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Global Carleman estimates for the linear stochastic Kuramoto–Sivashinsky equations and their applications ☆

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A B S T R A C T

In this paper, we establish two new global Carleman estimates for the linear stochastic Kuramoto–Sivashinsky equations. The first one is for the backward linear stochastic Kuramoto–Sivashinsky equation. Based on this estimate and the duality argument, we obtain the null controllability of the forward linear stochastic Kuramoto–Sivashinsky equation by three controls, one is an internal control in the diffusion term and the others are boundary controls. The second one is for the forward linear stochastic Kuramoto–Sivashinsky equation. According to this estimate, we obtain a unique continuation property for the forward linear stochastic Kuramoto–Sivashinsky equation.

1. Introduction

The stochastic Kuramoto–Sivashinsky (KS) equation appears in the study of dynamic roughening in sputter-eroded surfaces and, in principle, in any physical system modeled by the deterministic KS equation in which the relevance of time-dependent noise as, e.g., fluctuations in a flux or thermal fluctuations, can be argued for. In [7] the early and late time dynamics of the erosion model were numerically studied with the conclusion that they are the same as those obtained from the stochastic KS equation [15]. In [17], the authors confirmed this result by showing analytically that the stochastic KS equation yields the continuum description of the erosion model.

In this paper, we establish new global Carleman estimates for the backward linear stochastic KS equation

$$\begin{cases} dz - (kz_{xx} + z_{xxxx})dt = (pz + qZ + h)dt + Zdw & \text{in } Q, \\ z(0,t) = 0 = z(1,t) & \text{in } (0,T), \\ z_x(0,t) = 0 = z_x(1,t) & \text{in } (0,T), \\ z(x,T) = z_T(x) & \text{in } I \end{cases} \tag{1.1}$$

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and the forward linear stochastic KS equation

$$\begin{cases} dz + (kz_{xx} + z_{xxxx})dt = (pz + f)dt + (qz + g)dw & \text{in } Q, \\ z(0,t) = 0 = z(1,t) & \text{in } (0,T), \\ z_x(0,t) = 0 = z_x(1,t) & \text{in } (0,T), \\ z(x,0) = z_0(x) & \text{in } I \end{cases} \tag{1.2}$$

with suitable coefficients p and q. In (1.1) and (1.2), $I = (0,1), T > 0, Q = I \times (0,T), k > 0$ is the antidiffusion parameter.

In recent years, many efforts have been devoted to studying the Carleman estimates for stochastic partial differential equations:

- stochastic transport equations [24],
- stochastic heat equation [2,29,21,20],
- stochastic wave equation [31],
- stochastic KdV equation [9],
- stochastic Kuramoto–Sivashinsky equations [14],
- stochastic Schrödinger equation [23],
- stochastic fourth order Schrödinger equations [13]
- stochastic Kawahara equation [11]
- $\cdots\cdots$

Through this paper, we make the following assumptions:

(H1) We denote by $L^2(I)$ the space of all Lebesgue square integrable functions on I. The inner product on $L^2(I)$ is

$$(u,v)_{L^2(I)} = \int_I uv dx,$$

for any $u, v \in L^2(I)$.

$H^s(I)(s \geq 0)$ are the classical Sobolev spaces of functions on I. The definition of $H^s(I)$ can be found in [18].

(H2) Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{w(t)\}_{t\geq 0}$ is defined such that $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by $w(\cdot)$, augmented by all the P-null sets in $\mathcal{F}$. Let H be a Banach space, and let $C([0,T];H)$ be the Banach space of all H-valued strongly continuous functions defined on $[0,T]$. We denote by $L^2_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$ such that $E(\|X(\cdot)\|^2_{L^2(0,T;H)}) < \infty$, with the canonical norm; by $L^\infty_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted bounded processes; and by $L^2_{\mathcal{F}}(\Omega; C([0,T];H))$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted continuous processes $X(\cdot)$ such that $E(\|X(\cdot)\|^2_{C([0,T];H)}) < \infty$, with the canonical norm.

(H3) Unless otherwise stated, C stands for a generic positive constant whose value can change from line to line. If it is essential, the dependence of a constant C on some parameters, say "$\cdot$", will be written by $C(\cdot)$.

(H4) Let $\psi(x) = (x - x_0)^2 + \delta_0$, where δ_0 is a positive constant such that $\psi \geq \frac{3}{4}\|\psi\|_{L^\infty(I)}$ and $x_0 > 1$. For any given positive constants λ and μ, we set $\rho(x,t) = \frac{e^{\mu\psi(x)} - e^{\frac{3}{2}\|\psi\|_{L^\infty(I)}\mu}}{t(T-t)}$, $l = \lambda\rho$, $\theta = e^l$ and $\varphi(x,t) = \frac{e^{\mu\psi(x)}}{t(T-t)}, \forall (x,t) \in Q$.

(H5) $p, q, a, b \in L^\infty_{\mathcal{F}}(0,T;L^\infty(I))$.

1.1. Global Carleman estimate for the backward linear stochastic Kuramoto–Sivashinsky equation and its application

The first result in this paper is the following new global Carleman estimate:

Theorem 1.1. *There are two positive constants λ_0 and C such that for all $(z, Z) \in (L^2_{\mathcal{F}}(\Omega; C([0,T]; H^2_0(I))) \cap L^2_{\mathcal{F}}(0,T; H^4(I))) \times L^2_{\mathcal{F}}(0,T; H^2_0(I))$, $h \in L^2_{\mathcal{F}}(0,T; L^2(I))$, satisfying (1.1) and all*

$$\lambda \geq \lambda_0,$$

it holds that

$$\begin{aligned}
&E\int_Q [\lambda\varphi\theta^2 z_{xxx}^2 + \lambda^3\varphi^3\theta^2 z_{xx}^2 + \lambda^5\varphi^5\theta^2 z_x^2 + \lambda^7\varphi^7\theta^2 z^2]dxdt \\
&\leq C\Big[E\int_0^T (\lambda^3\varphi^3(0,t)\theta^2(0,t)z_{xx}^2(0,t) + \lambda\varphi(0,t)\theta^2(0,t)z_{xxx}^2(0,t))dt \\
&\quad + E\int_Q (\theta^2 h^2 + \lambda^4\varphi^4\theta^2 Z^2)dxdt\Big].
\end{aligned} \tag{1.3}$$

Remark 1.1. The well-posedness of (1.1) is established in Proposition 2.1.

Based on the above results, we can consider the null controllability of the following forward linear stochastic KS equation

$$\begin{cases}
dy + (ky_{xx} + y_{xxxx})dt = aydt + (by + g)dw & \text{in } Q, \\
y(0,t) = u_1(t), y(1,t) = 0 & \text{in } (0,T), \\
y_x(0,t) = u_2(t), y_x(1,t) = 0 & \text{in } (0,T), \\
y(x,0) = y_0(x) & \text{in } I
\end{cases} \tag{1.4}$$

where y_0 is the initial state, y is a state variable and the control variable consists of the pair (u_1, u_2, g).

The control problem of deterministic KS equation has been studied in several papers [5,10,1,6,4,26,12,27]. However, there are very few results for stochastic KS equation. To our best knowledge, the only known result is due to [14], [14] obtained the internal null controllability of forward and backward linear stochastic KS equations.

By means of the duality argument, we can obtain the following null controllability result for system (1.4).

Corollary 1.1. *System (1.4) is null controllable. Namely, for any given $y_0 \in H^{-2}(I)$, one can find controls $(u_1, u_2, g) \in L^2_{\mathcal{F}}(\Omega; L^2(0,T)) \times L^2_{\mathcal{F}}(\Omega; L^2(0,T)) \times L^2_{\mathcal{F}}(0,T; L^2(I))$ such that the solution of (1.4) satisfying $y(T) = 0$ in I, P-a.s.*

Further, we have the lack of null controllability for (1.4) with only one control g.

Proposition 1.1. *If $u_1 = u_2 \equiv 0$ in (1.4), then system (1.4) is not null controllable at any time T.*

1.2. Global Carleman estimate for the forward linear stochastic Kuramoto–Sivashinsky equation and its application

For (1.2), we have the following global Carleman estimate.

Theorem 1.2. *There are two positive constants λ_0 and C such that for all $y \in (L^2_{\mathcal{F}}(\Omega; C([0,T]; H^2_0(I))) \cap L^2_{\mathcal{F}}(0,T; H^4(I)))$, $f \in L^2_{\mathcal{F}}(0,T; L^2(I)), g \in L^2_{\mathcal{F}}(0,T; L^2(I))$ satisfying (1.2) and all*

$$\lambda \geq \lambda_0,$$

it holds that

$$\begin{aligned} &E\int_Q [\lambda\varphi\theta^2 z_{xxx}^2 + \lambda^3\varphi^3\theta^2 z_{xx}^2 + \lambda^5\varphi^5\theta^2 z_x^2 + \lambda^7\varphi^7\theta^2 z^2]dxdt \\ &\leq C\Big[E\int_0^T (\lambda^3\varphi^3(0,t)\theta^2(0,t)z_{xx}^2(0,t) + \lambda\varphi(0,t)\theta^2(0,t)z_{xxx}^2(0,t))dt + E\int_Q (\theta^2 f^2 + \lambda^4\mu^4\varphi^4\theta^2 g^2)dxdt\Big]. \end{aligned} \tag{1.5}$$

Remark 1.2. The well-posedness of (1.2) is established in Proposition 2.2.

Applying Theorem 1.2, we can obtain the following unique continuation property:

Corollary 1.2. *Let z solve*

$$\begin{cases} dz + (kz_{xx} + z_{xxxx})dt = azdt + bzdw & \text{in } Q, \\ z(0,t) = 0 = z(1,t) & \text{in } (0,T), \\ z_x(0,t) = 0 = z_x(1,t) & \text{in } (0,T), \\ z(x,0) = z_0(x) & \text{in } I \end{cases} \tag{1.6}$$

with the initial state $z_0 \in H^2_0(I)$. If

$$z_{xx}(0,t) = z_{xxx}(0,t) = 0 \ \text{ in } \ (0,T), \quad P-a.s.,$$

we have

$$z \equiv 0 \ \text{ in } \ Q, \quad P-a.s.$$

This paper is organized as follows. In Section 2, we establish the well-posedness of (1.1), (1.2) and (1.4). Section 3 is devoted to the proof of Theorem 1.1 and Theorem 1.2. In Section 4 we prove Corollary 1.1, Proposition 1.1 and Corollary 1.2.

2. Preliminaries

In this section we prove the well-posedness results we need along this paper.

Definition 2.1. A pair of stochastic processes (z, Z) is said to be a solution of (1.1) if

$$\begin{aligned} &(z,Z) \text{ is } L^2(I)\times L^2(I) - \text{valued and } \mathcal{F}_t - \text{measurable for each } t\in[0,T], \\ &(z,Z) \in (L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(I))) \cap L^2_{\mathcal{F}}(0,T; H^2_0(I))) \times L^2_{\mathcal{F}}(0,T; L^2(I)), \\ &z(T) = z_T \text{ in } I, P-a.s. \end{aligned}$$

and

$$\begin{aligned}\int_I z_T v dx = \int_I z(t) v dx + \int_t^T \int_I z_{xx}(s) v_{xx} dx ds \\ + \int_t^T \int_I (k z_{xx} + pz + qZ + h) v dx ds + \int_t^T \int_I Z(s) v dx dw\end{aligned} \tag{2.1}$$

holds for all $t \in [0, T]$ and all $v \in C_0^\infty(I)$, for almost all $\omega \in \Omega$.

Consider the one-dimensional fourth order elliptic operator Λ on $L^2(I)$ as follows

$$\begin{cases} \mathcal{D}(\Lambda) = H_0^2(I) \cap H^4(I), \\ \Lambda y = y_{xxxx} \qquad \forall y \in \mathcal{D}(\Lambda). \end{cases}$$

Let $\{\varphi_k\}_{k=1}^\infty$ be the corresponding eigenfunctions of Λ such that $\|\varphi_k\|_{L^2(I)} = 1$ $(k = 1, 2, 3, \cdots)$, which serves as an orthonormal basis of $L^2(I)$ (see [25, Theorem 8.94]), $\{\lambda_k\}_{k=1}^\infty$ be the corresponding eigenvalues of Λ such that $\Lambda\varphi_k = \lambda_k\varphi_k$ $(k = 1, 2, 3, \cdots)$.

Proposition 2.1. *If $p, q \in L_{\mathcal{F}}^\infty(0, T; L^\infty(I)), z_T \in L^2(\Omega, \mathcal{F}_T, P; H_0^2(I)), h \in L_{\mathcal{F}}^2(0, T; L^2(I))$ are given. (1.1) admits a unique solution (z, Z) such that*

$$\begin{aligned} & \|z\|_{L_{\mathcal{F}}^2(\Omega; C([0,T]; H_0^2(I))) \bigcap L_{\mathcal{F}}^2(0,T; H^4(I))} + \|Z\|_{L_{\mathcal{F}}^2(0,T; H_0^2(I))} \\ \leq \quad & C[\|z_T\|_{L^2(\Omega, \mathcal{F}_T, P; H_0^2(I))} + \|h\|_{L_{\mathcal{F}}^2(0,T; L^2(I))}], \end{aligned} \tag{2.2}$$

where $C = C(p, q, k, T)$. Moreover, for any $0 \leq t_1 \leq t_2 \leq T$, it holds that

$$E\|z(t_1)\|_{H_0^2(I)}^2 \leq C(E\|z(t_2)\|_{H_0^2(I)}^2 + E\int_{t_1}^{t_2} \|h(s)\|_{L^2(I)}^2 ds). \tag{2.3}$$

Proof. Inspired by [28, P1220] and [16], we use the Galerkin method.

Let

$$P_m : L^2(I) \to H_m$$

be the projection where $H_m = span\{\varphi_1, \varphi_2, \cdots, \varphi_m\}$.

We consider the system

$$\begin{cases} dc_i^m = (\sum_{j=1}^m a_{ij} c_j^m + \sum_{j=1}^m b_{ij} C_j^m + h_i) dt + C_i^m dw, \\ c_i^m(T) = (P_m z_T, \varphi_i)_{L^2(I)} \end{cases} \tag{2.4}$$

where

$$\begin{aligned} a_{ij} &= [(p\varphi_j, \varphi_i)_{L^2(I)} + (k\varphi_{jxx} + \varphi_{jxxxx}, \varphi_i)_{L^2(I)}], \\ b_{ij} &= (q\varphi_j, \varphi_i)_{L^2(I)}, \\ h_i &= (h, \varphi_i)_{L^2(I)} \end{aligned}$$

for $i, j = 1, \ldots, m$.

It follows from the classical theory of backward stochastic differential equations ([30, P355, Theorem 3.2]) and ([3, P119, Theorem 4.2]) that (2.4) admits a unique solution $(c_i^m(t), C_i^m(t))$, such that

$$E\sup_{0\leq t\leq T}|c_i^m(t)|^2+E\int_0^T|C_i^m(t)|^2dt\leq C(p,q,h,m,i). \tag{2.5}$$

Let us write

$$z^m=\sum_{i=1}^m c_i^m\varphi_i, Z^m=\sum_{i=1}^m C_i^m\varphi_i,$$

it follows from (2.4) that (z^m, Z^m) satisfies the following equations

$$\begin{aligned} d(z^m,\varphi_k)_{L^2(I)}-[(kz^m_{xx},\varphi_k)_{L^2(I)}+(z^m_{xxxx},\varphi_k)_{L^2(I)}]dt \\ =(pz^m+qZ^m+h^m,\varphi_k)_{L^2(I)}dt+(Z^m,\varphi_k)_{L^2(I)}dw. \end{aligned} \tag{2.6}$$

By Itô's rule, we have

$$\begin{aligned} &\|z^m_{xx}(t)\|^2_{L^2(I)}+\int_t^T[2\|z^m_{xxxx}\|^2_{L^2(I)}+\|Z^m_{xx}\|^2_{L^2(I)}]ds \\ &=\|(P_mz_T)_{xx}\|^2_{L^2(I)}-\int_t^T 2[(kz^m_{xx},z^m_{xxxx})_{L^2(I)} \\ &\quad+(pz^m,z^m_{xxxx})_{L^2(I)}+(qZ^m,z^m_{xxxx})_{L^2(I)}+(z^m_{xxxx},h^m)_{L^2(I)}]ds-2\int_t^T(z^m_{xxxx},Z^m)_{L^2(I)}dw. \end{aligned} \tag{2.7}$$

Let $E^{\mathcal{F}_r}X$ be the conditional expectation $E(X|\mathcal{F}_r)$.

By taking the conditional expectation in (2.7) for $0<r\leq t\leq T$, we get

$$\begin{aligned} &E^{\mathcal{F}_r}\|z^m_{xx}(t)\|^2_{L^2(I)}+E^{\mathcal{F}_r}\int_t^T[2\|z^m_{xxxx}\|^2_{L^2(I)}+\|Z^m_{xx}\|^2_{L^2(I)}]ds \\ &=E^{\mathcal{F}_r}\|(P_mz_T)_{xx}\|^2_{L^2(I)}-E^{\mathcal{F}_r}\int_t^T 2[(kz^m_{xx},z^m_{xxxx})_{L^2(I)} \\ &\quad+(pz^m,z^m_{xxxx})_{L^2(I)}+(qZ^m,z^m_{xxxx})_{L^2(I)}+(z^m_{xxxx},h^m)_{L^2(I)}]ds. \end{aligned} \tag{2.8}$$

It follows from the Hölder inequality, Poincaré's inequality, Cauchy inequality and (2.8) that

$$\begin{aligned} &E^{\mathcal{F}_r}\|z^m_{xx}(t)\|^2_{L^2(I)}+E^{\mathcal{F}_r}\int_t^T[2\|z^m_{xxxx}\|^2_{L^2(I)}+\|Z^m_{xx}\|^2_{L^2(I)}]ds \\ &\leq E^{\mathcal{F}_r}\|(P_mz_T)_{xx}\|^2_{L^2(I)}+C(p,q,k,\varepsilon)E^{\mathcal{F}_r}\int_t^T(\|z^m_{xx}\|^2_{L^2(I)}+\|h^m\|^2_{L^2(I)}+\|Z^m\|^2_{L^2(I)})ds \\ &\quad+\varepsilon E^{\mathcal{F}_r}\int_t^T\|z^m_{xxxx}\|^2_{L^2(I)}ds \end{aligned}$$

for $\forall\varepsilon>0$. If we take $\varepsilon<<1$ in the above inequality, we have

$$E^{\mathcal{F}_r}\|z^m_{xx}(t)\|^2_{L^2(I)}+E^{\mathcal{F}_r}\int_t^T[\|z^m_{xxxx}\|^2_{L^2(I)}+\|Z^m_{xx}\|^2_{L^2(I)}]ds$$

$$\leq C(p,q,k)E^{\mathcal{F}_r}[\|(P_mz_T)_{xx}\|^2_{L^2(I)}+\int_t^T(\|z^m_{xx}\|^2_{L^2(I)}+\|h^m\|^2_{L^2(I)}+\|Z^m\|^2_{L^2(I)})ds].$$

By taking the expectation on both sides, one gets

$$E\|z^m_{xx}(t)\|^2_{L^2(I)}+E\int_t^T[\|z^m_{xxxx}\|^2_{L^2(I)}+\|Z^m_{xx}\|^2_{L^2(I)}]ds$$

$$\leq C(p,q,k)E[\|(P_mz_T)_{xx}\|^2_{L^2(I)}+\int_t^T(\|z^m_{xx}\|^2_{L^2(I)}+\|h^m\|^2_{L^2(I)}+\|Z^m\|^2_{L^2(I)})ds]. \tag{2.9}$$

By the same method as above, we can obtain that

$$E\|z^m(t)\|^2_{L^2(I)}+E\int_t^T[2\|z^m_{xx}\|^2_{L^2(I)}+\|Z^m\|^2_{L^2(I)}]ds=E\|P_mz_T\|^2_{L^2(I)}$$

$$-E\int_t^T 2[(kz^m_{xx},z^m)_{L^2(I)}+(pz^m,z^m)_{L^2(I)}+(qZ^m,z^m)_{L^2(I)}+(z^m,h^m)_{L^2(I)}]ds.$$

According to Cauchy inequality and Poincaré's inequality, it holds that

$$E\|z^m(t)\|^2_{L^2(I)}+E\int_t^T[\|z^m_{xx}\|^2_{L^2(I)}+\|Z^m\|^2_{L^2(I)}]ds$$

$$\leq C(p,q,k,\varepsilon)E[\|P_mz_T\|^2_{L^2(I)}+\int_t^T(\|z^m\|^2_{L^2(I)}+\|h^m\|^2_{L^2(I)})ds]$$

$$+\varepsilon E\int_t^T(\|z^m_{xx}\|^2_{L^2(I)}+\|Z^m\|^2_{L^2(I)})ds.$$

If we take $\varepsilon << 1$ in the above inequality, we have

$$E\|z^m(t)\|^2_{L^2(I)}+E\int_t^T[\|z^m_{xx}\|^2_{L^2(I)}+\|Z^m\|^2_{L^2(I)}]ds$$

$$\leq C(p,q,k)E[\|P_mz_T\|^2_{L^2(I)}+\int_t^T(\|z^m\|^2_{L^2(I)}+\|h^m\|^2_{L^2(I)})ds]. \tag{2.10}$$

Applying the Gronwall inequality, we can obtain

$$E\|z^m(t)\|^2_{L^2(I)}\leq C(p,q,k,T)E(\|P_mz_T\|^2_{L^2(I)}+\int_t^T\|h^m\|^2_{L^2(I)}ds).$$

Combined this inequality with (2.10), we have

$$E\int_0^T [\|z_{xx}^m\|_{L^2(I)}^2 + \|Z^m\|_{L^2(I)}^2]ds \leq C(p,q,k,T)E[\|P_m z_T\|_{L^2(I)}^2 + \int_0^T \|h^m\|_{L^2(I)}^2 ds]. \tag{2.11}$$

Applying (2.11) to (2.9), it follows from the Poincaré's inequality that

$$\begin{aligned} &E\|z_{xx}^m(t)\|_{L^2(I)}^2 + E\int_t^T [\|z_{xxxx}^m\|_{L^2(I)}^2 + \|Z_{xx}^m\|_{L^2(I)}^2]ds \\ &\leq C(p,q,k,T)E[\|(P_m z_T)_{xx}\|_{L^2(I)}^2 + \int_0^T \|h^m\|_{L^2(I)}^2 ds]. \end{aligned} \tag{2.12}$$

It is easy to see that

$$E\|z_{xx}^m(t)\|_{L^2(I)}^2 \leq C(p,q,k,T)E(\|(P_m z_T)_{xx}\|_{L^2(I)}^2 + \int_0^T \|h^m\|_{L^2(I)}^2 ds), \tag{2.13}$$

and

$$\begin{aligned} &E\int_0^T [\|z_{xxxx}^m\|_{L^2(I)}^2 + \|Z_{xx}^m\|_{L^2(I)}^2]ds \\ &\leq C(p,q,k,T)E(\|(P_m z_T)_{xx}\|_{L^2(I)}^2 + \int_0^T \|h^m\|_{L^2(I)}^2 ds). \end{aligned} \tag{2.14}$$

Moreover, we have

$$\sup_{0\leq t\leq T} E\|z_{xx}^m(t)\|_{L^2(I)}^2 \leq C(p,q,k,T)E(\|(P_m z_T)_{xx}\|_{L^2(I)}^2 + \int_0^T \|h^m\|_{L^2(I)}^2 ds).$$

It follows from the Burkholder–Davis–Gundy inequality, Cauchy inequality and (2.7) that

$$\begin{aligned} &E\sup_{0\leq t\leq T} \|z_{xx}^m(t)\|_{L^2(I)}^2 + E\int_0^T [\|z_{xxxx}^m\|_{L^2(I)}^2 + \|Z_{xx}^m\|_{L^2(I)}^2]dt \\ &\leq \varepsilon E\Big(\sup_{0\leq t\leq T} \|z_{xx}^m(t)\|_{L^2(I)}^2\Big) \\ &\quad + C(p,q,k,T,\varepsilon)E[\|(P_m z_T)_{xx}\|_{L^2(I)}^2 + \int_0^T (\|Z_{xx}^m(t)\|_{L^2(I)}^2 + \|z_{xxxx}^m\|_{L^2(I)}^2 + \|h^m\|_{L^2(I)}^2)dt] \end{aligned}$$

for $\forall \varepsilon > 0$. Thus, if we take ε small enough, we have

$$E\sup_{0\le t\le T}\|z^m_{xx}(t)\|^2_{L^2(I)}+E\int_0^T[\|z^m_{xxxx}\|^2_{L^2(I)}+\|Z^m_{xx}\|^2_{L^2(I)}]dt$$
$$\le C(p,q,k,T)E[\|(P_mz_T)_{xx}\|^2_{L^2(I)}+\int_0^T(\|Z^m_{xx}\|^2_{L^2(I)}+\|z^m_{xxxx}\|^2_{L^2(I)}+\|h^m\|^2_{L^2(I)})dt]. \tag{2.15}$$

Applying (2.14) to (2.15), we have

$$E\sup_{0\le t\le T}\|z^m_{xx}(t)\|^2_{L^2(I)}+E\int_0^T[\|z^m_{xxxx}\|^2_{L^2(I)}+\|Z^m_{xx}\|^2_{L^2(I)}]dt$$
$$\le C(p,q,k,T)E[\|(P_mz_T)_{xx}\|^2_{L^2(I)}+\int_0^T\|h^m\|^2_{L^2(I)}dt].$$

It follows from the Poincaré's inequality and Gagliardo–Nirenberg inequality (see [8]) that

$$E\sup_{0\le t\le T}\|z^m(t)\|^2_{H^2_0(I)}+E\int_0^T[\|z^m\|^2_{H^4(I)}+\|Z^m\|^2_{H^2_0(I)}]dt$$
$$\le C(p,q,k,T)E[\|P_mz_T\|^2_{H^2_0(I)}+\int_0^T\|h^m\|^2_{L^2(I)}dt]. \tag{2.16}$$

By the same argument, we also have, for $m\ge n\ge 1$,

$$E\sup_{0\le t\le T}\|z^m(t)-z^n(t)\|^2_{H^2_0(I)}+E\int_0^T[\|z^m-z^n\|^2_{H^4(I)}+\|Z^m-Z^n\|^2_{H^2_0(I)}]dt$$
$$\le C(p,q,k,T)E[\|P_mz_T-P_nz_T\|^2_{H^2_0(I)}+\int_0^T\|h^m-h^n\|^2_{L^2(I)}dt]. \tag{2.17}$$

Next we observe that the right-hand side of (2.17) converges to zero as $n,m\to\infty$. Hence, it follows that $\{(z^m,Z^m)\}_{m=1}^{\infty}$ is a Cauchy sequence that converges strongly in $(L^2_{\mathcal{F}}(\Omega;C([0,T];H^2_0(I)))\cap L^2_{\mathcal{F}}(0,T;H^4(I)))\times L^2_{\mathcal{F}}(0,T;H^2_0(I))$. Let (z,Z) be the limit. It is apparent that (z,Z) satisfies the terminal and boundary conditions in (1.1), and $(z(t),Z(t))$ is $\mathcal{F}_t$-adapted for each $t\in[0,T]$. Also, it follows from (2.6) that (2.1) holds. By passing $m\to\infty$ in (2.16), we arrive at (2.2).

For the uniqueness of the solution, we suppose that (z_1,Z_1) and (z_2,Z_2) are two solutions of (1.1). Let $z=z_1-z_2, Z=Z_1-Z_2$. Then

$$E\sup_{0\le t\le T}\|z(t)\|^2_{H^2_0(I)}+E\int_0^T[\|z\|^2_{H^4(I)}+\|Z\|^2_{H^2_0(I)}]dt\le 0,$$

thus $z\equiv Z\equiv 0$ for any $t\in[0,T]$, for almost all $\omega\in\Omega$.

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By the same method in the proof of (2.13), we can obtain

$$E\|z_{xx}^m(t_1)\|_{L^2(I)}^2 \le C(p,q,k,T)E(\|z_{xx}^m(t_2)\|_{L^2(I)}^2 + \int_{t_1}^{t_2} \|h^m\|_{L^2(I)}^2 ds)$$

for any $0 \le t_1 \le t_2 \le T$. It follows from the Poincaré's inequality that

$$E\|z^m(t_1)\|_{H_0^2(I)}^2 \le C(p,q,k,T)E(\|z^m(t_2)\|_{H_0^2(I)}^2 + \int_{t_1}^{t_2} \|h^m\|_{L^2(I)}^2 ds). \tag{2.18}$$

By passing $m \to \infty$ in (2.18), we arrive at (2.3). □

In order to deal with L^2-regular boundary data, we present the next regularity result. Taking into account the continuous embedding $H^4(I) \subset C^3(\overline{I})$, from Proposition 2.1, we directly obtain:

Corollary 2.1. *There exists a constant $C = C(p,q,k,T)$ such that for any solution (z, Z) of (1.1)*

$$\begin{aligned} &\|z_{xx}(0,\cdot)\|_{L_{\mathcal{F}}^2(\Omega,L^2(0,T))} + \|z_{xxx}(0,\cdot)\|_{L_{\mathcal{F}}^2(\Omega,L^2(0,T))} \\ &\le C[\|z_T\|_{L^2(\Omega,\mathcal{F}_T,P;H_0^2(I))} + \|h\|_{L_{\mathcal{F}}^2(0,T;L^2(I))}], \end{aligned} \tag{2.19}$$

where $z_T \in L^2(\Omega, \mathcal{F}_T, P; H_0^2(I))$ and $h \in L_{\mathcal{F}}^2(0,T;L^2(I))$.

By the same methods in Proposition 2.1 and Corollary 2.1, we have the following results.

Proposition 2.2. *If $p, q \in L_{\mathcal{F}}^\infty(0,T;W^{2,\infty}(I))$, $z_0 \in L^2(\Omega,\mathcal{F}_0,P;H_0^2(I))$, $f \in L_{\mathcal{F}}^2(0,T;L^2(I))$, $g \in L_{\mathcal{F}}^2(0,T;H_0^2(I))$ are given. Then (1.2) admits a unique solution z such that*

$$\begin{aligned} &\|z\|_{L_{\mathcal{F}}^2(\Omega;C([0,T];H_0^2(I)))\cap L_{\mathcal{F}}^2(0,T;H^4(I))} \\ &\le C[\|z_0\|_{L^2(\Omega,\mathcal{F}_0,P;H_0^2(I))} + \|f\|_{L_{\mathcal{F}}^2(0,T;L^2(I))} + \|g\|_{L_{\mathcal{F}}^2(0,T;H_0^2(I))}], \end{aligned} \tag{2.20}$$

where $C = C(p,q,k,T)$. Moreover, for any $0 \le t_1 \le t_2 \le T$, it holds that

$$E\|z(t_2)\|_{H_0^2(I)}^2 \le CE(\|z(t_1)\|_{H_0^2(I)}^2 + \int_{t_1}^{t_2} (\|f(s)\|_{L^2(I)}^2 + \|g(s)\|_{H_0^2(I)}^2) ds]. \tag{2.21}$$

Corollary 2.2. *There a constant $C = C(p,q,k,T)$ such that for any solution z of (1.2)*

$$\begin{aligned} &\|z_{xx}(0,\cdot)\|_{L_{\mathcal{F}}^2(\Omega,L^2(0,T))} + \|z_{xxx}(0,\cdot)\|_{L_{\mathcal{F}}^2(\Omega,L^2(0,T))} \\ &\le C[\|z_0\|_{L^2(\Omega,\mathcal{F}_0,P;H_0^2(I))} + \|f\|_{L_{\mathcal{F}}^2(0,T;L^2(I))} + \|g\|_{L_{\mathcal{F}}^2(0,T;H_0^2(I))}], \end{aligned} \tag{2.22}$$

where $z_0 \in L^2(\Omega, \mathcal{F}_0, P; H_0^2(I)), f \in L_{\mathcal{F}}^2(0,T;L^2(I))$ and $g \in L_{\mathcal{F}}^2(0,T;H_0^2(I))$.

Now, referring to [22,18,19] for the transposition method, we can give a meaning to the solution of (1.4).

Definition 2.2. A stochastic process $y \in C_{\mathcal{F}}([0,T]; L^2(\Omega; H^{-2}(I)))$ is said to be a solution of

$$\begin{cases} dy + (ky_{xx} + y_{xxxx})dt = (ay + f)dt + (by + g)dw & \text{in } Q, \\ y(0,t) = u_1(t), y(1,t) = 0 & \text{in } (0,T), \\ y_x(0,t) = u_2(t), y_x(1,t) = 0 & \text{in } (0,T), \\ y(x,0) = y_0(x) & \text{in } I, \end{cases} \tag{2.23}$$

if for every $\tau \in [0,T]$ and every $z_\tau \in L^2(\Omega, \mathcal{F}_\tau, P; H_0^2(I))$ it holds that

$$\begin{aligned} &E(y(\tau), z_\tau)_{H^{-2}(I),H_0^2(I)} - E(y_0, z(\cdot,0))_{H^{-2}(I),H_0^2(I)} \\ &= E\int_0^\tau (-u_1(t)z_{xxx}(0,t) + u_2(t)z_{xx}(0,t))dt \\ &\quad + E\int_0^\tau [(f,z)_{H^{-2}(I),H_0^2(I)} + (g,Z)_{H^{-2}(I),H_0^2(I)}]dt, \end{aligned}$$

where (z, Z) is the solution to (4.1) with terminal state z_τ in the domain $I \times (0,\tau)$.

For (2.23), we have the following well-posedness result.

Proposition 2.3. *Let $y_0 \in H^{-2}(I), u_1, u_2 \in L^2_{\mathcal{F}}(\Omega, L^2(0,T)), f \in L^1_{\mathcal{F}}(0,T; H^{-2}(I))$ and $g \in L^2_{\mathcal{F}}(0,T; H^{-2}(I))$ be given. Then (2.23) admits a unique solution $y \in C_{\mathcal{F}}([0,T]; L^2(\Omega; H^{-2}(I)))$ such that*

$$\begin{aligned} \|y\|_{C_{\mathcal{F}}([0,T];L^2(\Omega;H^{-2}(I)))} \le C(&\|y_0\|_{H^{-2}(I)} + \|f\|_{L^1_{\mathcal{F}}(0,T;H^{-2}(I))} + \|g\|_{L^2_{\mathcal{F}}(0,T;H^{-2}(I))} \\ &+ \|u_1\|_{L^2_{\mathcal{F}}(\Omega,L^2(0,T))} + \|u_2\|_{L^2_{\mathcal{F}}(\Omega,L^2(0,T))}), \end{aligned} \tag{2.24}$$

where $C = C(a, b, k, T)$.

Proof. The main idea in this part comes from [22,5].

Let us define a linear functional F on $L^2(\Omega, \mathcal{F}_\tau, P; H_0^2(I))$ as

$$\begin{aligned} F(z_\tau) = E(y_0, z(\cdot,0))_{H^{-2}(I),H_0^2(I)} &+ E\int_0^\tau (-u_1(t)z_{xxx}(0,t) + u_2(t)z_{xx}(0,t))dt \\ + E\int_0^\tau &[(f,z)_{H^{-2}(I),H_0^2(I)} + (g,Z)_{H^{-2}(I),H_0^2(I)}]dt. \end{aligned}$$

Applying Proposition 2.1 and Corollary 2.1, we can obtain that the solution (z, Z) for (4.1) satisfies

$$\begin{aligned} &\|z\|_{L^2_{\mathcal{F}}(\Omega;C([0,\tau];H_0^2(I)))} + \|Z\|_{L^2_{\mathcal{F}}(0,\tau;H_0^2(I))} \le C\|z_\tau\|_{L^2(\Omega,\mathcal{F}_\tau,P;H_0^2(I))}, \\ &\|z_{xx}(0,\cdot)\|_{L^2_{\mathcal{F}}(\Omega,L^2(0,\tau))} + \|z_{xxx}(0,\cdot)\|_{L^2_{\mathcal{F}}(\Omega,L^2(0,\tau))} \le C\|z_\tau\|_{L^2(\Omega,\mathcal{F}_\tau,P;H_0^2(I))}. \end{aligned}$$

Thus

$$\begin{aligned} |F(z_\tau)| \le C(&\|y_0\|_{H^{-2}(I)} + \|f\|_{L^1_{\mathcal{F}}(0,T;H^{-2}(I))} + \|g\|_{L^2_{\mathcal{F}}(0,T;H^{-2}(I))} \\ &+ \|u_1\|_{L^2_{\mathcal{F}}(\Omega,L^2(0,T))} + \|u_2\|_{L^2_{\mathcal{F}}(\Omega,L^2(0,T))})\|z_\tau\|_{L^2(\Omega,\mathcal{F}_\tau,P;H_0^2(I))}. \end{aligned}$$

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Hence, we get that F is a bounded linear functional on $L^2(\Omega,\mathcal{F}_\tau,P;H_0^2(I))$. It follows from the Riesz Representation Theorem that there exists a unique $y_\tau\in L^2(\Omega,\mathcal{F}_\tau,P;H^{-2}(I))$ such that

$$F(z_\tau)=E(y_\tau,z_\tau)_{H^{-2}(I),H_0^2(I)}$$

for any $z_\tau\in L^2(\Omega,\mathcal{F}_\tau,P;H_0^2(I))$ and

$$\begin{aligned}\|y_\tau\|_{L^2(\Omega,\mathcal{F}_\tau,P;H^{-2}(I))}\leq C(&\|y_0\|_{H^{-2}(I)}+\|f\|_{L^1_{\mathcal{F}}(0,T;H^{-2}(I))}+\|g\|_{L^2_{\mathcal{F}}(0,T;H^{-2}(I))}\\&+\|u_1\|_{L^2_{\mathcal{F}}(\Omega,L^2(0,T))}+\|u_2\|_{L^2_{\mathcal{F}}(\Omega,L^2(0,T))})\end{aligned}$$

for any $\tau\in(0,T)$.

Define a process $y(\cdot)$ by

$$y(\tau)=y_\tau$$

for any $\tau\in(0,T)$.

Now we prove that $y\in C_{\mathcal{F}}([0,T];L^2(\Omega;H^{-2}(I)))$.

Indeed, let $\tau\in[0,T)$ and $\xi\in L^2(\Omega,\mathcal{F}_T,P;H_0^2(I))$. Consider the following random KS equation

$$\begin{cases}\widetilde{z}_t-(k\widetilde{z}_{xx}+\widetilde{z}_{xxxx})=-a\widetilde{z} & \text{in } I\times(\tau,\tau+\delta),\\ \widetilde{z}(0,t)=0=\widetilde{z}(1,t) & \text{in } (\tau,\tau+\delta),\\ \widetilde{z}_x(0,t)=0=\widetilde{z}_x(1,t) & \text{in } (\tau,\tau+\delta),\\ \widetilde{z}(\tau)=E(\xi\mid\mathcal{F}_\tau) & \text{in } I,\end{cases}\tag{2.25}$$

with $\delta>0$ satisfying that $\tau+\delta<T$.

It is easy to see that

$$\lim_{\delta\to0^+}E\|\widetilde{z}(\tau+\delta)-\widetilde{z}(\tau)\|^2_{H_0^2(I)}=0$$

and

$$\lim_{\delta\to0^+}E\|E(\xi\mid\mathcal{F}_{\tau+\delta})-E(\xi\mid\mathcal{F}_\tau)\|^2_{H_0^2(I)}=0.$$

Thus we have

$$\lim_{\delta\to0^+}E\|\widetilde{z}(\tau+\delta)-E(\xi\mid\mathcal{F}_{\tau+\delta})\|^2_{H_0^2(I)}=0.\tag{2.26}$$

Let $(z_1,Z_1),(z_2,Z_2)$ and (z_3,Z_3) satisfy

$$\begin{cases}dz_1-(kz_{1xx}+z_{1xxxx})dt=(-az_1-bZ_1)dt+Z_1dw & \text{in } I\times(0,\tau+\delta),\\ z_1(0,t)=0=z_1(1,t) & \text{in } (0,\tau+\delta),\\ z_{1x}(0,t)=0=z_{1x}(1,t) & \text{in } (0,\tau+\delta),\\ z_1(x,\tau+\delta)=E(\xi\mid\mathcal{F}_{\tau+\delta}) & \text{in } I,\end{cases}$$

$$\begin{cases}dz_2-(kz_{2xx}+z_{2xxxx})dt=(-az_2-bZ_2)dt+Z_2dw & \text{in } I\times(0,\tau+\delta),\\ z_2(0,t)=0=z_2(1,t) & \text{in } (0,\tau+\delta),\\ z_{2x}(0,t)=0=z_{2x}(1,t) & \text{in } (0,\tau+\delta),\\ z_2(x,\tau+\delta)=\widetilde{z}(\tau+\delta) & \text{in } I\end{cases}$$

and

$$\begin{cases} dz_3 - (kz_{3xx} + z_{3xxxx})dt = (-az_3 - bZ_3)dt + Z_3 dw & \text{in } I \times (0,\tau), \\ z_3(0,t) = 0 = z_3(1,t) & \text{in } (0,\tau), \\ z_{3x}(0,t) = 0 = z_{3x}(1,t) & \text{in } (0,\tau), \\ z_3(x,\tau) = E(\xi \mid \mathcal{F}_\tau) & \text{in } I. \end{cases}$$

It follows from (2.26), Proposition 2.1 and Corollary 2.1 that

$$\begin{aligned} &\lim_{\delta\to 0^+} \|z_1 - z_2\|_{L^\infty_{\mathcal{F}}(0,\tau;H_0^2(I))} = 0, \\ &\lim_{\delta\to 0^+} \|Z_1 - Z_2\|_{L^2_{\mathcal{F}}(0,\tau;H_0^2(I))} = 0, \\ &\lim_{\delta\to 0^+} \|z_{1xx}(0,\cdot) - z_{2xx}(0,\cdot)\|_{L^2_{\mathcal{F}}(\Omega,L^2(0,\tau))} = 0, \\ &\lim_{\delta\to 0^+} \|z_{1xxx}(0,\cdot) - z_{2xxx}(0,\cdot)\|_{L^2_{\mathcal{F}}(\Omega,L^2(0,\tau))} = 0, \\ &\lim_{\delta\to 0^+} \|z_1(\cdot,0) - z_2(\cdot,0)\|_{L^2_{\mathcal{F}}(\Omega,H_0^2(I))} = 0. \end{aligned} \tag{2.27}$$

By the uniqueness of the solution to (2.25) and (4.1), we have

$$\begin{aligned} z_3 = z_2 \quad & \text{in } I \times (0,\tau), \\ Z_3 = Z_2 \quad & \text{in } I \times (0,\tau). \end{aligned} \tag{2.28}$$

From the definition of $y(\tau)$, we have

$$\begin{aligned} &E(y(\tau+\delta) - y(\tau), \xi)_{H^{-2}(I),H_0^2(I)} \\ &= E(y(\tau+\delta), \xi)_{H^{-2}(I),H_0^2(I)} - E(y(\tau), \xi)_{H^{-2}(I),H_0^2(I)} \\ &= E(y(\tau+\delta), E(\xi \mid \mathcal{F}_{\tau+\delta}))_{H^{-2}(I),H_0^2(I)} - E(y(\tau), E(\xi \mid \mathcal{F}_\tau))_{H^{-2}(I),H_0^2(I)} \\ &= E(y_0, z_1(0) - z_3(0))_{H^{-2}(I),H_0^2(I)} + E\int_0^\tau [(f, z_1 - z_3)_{H^{-2}(I),H_0^2(I)} + (g, Z_1 - Z_3)_{H^{-2}(I),H_0^2(I)}]dt \\ &+ E\int_0^\tau [-u_1(t)(z_{1xxx}(0,t) - z_{3xxx}(0,t)) + u_2(t)(z_{1xx}(0,t) - z_{3xx}(0,t))]dt \\ &+ E\int_\tau^{\tau+\delta} (-u_1(t) z_{1xxx}(0,t) + u_2(t) z_{1xx}(0,t))dt + E\int_\tau^{\tau+\delta} [(f, z_1)_{H^{-2}(I),H_0^2(I)} + (g, Z_1)_{H^{-2}(I),H_0^2(I)}]dt. \end{aligned}$$

It follows from (2.27) and (2.28) that

$$\lim_{\delta\to 0^+} E(y(\tau+\delta) - y(\tau), \xi)_{H^{-2}(I),H_0^2(I)} = 0,$$

for any $\xi \in L^2(\Omega, \mathcal{F}_T, P; H_0^2(I))$.

Similarly, we can show that for any $\tau \in (0,T]$,

$$\lim_{\delta\to 0^-} E(y(\tau+\delta) - y(\tau), \xi)_{H^{-2}(I),H_0^2(I)} = 0,$$

for any $\xi \in L^2(\Omega, \mathcal{F}_T, P; H_0^2(I))$.

Hence, we have $y \in C_{\mathcal{F}}([0,T]; L^2(\Omega; H^{-2}(I)))$.

Further, we prove the uniqueness of the solution.

Indeed, let $y_1(\cdot)$ and $y_2(\cdot)$ be the solutions of (2.23), we have

$$E\int_I y_1(x,\tau)z_\tau(x)dx = E\int_I y_2(x,\tau)z_\tau(x)dx$$

for all $\tau \in [0,T]$ and $z_\tau \in L^2(\Omega,\mathcal{F}_\tau,P;H_0^2(I))$. This concludes that $y_1 = y_2$. □

3. Proof of Theorem 1.1 and Theorem 1.2

In this section, we first obtain an identity for a stochastic fourth order parabolic operator, which plays a key role in the proof of Theorem 1.1 and Theorem 1.2.

Proposition 3.1. *Let $l \in C^\infty(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $\theta = e^l$. Assume that y is a continuous $H^4(\mathbb{R})$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted semi-martingale. Put*

$$\begin{aligned}
&Ly = dy + \delta y_{xxxx}dt,\\
&u = \theta y,\\
&\theta Ly = \theta(dy + \delta y_{xxxx}dt) = du + (A_0u + A_1u_x + A_2u_{xx} + A_3u_{xxx} + \delta u_{xxxx})dt = P_1 + P_2,\\
&P_1 = [\delta u_{xxxx} + (B_2u_x)_x + (B_0 - l_t)u]dt,\\
&P_2 = du + (C_0u + C_1u_x + C_2u_{xx} + C_3u_{xxx})dt,\\
&P = \delta u_{xxxx} + (B_2u_x)_x + B_0u,
\end{aligned}$$

where

$$\begin{aligned}
&A_0 = \delta(l_x^4 + 4l_xl_{xxx} - l_{xxxx} - 6l_x^2l_{xx} + 3l_{xx}^2) - l_t,\\
&A_1 = 4\delta(-l_x^3 + 3l_xl_{xx} - l_{xxx}),\\
&A_2 = 6\delta(l_x^2 - l_{xx}),\\
&A_3 = -4\delta l_x,
\end{aligned}$$

δ is a constant and the coefficients $B_0, B_2, C_0, C_1, C_2, C_3$ are functions. Then for a.e. $x \in \mathbb{R}$ and P-a.s. $\omega \in \Omega$, it holds that

$$\begin{aligned}
2P\theta Ly \;=\;& 2PP_1 + (u^2\{\cdot\} + u_x^2\{\cdot\} + u_{xx}^2\{\cdot\} + u_{xxx}^2\{\cdot\})dt\\
&+(\{\cdot\}_x + \{\cdot\}_{xx} + \{\cdot\}_{xxx})dt + dM\\
&+(-\delta)(du_{xx})^2 + B_2(du_x)^2 - B_0(du)^2,
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
&u^2\{\cdot\} = u^2\{\delta C_{0xxxx} - (B_{2x}C_0)_x + (B_2C_0)_{xx} - B_{0t} + 2B_0C_0 - (B_0C_1)_x\\
&\qquad + (B_0C_2)_{xx} - (B_0C_3)_{xxx}\},\\
&u_x^2\{\cdot\} = u_x^2\{-4\delta C_{0xx} - \delta C_{1xxx} + B_{2t} - 2B_2C_0 + 2B_{2x}C_1 - (B_2C_1)_x - (B_{2x}C_2)_x + (B_{2x}C_3)_{xx}\\
&\qquad - 2B_0C_2 + 3(B_0C_3)_x\},\\
&u_{xx}^2\{\cdot\} = u_{xx}^2\{2\delta C_0 + 3\delta C_{1x} + \delta C_{2xx} + 2B_2C_2 - 2B_{2x}C_3 - (B_2C_3)_x\},\\
&u_{xxx}^2\{\cdot\} = u_{xxx}^2\{-2\delta C_2 - \delta C_{3x}\},\\
&\{\cdot\}_x = \{2\delta u_{xxx}du - 2\delta u_{xx}du_x + 2\delta C_0uu_{xxx} + 2\delta C_{0x}u_x^2 - 3\delta C_{0xxx}u^2 + 3\delta C_{0x}u_x^2\\
&\qquad + 3\delta C_{1xx}u_x^2 - 3\delta C_1u_{xx}^2 - 2\delta C_{2x}u_{xx}^2 + \delta C_3u_{xxx}^2 + 2B_2u_xdu + B_{2x}C_0u^2\\
&\qquad - 2(B_2C_0)_xu^2 + B_2C_1u_x^2 + B_{2x}C_2u_x^2 - 2(B_{2x}C_3)_xu_x^2 + B_2C_3u_{xx}^2 + B_0C_1u^2\\
&\qquad - 2(B_0C_2)_xu^2 + 3(B_0C_3)_{xx}u^2 - 3B_0C_3u_x^2\}_x,\\
&\{\cdot\}_{xx} = \{-\delta C_0u_x^2 + 3\delta C_{0xx}u^2 - 3\delta C_{1x}u_x^2 + \delta C_2u_{xx}^2 + B_2C_0u^2 + B_{2x}C_3u_x^2\\
&\qquad + B_0C_2u^2 - 3(B_0C_3)_xu^2\}_{xx},\\
&\{\cdot\}_{xxx} = \{-\delta C_{0x}u^2 + \delta C_1u_x^2 + B_0C_3u^2\}_{xxx},\\
&M = \delta u_{xx}^2 - B_2u_x^2 + B_0u^2.
\end{aligned}$$

Proof. According to

$$2P\theta Ly = 2PP_1 + 2PP_2,$$

we need to compute

$$2PP_2.$$

First, we consider

$$2\delta u_{xxxx}P_2. \tag{3.2}$$

Each term in (3.2) can be computed as follows:

$$\begin{aligned}
&2\delta u_{xxxx}du = 2\delta(u_{xxx}du - u_{xx}du_x)_x + \delta(du_{xx}^2 - (du_{xx})^2),\\
&2\delta u_{xxxx}C_0u = 2\delta(C_0uu_{xxx})_x - \delta[(C_0u_x^2)_{xx} - 2(C_{0x}u_x^2)_x - 2C_0u_{xx}^2 + C_{0xx}u_x^2 + (C_{0x}u^2)_{xxx}\\
&\qquad - 3(C_{0xx}u^2)_{xx} + 3(C_{0xxx}u^2 - C_{0x}u_x^2)_x + 3C_{0xx}u_x^2 - C_{0xxxx}u^2],\\
&2\delta u_{xxxx}C_1u_x = \delta[(C_1u_x^2)_{xxx} - 3(C_{1x}u_x^2)_{xx} + 3(C_{1xx}u_x^2 - C_1u_{xx}^2)_x + 3C_{1x}u_{xx}^2 - C_{1xxx}u_x^2],\\
&2\delta u_{xxxx}C_2u_{xx} = \delta[(C_2u_{xx}^2)_{xx} - 2(C_{2x}u_{xx}^2)_x - 2C_2u_{xxx}^2 + C_{2xx}u_{xx}^2],\\
&2\delta u_{xxxx}C_3u_{xxx} = \delta[(C_3u_{xxx}^2)_x - C_{3x}u_{xxx}^2].
\end{aligned}$$

By a similar argument, calculating each term in $2(B_2u_x)_xP_2$ and $2B_0uP_2$, we obtain

$$\begin{aligned}
&2(B_2u_x)_xdu = 2(B_2u_xdu)_x + B_{2t}u_x^2dt + B_2(du_x)^2 - d(B_2u_x^2),\\
&2(B_2u_x)_xC_0u = (B_{2x}C_0u^2)_x - (B_{2x}C_0)_xu^2 + (B_2C_0u^2)_{xx} - 2((B_2C_0)_xu^2)_x - 2B_2C_0u_x^2\\
&\qquad + (B_2C_0)_{xx}u^2,\\
&2(B_2u_x)_xC_1u_x = 2B_{2x}C_1u_x^2 + (B_2C_1u_x^2)_x - (B_2C_1)_xu_x^2,\\
&2(B_2u_x)_xC_2u_{xx} = 2B_2C_2u_{xx}^2 + (B_{2x}C_2u_x^2)_x - (B_{2x}C_2)_xu_x^2,\\
&2(B_2u_x)_xC_3u_{xxx} = (B_{2x}C_3u_x^2)_{xx} - 2((B_{2x}C_3)_xu_x^2)_x - 2B_{2x}C_3u_{xx}^2 + (B_{2x}C_3)_{xx}u_x^2\\
&\qquad + (B_2C_3u_{xx}^2)_x - (B_2C_3)_xu_{xx}^2,\\
&2B_0udu = d(B_0u^2) - B_{0t}u^2dt - B_0(du)^2,\\
&2B_0uC_0u = 2B_0C_0u^2,\\
&2B_0uC_1u_x = (B_0C_1u^2)_x - (B_0C_1)_xu^2,\\
&2B_0uC_2u_{xx} = (B_0C_2u^2)_{xx} - 2((B_0C_2)_xu^2)_x - 2B_0C_2u_x^2 + (B_0C_2)_{xx}u^2,\\
&2B_0uC_3u_{xxx} = (B_0C_3u^2)_{xxx} - 3((B_0C_3)_xu^2)_{xx} + (3(B_0C_3)_{xx}u^2 - 3B_0C_3u_x^2)_x\\
&\qquad + 3(B_0C_3)_xu_x^2 - (B_0C_3)_{xxx}u^2.
\end{aligned}$$

From the above identities, we can obtain (3.1). □

Now, we can prove the global Carleman estimate in Theorem 1.1.

Proof of Theorem 1.1. Similar as in [10], it is enough to prove (1.3) for all $(z, Z) \in (L^2_{\mathcal{F}}(\Omega; C([0,T]; H_0^2(I)))\cap L^2_{\mathcal{F}}(0,T; H^4(I))) \times L^2_{\mathcal{F}}(0,T; H_0^2(I))$, $h \in L^2_{\mathcal{F}}(0,T; L^2(I))$, satisfying

$$dz - z_{xxxx}dt = hdt + Zdw. \tag{3.3}$$

In fact, assume that we have proved (1.3) for (3.3), then for (1.1), we have

$$E\int_Q (\lambda\varphi\theta^2z_{xxx}^2 + \lambda^3\varphi^3\theta^2z_{xx}^2 + \lambda^5\varphi^5\theta^2z_x^2 + \lambda^7\varphi^7\theta^2z^2)dxdt$$

$$\leq C\Big(E\int\limits_Q \theta^2|kz_{xx}+pz+qZ+h|^2dxdt+E\int\limits_Q \lambda^4\varphi^4\theta^2Z^2dxdt$$
$$+E\int\limits_0^T(\lambda^3\varphi^3(0,t)\theta^2(0,t)z_{xx}^2(0,t)+\lambda\varphi(0,t)\theta^2(0,t)z_{xxx}^2(0,t))dt\Big)$$
$$\leq C\Big(E\int\limits_Q(\theta^2z_{xx}^2+\theta^2z^2+\lambda^4\varphi^4\theta^2Z^2+\theta^2h^2)dxdt$$
$$+E\int\limits_0^T(\lambda^3\varphi^3(0,t)\theta^2(0,t)z_{xx}^2(0,t)+\lambda\varphi(0,t)\theta^2(0,t)z_{xxx}^2(0,t))dt\Big), \tag{3.4}$$

thus when λ is large enough, it is possible to absorb $E\int\limits_Q\theta^2z_{xx}^2dxdt$, $E\int\limits_Q\theta^2z^2dxdt$ with the left-hand side of (3.4), concluding that (1.3) also holds for (1.1).

Now, we prove (1.3) holds for (3.3).

We apply Proposition 3.1 with $\delta=-1, y=z$ and set

$$B_0=-l_x^4-4l_xl_{xxx}+2l_x^2l_{xx}-3l_{xx}^2,$$
$$B_2=7l_{xx}-6l_x^2,$$

then

$$C_0=A_0+l_t-B_0=-l_{xxxx}+4l_x^2l_{xx},$$
$$C_1=A_1-B_{2x}=4l_x^3-3l_{xxx},$$
$$C_2=A_2-B_2=-l_{xx},$$
$$C_3=A_3=4l_x.$$

According to (3.1), we have

$$2\int\limits_0^T P\theta Lz$$
$$=2\int\limits_0^T P(P_1+P_2)$$
$$\geq\int\limits_0^T P^2dt-\int\limits_0^T l_t^2u^2dt$$
$$+\int\limits_0^T(u^2\{\cdots\}+u_x^2\{\cdots\}+u_{xx}^2\{\cdots\}+u_{xxx}^2\{\cdots\})dt$$
$$+\int\limits_0^T(\{\cdot\}_x+\{\cdot\}_{xx}+\{\cdot\}_{xxx})dt+\int\limits_0^T dM$$

$$+\int_0^T (du_{xx})^2 + \int_0^T B_2(du_x)^2 - \int_0^T B_0(du)^2. \tag{3.5}$$

By the definitions of ρ, φ, ψ, it is obvious that for $n \in \mathbb{N}$

$$|\partial_x^n \rho| \le C(\psi)\mu^n \varphi, \quad |\partial_x^n \rho_t| \le C(\psi)T\mu^n \varphi^2,$$
$$|\rho_t| \le CT\varphi^2, \quad |\rho_{tt}| \le CT^2\varphi^3.$$

Observe that $\varphi \le \frac{T^2}{4}\varphi^2 \le \frac{T^4}{16}\varphi^3 \le \frac{T^6}{64}\varphi^4 \le \frac{T^8}{256}\varphi^5 \le \frac{T^{10}}{1024}\varphi^6$.

For the term $u^2\{\cdots\} - l_t^2 u^2$ in (3.5), if we choose $\lambda \ge \mu C(\psi)(T+T^2)$ with $C(\psi)$ large enough, then it holds that

$$\begin{aligned}&-C_{0xxxx} - (B_{2x}C_0)_x + (B_2C_0)_{xx} - B_{0t} + 2B_0C_0 - (B_0C_1)_x \\ &\quad + (B_0C_2)_{xx} - (B_0C_3)_{xxx} - l_t^2 \\ &= 20\lambda^7\mu^8\varphi^7\psi_x^8 + R_0,\end{aligned}$$

where $|R_0| \le C(\psi)\lambda^7\mu^7\varphi^7$.

Namely

$$u^2\{\cdots\} - l_t^2 u^2 = 20\lambda^7\mu^8\varphi^7\psi_x^8 u^2 + R_0 u^2. \tag{3.6}$$

Using the same method, we can obtain that

$$\begin{aligned}u_x^2\{\cdots\} &= 10\lambda^5\mu^6\varphi^5\psi_x^6 u_x^2 + R_1 u_x^2, \\ u_{xx}^2\{\cdots\} &= 136\lambda^3\mu^4\varphi^3\psi_x^4 u_{xx}^2 + R_2 u_{xx}^2, \\ u_{xxx}^2\{\cdots\} &= 2\lambda\mu^2\varphi\psi_x^2 u_{xxx}^2 + R_3 u_{xxx}^2,\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}|R_1| &\le C(\psi)\lambda^5\mu^5\varphi^5, \\ |R_2| &\le C(\psi)\lambda^3\mu^3\varphi^3, \\ |R_3| &\le C(\psi)\lambda\mu\varphi.\end{aligned} \tag{3.8}$$

By the Cauchy inequality, we know that there exists a positive constant C such that

$$\begin{aligned}&\int_0^T (du_{xx})^2 + B_2(du_x)^2 - B_0(du)^2 \\ &\ge \frac{1}{2}\int_0^T \theta^2 Z_{xx}^2 dt - C\int_0^T \lambda^2\varphi^2\theta^2 Z_x^2 dt - C\int_0^T \lambda^4\varphi^4\theta^2 Z^2 dt.\end{aligned}$$

It follows from the interpolation inequality that

$$\int_I \lambda^2\varphi^2\theta^2 Z_x^2 dx \le \varepsilon \int_I \theta^2 Z_{xx}^2 dx + C(\varepsilon)\int_I \lambda^4\varphi^4\theta^2 Z^2 dx,$$

this leads to

$$\begin{aligned}&\int_Q [(du_{xx})^2 + B_2(du_x)^2 - B_0(du)^2]dx\\ &\geq (\frac{1}{2}-\varepsilon)\int_Q \theta^2 Z_{xx}^2 dxdt - C\int_Q \lambda^4\varphi^4\theta^2 Z^2 dxdt.\end{aligned}$$

By the equation $dz - z_{xxxx}dt = Lz = hdt + Zdw$, we have

$$\begin{aligned}&2E\int_Q P\theta Lz dx\\ =\ &2E\int_Q P\theta(hdt + Zdw)dx\\ =\ &2E\int_Q P\theta h dxdt\\ \leq\ &E\int_Q P^2 dxdt + E\int_Q \theta^2 h^2 dxdt,\end{aligned}$$

it follows that

$$\begin{aligned}&E\int_Q [(u^2\{\cdots\} - l_t^2 u^2) + u_x^2\{\cdots\} + u_{xx}^2\{\cdots\} + u_{xxx}^2\{\cdots\} + (\frac{1}{2}-\varepsilon)\theta^2 Z_{xx}^2]dxdt\\ &+ E\int_Q (\{\cdot\}_x + \{\cdot\}_{xx} + \{\cdot\}_{xxx})dxdt + E\int_I\int_0^T dM dx\\ &\leq CE\int_Q (\theta^2 h^2 + \lambda^4\varphi^4\theta^2 Z^2)dxdt.\end{aligned}$$

Taking $\varepsilon << 1$, we obtain

$$\begin{aligned}&E\int_Q [(u^2\{\cdots\} - l_t^2 u^2) + u_x^2\{\cdots\} + u_{xx}^2\{\cdots\} + u_{xxx}^2\{\cdots\}]dxdt\\ &+ E\int_Q (\{\cdot\}_x + \{\cdot\}_{xx} + \{\cdot\}_{xxx})dxdt + E\int_I\int_0^T dM dx\\ &\leq CE\int_Q (\theta^2 h^2 + \lambda^4\varphi^4\theta^2 Z^2)dxdt. \qquad (3.9)\end{aligned}$$

Now, we estimate the term $E\int_Q (\{\cdot\}_x + \{\cdot\}_{xx} + \{\cdot\}_{xxx})dxdt$ in (3.9).

Indeed, noting that $z(0,t) = z(1,t) = z_x(0,t) = z_x(1,t) = 0$, we have

$$u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = 0 \quad \forall\ t \in (0,T).$$

Thus

$$\begin{aligned}
&E\int_Q (\{\cdot\}_x+\{\cdot\}_{xx}+\{\cdot\}_{xxx})dxdt\\
&\quad = E\int_Q \{u_{xx}^2(-20\lambda^3\mu^3\varphi^3\psi_x^3+r_1)+u_{xxx}^2(-4\lambda\mu\varphi\psi_x)+u_{xx}u_{xxx}(2\lambda\mu^2\varphi\psi_x^2+r_2)\}_x dxdt\\
&\triangleq V(1)-V(0),
\end{aligned}$$

where

$$\begin{aligned}
&|r_1|\le C\lambda^2\mu^3\varphi^2,\\
&|r_2|\le C\lambda\mu\varphi.
\end{aligned}$$

It holds that for any $\varepsilon>0$, if we choose $\lambda\ge\mu C(\varepsilon,\psi)(T+T^2)$ with $C(\varepsilon,\psi)$ large enough, then

$$\begin{aligned}
&|u_{xxx}u_{xx}(2\lambda\mu^2\varphi\psi_x^2+r_2)(1,t)|\\
&\le \varepsilon\lambda^3\mu^3\varphi^3(1,t)u_{xxx}^2(1,t)+\varepsilon\lambda\mu\varphi(1,t)u_{xx}^2(1,t).
\end{aligned}$$

Note that $\psi_x(1)<0,\ \psi_x(0)<0$, if we choose ε small sufficiently and $\lambda\ge\mu C(\varepsilon,\psi)(T+T^2)$, then there exist positive constants N,K such that

$$\begin{aligned}
V(1)&=E\int_0^T [u_{xx}^2(-20\lambda^3\mu^3\varphi^3\psi_x^3+r_1)+u_{xxx}^2(-4\lambda\mu\varphi\psi_x)+u_{xxx}u_{xx}(2\lambda\mu^2\varphi\psi_x^2+r_2)](1,t)dt\\
&\ge E\int_0^T (-N\lambda^3\mu^3\varphi^3(1,t)\psi_x^3(1)|u_{xx}(1,t)|^2-K\lambda\mu\varphi(1,t)\psi_x(1)|u_{xxx}(1,t)|^2)dt\\
&\ge 0.
\end{aligned}\tag{3.10}$$

Noting that $\lim\limits_{t\to 0^+}\rho(\cdot,t)=\lim\limits_{t\to T^-}\rho(\cdot,t)=-\infty$, we have

$$u(x,0)=u(x,T)=u_x(x,0)=u_x(x,T)=0\quad \forall\, x\in I.$$

It is obvious that

$$E\int_I\int_0^T dMdx=0.\tag{3.11}$$

From (3.6)–(3.11), if we choose $\lambda\ge\mu C(\psi)(T+T^2)$ with $C(\psi)$ large enough, then it holds that

$$\begin{aligned}
&E\int_Q(\lambda\mu^2\varphi\psi_x^2u_{xxx}^2+\lambda^3\mu^4\varphi^3\psi_x^4u_{xx}^2+\lambda^5\mu^6\varphi^5\psi_x^6u_x^2+\lambda^7\mu^8\varphi^7\psi_x^8u^2)dxdt\\
&\le C\Big[V(0)+E\int_Q(\theta^2h^2+\lambda^4\varphi^4\theta^2Z^2)dxdt\\
&\quad +E\int_Q(\lambda\mu\varphi u_{xxx}^2+\lambda^3\mu^3\varphi^3u_{xx}^2+\lambda^5\mu^5\varphi^5u_x^2+\lambda^7\mu^7\varphi^7u^2)dxdt\Big].
\end{aligned}$$

Recall that $|\psi_x| > 0$ in $\overline{I}$, it follows that

$$\begin{aligned}
&E\int_Q (\lambda^7\mu^8\varphi^7u^2 + \lambda^5\mu^6\varphi^5u_x^2 + \lambda^3\mu^4\varphi^3u_{xx}^2 + \lambda\mu^2\varphi u_{xxx}^2)dxdt\\
&\le C(\psi)\Big[V(0) + E\int_Q (\theta^2h^2 + \lambda^4\varphi^4\theta^2Z^2\\
&\qquad + \lambda^7\mu^7\varphi^7u^2 + \lambda^5\mu^5\varphi^5u_x^2 + \lambda^3\mu^3\varphi^3u_{xx}^2 + \lambda\mu\varphi u_{xxx}^2)dxdt\Big],
\end{aligned}$$

from which if we choose $\mu_0 = C(\psi) + 1$, then it holds that

$$\begin{aligned}
&E\int_Q (\lambda^7\mu^7\varphi^7u^2 + \lambda^5\mu^5\varphi^5u_x^2 + \lambda^3\mu^3\varphi^3u_{xx}^2 + \lambda\mu\varphi u_{xxx}^2)dxdt\\
&\le C_1(\psi)\Big[V(0) + E\int_Q (\theta^2h^2 + \lambda^4\varphi^4\theta^2Z^2)dxdt\Big].
\end{aligned}$$

Then

$$\begin{aligned}
&E\int_Q (\lambda^7\mu^7\varphi^7u^2 + \lambda^5\mu^5\varphi^5u_x^2 + \lambda^3\mu^3\varphi^3u_{xx}^2 + \lambda\mu\varphi u_{xxx}^2)dxdt\\
&\le C\Big[E\int_0^T (\lambda^3\mu^3\varphi^3(0,t)|u_{xx}(0,t)|^2 + \lambda\mu\varphi(0,t)|u_{xxx}(0,t)|^2)dt + E\int_Q (\theta^2h^2 + \lambda^4\varphi^4\theta^2Z^2)dxdt\Big],
\end{aligned}$$

from which it holds that

$$\begin{aligned}
&E\int_Q (\lambda^7\varphi^7u^2 + \lambda^5\varphi^5u_x^2 + \lambda^3\varphi^3u_{xx}^2 + \lambda\varphi u_{xxx}^2)dxdt\\
&\le C(\mu)\Big[E\int_0^T (\lambda^3\varphi^3(0,t)u_{xx}^2(0,t) + \lambda\varphi(0,t)u_{xxx}^2(0,t))dt + E\int_Q (\theta^2h^2 + \lambda^4\varphi^4\theta^2Z^2)dxdt\Big].
\end{aligned}$$

Returning u to θz, we can obtain (1.3) for (3.3).

Thus, we have proved (1.3) for (3.3).

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. By using the dual method in [20, P3, Theorem 1.1], we can prove Theorem 1.2. □

4. Proofs of Corollary 1.1, Proposition 1.1 and Corollary 1.2

First, we prove Corollary 1.1.

Proof of Corollary 1.1. Step 1. We should prove a observability estimate: if (z, Z) solve

$$\begin{cases}
dz - (kz_{xx} + z_{xxxx})dt = (-az - bZ)dt + Zdw & \text{in } Q,\\
z(0,t) = 0 = z(1,t) & \text{in } (0,T),\\
z_x(0,t) = 0 = z_x(1,t) & \text{in } (0,T),\\
z(x,T) = z_T(x) & \text{in } I,
\end{cases} \tag{4.1}$$

with the terminal state $z_T \in L^2(\Omega, \mathcal{F}_T, P; H_0^2(I))$, we have

$$\|z(\cdot,0)\|_{H_0^2(I)} \le C\Big[\|z_{xx}(0,\cdot)\|_{L^2_{\mathcal{F}}(\Omega,L^2(0,T))} + \|z_{xxx}(0,\cdot)\|_{L^2_{\mathcal{F}}(\Omega,L^2(0,T))} + \|Z\|_{L^2_{\mathcal{F}}(0,T;L^2(I))}\Big], \tag{4.2}$$

where $C = C(a,b,k,T)$.

Indeed, taking $p=-a, q=-b, h=0$ in Theorem 1.1, we have

$$\begin{aligned} &E\int_Q \varphi^3\theta^2 z_{xx}^2 dxdt + E\int_Q \varphi^5\theta^2 z_x^2 dxdt + E\int_Q \varphi^7\theta^2 z^2 dxdt \\ &\le C\Big(E\int_Q \varphi^4\theta^2 Z^2 dxdt \\ &\quad + E\int_0^T (\varphi^3(0,t)\theta^2(0,t)z_{xx}^2(0,t) + \varphi(0,t)\theta^2(0,t)z_{xxx}^2(0,t))dt\Big), \end{aligned}$$

this implies

$$\begin{aligned} E\int_{\frac{T}{4}}^{\frac{3T}{4}} \|z(t)\|^2_{H_0^2(I)} dt \le C &\frac{\max\limits_{(x,t)\in Q}\varphi^4(x,t)\theta^2(x,t) + \max\limits_{t\in[0,T]}(\varphi^3(0,t)\theta^2(0,t) + \varphi(0,t)\theta^2(0,t))}{\min\{\min\limits_{x\in I}\varphi^7(x,\frac{T}{2})\theta^2(x,\frac{T}{4}), \min\limits_{x\in I}\varphi^5(x,\frac{T}{2})\theta^2(x,\frac{T}{4}), \min\limits_{x\in I}\varphi^3(x,\frac{T}{2})\theta^2(x,\frac{T}{4})\}} \\ &\cdot \Big(E\int_Q Z^2 dxdt + E\int_0^T (z_{xx}^2(0,t) + z_{xxx}^2(0,t))dt\Big). \end{aligned}$$

It follows from Proposition 2.1 that

$$E\|z(t_1)\|^2_{H_0^2(I)} \le CE\|z(t_2)\|^2_{H_0^2(I)},$$

for any $0 \le t_1 \le t_2 \le T$. Then, it holds that

$$E\|z(0)\|^2_{H_0^2(I)} \le C\Big(E\int_Q Z^2 dxdt + E\int_0^T (z_{xx}^2(0,t) + z_{xxx}^2(0,t))dt\Big),$$

namely, (4.2) holds.

Step 2. We should prove Corollary 1.1.

It is easy to see the dual system of (1.4) is (4.1).

Let us set

$$\mathcal{W} = \{(z_{xx}(0,t), z_{xxx}(0,t), Z) \mid (z,Z) \text{ solves } (4.1) \text{ with some } z_T \in L^2(\Omega,\mathcal{F}_T,P;H_0^2(I))\}.$$

Clearly, $\mathcal{W}$ is a linear subspace of $L^2_{\mathcal{F}}(\Omega, L^2(0,T)) \times L^2_{\mathcal{F}}(\Omega, L^2(0,T)) \times L^2_{\mathcal{F}}(0,T;L^2(I))$. Let us define a linear functional $\mathcal{L}$ on $\mathcal{W}$ as follows:

$$\mathcal{L}((z_{xx}(0,t), z_{xxx}(0,t), Z)) = -E(y_0, z(\cdot,0))_{H^{-2}(I),H_0^2(I)}.$$

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From Theorem 1.1, we see that $\mathcal{L}$ is a bounded linear functional on $\mathcal{W}$. By means of the Hahn–Banach Theorem, $\mathcal{L}$ can be extended to be a bounded linear functional on the space $L^2_{\mathcal{F}}(\Omega, L^2(0,T)) \times L^2_{\mathcal{F}}(\Omega, L^2(0,T)) \times L^2_{\mathcal{F}}(0,T; L^2(I))$. For simplicity, we still use $\mathcal{L}$ to denote this extension. Now, by the Riesz Representation Theorem, we know that there is a random fields

$$(u_1, u_2, g) \in L^2_{\mathcal{F}}(\Omega, L^2(0,T)) \times L^2_{\mathcal{F}}(\Omega, L^2(0,T)) \times L^2_{\mathcal{F}}(0,T; L^2(I))$$

such that

$$\begin{aligned} & -E(y_0, z(\cdot,0))_{H^{-2}(I),H_0^2(I)} \\ & \quad = E\int_0^T [-u_1(t)z_{xxx}(0,t) + u_2(t)z_{xx}(0,t) + (g,Z)_{H^{-2}(I),H_0^2(I)}]dt. \end{aligned} \tag{4.3}$$

We claim that this random fields (u_1, u_2, g) is the control we need.

In fact, from the definition of the solution to (1.4), we have

$$\begin{aligned} & E(y(T), z_T)_{H^{-2}(I),H_0^2(I)} - E(y_0, z(\cdot,0))_{H^{-2}(I),H_0^2(I)} \\ & \quad = E\int_0^T [-u_1(t)z_{xxx}(0,t) + u_2(t)z_{xx}(0,t) + (g,Z)_{H^{-2}(I),H_0^2(I)}]dt, \end{aligned}$$

thus, it follows from (4.3) that

$$E(y(T), z_T)_{H^{-2}(I),H_0^2(I)} = 0. \tag{4.4}$$

Since z_T can be arbitrary element in $L^2(\Omega, \mathcal{F}_T, P; H_0^2(I))$, from the equality (4.4), we get $y(T) = 0$ in I, P-a.s.

This completes the proof of Corollary 1.1. $\square$

Then, we prove Proposition 1.1. The main idea comes from [22].

Proof of Proposition 1.1. The proof of Proposition 1.1 is achieved by the contradiction argument.

We take $u_1 = u_2 \equiv 0$ in (1.4), (1.4) becomes

$$\begin{cases} dy + (ky_{xx} + y_{xxxx})dt = aydt + (by+g)dw & \text{in } Q, \\ y(0,t) = 0 = y(1,t) & \text{in } (0,T), \\ y_x(0,t) = 0 = y_x(1,t) & \text{in } (0,T), \\ y(x,0) = y_0(x) & \text{in } I. \end{cases} \tag{4.5}$$

The solution of system (4.5) is

$$y(T) = S(T)y_0 + \int_0^T S(T-t)aydt + \int_0^T S(T-t)(by+g)dw,$$

where $\{S(t)\}_{t\ge 0}$ is the semigroup generated by Λ.

Taking $y_0 = 1$, if we can choose $g \in L^2_{\mathcal{F}}(0,T;L^2(I))$ such that $y(T) = 0$, then we have

$$0 = ES(T)y_0 + E\int_0^T S(T-t)aydt. \tag{4.6}$$

In order to present the key idea in the simplest way, we only consider a very special case of system (4.5), that is, $a = 0$. (4.6) becomes

$$0 = S(T)y_0. \tag{4.7}$$

However, it follows from the property of $\{S(t)\}_{t\geq 0}$ that

$$S(T)y_0 > 0,$$

this contradicts (4.7).

This completes the proof of Proposition 1.1. □

At last, we prove Corollary 1.2.

Proof of Corollary 1.2. According to Theorem 1.2, if λ is large enough, it holds that

$$\begin{aligned} &E\int_Q [\lambda\varphi\theta^2 z^2_{xxx} + \lambda^3\varphi^3\theta^2 z^2_{xx} + \lambda^5\varphi^5\theta^2 z^2_x + \lambda^7\varphi^7\theta^2 z^2]dxdt \\ &\leq CE\int_0^T (\lambda^3\varphi^3(0,t)\theta^2(0,t)z^2_{xx}(0,t) + \lambda\varphi(0,t)\theta^2(0,t)z^2_{xxx}(0,t))dt. \end{aligned}$$

It is easy to see $E\int_Q [\lambda\varphi\theta^2 z^2_{xxx} + \lambda^3\varphi^3\theta^2 z^2_{xx} + \lambda^5\varphi^5\theta^2 z^2_x + \lambda^7\varphi^7\theta^2 z^2]dxdt = 0$, thus,

$$z \equiv 0 \quad \text{in} \quad Q, \quad P-a.s. \quad \square$$

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