq 1\}$ instead of $\{X_n; n \geq 1\}$ in (3.3), we can obtain (3.4). It is easy to prove that (3.2) is equivalent to for any $c > 0$,

$$\sum_{n=1}^{\infty} \mathbb{V}(|X_1| > ca_n) < \infty. \tag{3.9}$$

Note that

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbb{V}(|X_1| > ca_n) &\geq \sum_{k=1}^{\infty} \sum_{2^{k-1}\leq n\leq 2^k} \mathbb{V}(|X_1| > ca_n) \geq \sum_{k=1}^{\infty} \sum_{2^{k-1}\leq n\leq 2^k} \mathbb{V}(|X_1| > ca_{2^k}) \\ &= 2^{-1} \sum_{k=1}^{\infty} 2^k \mathbb{V}(|X_1| > ca_{2^k}).\end{aligned}$$

Hence, (3.9) implies

$$\sum_{k=1}^{\infty} 2^k \mathbb{V}(|X_1| > c \cdot a_{2^k}) < \infty, \quad \forall c > 0. \tag{3.10}$$

For every n, there exists a k such that $2^{k-1} \leq n < 2^k$, thus,

$$\begin{aligned}\frac{S_n - n\hat{\mathbb{E}}X_1(a_n)}{a_n} &= \frac{(S_n - n\hat{\mathbb{E}}X_1(a_{2^k})) + n(\hat{\mathbb{E}}X_1(a_{2^k}) - \hat{\mathbb{E}}X_1(a_n))}{a_n} \\ &\leq \frac{\left|\max_{n<2^k}(S_n - n\hat{\mathbb{E}}X_1(a_{2^k}))\right|}{a_{2^{k-1}}} + \frac{\max_{2^{k-1}\leq n<2^k} n\hat{\mathbb{E}}\,|X_1(a_{2^k}) - X_1(a_n)|}{a_{2^{k-1}}} \\ &=: I_1 + I_2.\end{aligned}$$

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Thus, to prove (3.3), it suffices to verify that

$$\limsup_{k\to\infty} I_1 \le 0 \quad \text{a.s.} \quad \mathbb{V} \quad \text{and} \quad I_2 \to 0, \quad k \to \infty. \tag{3.11}$$

By $\sup_{n\ge1}\frac{a_{2n}}{a_n} < \infty$, (3.10), $g(x)\downarrow$, and $\left|g\left(\frac{|X_1|}{a_{2^k}}\right) - g\left(\frac{|X_1|}{a_{2^{k-1}}}\right)\right| \le I(\mu a_{2^{k-1}} < |X_1| \le a_{2^k})$ from (3.1),

$$\begin{aligned} I_2 &\le a_{2^{k-1}}^{-1} 2^k \hat{\mathbb{E}}\left(|X_1|\left|g\left(\frac{|X_1|}{a_{2^k}}\right) - g\left(\frac{|X_1|}{a_{2^{k-1}}}\right)\right|\right) \\ &\le a_{2^{k-1}}^{-1} 2^k a_{2^k} \mathbb{V}(|X_1| > \mu a_{2^{k-1}}) \\ &\ll \sup_{n\ge1}\frac{a_{2n}}{a_n} 2^{k-1} \mathbb{V}(|X_1| > \mu a_{2^{k-1}}) \\ &\to 0, k \to \infty. \end{aligned} \tag{3.12}$$

Define

$$Y_i = X_i(a_{2^k}) - \hat{\mathbb{E}} X_i(a_{2^k}) \quad \text{and} \quad T_j = \sum_{i=1}^{j} Y_i.$$

Let $g_j(x) \in C_{l,Lip}(\mathbb{R}), j \ge 1$ such that $0 \le g_j(x) \le 1$ for all x and $g_j\left(\frac{x}{a_{2^j}}\right) = 1$ if $a_{2^{j-1}} < x \le a_{2^j}$, $g_j\left(\frac{x}{a_{2^j}}\right) = 0$ if $x \le \mu a_{2^{j-1}}$ or $x > (1+\mu)a_{2^j}$. Then

$$|X_1|^q g\left(\frac{|X_1|}{a_{2^k}}\right) \le a_1^q + \sum_{j=1}^{k} |X_1|^q g_j\left(\frac{|X_1|}{a_{2^j}}\right), \quad q > 0, \quad g_j\left(\frac{|X_1|}{a_{2^j}}\right) \le I(\mu a_{2^{j-1}} < |X_1| \le (1+\mu)a_{2^j}), \tag{3.13}$$

and by (3.1),

$$I(|X_i| > a_{2^k}) \le 1 - g\left(\frac{|X_i|}{a_{2^k}}\right) \le I(|X_i| > \mu a_{2^k}). \tag{3.14}$$

It shall be noted that the identical distribution is defined under $\hat{\mathbb{E}}$, not under $\mathbb{V}$ (see Definition 2.3). X_i identical distribution implies $\hat{\mathbb{E}}(f(X_i)) = \hat{\mathbb{E}}(f(X_1))$ for $f(\cdot) \in C_{l,Lip}(\mathbb{R})$, but does not imply $\mathbb{V}(f(X_i) \in A) = \mathbb{V}(f(X_1) \in A)$. Therefore, in the calculation of $\mathbb{V}(f(X_i) \in A)$, we need to convert $\mathbb{V}$ to $\hat{\mathbb{E}}$.

On the other hand, by $g(\cdot) \in C_{l,Lip}(\mathbb{R})$, so, $\{Y_i; i \ge 1\}$ is also a sequence of extended independent and identically distributed random variables with $\hat{\mathbb{E}} Y_i = 0$. Using (2.2) in Lemma 2.6 for $\{Y_i; i \ge 1\}$, for every $\epsilon > 0$, by the Markov inequality, (2.1), (3.10), (3.13), and (3.14), we obtain

$$\begin{aligned} &\sum_{k=1}^{\infty} \mathbb{V}\left(\left|\max_{n<2^k}(S_n - n\hat{\mathbb{E}} X_1(a_{2^k}))\right| \ge \epsilon a_{2^{k-1}}\right) \\ &\le \sum_{k=1}^{\infty} \mathbb{V}\left(\bigcup_{i=1}^{2^k}(|X_i| > a_{2^k})\right) + \sum_{k=1}^{\infty} \mathbb{V}\left(\left|\max_{n<2^k} T_n\right| \ge \epsilon a_{2^{k-1}}\right) \\ &\ll \sum_{k=1}^{\infty}\sum_{i=1}^{2^k} \mathbb{V}(|X_i| > a_{2^k}) + \sum_{k=1}^{\infty} a_{2^{k-1}}^{-2} \hat{\mathbb{E}}\left|\max_{n<2^k} T_n\right|^2 \end{aligned}$$

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$$
\begin{aligned}
&\ll \sum_{k=1}^{\infty}\sum_{i=1}^{2^k}\hat{\mathbb{E}}\left(1-g\left(\frac{|X_i|}{a_{2^k}}\right)\right)+\sum_{k=1}^{\infty}a_{2^{k-1}}^{-2}\sum_{i=1}^{2^k}\hat{\mathbb{E}}(Y_i^2)\\
&=\sum_{k=1}^{\infty}2^k\hat{\mathbb{E}}\left(1-g\left(\frac{|X_1|}{a_{2^k}}\right)\right)+\sum_{k=1}^{\infty}a_{2^{k-1}}^{-2}2^k\hat{\mathbb{E}}\left(X_1^2g\left(\frac{|X_1|}{a_{2^k}}\right)\right)\\
&\le \sum_{k=1}^{\infty}2^k\mathbb{V}(|X_1|>\mu a_{2^k})+\sum_{k=1}^{\infty}a_{2^{k-1}}^{-2}2^k\left(a_1^2+\sum_{j=1}^{k}\hat{\mathbb{E}}X_1^2g_j\left(\frac{|X_1|}{a_{2^j}}\right)\right)\\
&\ll \sum_{k=1}^{\infty}a_{2^{k-1}}^{-2}2^k\sum_{j=1}^{k}(1+\mu)^2a_{2^j}^2\mathbb{V}(|X_1|>\mu a_{2^{j-1}})\\
&\ll \sum_{j=1}^{\infty}a_{2^j}^2\mathbb{V}(|X_1|>\mu a_{2^{j-1}})\sum_{k=j}^{\infty}\frac{2^k}{a_{2^{k-1}}^2}. \qquad (3.15)
\end{aligned}
$$

Noting that $\beta<2$, and $\frac{a_n}{n^{1/\beta}}\uparrow$, we have

$$
\begin{aligned}
\sum_{k=j}^{\infty}\frac{2^k}{a_{2^{k-1}}^2}&=\sum_{k=j}^{\infty}\left(\frac{2^{(k-1)/\beta}}{a_{2^{k-1}}}\right)^2\frac{1}{2^{(2/\beta-1)k}2^{-2/\beta}}\\
&\le \frac{2^{2j/\beta}}{a_{2^{j-1}}^2}\sum_{k=j}^{\infty}\frac{1}{2^{(2/\beta-1)k}}\ll\frac{2^{2j/\beta}}{a_{2^{j-1}}^2}\frac{1}{2^{(2/\beta-1)j}}\\
&=\frac{2^j}{a_{2^{j-1}}^2}.
\end{aligned}
$$

Substitute the above inequality in (3.15), by (3.10),

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\mathbb{V}\left(\left|\max_{n<2^k}(S_n-n\hat{\mathbb{E}}X_1(a_{2^k}))\right|\ge\varepsilon a_{2^{k-1}}\right)\\
&\ll\left(\sup\frac{a_{2n}}{a_n}\right)^2\sum_{j=2}^{\infty}2^j\mathbb{V}\left(|X_1|>\mu a_{2^{j-1}}\right)\\
&<\infty.
\end{aligned}
$$

Thus, $\limsup_{k\to\infty} I_1\le 0$ a.s. $\mathbb{V}$ follows from the Borel–Cantelli lemma and continuity of $\mathbb{V}$ ($\mathbb{V}$ continuous implies $\mathbb{V}$ countably sub-additive). Thus, combining with (3.11) and (3.12), we get that (3.3) holds for $c_n=n\hat{\mathbb{E}}X_1(a_n)$. Next, we prove (3.3) in following two particular cases.

(1) If $\frac{a_n}{n}\uparrow\infty$, we only need to prove

$$
\lim_{n\to\infty}\frac{n\hat{\mathbb{E}}X(a_n)}{a_n}=0. \qquad (3.16)
$$

Since for $2^{k-1}\le n<2^k$, by $\sup_{n\ge1}\frac{a_{2n}}{a_n}<\infty$, $g(x)\downarrow$, $\frac{n}{a_n}\downarrow 0$ and (3.13),

$$
\begin{aligned}
\frac{|n\hat{\mathbb{E}}X_1(a_n)|}{a_n}&\le\frac{2^k}{a_{2^{k-1}}}\hat{\mathbb{E}}\left(|X_1|g\left(\frac{|X_1|}{a_{2^k}}\right)\right)\\
&\le\frac{2^k}{a_{2^{k-1}}}\left[a_1+\sum_{j=1}^{k}\hat{\mathbb{E}}\left(|X_1|g_j\left(\frac{|X_1|}{a_{2^j}}\right)\right)\right]
\end{aligned}
$$

$$\ll \frac{2^k}{a_{2^k}} + \frac{2^k}{a_{2^k}} \sum_{j=1}^{k} a_{2^j} \mathbb{V}\left(|X_1| > \mu a_{2^{j-1}}\right).$$

Noting that $\sum_{j=1}^{\infty} \frac{2^j}{a_{2^j}} a_{2^j} \mathbb{V}(|X_1| > \mu a_{2^j}) < \infty$ and $\frac{a_n}{n} \uparrow \infty$, by the Kronecker lemma,

$$\frac{2^k}{a_{2^k}} \sum_{j=1}^{k} a_{2^j} \mathbb{V}(|X_1| > \mu a_{2^j}) \to 0 \quad \text{as} \quad k \to \infty.$$

Therefore (3.16) holds.

(2) If $\frac{n}{a_n} \uparrow, \hat{\mathbb{E}}|X_1| < \infty$, and $\hat{\mathbb{E}}$ is countably sub-additive, we only need to prove

$$\lim_{n\to\infty} \frac{n(\hat{\mathbb{E}}X_1 - \hat{\mathbb{E}}X_1(a_n))}{a_n} = 0. \tag{3.17}$$

By the definition of g, g_j, we have

$$1 - g\left(\frac{|X_1|}{a_{2^{k-1}}}\right) \le \sum_{j=k-1}^{\infty} g_j\left(\frac{|X_1|}{a_{2^j}}\right).$$

Thus, for $2^{k-1} \le n < 2^k$, by $\sup_{n\ge 1} \frac{a_{2n}}{a_n} < \infty$, (3.10), and $\hat{\mathbb{E}}$ being countably sub-additive, we get

$$\begin{aligned}
\frac{n|\hat{\mathbb{E}}X_1 - \hat{\mathbb{E}}X_1(a_n)|}{a_n} &\le \frac{2^k}{a_{2^{k-1}}} \hat{\mathbb{E}}\left(|X_1|\left(1 - g\left(\frac{|X_1|}{a_{2^{k-1}}}\right)\right)\right) \\
&\le \frac{2^k}{a_{2^{k-1}}} \sum_{j=k-1}^{\infty} \hat{\mathbb{E}}\left(|X_1| g_j\left(\frac{|X_1|}{a_{2^j}}\right)\right) \\
&\ll \frac{2^{k-2}}{a_{2^{k-2}}} \sum_{j=k-1}^{\infty} a_{2^j} \mathbb{V}\left(|X_1| > \mu a_{2^{j-1}}\right) \\
&\ll \sum_{j=k}^{\infty} \frac{2^j}{a_{2^j}} a_{2^j} \mathbb{V}\left(|X_1| > \mu a_{2^j}\right) \\
&\to 0, \quad \text{as} \quad k \to \infty.
\end{aligned}$$

Thus, (3.17) holds.

(ii) Suppose that (3.5) holds. For any $M > 0$, let

$$\xi_j = 1 - g\left(\frac{|X_j|}{M a_j}\right), \quad \eta_n = \sum_{j=1}^{n} \xi_j \quad \text{and} \quad b_n = \sum_{j=1}^{n} \hat{\mathbb{E}}\xi_j.$$

By $1 - g(x) \in C_{l,Lip}$, so $\{\xi_j; j \ge 1\}$ is also extended independent. Since $1 - g(|x|) \ge I(|x| > 1)$, by (2.1) and (3.5),

$$\begin{aligned}
\sum_{j=1}^{\infty} \hat{\mathbb{E}}\xi_j &= \sum_{j=1}^{\infty} \hat{\mathbb{E}}\left(1 - g\left(\frac{|X_j|}{M a_j}\right)\right) = \sum_{j=1}^{\infty} \hat{\mathbb{E}}\left(1 - g\left(\frac{|X_1|}{M a_j}\right)\right) \\
&\ge \sum_{j=1}^{\infty} \mathbb{V}(|X_1| > M a_j) = \infty, \qquad \forall M > 0.
\end{aligned}$$

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This implies $b_n \to \infty$ as $n \to \infty$. For any $0 < \delta < \epsilon < 1$ and $t > 0$, we get

$$\begin{aligned}
I\left(\frac{\eta_n - b_n}{b_n} \geq -\epsilon\right) &\geq I\left(-\epsilon \leq \frac{\eta_n - b_n}{b_n} < \delta\right) \\
&\geq \mathrm{e}^{-t\delta} \exp\left\{t\frac{\eta_n - b_n}{b_n}\right\} I\left(-\epsilon \leq \frac{\eta_n - b_n}{b_n} < \delta\right) \\
&= \begin{cases} 0 \geq \mathrm{e}^{-t\delta}\left(\exp\left\{t\frac{\eta_n - b_n}{b_n}\right\} - \mathrm{e}^{-t\epsilon}\right), & \frac{\eta_n - b_n}{b_n} < -\epsilon, \\ \mathrm{e}^{-t\delta}\exp\left\{t\frac{\eta_n - b_n}{b_n}\right\} \geq \mathrm{e}^{-t\delta}\left(\exp\left\{t\frac{\eta_n - b_n}{b_n}\right\} - \mathrm{e}^{-t\epsilon}\right), & -\epsilon \leq \frac{\eta_n - b_n}{b_n} < \delta \end{cases} \\
&\geq \mathrm{e}^{-t\delta}\left(\exp\left\{t\frac{\eta_n - b_n}{b_n}\right\} - \mathrm{e}^{-t\epsilon}\right) I\left(\frac{\eta_n - b_n}{b_n} < \delta\right) \\
&\geq \mathrm{e}^{-t\delta}\left(\exp\left\{t\frac{\eta_n - b_n}{b_n}\right\} - \mathrm{e}^{-t\epsilon}\right) - \mathrm{e}^{-t\delta - t}\exp\left\{t\frac{\eta_n}{b_n}\right\} I(\frac{\eta_n - b_n}{b_n} \geq \delta) \\
&\geq \mathrm{e}^{-t\delta}\left(\exp\left\{t\frac{\eta_n - b_n}{b_n}\right\} - \mathrm{e}^{-t\epsilon}\right) - \mathrm{e}^{-t\delta - t}\exp\left\{t\frac{\eta_n}{b_n}\right\}\left(1 - g\left(\frac{\eta_n - b_n}{\delta b_n}\right)\right).
\end{aligned}$$

So, using $\hat{\mathbb{E}}(X - Y) \geq \hat{\mathbb{E}}X - \hat{\mathbb{E}}Y$,

$$\begin{aligned}
\mathbb{V}\left(\frac{\eta_n - b_n}{b_n} \geq -\epsilon\right) \geq \mathrm{e}^{-t\delta}&\left(\hat{\mathbb{E}}\exp\left\{t\frac{\eta_n - b_n}{b_n}\right\} - \mathrm{e}^{-t\epsilon}\right) \\
&-\mathrm{e}^{-t\delta - t}\hat{\mathbb{E}}\left(\exp\left\{t\frac{\eta_n}{b_n}\right\}\left(1 - g\left(\frac{\eta_n - b_n}{\delta b_n}\right)\right)\right).
\end{aligned} \tag{3.18}$$

By the extended independence of $\{\xi_i; i \geq 1\}$ and the fact that $\mathrm{e}^x \geq 1 + x$,

$$\hat{\mathbb{E}}\left(\exp\left\{t\frac{\eta_n - b_n}{b_n}\right\}\right) = \prod_{j=1}^{n}\hat{\mathbb{E}}\left(\exp\left\{t\frac{\xi_j - \hat{\mathbb{E}}\xi_j}{b_n}\right\}\right) \geq \prod_{j=1}^{n}\hat{\mathbb{E}}\left(t\frac{\xi_j - \hat{\mathbb{E}}\xi_j}{b_n} + 1\right) = 1. \tag{3.19}$$

On the other hand, by noting $\mathrm{e}^x \leq 1 + |x|\mathrm{e}^{|x|}$, $\mathrm{e}^x \geq 1 + x$ and $0 \leq \xi_j \leq 1$,

$$\begin{aligned}
\hat{\mathbb{E}}\left(\exp\left\{2t\frac{\eta_n}{b_n}\right\}\right) &= \prod_{j=1}^{n}\hat{\mathbb{E}}\left(\exp\left\{2t\frac{\xi_j}{b_n}\right\}\right) \leq \prod_{j=1}^{n}\hat{\mathbb{E}}\left(1 + t\frac{2\xi_j}{b_n}\mathrm{e}^{2t/b_n}\right) \\
&= \prod_{j=1}^{n}\left(1 + 2t\frac{\hat{\mathbb{E}}\xi_j}{b_n}\mathrm{e}^{2t/b_n}\right) \leq \prod_{j=1}^{n}\exp\left(2t\frac{\hat{\mathbb{E}}\xi_j}{b_n}\mathrm{e}^{2t/b_n}\right) \\
&= \exp\left(2t\frac{\sum_{j=1}^{n}\hat{\mathbb{E}}\xi_j}{b_n}\mathrm{e}^{2t/b_n}\right) = \exp\left(2t\mathrm{e}^{2t/b_n}\right).
\end{aligned} \tag{3.20}$$

Also, by (2.3) in Lemma 2.6, $b_n \to \infty$ and $\hat{\mathbb{E}}(\xi_j - \hat{\mathbb{E}}\xi_j)^2 \leq 2\hat{\mathbb{E}}(\xi_j)$,

$$\begin{aligned}
\mathbb{V}\left(\frac{\eta_n - b_n}{b_n} \geq \mu\delta\right) &= \mathbb{V}\left(\sum_{j=1}^{n}(\xi_j - \hat{\mathbb{E}}\xi_j) \geq \mu\delta\right) \\
&\ll \frac{\sum_{j=1}^{n}\hat{\mathbb{E}}(\xi_j - \hat{\mathbb{E}}\xi_j)^2}{b_n^2} \ll \frac{\sum_{j=1}^{n}\hat{\mathbb{E}}\xi_j}{b_n^2} \\
&= \frac{1}{b_n} \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

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Thus, by the Hölder inequality and $I(x>1) \le 1-g(x) \le I(x>\mu)$, it follows that

$$\begin{aligned}\hat{\mathbb{E}}\left[\exp\left\{t\frac{\eta_n}{b_n}\right\}\left(1-g\left(\frac{\eta_n-b_n}{\delta b_n}\right)\right)\right] &\le \left\{\hat{\mathbb{E}}\left[\exp\left\{2t\frac{\eta_n}{b_n}\right\}\right]\hat{\mathbb{E}}\left[1-g\left(\frac{\eta_n-b_n}{\delta b_n}\right)\right]^2\right\}^{1/2}\\ &\le \exp\left(t\mathrm{e}^{2t/b_n}\right)\left(\mathbb{V}\left(\frac{\eta_n-b_n}{b_n}\ge\mu\delta\right)\right)^{1/2}\\ &\le \exp\left(t\mathrm{e}^{2t/b_n}\right)\frac{1}{b_n^{1/2}}\to 0. \end{aligned} \tag{3.21}$$

Hence, substitute (3.19)–(3.21) in (3.18), we have

$$\liminf_{n\to\infty}\mathbb{V}\left(\frac{\eta_n-b_n}{b_n}\ge-\epsilon\right)\ge \mathrm{e}^{-t\delta}(1-\mathrm{e}^{-t\epsilon}).$$

Letting $\delta\to 0$ and then $t\to\infty$, we get

$$\lim_{n\to\infty}\mathbb{V}\left(\frac{\eta_n-b_n}{b_n}\ge-\epsilon\right)=1.$$

Now, choose $\epsilon=1-\mu>0$. Noting that

$$\left(\frac{\eta_n}{b_n}>\mu;\text{i.o.}\right)\subset\left(\sum_{j=1}^{\infty}\xi_j=\infty\right)\subset(\xi_j\ne 0;\text{i.o.})\subset\left(\frac{|X_n|}{Ma_n}>\mu;\text{i.o.}\right)$$

and the continuity of $\mathbb{V}$, we get

$$\begin{aligned}\mathbb{V}\left(\limsup_{n\to\infty}\frac{|X_n|}{a_n}>\mu M\right)&=\mathbb{V}\left(\frac{|X_n|}{Ma_n}>\mu;\text{i.o.}\right)\ge\mathbb{V}\left(\sum_{j=1}^{\infty}\left(1-g\left(\frac{|X_j|}{Ma_j}\right)\right)=\infty\right)\\ &\ge\mathbb{V}\left(\frac{\eta_n}{b_n}>\mu;\text{i.o.}\right)=\mathbb{V}\left(\frac{\eta_n-b_n}{b_n}>-(1-\mu);\text{i.o.}\right)\\ &\ge\limsup_{n\to\infty}\mathbb{V}\left(\frac{\eta_n-b_n}{b_n}>-(1-\mu)\right)=1.\end{aligned}$$

Hence, for any $\{d_n; n\ge 1\}$ which satisfy (3.7),

$$\begin{aligned}\limsup_{n\to\infty}\frac{|X_n|}{a_n}&=\limsup_{n\to\infty}\frac{|(S_n-d_n)-(S_{n-1}-d_{n-1})+(d_n-d_{n-1})|}{a_n}\\ &\le 2\limsup_{n\to\infty}\frac{|S_n-d_n|}{a_n}+\limsup_{n\to\infty}\frac{|d_n-d_{n-1}|}{a_n}\\ &=2\limsup_{n\to\infty}\frac{|S_n-d_n|}{a_n}+c.\end{aligned}$$

From the arbitrariness of M, it follows that

$$\mathbb{V}\left(\limsup_{n\to\infty}\frac{|S_n-d_n|}{a_n}>M_1\right)=1,\qquad\forall M_1>0.$$

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Therefore,

$$\mathbb{V}\left(\limsup_{n\to\infty}\frac{|S_n-d_n|}{a_n}=\infty\right)=\lim_{M_1\to\infty}\mathbb{V}\left(\limsup_{n\to\infty}\frac{|S_n-d_n|}{a_n}>M_1\right)=1.$$

That is

$$\limsup_{n\to\infty}\frac{|S_n-d_n|}{a_n}=\infty \quad \text{a.s.} \quad \nu$$

Thus (3.6) is proved.

Proof of Theorem 3.6. Note that

$$C_{\mathbb{V}}(|X|^p)=\int_0^\infty \mathbb{V}(|X|>x^{1/p})\mathrm{d}x<\infty \Leftrightarrow \sum_{n=1}^\infty \mathbb{V}(|X|>n^{1/p})<\infty,$$

and for $p\geq 1$, $\hat{\mathbb{E}}(|X|)\leq C_{\mathbb{V}}(|X|)<\infty$ from $\hat{\mathbb{E}}$ being countably sub-additive and Lemma 4.5 in Zhang (2016 [24]). Therefore, by Theorem 3.3, we can prove Theorem 3.6.

4. Chover's law of the iterated logarithm

For traditional probability space, Chover (1966 [6]) established the following strong limit theorem which is refereed as the classical Chover LIL for sequence of independent and identically distributed random variables with a stable distribution for an exponent in $(0,2)$:

$$\limsup_{n\to\infty}\left|\frac{S_n}{n^{1/\alpha}}\right|^{\frac{1}{\log\log n}}=\mathrm{e}^{\frac{1}{\alpha}} \quad \text{a.s.}$$

Some Chover's LIL type results obtained by Mikosch (1984 [12]). Under general assumption, Qi and Cheng (1996 [18]) proved that there exist some constants $A_n\in\mathbb{R}$, $B_n>0$ such that

$$\limsup_{n\to\infty}\left|\frac{S_n-A_n}{B_n}\right|^{\frac{1}{\log\log n}}=\mathrm{e}^{\frac{1}{\alpha}} \quad \text{a.s.} \tag{4.1}$$

Vasudeva (1984 [20]) also obtained a similar result. Some papers have been devoted to the study of Chover's LIL for dependent sequence. We refer the reader to Chen (2005 [2]), Cai (2006 [1]), Wu and Jiang (2010 [21], 2010 [22]).

In this section we studied and extended the Chover's law of the iterated logarithm from sequences of probability space to sequences of the sub-linear expectations. The corresponding results for traditional probability space by obtained Mikosch (1984 [12]), Vasudeva (1984 [20]), Qi and Cheng (1996 [18]), and Wu and Jiang (2010 [21], 2010 [22]) are extended or improved.

A sequence of identically distributed random variables $\{X_n; n\geq 1\}$ is called to be a stable distribution with exponent $\alpha\in(0,2)$, if

$$\mathbb{V}(|X_1|>x)=\frac{c(x)l(x)}{x^\alpha} \quad \text{for} \quad x>0, \tag{4.2}$$

where, $c(x)\geq 0$, $\lim_{x\to\infty}c(x)=c>0$, and $l(x)\geq 0$ is a slowly varying function, i.e.,

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$$\lim_{t\to\infty}\frac{l(tx)}{l(t)}=1 \quad \text{for} \quad x>0.$$

From Seneta (1976 [19]), $l(x)$ is a slowly varying function if and only if

$$l(x)=c_1(x)\exp\left\{\int_1^x\frac{f(u)}{u}\mathrm{d}u\right\},x>0,$$

where $c_1(x)\geq 0$, $\lim\limits_{x\to\infty}c_1(x)=c_1>0$, and $\lim\limits_{x\to\infty}f(x)=0$.

We will prove that (4.1) is extended as follows

$$\limsup_{n\to\infty}\left|\frac{S_n-A_n}{B_n}\right|^{\frac{1}{\lg_{k+2}n}}=\mathrm{e}^{\frac{1}{\alpha}} \quad \text{a.s.} \quad \nu$$

under the sub-linear expectations.

Set $\lg_0 x=x$ and denote $\lg_j x=\ln\{\max(\mathrm{e},\lg_{j-1}x)\}$ for $j\geq 1$, $\lg x=\lg_1 x$. We also assume $k\geq 0$ is fixed integer. For convenience, the product $\prod\limits_{j=1}^{k}(\cdot)$ is defined as 1 if $k<1$. $A\sim B$ denotes $A/B\to 1$.

Set $G(x)=\mathbb{V}(|X_1|\geq x)$, and define

$$B(x)=\inf\{y;G(y)\leq 1/x\} \quad \text{for} \quad x>0.$$

From (4.2), $G(x)$ is a regularly varying function with index $-\alpha$ at infinity. Hence, from De Haan (1970 [7]), $B(x)$ is a regularly varying function with index $1/\alpha$ at infinity, and by Karamata's representation,

$$B(x)=x^{1/\alpha}c(x)\exp\left\{\int_1^x\frac{b_1(u)}{u}\mathrm{d}u\right\}\hat{=}x^{1/\alpha}l_1(x), \tag{4.3}$$

where $l_1(x)$ is a slowly varying function, $\lim\limits_{x\to\infty}c(x)=c\in(0,\infty)$ and $\lim\limits_{x\to\infty}b_1(x)=0$.

Theorem 4.1. *Assume that $\{X_n;n\geq 1\}$ is a sequence of extended independent and identically distributed random variables, (4.2) holds with some $\alpha\in(0,2)$. For $\alpha\geq 1$, suppose that $\hat{\mathbb{E}}$ is countably sub-additive and $\hat{\mathbb{E}}X_1=\hat{\varepsilon}X_1$, for $\alpha=1$, further suppose that $f_nl_1(n)\downarrow$. Let $\{f_n\}$ be a sequence of positive nondecreasing numbers. Set*

$$\limsup_{n\to\infty}\frac{|S_n-c_n|}{B(nf_n)}\begin{cases}=0 & a.s. \quad \mathbb{V},\\ =\infty & a.s. \quad \nu\end{cases} \tag{4.4}$$

where $c_n=0$ for $0<\alpha<1$, and $c_n=n\hat{\mathbb{E}}X_1$ for $1\leq\alpha<2$.

$$\sum_{n=1}^{\infty}\mathbb{V}\left(|X_1|>B(nf_n)\right)\begin{cases}<\infty,\\ =\infty.\end{cases} \tag{4.5}$$

$$\sum_{n=1}^{\infty}\frac{1}{nf_n}\begin{cases}<\infty,\\ =\infty.\end{cases} \tag{4.6}$$

Then (4.5) is equivalent to (4.6), and (4.5) implies (4.4).

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Theorem 4.2. *Under the conditions of Theorem 4.1,*

$$\limsup_{n\to\infty}\left(\frac{|S_n-c_n|}{B\left(n\prod_{j=1}^{k}\lg_j n\right)}\right)^{\frac{1}{\lg_{k+2} n}}\leq \mathrm{e}^{\frac{1}{\alpha}}\qquad a.s.\quad \mathbb{V}, \tag{4.7}$$

and

$$\limsup_{n\to\infty}\left(\frac{|S_n-c_n|}{B\left(n\prod_{j=1}^{k}\lg_j n\right)}\right)^{\frac{1}{\lg_{k+2} n}}\geq \mathrm{e}^{\frac{1}{\alpha}}\qquad a.s.\quad \nu. \tag{4.8}$$

This implies

$$\limsup_{n\to\infty}\left(\frac{|S_n-c_n|}{B\left(n\prod_{j=1}^{k}\lg_j n\right)}\right)^{\frac{1}{\lg_{k+2} n}}= \mathrm{e}^{\frac{1}{\alpha}}\qquad a.s.\quad \nu,$$

where c_n is defined by Theorem 4.1.

Proof of Theorem 4.1. Firstly, we prove that (4.5) $\Leftrightarrow$ (4.6). Using the properties of regular variation (see Seneta (1976 [19])), we have

$$G(B(x))\sim x^{-1},\ x\to\infty. \tag{4.9}$$

From (4.2) and (4.9),

$$\mathbb{V}(|X_1|>B(nf_n))=G(B(nf_n))\sim\frac{1}{nf_n}.$$

Hence, (4.5) is equivalent (4.6).

Now, we prove that (4.5)$\Rightarrow$(4.4).

Set now $l_2(x)=c\exp\left\{\int_1^x\frac{b_1(u)}{u}\mathrm{d}u\right\}$, and $b(x)=x^{1/\alpha}l_2(x)$. By (4.4), it is easy to see that

$$B(x)\sim b(x),\quad x\to\infty.$$

Therefore, $B(nf_n)$ in (4.4)$\sim$(4.5) can be replaced by $b(nf_n)$ and that for any given $r\in(\alpha,2)$, there exists an $x_r>0$ such that $b(x)/x^{1/r}$ is increasing in (x_r,∞). By Lemma 2.7, without loss of generality, assume f_n is slowly varying and $\sup_{n\geq1}\frac{f_{2n}}{f_n}<\infty$. Thus, let $a_n=b(nf_n)$, we have

$$\frac{a_n}{n^{1/r}}=\frac{b(nf_n)}{n^{1/r}}=\frac{b(nf_n)}{(nf_n)^{1/r}}f_n^{1/r}\uparrow,$$

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and

$$\sup_{n\geq 1}\frac{a_{2n}}{a_n}=\sup_{n\geq 1}\frac{b(2nf_{2n})}{b(nf_n)}=\sup_{n\geq 1}\frac{(2nf_{2n})^{1/r}l_2(2nf_{2n})}{(nf_n)^{1/r}l_2(nf_n)}\ll\sup_{n\geq 1}\left(\frac{f_{2n}}{f_n}\right)^{1/r}\frac{l_2(nf_{2n})}{l_2(nf_n)}<\infty.$$

Therefore the conditions of Theorem 3.3 are satisfied by the sequence $\{a_n=b(nf_n)\}$.

For $0<\alpha<1$, we have $\frac{a_n}{n}\sim n^{1/\alpha-1}f_n^{1/\alpha}l_1(n)\to\infty$ and $\frac{a_n}{n}$ is quasi-nondecreasing from the properties of slowly varying functions. Without loss of generality, we can suppose $\frac{a_n}{n}\uparrow\infty$.

For $\alpha>1$, $\frac{n}{a_n}\sim\frac{n^{1/\alpha-1}}{f_n^{1/\alpha}l_1(n)}$ is quasi-nondecreasing from the properties of slowly varying functions. Without loss of generality, we can suppose $\frac{n}{a_n}\uparrow$.

For $\alpha=1$, by assumption, $\frac{n}{a_n}\sim\frac{1}{f_nl_1(n)}\uparrow$.

Therefore, from Theorem 3.3 and assumptions of the Theorem 4.1, we obtain that (4.5) implies (4.4).

Proof of Theorem 4.2. For $\forall\varepsilon>0$, let $f_n=\prod_{j=1}^{k}\lg_j n\lg_{k+1}^{1+\varepsilon}n$. By (4.2) and (4.9), we obtain

$$\mathbb{V}\Big(|X_1|>B(nf_n)\Big)=G\big(B(nf_n)\big)\sim\frac{1}{nf_n}=\frac{1}{n\prod_{j=1}^{k}\lg_j n\lg_{k+1}^{1+\varepsilon}n}.$$

Therefore

$$\sum_{n=1}^{\infty}\mathbb{V}\big(|X_1|>B(nf_n)\big)<\infty.$$

Applying Theorem 4.1, we get

$$\limsup_{n\to\infty}\frac{|S_n-c_n|}{B(nf_n)}=0\qquad\text{a.s.}\quad\mathbb{V}.$$

Thus, let $\delta=\dfrac{\varepsilon}{(1+\varepsilon)\alpha}>0$ in Lemma 2.4, for sufficiently large n, by combining (4.3) we have

$$\begin{aligned}\frac{|S_n-c_n|}{B\Big(n\prod_{j=1}^{k}\lg_j n\Big)}&=\frac{|S_n-c_n|}{B(nf_n)}\frac{B\Big(n\prod_{j=1}^{k}\lg_j n\lg_{k+1}^{1+\varepsilon}n\Big)}{B\Big(n\prod_{j=1}^{k}\lg_j n\Big)}\leq\frac{B\Big(n\prod_{j=1}^{k}\lg_j n\lg_{k+1}^{1+\varepsilon}n\Big)}{B\Big(n\prod_{j=1}^{k}\lg_j n\Big)}\\&=\frac{\Big(n\prod_{j=1}^{k}\lg_j n\lg_{k+1}^{1+\varepsilon}n\Big)^{1/\alpha}l_1\Big(n\prod_{j=1}^{k}\lg_j n\lg_{k+1}^{1+\varepsilon}n\Big)}{\Big(n\prod_{j=1}^{k}\lg_j n\Big)^{1/\alpha}l_1\Big(n\prod_{j=1}^{k}\lg_j n\Big)}\\&\leq\Big(\lg_{k+1}n\Big)^{(1+\varepsilon)/\alpha}\lg_{k+1}^{(1+\varepsilon)\delta}n=\big(\lg_{k+1}n\big)^{(1+2\varepsilon)/\alpha}\qquad\text{a.s.}\quad\mathbb{V}.\end{aligned}$$

Therefore

$$\left(\frac{|S_n-c_n|}{B\Big(n\prod_{j=1}^{k}\lg_j n\Big)}\right)^{\frac{1}{\lg_{k+2}n}}\leq\Big(\lg_{k+1}n\Big)^{\frac{1+2\varepsilon}{\alpha\lg_{k+2}n}}=\mathrm{e}^{\frac{1+2\varepsilon}{\alpha}}\qquad\text{a.s.}\quad\mathbb{V},$$

i.e.

$$\limsup_{n\to\infty}\left(\frac{|S_n-c_n|}{B\Big(n\prod_{j=1}^{k}\lg_j n\Big)}\right)^{\frac{1}{\lg_{k+2} n}}\le \mathrm{e}^{\frac{1+2\varepsilon}{\alpha}}\quad \text{a.s.}\quad \mathbb{V} \tag{4.10}$$

Similarly, for $\varepsilon\in(0,1/2)$, let $f_n=\prod_{j=1}^{k}\lg_j n\lg_{k+1}^{1-\varepsilon} n$, then

$$\sum_{n=1}^{\infty}\mathbb{V}\Big(|X_1|>B(nf_n)\Big)=\infty$$

from $\mathbb{V}\Big(|X_1|>B(nf_n)\Big)\sim\dfrac{1}{n\prod_{j=1}^{k}\lg_j n\lg_{k+1}^{1-\varepsilon} n}$.

Applying Theorem 4.1, we have

$$\limsup_{n\to\infty}\frac{|S_n-c_n|}{B(nf_n)}=\infty\qquad \text{a.s.}\quad \nu$$

Thus, there exist infinite indices n such that

$$\frac{|S_n-c_n|}{B(nf_n)}\ge 1\qquad \text{a.s.}\quad \nu.$$

Let $\delta=\dfrac{\varepsilon}{(1-\varepsilon)\alpha}>0$ in Lemma 2.4, for these n, combing (4.3), we obtain

$$\begin{aligned}\frac{|S_n-c_n|}{B\Big(n\prod_{j=1}^{k}\lg_j n\Big)}&=\frac{|S_n-c_n|}{B(nf_n)}\frac{B\Big(n\prod_{j=1}^{k}\lg_j n\lg_{k+1}^{1-\varepsilon} n\Big)}{B\Big(n\prod_{j=1}^{k}\lg_j n\Big)}\\&\ge\frac{\Big(n\prod_{j=1}^{k}\lg_j n\lg_{k+1}^{1-\varepsilon} n\Big)^{1/\alpha}l_1\Big(n\prod_{j=1}^{k}\lg_j n\lg_{k+1}^{1-\varepsilon} n\Big)}{\Big(n\prod_{j=1}^{k}\lg_j n\Big)^{1/\alpha}l_1\Big(n\prod_{j=1}^{k}\lg_j n\Big)}\\&\ge\Big(\lg_{k+1} n\Big)^{(1-\varepsilon)/\alpha}\lg_{k+1}^{-(1-\varepsilon)\delta} n=\Big(\lg_{k+1} n\Big)^{\frac{1-2\varepsilon}{\alpha}}\qquad \text{a.s.}\quad \nu\end{aligned}$$

Therefore

$$\limsup_{n\to\infty}\left(\frac{|S_n-c_n|}{B\Big(n\prod_{j=1}^{k}\lg_j n\Big)}\right)^{\frac{1}{\lg_{k+2} n}}\ge \mathrm{e}^{\frac{1-2\varepsilon}{\alpha}}\qquad \text{a.s.}\quad \nu \tag{4.11}$$

Since $\varepsilon\in(0,1/2)$ is arbitrary, we get (4.7) and (4.8) from (4.10) and (4.11).

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Author contributions

QW conceived of the study, drafted, complete the final manuscript.
YJ conceived of the study, complete, read and approved the final manuscript.

Acknowledgments

We are very grateful to the referees and the editors for their valuable comments and some helpful suggestions that improved the clarity and readability of the paper.

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