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Global existence and blow-up of solutions to a nonlocal parabolic equation with singular potential ☆

Min Feng, Jun Zhou *

School of Mathematics and Statistics, Southwest University, Chongqing 400715, People's Republic of China

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A B S T R A C T

In this paper, we study a nonlocal parabolic equation with singular potential on a bounded smooth domain with homogeneous Neumann boundary condition. Firstly, we find a threshold of global existence and blow-up to the solutions of the problem when the initial data is at the low energy level, i.e., $J(u_0) \le d$, where $J(u_0)$ is the initial energy and d is the mountain-pass level. Moreover, when $J(u_0) < d$, the vacuum isolating behavior of the solutions is also discussed. Secondly, we prove that there exist solutions of the problem with arbitrary initial energy that blow up in finite time. We also obtain the upper bounds of the blow-up time for blow-up solutions.

1. Introduction

In this paper, we consider the following nonlocal parabolic equation with a singular potential:

$$\begin{cases} |x|^{-s}u_t - \Delta u = |u|^{p-1}u - \dfrac{|x|^{-s}}{\int_\Omega |x|^{-s}dx}\displaystyle\int_\Omega |u|^{p-1}u dx, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded connected domain in $\mathbb{R}^n$ $(n \ge 3)$ with smooth boundary $\partial\Omega$, ν is the unit outer vector on $\partial\Omega$, $x = (x_1, x_2, \cdots, x_n) \in \Omega \subset \mathbb{R}^n$ and

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* Corresponding author.
E-mail addresses: fm13110287656@163.com (M. Feng), jzhouwm@163.com (J. Zhou).

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$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}.$$

The parameters p and s satisfy

$$\begin{cases} 1 < p < \frac{n+2}{n-2}, \\ 0 \le s < \frac{n(p-1)}{p+1}, \end{cases} \tag{1.2}$$

and $0 \not\equiv u_0(x) \in W$, where

$$W := \left\{ \phi \in H^1(\Omega) : \int\limits_\Omega |x|^{-s}\phi(x)dx = 0 \right\}. \tag{1.3}$$

Throughout the paper, we denote the norm of $L^q(\Omega)$ for $1 \le q \le \infty$ by $\|\cdot\|_q$ and the norm of $H^1(\Omega)$ by $\|\cdot\|_{H^1(\Omega)}$. That is, for any $\phi \in L^q(\Omega)$,

$$\|\phi\|_q = \begin{cases} \left(\int\limits_\Omega |\phi(x)|^q dx \right)^{\frac{1}{q}}, & \text{if } 1 \le q < \infty; \\ \text{ess sup}_{x\in\Omega}|\phi(x)|, & \text{if } q = \infty, \end{cases}$$

and

$$\|\phi\|_{H^1(\Omega)} = \sqrt{\|\phi\|_2^2 + \|\nabla\phi\|_2^2}, \quad \forall \phi \in H^1(\Omega).$$

We also denote the maximal existence time of the solution for problem (1.1) by $T_{\max}$ and let

$$R := \sup_{x\in\Omega} |x|. \tag{1.4}$$

Remark 1.1. Before introducing the previous studies, let's firstly give some instructions about (1.1) and (1.3), which indicate the term $\left(\int_\Omega |x|^{-s}dx\right)^{-1}$ and the set W are well-defined.

(1) When $n \ge 3$ and s, p satisfy the assumption of (1.2), we have

$$\begin{aligned} 0 < \int\limits_\Omega |x|^{-s}dx &\le \int\limits_{B(0,R)} |x|^{-s}dx \\ &= \int\limits_0^R \left[\int\limits_{\partial B(0,r)} |x|^{-s}dS(x) \right] dr \\ &= \omega_n \int\limits_0^R r^{-s}r^{n-1}dr \\ &= \frac{\omega_n}{n-s}R^{n-s} \\ &< \infty, \end{aligned}$$

where

$$\omega_n = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

is the surface area of the unit sphere $\partial B(0,1)$ (see [7, p. 699]), which means the term $\left(\int_\Omega |x|^{-s}dx\right)^{-1}$ of (1.1) makes sense.

(2) The term $\int_\Omega |x|^{-s}\phi(x)dx$ in (1.3) is well defined for any $\phi \in H^1(\Omega)$. In fact, by using the condition of $1 < p < \frac{n+2}{n-2}$, one can see that $H^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ continuously, i.e., there exists a positive constant C depending only on p, n and Ω such that

$$\|\phi\|_{p+1} \le C\|\phi\|_{H^1(\Omega)}, \quad \forall \phi \in H^1(\Omega).$$

Hence, for any $\phi \in H^1(\Omega)$, by using Hölder's inequality we have

$$\begin{aligned}
\left|\int_\Omega |x|^{-s}\phi(x)dx\right| &\le \int_\Omega |x|^{-s}|\phi(x)|dx \\
&\le \left(\int_\Omega |x|^{\frac{-s(p+1)}{p}}dx\right)^{\frac{p}{p+1}} \left(\int_\Omega |\phi(x)|^{p+1}dx\right)^{\frac{1}{p+1}} \\
&= \left(\int_\Omega |x|^{\frac{-s(p+1)}{p}}dx\right)^{\frac{p}{p+1}} \|\phi(x)\|_{p+1} \qquad (1.5)\\
&\le C\left(\int_\Omega |x|^{\frac{-s(p+1)}{p}}dx\right)^{\frac{p}{p+1}} \|\phi(x)\|_{H^1(\Omega)} \\
&= C\widetilde{C}\|\phi(x)\|_{H^1(\Omega)},
\end{aligned}$$

where

$$\widetilde{C} := \left(\int_\Omega |x|^{\frac{-s(p+1)}{p}}dx\right)^{\frac{p}{p+1}} < \infty.$$

In fact, since $0 \le s < \frac{n(p-1)}{p+1} < \frac{np}{p+1}$, we have

$$\begin{aligned}
\left(\int_\Omega |x|^{\frac{-s(p+1)}{p}}dx\right)^{\frac{p}{p+1}} &\le \left(\int_{B(0,R)} |x|^{\frac{-s(p+1)}{p}}dx\right)^{\frac{p}{p+1}} \\
&= \left[\int_0^R \left(\int_{\partial B(0,r)} |x|^{\frac{-s(p+1)}{p}}dS(x)\right)dr\right]^{\frac{p}{p+1}} \\
&= \left(\omega_n \int_0^R r^{\frac{-s(p+1)}{p}} r^{n-1}dr\right)^{\frac{p}{p+1}} \qquad (1.6)
\end{aligned}$$

$$= \left(\frac{\omega_n}{n - \frac{s(p+1)}{p}} R^{n - \frac{s(p+1)}{p}} \right)^{\frac{p}{p+1}}$$

$$< \infty.$$

(3) Actually, when p, s satisfy (1.2) one can see that $(W, \|\cdot\|_{H^1(\Omega)})$ is a Banach space (see Lemma 3.1), and $\|\nabla(\cdot)\|_2$ is an equivalent norm to the norm $\|\cdot\|_{H^1(\Omega)}$ (see Remark 3.2). So in the following, we equip W with the norm

$$\|\phi\| := \|\nabla \phi\|_2, \tag{1.7}$$

then $(W, \|\cdot\|)$ is a Banach space. Furthermore, $W \hookrightarrow L^{p+1}(\Omega)$ continuously if p satisfies (1.2). We denote by B the optimal embedding constant, i.e.

$$\|\phi\|_{p+1} \leq B\|\phi\|, \quad \forall \phi \in W, \tag{1.8}$$

and

$$B^{-1} = \inf_{\phi \in W \setminus \{0\}} \frac{\|\phi\|}{\|\phi\|_{p+1}}.$$

Next we review some previous studies on problem (1.1). It is well known that nonlocal parabolic type equations like (1.1) can be used in ecology, population dynamics, thermal physics and so on (see [9,12,22,23,34,41] and the references therein). If we ignore the nonlocal term of problem (1.1), we get the following model

$$|x|^{-s} u_t - \Delta u = |u|^{p-1} u. \tag{1.9}$$

Problem (1.9) with $s = 0$ in bounded domain with homogenous Dirichlet boundary condition or in $\mathbb{R}^n$ has been studied by many authors. In particular, we mention critical-point theory and the mountain pass theorem by Ambrosetti–Rabinowitz [1], the potential well theory which started with the paper by Tsutsumi [45] (see also [36]), semigroup theory for which the starting point seems the paper by Weissler [49], classical tools such as smoothing effects and comparison methods revisited in a new functional analytic framework as in the paper by Hoshino–Yamada [17] (see also previous work in the monograph by Henry [16]). For the studies of global existence and blow-up, we refer to [8,11,15,39,50] and references therein. For $s > 0$, problem (1.9) in bounded domain with homogenous Dirichlet boundary condition has been studied in [2,14,44,47,54,55], and the global existence and blow-up conditions were gotten by means of potential well method.

If we denote the source term (i.e., the right-hand side of problem (1.1)) by $F(x, t)$, then it is easy to see

$$\int_\Omega F(x, t) dx = 0, \quad \forall t \geq 0,$$

so that the quantity $\int_\Omega |x|^{-s} u(t) dx$ is conserved by using the homogeneous Neumann condition. This type of source is related to Navier–Stokes equations on an infinite slab [42] (also see other reasons explained in [3,18]), which has been used extensively in evolution equations with conserved quantity (see [13,29,33,38] for p-Laplace equation, [20,27] for Gierer–Meinhardt system, [4,6,28,37,43,52] for Thin-film equation). Especially, problem (1.1) with $s = 0$, i.e., the following

$$u_t - \Delta u = |u|^{p-1} u - \frac{1}{|\Omega|} \int_\Omega |u|^{p-1} u dx$$

in bounded domain Ω with homogeneous Neumann boundary condition was studied in [3,10,18,19,21,42,48], where the local existence and uniqueness of weak solutions, the conditions on global existence and blow-up, the blow-up rate and the estimates of the blow-up time, the asymptotic behavior of the global solutions were established.

In this paper, we mainly study the global existence and blow-up conditions for the solutions of problem (1.1). Our proof is based on the potential well method which was firstly introduced by Sattinger [40] (or see [36]) and the concavity method which was established by Levine [24,25]. Furthermore, we will discuss the vacuum isolating behavior of the solutions, which means that there is a region which does not contain any low energy solutions. The vacuum isolating phenomena was firstly studied by Liu and Zhao [32] with the help of a family of potential wells. Now it has been used to various kinds of evolution equations with variational structures (see [5,26,30,35,51,53] and references therein).

The remaining parts of this paper are organized as follows. In Section 2, we will state the main results of this paper. In Section 3, we will give some preliminaries, and then we will prove the main results in Section 4.

2. Main results

In this section, we will give the main results of this paper. Firstly, we introduce some functionals and notations. Let W be the set defined in (1.3). For any $\phi \in W$, we define two functionals $J(\phi)$ and $I(\phi)$ by

$$J(\phi) := \frac{1}{2}\|\phi\|^2 - \frac{1}{p+1}\|\phi\|_{p+1}^{p+1} \tag{2.1}$$

and

$$I(\phi) := \langle J'(\phi), \phi\rangle = \|\phi\|^2 - \|\phi\|_{p+1}^{p+1}. \tag{2.2}$$

Then we can define the Nehari manifold by

$$N := \{\phi \in W \setminus \{0\} : I(\phi) = 0\}. \tag{2.3}$$

Furthermore, the mountain passing level d can be defined by

$$d := \inf_{\phi \in N} J(\phi). \tag{2.4}$$

By Lemma 3.3, we know that

$$d = \left(\frac{1}{2} - \frac{1}{p+1}\right)\alpha_1^2, \tag{2.5}$$

where

$$\alpha_1 := B^{-\frac{p+1}{p-1}}. \tag{2.6}$$

For $\delta > 0$, we let

$$\begin{aligned} I_\delta(\phi) &:= \delta\|\phi\|^2 - \|\phi\|_{p+1}^{p+1}, \\ N_\delta &:= \{\phi \in W \setminus \{0\} : I_\delta(\phi) = 0\}, \\ d(\delta) &:= \inf_{\phi \in N_\delta} J(\phi). \end{aligned} \tag{2.7}$$

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Now, we can state the main results of this paper. At first, we consider the case of $J(u_0) < d$, and the results are presented as the following three theorems, which are concerned with blow-up (Theorem 2.1), global existence (Theorem 2.3) and vacuum region (Theorem 2.4).

Theorem 2.1. *Assume the constants p, s satisfy (1.2) and $u_0 \in W$ satisfies*

$$J(u_0) < d \tag{2.8}$$

and

$$I(u_0) < 0 \ (\text{or equivalently } \|u_0\| > \alpha_1 \ (\text{see Lemma 3.5})). \tag{2.9}$$

Let $u = u(t)$ be a corresponding solution to problem (1.1). Then $T_{\max} < \infty$ and

$$\lim_{t \to T_{\max}^-} \left\| |x|^{-\frac{s}{2}} u(t) \right\|_2 = \infty.$$

Furthermore, $T_{\max}$ can be estimated by

$$T_{\max} \leq \begin{cases} \dfrac{(p+1) C_0^{\frac{p+1}{2}} \||x|^{-\frac{s}{2}} u_0\|_2^{-(p-1)}}{(p-1)^2 \left[1 - \left(\dfrac{p+1}{2}\right)^{-\frac{p+1}{p-1}}\right]}, & \text{if } J(u_0) \leq 0; \\ \dfrac{(p+1) C_0^{\frac{p+1}{2}} \||x|^{-\frac{s}{2}} u_0\|_2^{-(p-1)}}{(p-1)^2 \left\{1 - \left[(p+1)\left(\dfrac{1}{2} - \dfrac{J(u_0)}{\alpha_1^2}\right)\right]^{-\frac{p+1}{p-1}}\right\}}, & \text{if } 0 < J(u_0) < d, \end{cases} \tag{2.10}$$

where

$$C_0 := \left(\int_\Omega |x|^{\frac{-s(p+1)}{p-1}} dx \right)^{\frac{p-1}{p+1}}. \tag{2.11}$$

Remark 2.2. We give some remarks on the above theorem.

(1) Similar to [54, Theorem 1.1] or [55, Theorem 1.4], one can prove problem (1.1) admits a local (weak) solution if (1.2) holds. That is, there exists a function $u(t) \in L^\infty(0, T_{\max}; W)$ with $|x|^{-\frac{s}{2}} u_t \in L^2(0, T_{\max}; L^2(\Omega))$ such that $u(t)$ satisfies (1.1) in the distribution sense.
(2) C_0 is a positive bounded constant and satisfies

$$C_0 \leq \underbrace{\left(\frac{\omega_n}{n - \frac{s(p+1)}{p-1}} R^{n - \frac{s(p+1)}{p-1}} \right)^{\frac{p-1}{p+1}} < \infty}_{\text{since (1.2)}},$$

where R is the positive constant given in (1.4). In fact, since $\Omega \subset B(0, R)$, by (1.2), we have

$$\int_\Omega |x|^{\frac{-s(p+1)}{p-1}} dx \leq \int_{B(0,R)} |x|^{\frac{-s(p+1)}{p-1}} dx$$

$$= \int_0^R \left[\int_{\partial B(0,r)} |x|^{\frac{-s(p+1)}{p-1}} dS(x) \right] dr$$

$$= \omega_n \int_0^R r^{\frac{-s(p+1)}{p-1}} r^{n-1} dr$$

$$= \frac{\omega_n}{n - \frac{s(p+1)}{p-1}} R^{n-\frac{s(p+1)}{p-1}}.$$

(3) For any $u \in W$, by using Hölder's inequality and the above discussions we can estimate

$$\begin{aligned} \left\| |x|^{-\frac{s}{2}} u \right\|_2^2 &= \int_\Omega |x|^{-s} u^2 dx \\ &\leq \left(\int_\Omega (u^2)^{\frac{p+1}{2}} dx \right)^{\frac{2}{p+1}} \left(\int_\Omega (|x|^{-s})^{\frac{p+1}{p-1}} dx \right)^{\frac{p-1}{p+1}} \\ &= \left(\int_\Omega |x|^{\frac{-s(p+1)}{p-1}} dx \right)^{\frac{p-1}{p+1}} \|u\|_{p+1}^2 \\ &= C_0 \|u\|_{p+1}^2 \\ &\underbrace{\leq C_0 B^2 \|u\|^2}_{\text{since (1.8)}} \\ &< \infty. \end{aligned} \tag{2.12}$$

Thus $\||x|^{-\frac{s}{2}} u(t)\|_2$ is well-defined for $0 \leq t < T_{\max}$ since $u(t) \in W$.

(4) For $u_0 \in W$ satisfying (2.8) and (2.9), we claim $\||x|^{-\frac{s}{2}} u_0\|_2 > 0$. In fact, if $\||x|^{-\frac{s}{2}} u_0\|_2 = 0$, then $u_0(x) = 0$ for a.e. $x \in \Omega$. Since $u_0 \in W$, we get u_0 is the zero element of W, and then $I(u_0) = 0$, which contradicts $I(u_0) < 0$. Furthermore, by Lemma 3.7 we know that

$$(p+1)\left(\frac{1}{2} - \frac{J(u_0)}{\alpha_1^2} \right) > 1.$$

So the right-hand side of (2.10) is well-defined.

The next theorem is about the conditions of global existence.

Theorem 2.3. *Assume the constants p, s satisfy (1.2) and $u_0 \in W$ satisfies*

$$J(u_0) < d$$

and

$$I(u_0) > 0 \ (\textit{or equivalently } \|u_0\| < \alpha_1 \ (\textit{see Lemma } 3.5)).$$

Let $u = u(t)$ be a corresponding solution to problem (1.1). Then $u(t)$ exists globally.

Next, we give the results about the vacuum isolating behavior of the solutions. To this end, we need some properties of $d(\delta)$ given in Lemma 3.3. That is, for any $e \in (0, d)$, the equation $d(\delta) = e$ admits two positive roots δ_1 and δ_2, which satisfy

$$0 < \delta_1 < 1 < \delta_2 < \frac{p+1}{2}.$$

Theorem 2.4. *Assume the constants p and s satisfy (1.2). Let δ_1 and δ_2 be the two roots of the equation $d(\delta) = e$ for $e \in (0, d)$. Then there is no solution of (1.1) in U_e with $u_0 \in W$ satisfying $J(u_0) \le e$, where*

$$U_e := \left\{ \phi \in W : \alpha_1 \delta_1^{\frac{1}{p-1}} < \|\phi\| < \alpha_1 \delta_2^{\frac{1}{p-1}} \right\}. \tag{2.13}$$

Remark 2.5. Since there is no solution in U_e, the space W can be written as $W = G_e \cup U_e \cup B_e$ such that G_e and B_e are both invariant sets for the solutions of problem (1.1), where

$$G_e := \left\{ \phi \in W : \|\phi\| \le \alpha_1 \delta_1^{\frac{1}{p-1}} \right\} \tag{2.14}$$

and

$$B_e := \left\{ \phi \in W : \|\phi\| \ge \alpha_1 \delta_2^{\frac{1}{p-1}} \right\}. \tag{2.15}$$

Moreover, since $J(u_0) \le e < d$, by the definition of the d in (2.4), we have $I(u_0) < 0$ or $I(u_0) > 0$ when u_0 is nontrivial. In view of Theorems 2.1 and 2.3, we have:

- If $J(u_0) < d$ and $I(u_0) < 0$ then $u(t) \in B_e$ for all $t \in [0, T_{\max})$ and $T_{\max} < \infty$;
- If $J(u_0) < d$ and $I(u_0) > 0$ then $u(t) \in G_e$ for all $t \in [0, T_{\max})$ and $T_{\max} = \infty$.

Secondly, we consider the case of $J(u_0) = d$, and the results are the following three theorems, which are concerned with global existence (Theorems 2.6 and 2.8), blow-up (Theorem 2.7).

Theorem 2.6. *Assume the constants p, s satisfy (1.2) and $u_0 \in W$ satisfies*

$$J(u_0) = d \tag{2.16}$$

and

$$I(u_0) > 0 \ (\text{or equivalently } \|u_0\| < \alpha_1 \ (\text{see Lemma } 3.5)). \tag{2.17}$$

Then problem (1.1) admits a global solution.

Theorem 2.7. *Assume the constants p, s satisfy (1.2) and $u_0 \in W$ satisfies*

$$J(u_0) = d \tag{2.18}$$

and

$$I(u_0) < 0 \ (\text{or equivalently } \|u_0\| > \alpha_1 \ (\text{see Lemma } 3.5)). \tag{2.19}$$

Let $u = u(t)$ be a corresponding solution to problem (1.1). Then $T_{\max} < \infty$ and

$$\lim_{t \to T_{\max}^-} \left\| |x|^{-\frac{s}{2}} u(t) \right\|_2 = \infty.$$

Theorem 2.8. *Assume the constants p, s satisfy (1.2) and $u_0 \in W$ satisfies*

$$J(u_0) = d \tag{2.20}$$

and

$$I(u_0) = 0 \ (\text{or equivalently } \|u_0\| = \alpha_1 \ (\text{see Lemma 3.5})). \tag{2.21}$$

Then u_0 is a stationary solution of problem (1.1), i.e., u_0 satisfies

$$\begin{cases} -\Delta u = |u|^{p-1}u - \dfrac{|x|^{-s}}{\int_\Omega |x|^{-s} dx} \displaystyle\int_\Omega |u|^{p-1} u dx, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \\ u \in W. \end{cases} \tag{2.22}$$

So, problem (1.1) admits a global solution $u(t) \equiv u_0$.

Finally, we give another blow-up result, which indicates the solution of problem (1.1) may blow up at arbitrary initial energy level. To this end, we denote by λ_1 the least eigenvalue of the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda |x|^{-s} u, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \\ u \in W, \end{cases} \tag{2.23}$$

i.e.,

$$\lambda_1 = \inf_{\phi \in W \setminus \{0\}} \frac{\|\phi\|^2}{\left\| |x|^{-\frac{s}{2}} \phi \right\|_2^2}. \tag{2.24}$$

By (2.12), we have

$$\left\| |x|^{-\frac{s}{2}} \phi \right\|_2^2 \le C_0 B^2 \|\phi\|^2.$$

Then

$$\lambda_1 \ge \frac{1}{C_0 B^2} > 0.$$

Theorem 2.9. *Assume the constants p, s satisfy (1.2) and $u_0 \in W$ satisfies*

$$J(u_0) < \frac{(p-1)\lambda_1}{2(p+1)} \left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2. \tag{2.25}$$

Let $u = u(t)$ be a corresponding solution to problem (1.1). Then $T_{\max} < \infty$ and

$$T_{\max} \leq \frac{8(p+1)\left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2}{(p-1)^2[(p-1)\lambda_1 \left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2 - 2(p+1)J(u_0)]}. \tag{2.26}$$

3. Preliminaries

In this section, we give some preparative lemmas, which are important to the proofs of the main theorems. The first lemma is concerned with the norms of the linear subspace W of $H^1(\Omega)$, which indicates $(W, \|\cdot\|)$ is a Banach space, where W and $\|\cdot\|$ are given in (1.3) and (1.7) respectively.

Lemma 3.1. *Assume p and s satisfy (1.2), then $\left(W, \|\cdot\|_{H^1(\Omega)}\right)$ is a Banach space and the Poincaré's inequality holds in W. That is to say, there exists a constant $\overline{C}$, depending only on n and Ω, such that*

$$\|\phi\|_2 \leq \overline{C}\|\nabla\phi\|_2 \tag{3.1}$$

for each function $\phi \in W$.

Remark 3.2. By using (3.1), it is easy to see $\|\cdot\|$ is an equivalent norm of $\|\cdot\|_{H^1(\Omega)}$ in W. Then by the above lemma, $(W, \|\cdot\|)$ is a Banach space.

Proof of Lemma 3.1. To prove $\left(W, \|\cdot\|_{H^1(\Omega)}\right)$ is a Banach space, it is sufficient to show W is a closed subspace of $H^1(\Omega)$. Obviously, W is a linear subspace of $H^1(\Omega)$. So, we only need to prove: for any sequence $\{\phi_n\}_{n=1}^{\infty} \subset W$ and $\phi_n \to \phi$ $(n \to \infty)$ in $H^1(\Omega)$, then $\phi \in W$. In fact, by $\phi_n \in W$, we have $\int_\Omega |x|^{-s}\phi_n dx = 0$. Then by a similar estimation as (1.5), we get

$$\begin{aligned} \left| \int_\Omega |x|^{-s}\phi dx \right| &= \left| \int_\Omega |x|^{-s}(\phi_n - \phi) dx \right| \\ &\leq C\widetilde{C}\|\phi_n - \phi\|_{H^1(\Omega)} \\ &\to 0 \ (n \to \infty), \end{aligned}$$

which implies $\int_\Omega |x|^{-s}\phi dx = 0$, i.e., $\phi \in W$.

To prove the Poincaré's inequality (3.1), we use a similar argument as in [7, p. 290]. We argue by contradiction. Suppose (3.1) is not true, then for each $k = 1, 2, \cdots$, there exists a function $\tilde{\phi}_k \in W$ satisfying

$$\|\tilde{\phi}_k\|_2 > k\|\nabla\tilde{\phi}_k\|_2. \tag{3.2}$$

We re-normalize by defining

$$\phi_k := \frac{\tilde{\phi}_k}{\|\tilde{\phi}_k\|_2}. \tag{3.3}$$

Then

$$\|\phi_k\|_2 = 1, \tag{3.4}$$

$$\int_\Omega |x|^{-s}\phi_k dx = 0, \tag{3.5}$$

and

$$\|\nabla \phi_k\|_2 < \frac{1}{k}, \quad k = 1, 2, \cdots . \tag{3.6}$$

In particular, (3.4) and (3.6) imply the functions $\{\phi_k\}_{k=1}^{\infty}$ are bounded in W. By the Rellich–Kondrachov Compactness Theorem (see [7, p. 286]), we know that $H^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ compactly when p satisfies (1.2). Hence, in view of $H^1(\Omega)$ is reflexive, there exists a subsequence, which we still denote by $\{\phi_k\}_{k=1}^{\infty}$ as well, and a function $\phi \in H^1(\Omega) \subset L^{p+1}(\Omega)$ such that

$$\phi_k \to \phi \text{ in } L^{p+1}(\Omega). \tag{3.7}$$

Then we have

- $\|\phi\|_2 = 1$. In fact, it follows from $L^{p+1}(\Omega) \hookrightarrow L^2(\Omega)$, (3.4) and (3.7) that

$$\|\phi\|_2 \le \|\phi - \phi_k\|_2 + \|\phi_k\|_2 \to 1 \ (k \to \infty)$$

  and

$$\|\phi\|_2 \ge -\|\phi - \phi_k\|_2 + \|\phi_k\|_2 \to 1 \ (k \to \infty).$$

- $\phi \in W$. In fact, by Hölder's inequality, (3.5) and (3.7) we have

$$\begin{aligned} \left|\int_\Omega |x|^{-s}\phi dx\right| &= \left|\int_\Omega |x|^{-s}(\phi - \phi_k) dx\right| \\ &\le \int_\Omega |x|^{-s} |\phi_k - \phi| \, dx \\ &\le \left(\int_\Omega |x|^{\frac{-s(p+1)}{p}} dx\right)^{\frac{p}{p+1}} \|\phi_k - \phi\|_{p+1} \\ &\to 0 \ (k \to \infty), \end{aligned} \tag{3.8}$$

  where we have used (1.6) to show the constant

$$\left(\int_\Omega |x|^{\frac{-s(p+1)}{p}} dx\right)^{\frac{p}{p+1}}$$

  is bounded.

Furthermore, (3.6) and (3.7) imply for every $i = 1, \cdots, n$ and $\psi \in C_0^\infty(\Omega)$ that

$$\int_\Omega \phi \psi_{x_i} dx = \lim_{k\to\infty} \int_\Omega \phi_k \psi_{x_i} dx = -\lim_{k\to\infty} \int_\Omega \phi_{k_{x_i}} \psi dx = 0.$$

Consequently $\phi \in W$, and $\nabla \phi = 0$ a.e. thus ϕ is a constant, since Ω is connected (see [7, p. 307, problem 11]). However, this conclusion is inconsistent with $\|\phi\|_2 = 1$: since ϕ is a constant and $\int_\Omega |x|^{-s}\phi dx = 0$, there must be $\phi \equiv 0$. □

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The next three lemmas of this section will study the properties of $d(\delta)$ and $I_\delta(\phi)$, which were given in (2.7).

Lemma 3.3. *For p satisfying (1.2) and $\delta > 0$, the exact value of $d(\delta)$ can be given by*

$$d(\delta) = \left(\frac{1}{2}\delta^{\frac{2}{p-1}} - \frac{1}{p+1}\delta^{\frac{p+1}{p-1}}\right) B^{\frac{-2(p+1)}{p-1}}, \tag{3.9}$$

where B is the positive constant defined by (1.8). Furthermore,

(i) $d(\delta) > 0$ *for* $0 < \delta < \frac{p+1}{2}$;
(ii) $\lim\limits_{\delta\to 0} d(\delta) = \lim\limits_{\delta\to\frac{p+1}{2}} d(\delta) = 0$;
(iii) $d(\delta)$ *is strictly increasing on* $(0, 1)$, *strictly decreasing on* $(1, \frac{p+1}{2})$ *and takes its maximum* $d := d(1)$ *at* $\delta = 1$.

Proof. For any $\phi \in W \setminus \{0\}$, it is easy to see that $\lambda_\phi\phi \in N_\delta$, where the set N_δ is given in (2.7) and

$$\lambda_\phi := \left(\frac{\delta\|\phi\|^2}{\|\phi\|_{p+1}^{p+1}}\right)^{\frac{1}{p-1}}.$$

So,

$$\Phi := \{\lambda_\phi\phi : \phi \in W \setminus \{0\}\} \subset N_\delta. \tag{3.10}$$

On the other hand, for any $\phi \in N_\delta$, it follows from the definition of N_δ that $\phi \in W \setminus \{0\}$ and $\lambda_\phi = 1$, so $\phi = \lambda_\phi\phi \in \Phi$, i.e.

$$N_\delta \subset \Phi. \tag{3.11}$$

Then we get from (3.10) and (3.11) that $N_\delta = \Phi$. So, it follows from the definitions of B in (1.8), J in (2.1) and $d(\delta)$ in (2.7) that

$$\begin{aligned}
d(\delta) &= \inf_{\phi\in N_\delta} J(\phi) \\
&= \inf_{\phi\in\Phi} J(\phi) \\
&= \inf_{\phi\in W\setminus\{0\}} J(\lambda_\phi\phi) \\
&= \left(\frac{1}{2}\delta^{\frac{2}{p-1}} - \frac{1}{p+1}\delta^{\frac{p+1}{p-1}}\right) \inf_{\phi\in W\setminus\{0\}} \left(\frac{\|\phi\|}{\|\phi\|_{p+1}}\right)^{\frac{2(p+1)}{p-1}} \\
&= \left(\frac{1}{2}\delta^{\frac{2}{p-1}} - \frac{1}{p+1}\delta^{\frac{p+1}{p-1}}\right) B^{\frac{-2(p+1)}{p-1}}.
\end{aligned}$$

By using (3.9), it is easy to see that (i)–(iii) hold. □

Lemma 3.4. *Assume $0 < e < d$ and δ_1, δ_2 are the two roots of equation $d(\delta) = e$. If $\phi \in W \setminus \{0\}$ and $J(\phi) \le e$, then the sign of $I_\delta(\phi)$ remains unchanged on (δ_1, δ_2).*

Proof. Suppose $I_\delta(\phi)$ changes its sign on (δ_1, δ_2), then there must be a $\delta_0 \in (\delta_1, \delta_2)$ such that $I_{\delta_0}(\phi) = 0$, i.e. $\phi \in N_{\delta_0}$ and then $J(\phi) \geq d(\delta_0)$. On the other hand, by Lemma 3.3, we have $J(\phi) \geq d(\delta_0) > d(\delta_1) = d(\delta_2) = e$, which contradicts the assumption that $J(\phi) \leq e$. □

Lemma 3.5. *Let $0 < \delta < \frac{p+1}{2}$ and $\phi \in W$ satisfying $J(\phi) \leq d(\delta)$, then*

(i) $I_\delta(\phi) > 0$ *if and only if*

$$0 < \|\phi\| < \alpha_1 \delta^{\frac{1}{p-1}}; \tag{3.12}$$

(ii) $I_\delta(\phi) < 0$ *if and only if*

$$\|\phi\| > \alpha_1 \delta^{\frac{1}{p-1}}; \tag{3.13}$$

(iii) $I_\delta(\phi) = 0$ *and* $\phi \neq 0$, *if and only if*

$$\|\phi\| = \alpha_1 \delta^{\frac{1}{p-1}}, \tag{3.14}$$

where α_1 is the constant given in (2.6).

Proof. (i) If (3.12) holds, then by (1.8) and (2.6) we have

$$\begin{aligned} \|\phi\|_{p+1}^{p+1} &\leq B^{p+1}\|\phi\|^{p+1} \\ &= \alpha_1^{1-p}\|\phi\|^{p-1}\|\phi\|^2 \\ &< \delta\|\phi\|^2, \end{aligned}$$

which implies $I_\delta(\phi) > 0$. On the other hand, if $I_\delta(\phi) > 0$, by the definition of $I_\delta(\phi)$ in (2.7) and $\phi \in W$, one can see that $\|\phi\| > 0$. Then from (2.1), (3.9), $J(\phi) \leq d(\delta)$ and the definition of $I_\delta(\phi)$, we have

$$\begin{aligned} \delta^{\frac{2}{p-1}} \left(\frac{1}{2} - \frac{\delta}{p+1}\right) B^{\frac{-2(p+1)}{p-1}} &= d(\delta) \\ &\geq J(\phi) \\ &= \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\|\phi\|^2 + \frac{1}{p+1} I_\delta(\phi) \\ &> \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\|\phi\|^2, \end{aligned} \tag{3.15}$$

which means $\|\phi\| < \delta^{\frac{1}{p-1}} B^{\frac{-(p+1)}{p-1}} = \alpha_1 \delta^{\frac{1}{p-1}}$.

(ii) Assume (3.13) holds, we have

$$\left(\frac{1}{2} - \frac{\delta}{p+1}\right)\|\phi\|^2 > \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\alpha_1^2 \delta^{\frac{2}{p-1}} = d(\delta).$$

Since $J(\phi) \leq d(\delta)$, then

$$I_\delta(\phi) = (p+1)\left[J(\phi) - \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\|\phi\|^2\right]$$

$$
\begin{aligned}
&\leq (p+1)\left[d(\delta) - \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\|\phi\|^2\right] \\
&< 0.
\end{aligned}
$$

On the other hand, if $I_\delta(\phi) < 0$, we also have $\|\phi\| > 0$. From (1.8), we can get

$$
\begin{aligned}
\delta\|\phi\|^2 &\underbrace{< \|\phi\|_{p+1}^{p+1}}_{\text{since } I_\delta(\phi)<0} \\
&\leq B^{p+1}\|\phi\|^{p+1} \\
&= \alpha_1^{1-p}\|\phi\|^{p-1}\|\phi\|^2,
\end{aligned}
$$

which, together with $\|\phi\| > 0$, implies (3.13).

(iii) We can obtain (iii) from (i) and (ii) immediately. □

The remaining lemmas are about some properties of the solution $u = u(t)$ to problem (1.1).

Lemma 3.6. *Let p and s satisfy (1.2), then $J(u(t))$ given in (2.1) is non-increasing with respect to t.*

Proof. By Remark 2.2, we have $u(t) \in W$, i.e.,

$$
\int_\Omega |x|^{-s}u(t)dx = 0.
$$

Then by the definition of $J(u(t))$ in (2.1), we obtain

$$
\begin{aligned}
\frac{d}{dt}J(u(t)) &= -\int_\Omega u_t(\Delta u + |u|^{p-1}u)dx \\
&= -\int_\Omega u_t\left(|x|^{-s}u_t + \frac{|x|^{-s}}{\int_\Omega |x|^{-s}dx}\int_\Omega |u|^{p-1}udx\right)dx \\
&= -\int_\Omega |x|^{-s}u_t^2dx - \frac{\left(\int_\Omega |x|^{-s}udx\right)_t}{\int_\Omega |x|^{-s}dx}\int_\Omega |u|^{p-1}udx \\
&= -\int_\Omega |x|^{-s}u_t^2dx \leq 0. \qquad \square
\end{aligned}
\tag{3.16}
$$

Lemma 3.7. *Assume the assumptions in Theorem 2.1 hold. Then there exists a positive constant $\alpha_2 > \alpha_1$ such that any solution $u = u(t)$ to problem (1.1) satisfies*

$$
\|u(t)\| \geq \alpha_2, \quad t \geq 0, \tag{3.17}
$$

and

$$
\|u(t)\|_{p+1} \geq B\alpha_2, \quad t \geq 0, \tag{3.18}
$$

where B and α_1 are the constants given in (1.8) and (2.6) respectively. More than that,

$$\frac{\alpha_2}{\alpha_1} \geq \begin{cases} \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} > 1, & \textit{if } J(u_0) \leq 0; \\ \left[(p+1)\left(\frac{1}{2} - \frac{J(u_0)}{\alpha_1^2}\right)\right]^{\frac{1}{p-1}} > 1, & \textit{if } 0 < J(u_0) < d. \end{cases} \tag{3.19}$$

Proof. We use a similar idea as in Liu and Wang [31] or Vitillaro [46]. By $I(u_0) < 0$ and (1.8), we have

$$\|u_0\|^2 < \|u_0\|_{p+1}^{p+1} \leq (B\|u_0\|)^{p+1},$$

i.e.,

$$\alpha_0 := \|u_0\| > B^{-\frac{(p+1)}{p-1}} = \alpha_1.$$

Let $\alpha := \|u\|$. We deduce from (1.8) and (2.1) that

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{p+1}B^{p+1}\|u\|^{p+1} \\ &= \frac{1}{2}\alpha^2 - \frac{1}{p+1}(B\alpha)^{p+1} \\ &=: g(\alpha). \end{aligned} \tag{3.20}$$

We can easily find that $g'(\alpha_1) = 0$ and $d = g(\alpha_1)$. Moreover, $g(\alpha)$ is increasing in $[0, \alpha_1]$ and decreasing in $[\alpha_1, \infty)$; $g(0) = 0$, $\lim\limits_{\alpha\to\infty} g(\alpha) = -\infty$. Since $J(u_0) < d$, there is a positive constant $\alpha_2 > \alpha_1$ such that $J(u_0) = g(\alpha_2)$.

By (3.20) we have $g(\alpha_0) \leq J(u_0) = g(\alpha_2)$, which implies that $\alpha_0 \geq \alpha_2$ since $\alpha_0, \alpha_2 > \alpha_1$. So (3.17) holds for $t = 0$. Actually, (3.17) also holds for $t > 0$, otherwise, there exists some $t_0 > 0$, such that $\|u(t_0)\| < \alpha_2$. By the continuity of $\|u(t)\|$ we can choose t_0 such that $\|u(t_0)\| > \alpha_1$. Then it follows from (3.20) that

$$J(u_0) = g(\alpha_2) < g(\|u(t_0)\|) \leq J(u(t_0)),$$

which contradicts Lemma 3.6. So (3.17) is proved.

From Lemma 3.6 and (2.1), we know that

$$J(u_0) \geq J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$

which implies that

$$\begin{aligned} \frac{1}{p+1}\|u\|_{p+1}^{p+1} &\geq \frac{1}{2}\|u\|^2 - J(u_0) \\ &\geq \frac{1}{2}\alpha_2^2 - J(u_0) \\ &= \frac{1}{2}\alpha_2^2 - g(\alpha_2) \\ &= \frac{1}{p+1}(B\alpha_2)^{p+1}. \end{aligned}$$

Then, (3.18) holds.

To prove the inequality (3.19), we denote $\beta = \frac{\alpha_2}{\alpha_1}$, then $\beta > 1$ since $\alpha_2 > \alpha_1$. By $J(u_0) = g(\alpha_2)$ and $\alpha_1 = B^{-\frac{p+1}{p-1}}$, we can get that

$$\begin{aligned} J(u_0) &= g(\beta\alpha_1) \\ &= (\beta\alpha_1)^2 \left(\frac{1}{2} - \frac{1}{p+1} B^{p+1} (\beta\alpha_1)^{p-1}\right) \\ &= (\beta\alpha_1)^2 \left(\frac{1}{2} - \frac{1}{p+1} \beta^{p-1}\right). \end{aligned}$$

When $J(u_0) \le 0$, it is obvious that $\frac{1}{2} - \frac{1}{p+1}\beta^{p-1} \le 0$, then

$$\beta \ge \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} > 1. \tag{3.21}$$

When $J(u_0) > 0$, from the above equality and $\beta > 1$, we get that

$$\frac{1}{2} - \frac{1}{p+1}\beta^{p-1} = \frac{J(u_0)}{(\beta\alpha_1)^2} \le \frac{J(u_0)}{\alpha_1^2},$$

then

$$\beta \ge \left[(p+1)\left(\frac{1}{2} - \frac{J(u_0)}{\alpha_1^2}\right)\right]^{\frac{1}{p-1}}. \tag{3.22}$$

By (2.5), we have $d = \frac{p-1}{2(p+1)}\alpha_1^2$, which, together with $J(u_0) < d$ implies

$$(p+1)\left(\frac{1}{2} - \frac{J(u_0)}{\alpha_1^2}\right) > (p+1)\left(\frac{1}{2} - \frac{p-1}{2(p+1)}\right) = 1. \tag{3.23}$$

Finally, (3.19) is proved by (3.21), (3.22) and (3.23). □

Lemma 3.8. *Assume the assumptions in Theorem 2.1 hold, then any solution $u = u(t)$ to problem (1.1) satisfies*

$$0 < H(u_0) \le H(u) \le \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{3.24}$$

where $H(u) := d - J(u)$.

Proof. By Lemma 3.6, it is clear that $H(u(t))$ is nondecreasing in t. Thus

$$H(u) \ge H(u_0) = d - J(u_0) > 0. \tag{3.25}$$

From (2.1), (3.17) with $\alpha_2 > \alpha_1$, we have

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$$\begin{aligned} H(u) &= d - \frac{1}{2}\|u\|^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &\le d - \frac{1}{2}\alpha_1^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &= \left(\frac{p-1}{2(p+1)} - \frac{1}{2}\right)\alpha_1^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &\le \frac{1}{p+1}\|u\|_{p+1}^{p+1}. \qquad \square \end{aligned} \tag{3.26}$$

4. Proofs of the main results

In this section, we will prove the main results listed in Section 2. Firstly, we consider Theorem 2.1.

Proof of Theorem 2.1. Let

$$M(t) := \frac{1}{2}\int_\Omega |x|^{-s}u(t)^2 dx.$$

By (3) of Remark 2.2, we know that the $M(t)$ is well defined since $u(t) \in W$. By differentiating M, we get

$$\begin{aligned} M'(t) &= \int_\Omega |x|^{-s}uu_t dx \\ &= \int_\Omega |x|^{-s}u\left(\Delta u + |u|^{p-1}u - \frac{|x|^{-s}}{\int_\Omega |x|^{-s}dx}\int_\Omega |u|^{p-1}udx\right)|x|^s dx \\ &= \int_\Omega |u|^{p+1}dx - \int_\Omega |\nabla u|^2 dx - \frac{\int_\Omega |x|^{-s}udx}{\int_\Omega |x|^{-s}dx}\int_\Omega |u|^{p-1}udx \\ &= \int_\Omega |u|^{p+1}dx - \int_\Omega |\nabla u|^2 dx \\ &= \int_\Omega |u|^{p+1}dx - 2J(u) - \frac{2}{p+1}\int_\Omega |u|^{p+1}dx \\ &= \frac{p-1}{p+1}\|u\|_{p+1}^{p+1} - 2d + 2H(u). \end{aligned} \tag{4.1}$$

By combining (2.5) and (3.18), we have

$$\begin{aligned} 2d &= \frac{p-1}{p+1}B^{-\frac{2(p+1)}{p-1}} = \frac{p-1}{p+1}\left(BB^{-\frac{p+1}{p-1}}\right)^{p+1} \\ &= \frac{p-1}{p+1}(B\alpha_1)^{p+1} = \frac{p-1}{p+1}\left(\frac{\alpha_1}{\alpha_2}\right)^{p+1}(B\alpha_2)^{p+1} \\ &\le \frac{p-1}{p+1}\left(\frac{\alpha_1}{\alpha_2}\right)^{p+1}\|u\|_{p+1}^{p+1}. \end{aligned} \tag{4.2}$$

Substituting (4.2) into (4.1) we get

$$M'(t) \ge C^*\|u\|_{p+1}^{p+1} + 2H(u), \tag{4.3}$$

where

$$C^* = \frac{p-1}{p+1}\left[1 - \left(\frac{\alpha_1}{\alpha_2}\right)^{p+1}\right] > 0.$$

On the other hand, we estimate $M^{\frac{p+1}{2}}(t)$: from (2.12) we have

$$2M(t) \le C_0\|u\|_{p+1}^2,$$

i.e.

$$M^{\frac{p+1}{2}}(t) \le \left(\frac{C_0}{2}\right)^{\frac{p+1}{2}} \|u\|_{p+1}^{p+1}. \tag{4.4}$$

So, by Lemma 3.8, (4.3) and (4.4) we have

$$M'(t) \ge \gamma M^{\frac{p+1}{2}}(t) \tag{4.5}$$

with

$$\gamma = 2^{\frac{p+1}{2}} C_0^{-\frac{p+1}{2}} C^* > 0,$$

which means

$$\begin{aligned} M(t) &\ge \left(M^{-\frac{p-1}{2}}(0) - \frac{p-1}{2}\gamma t\right)^{-\frac{2}{p-1}} \\ &= \left(2^{\frac{p-1}{2}} \||x|^{-\frac{s}{2}}u_0\|_2^{-(p-1)} - \frac{p-1}{2}\gamma t\right)^{-\frac{2}{p-1}}. \end{aligned}$$

Let

$$T^* := \frac{2^{\frac{p+1}{2}}}{\gamma(p-1)} \||x|^{-\frac{s}{2}}u_0\|_2^{-(p-1)} \in (0,\infty), \tag{4.6}$$

then $M(t)$ blows up at a finite time $T_{\max} \le T^*$.

By a simply computation, we obtain

$$T^* = \frac{(p+1)C_0^{\frac{p+1}{2}} \||x|^{-\frac{s}{2}}u_0\|_2^{-(p-1)}}{(p-1)^2\left[1 - \left(\frac{\alpha_1}{\alpha_2}\right)^{p+1}\right]},$$

then it follows from (3.19) that

$$T_{\max} \le T^* \le \begin{cases} \dfrac{(p+1)C_0^{\frac{p+1}{2}} \||x|^{-\frac{s}{2}}u_0\|_2^{-(p-1)}}{(p-1)^2\left[1 - \left(\frac{p+1}{2}\right)^{-\frac{p+1}{p-1}}\right]}, & \text{if } J(u_0) \le 0; \\[2ex] \dfrac{(p+1)C_0^{\frac{p+1}{2}} \||x|^{-\frac{s}{2}}u_0\|_2^{-(p-1)}}{(p-1)^2\left\{1 - \left[(p+1)\left(\frac{1}{2} - \frac{J(u_0)}{\alpha_1^2}\right)\right]^{-\frac{p+1}{p-1}}\right\}}, & \text{if } 0 < J(u_0) < d, \end{cases} \tag{4.7}$$

where C_0 is given in (2.11). □

Proof of Theorem 2.3. We claim $u(t) \in \Psi$ for $t \in [0, T_{\max})$, where

$$\Psi := \{\phi \in W : I(\phi) > 0, J(\phi) < d\} \cup \{0\}.$$

In fact, since $J(u(t))$ is non-increasing with respect to t (see Lemma 3.6), we get $J(u(t)) \leq J(u_0) < d$ for $t \in [0, T_{\max})$. Furthermore, in view of $u_0 \in \Psi$ and time continuity of $I(u(t))$, if the claim is not true, there must exist a $t_0 \in (0, T_{\max})$ such that $u(t_0) \neq 0$ and $I(u(t_0)) = 0$, i.e., $u(t_0) \in N$. Then it follows from the definition of d in (2.4) that $J(u(t_0)) \geq d$, which contradicts the fact that $J(u(t_0)) < d$. So the claim is true and we have $I(u(t)) \geq 0$ for $t \in [0, T_{\max})$.

Hence, we can get from the definition of I in (2.2) that

$$\|u(t)\|^2 \geq \|u(t)\|_{p+1}^{p+1}, \quad t \in [0, T_{\max}). \tag{4.8}$$

Since $J(u(t)) \leq J(u_0) < d$, it follows from the definition of J in (2.1) that

$$\frac{1}{2}\|u(t)\|^2 - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1} < d, \quad t \in [0, T_{\max}). \tag{4.9}$$

Substituting (4.8) into (4.9) we get

$$\left(\frac{1}{2} - \frac{1}{p+1}\right)\|u(t)\|^2 < d, \quad t \in [0, T_{\max}),$$

i.e.,

$$\|u(t)\| < \sqrt{\frac{2(p+1)d}{p-1}}, \quad t \in [0, T_{\max}).$$

Therefore, $u(t)$ exists globally and $T_{max} = \infty$. □

Proof of Theorem 2.4. Let $u(t)$ $(0 \leq t < T_{\max})$ be a solution of problem (1.1) with initial data u_0. By the definition of U_e in (2.13) and Lemma 3.5 we have

$$\begin{aligned} U_e &= \left\{\phi \in W : \alpha_1 \delta_1^{\frac{1}{p-1}} < \|\phi\| < \alpha_1 \delta_2^{\frac{1}{p-1}}\right\} \\ &= \bigcup_{\delta_1 < \delta < \delta_2} \left\{\phi \in W : \|\phi\| = \alpha_1 \delta^{\frac{1}{p-1}}\right\} \\ &= \bigcup_{\delta_1 < \delta < \delta_2} \left\{\phi \in W \setminus \{0\} : I_\delta(\phi) = 0\right\}. \end{aligned}$$

Therefore, if the conclusion does not hold, there exists a $t_0 \in [0, T_{\max})$ such that $u(t_0) \in U_e$. Then some $\delta \in (\delta_1, \delta_2)$ exists such that $I_\delta(u(t_0)) = 0$, which, together with $u(t_0) \neq 0$ (since $u(t_0) \in U_e$) implies $u(t_0) \in N_\delta$, and then we get $J(u(t_0)) \geq d(\delta) > d(\delta_1) = d(\delta_2) = e$. On the other hand, by Lemma 3.6, we get $e \geq J(u_0) \geq J(u(t_0))$, a contradiction. □

Proof of Theorem 2.6. Let $\lambda_k = 1 - 1/k$, $k = 1, 2, \cdots$. Consider the following problem:

$$\begin{cases} |x|^{-s}u_t - \Delta u = |u|^{p-1}u - \dfrac{|x|^{-s}}{\int_\Omega |x|^{-s}dx} \displaystyle\int_\Omega |u|^{p-1}udx, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0^k(x) := \lambda_k u_0(x), & x \in \Omega. \end{cases} \tag{4.10}$$

Since $I(u_0) = \|u_0\|^2 - \|u_0\|_{p+1}^{p+1} > 0$, by the proof of Lemma 3.3, there exists a unique λ_{u_0}, defined by

$$\lambda_{u_0} = \left(\frac{\|u_0\|^2}{\|u_0\|_{p+1}^{p+1}}\right)^{\frac{1}{p-1}} > 1,$$

such that $\lambda_{u_0}u_0 \in N$, i.e., $I(\lambda_{u_0}u_0) = 0$. Since $\lambda_k < 1 < \lambda_{u_0}$ and the function

$$\frac{\tau^2}{2}\|u_0\|^2 - \frac{\tau^{p+1}}{p+1}\|u_0\|_{p+1}^{p+1}$$

is strictly increasing with respect to τ for $\tau \in (0, \lambda_{u_0})$, we get

$$\begin{aligned} I\left(u_0^k\right) = I(\lambda_k u_0) &= I\left(\frac{\lambda_k}{\lambda_{u_0}}\lambda_{u_0}u_0\right) \\ &= \left(\frac{\lambda_k}{\lambda_{u_0}}\right)^2 \left[\|\lambda_{u_0}u_0\|^2 - \left(\frac{\lambda_k}{\lambda_{u_0}}\right)^{p-1} \|\lambda_{u_0}u_0\|_{p+1}^{p+1}\right] \\ &> \left(\frac{\lambda_k}{\lambda_{u_0}}\right)^2 I(\lambda_{u_0}u_0) = 0, \end{aligned}$$

and

$$J\left(u_0^k\right) = J(\lambda_k u_0) = \frac{\lambda_k^2}{2}\|u_0\|^2 - \frac{\lambda_k^{p+1}}{p+1}\|u_0\|_{p+1}^{p+1} < J(u_0) = d.$$

In view of Theorem 2.3, problem (4.10) admits a global solution $u^k(t)$ and $u^k(t) \in \Psi$ for $0 \le t < \infty$. Applying the arguments similar to the proof of [53, Theorem 13], we can see that there exists a subsequence of $\left\{u^k(t)\right\}_{k=1}^\infty$ and a function $u(t)$, such that $u(t)$ is a solution of (1.1) with $u(t) \in \overline{\Psi}$, i.e., $J(u(t)) \le d$ and $I(u(t)) \ge 0$ for $0 \le t < \infty$. Moreover, similar to the proof of Theorem 2.3, we can get

$$\|u(t)\| \le \sqrt{\frac{2(p+1)d}{p-1}}, \quad 0 \le t < \infty. \qquad \square$$

Proof of Theorem 2.7. We claim that

$$I(u(t)) < 0, \qquad \forall t \in [0, T_{\max}). \tag{4.11}$$

Assume it is not true, then by $I(u_0) < 0$, there must be a $t_0 > 0$ such that $I(u(t_0)) = 0$ and $I(u(t)) < 0$ for $t \in [0, t_0)$. On the one hand, we can obtain that $\|u(t)\| > \alpha_1$ on $[0, t_0)$ by using (ii) of Lemma 3.5, which indicates $u(t_0) \ne 0$. Thus we have $u(t_0) \in N$ and

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$$J(u(t_0)) \geq d \tag{4.12}$$

On the other hand, for $t \in [0, t_0)$, we also have $u_t \neq 0$ since $\int_\Omega |x|^{-s} u u_t dx = -I(u(t)) > 0$ on $[0, t_0)$ which implies $\int_0^{t_0} \||x|^{-\frac{s}{2}} u_t\|_2^2 d\tau > 0$. Integrating equation (3.16) over $[0, t_0)$, one can see

$$\begin{aligned} J(u(t_0)) &= J(u_0) - \int_0^{t_0} \||x|^{-\frac{s}{2}} u_\tau\|_2^2 d\tau \\ &< J(u_0) = d, \end{aligned}$$

which conflicts with (4.12).

Therefore, by (ii) of Lemma 3.5 and (4.11),

$$\|u(t)\| > \alpha_1, \qquad \forall t \in [0, T_{\max}). \tag{4.13}$$

To complete our proof, we argue by contradiction. Assume that $T_{\max} = \infty$. Let

$$G(t) := \frac{1}{2} \int_0^t \left\||x|^{-\frac{s}{2}} u(\tau)\right\|_2^2 d\tau.$$

Then we can get

$$G'(t) = \frac{1}{2} \left\||x|^{-\frac{s}{2}} u(t)\right\|_2^2 > 0 \tag{4.14}$$

and

$$G''(t) = \int_\Omega |x|^{-s} u u_t dx = -I(u(t)) > 0.$$

Choose $t_1 > 0$ such that

$$\begin{aligned} 0 < d_1 &= J(u(t_1)) \\ &= J(u_0) - \int_0^{t_1} \||x|^{-\frac{s}{2}} u_\tau\|_2^2 d\tau \\ &= d - \int_0^{t_1} \||x|^{-\frac{s}{2}} u_\tau\|_2^2 d\tau < d, \end{aligned}$$

where the last inequality can be gotten by using a similar argument as in the proof of (4.11). Then one has $J(u(t)) \leq d_1$ for each $t \geq t_1$. It follows from Lemma 3.4 and (4.11) that $I_\delta(u(t)) < 0$ for $\delta_1 < \delta < \delta_2$ and $t \geq t_1$, where δ_1, δ_2 are the two roots of the equation $d(\delta) = d_1$. So for any $\delta_0 \in (1, \delta_2)$, we have $I_{\delta_0}(u(t)) < 0$, $t \geq t_1$. By using (4.13), it is easy to find for all $t \geq t_1$,

$$\begin{aligned} G''(t) &= -I(u(t)) \\ &= (\delta_0 - 1)\|u(t)\|^2 - I_{\delta_0}(u(t)) \\ &> (\delta_0 - 1)\alpha_1^2 > 0, \end{aligned}$$

which shows that $G'(t) \to \infty$ and $G(t) \to \infty$ as $t \to \infty$.

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Next by (2.12), (3.16) and (4.14), we have the following estimation:

$$\begin{aligned} G''(t) &= -I(u(t)) \\ &= \frac{p-1}{2}\|u(t)\|^2 - (p+1)J(u(t)) \\ &\geq (p+1)\int_0^t \||x|^{-\frac{s}{2}}u_\tau\|_2^2 d\tau + (p-1)\lambda G'(t) - (p+1)J(u_0), \end{aligned} \tag{4.15}$$

where the constant $\lambda = \frac{1}{C_0 B^2}$.

Integrating $G''(t) = \int_\Omega |x|^{-s} u u_t dx$ over $(0,t)$, we get

$$G'(t) = G'(0) + \int_0^t \int_\Omega |x|^{-s} u u_\tau dx d\tau.$$

Thus,

$$\begin{aligned} (G'(t))^2 &= (G'(0))^2 + 2G'(0)(G'(t) - G'(0)) + \left(\int_0^t \int_\Omega |x|^{-s} u u_\tau dx d\tau\right)^2 \\ &= -(G'(0))^2 + 2G'(0)G'(t) + \left(\int_0^t \int_\Omega |x|^{-s} u u_\tau dx d\tau\right)^2 \\ &= -\frac{1}{4}\||x|^{-\frac{s}{2}}u_0\|_2^4 + G'(t)\left\||x|^{-\frac{s}{2}}u_0\right\|_2^2 + \left(\int_0^t \int_\Omega |x|^{-s} u u_\tau dx d\tau\right)^2 \\ &\leq G'(t)\left\||x|^{-\frac{s}{2}}u_0\right\|_2^2 + \left(\int_0^t \int_\Omega |x|^{-s} u u_\tau dx d\tau\right)^2. \end{aligned} \tag{4.16}$$

Then combining (4.15), (4.16) and using Schwarz's inequality, we get

$$\begin{aligned} GG'' - \frac{p+1}{2}G'^2 \geq{}& \frac{p+1}{2}\left[\int_0^t \||x|^{-\frac{s}{2}}u_\tau\|_2^2 d\tau \int_0^t \||x|^{-\frac{s}{2}}u\|_2^2 d\tau - \left(\int_0^t \int_\Omega |x|^{-s} u u_\tau dx d\tau\right)^2\right] \\ &+ (p-1)\lambda GG' - (p+1)GJ(u_0) - \frac{p+1}{2}G'\left\||x|^{-\frac{s}{2}}u_0\right\|_2^2 \\ \geq{}& (p-1)\lambda GG' - (p+1)GJ(u_0) - \frac{p+1}{2}G'\left\||x|^{-\frac{s}{2}}u_0\right\|_2^2. \end{aligned} \tag{4.17}$$

Since $G'(t) \to \infty$ and $G(t) \to \infty$ as $t \to \infty$, there must exist a $t_2 > 0$, such that

$$\frac{p-1}{2}\lambda G' > (p+1)J(u_0), \quad \frac{p-1}{2}\lambda G > \frac{p+1}{2}\left\||x|^{-\frac{s}{2}}u_0\right\|_2^2, \quad t > t_2.$$

So, from (4.17) we have

$$GG'' - \frac{p+1}{2}G'^2 > 0, \quad t > t_2.$$

Next, we take account of the function $G^{\frac{1-p}{2}}$. Let $\theta = \frac{p-1}{2}$, then $\theta > 0$ since $p > 1$. For any $t \in (t_2, \infty)$,

$$(G^{-\theta})' = -\theta G^{-\theta-1} G' < 0, \tag{4.18}$$

$$(G^{-\theta})'' = -\theta G^{-\theta-2}[GG'' - \frac{p+1}{2} G'^2] < 0, \tag{4.19}$$

which makes sure that the function $G^{\frac{1-p}{2}}$ is concave on (t_2, ∞). Obviously, there is a finite time $T > 0$, such that $\lim\limits_{t\to T} G^{\frac{1-p}{2}}(t) = 0$, i.e. $\lim\limits_{t\to T} G(t) = \infty$, which is inconsistent with the assumption that $T_{max} = \infty$. Hence our proof is finished. $\square$

Proof of Theorem 2.8. Since $J(u_0) = d$, we get $u_0 \neq 0$. Then it follows from the definitions of $\mathcal{N}$ in (2.3) and d in (2.5) that $u_0 \in \mathcal{N}$ and

$$J(u_0) = d = \inf_{u\in\mathcal{N}} J(u).$$

Hence, by the theory of Lagrange multipliers, there exists a $\mu \in \mathbb{R}$ such that

$$J'(u_0) = \mu I'(u_0). \tag{4.20}$$

Thus,

$$\langle J'(u_0), u_0\rangle = \mu\langle I'(u_0), u_0\rangle. \tag{4.21}$$

Since $u_0 \in \mathcal{N}$, we have

$$0 = I(u_0) = \langle J'(u_0), u_0\rangle = \|u_0\|^2 - \|u_0\|_{p+1}^{p+1}. \tag{4.22}$$

Then, it follows from (4.21) and the above equality that

$$\begin{aligned}\langle I'(u_0), u_0\rangle &= \underbrace{2\|u_0\|^2 - (p+1)\|u_0\|_{p+1}^{p+1}}_{\text{using the definition of } I \text{ in } (2.2)}\\ &= -(p-1)\|u_0\|^2\\ &< 0.\end{aligned}$$

So, by (4.21), (4.22) and the above inequality, we get $\mu = 0$. Therefore, it follows from (4.20) that $J'(u_0) = 0$, i.e., u_0 is a solution of problem (2.22). So, problem (1.1) obviously admits a global solution $u(t) \equiv u_0$. $\square$

Proof of Theorem 2.9. Firstly, we prove the solution will blow up in finite time.

If (2.25) holds, from (2.1), (2.2) and (2.24) we get

$$\begin{aligned}I(u_0) &= \frac{1-p}{2}\|u_0\|^2 + (p+1)J(u_0)\\ &\le \frac{(1-p)\lambda_1}{2}\left\||x|^{-\frac{s}{2}}u_0\right\|_2^2 + (p+1)J(u_0)\\ &< 0.\end{aligned} \tag{4.23}$$

Actually, we may claim that

$$I(u(t)) < 0, \quad \forall t \in [0, T_{\max}). \tag{4.24}$$

Otherwise, there is a $t_0 \in (0, T_{\max})$, such that

$$I(u(t_0)) = 0, \tag{4.25}$$

and

$$I(u(t)) < 0, \qquad t \in [0, t_0). \tag{4.26}$$

Then it follows from (4.1) that

$$\frac{d}{dt}\left(\frac{1}{2}\||x|^{-\frac{s}{2}}u\|_2^2\right) = \int_\Omega |x|^{-s} u u_t dx = -I(u(t)) > 0, \quad \forall t \in (0, t_0). \tag{4.27}$$

Thus, by (2.25), (4.26) and (4.27), we obtain

$$\begin{aligned} J(u_0) &< \frac{(p-1)\lambda_1}{2(p+1)} \left\||x|^{-\frac{s}{2}}u_0\right\|_2^2 \\ &\le \frac{(p-1)\lambda_1}{2(p+1)} \left\||x|^{-\frac{s}{2}}u(t_0)\right\|_2^2. \end{aligned} \tag{4.28}$$

Besides, it follows from (2.1), (2.2), (2.24), (3.16) and (4.25) that

$$\begin{aligned} \frac{(p-1)\lambda_1}{2(p+1)} \left\||x|^{-\frac{s}{2}}u(t_0)\right\|_2^2 &\le \frac{p-1}{2(p+1)}\|u(t_0)\|^2 \\ &= J(u(t_0)) \le J(u_0), \end{aligned} \tag{4.29}$$

which contradicts (4.28).

Furthermore, we can suppose that $J(u(t)) \ge 0$ for all $t \in [0, T_{\max})$. In fact, if there is a t_0 such that $J(u(t_0)) < 0$, combining with (4.24), we have $J(u(t_0)) < 0 < d$ and $I(u(t_0)) < 0$. Then we can regard $u(t_0)$ as the initial data, and by using Theorem 2.1, the solution $u = u(t)$ of problem (1.1) will blow up in finite time, and then the blow-up result have been obtained.

Next, we are going to prove the blow-up of the solution $u(t)$ by contradiction. Suppose $u(t)$ exists globally and let

$$\varphi(t) := \frac{1}{2}\left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2 - \frac{p+1}{(p-1)\lambda_1} J(u(t)), \qquad t \ge 0. \tag{4.30}$$

Then, it follows from (2.1), (2.2), (2.24), (3.16), (4.27) and $J(u(t)) \ge 0$ that

$$\begin{aligned} \varphi'(t) &= -I(u(t)) + \frac{p+1}{(p-1)\lambda_1} \left\||x|^{-\frac{s}{2}}u_t(t)\right\|_2^2 \\ &\ge \frac{p-1}{2}\|u(t)\|^2 - (p+1)J(u(t)) \\ &\ge \frac{(p-1)\lambda_1}{2} \left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2 - (p+1)J(u(t)) \\ &= (p-1)\lambda_1 \varphi(t) \end{aligned} \tag{4.31}$$

and

$$\varphi(0) = \frac{1}{2}\left\||x|^{-\frac{s}{2}}u_0\right\|_2^2 - \frac{p+1}{(p-1)\lambda_1} J(u_0) > 0, \tag{4.32}$$

which indicates

$$\frac{1}{2}\left\||x|^{-\frac{s}{2}}u\right\|_2^2 \geq \varphi(t) \geq \varphi(0)e^{(p-1)\lambda_1 t}, \qquad t \geq 0. \tag{4.33}$$

On the other hand, by $J(u(t)) \geq 0$, (3.16) and Hölder's inequality, we have

$$\begin{aligned}
\||x|^{-\frac{s}{2}}u(t)\|_2 &= \left\| |x|^{-\frac{s}{2}}u_0 + \int_0^t |x|^{-\frac{s}{2}}u_\tau d\tau \right\|_2 \\
&\leq \left\||x|^{-\frac{s}{2}}u_0\right\|_2 + \int_0^t \||x|^{-\frac{s}{2}}u_\tau\|_2 d\tau \\
&\leq \left\||x|^{-\frac{s}{2}}u_0\right\|_2 + t^{\frac{1}{2}}\left[\int_0^t \||x|^{-\frac{s}{2}}u_\tau\|_2^2 d\tau\right]^{\frac{1}{2}} \\
&= \left\||x|^{-\frac{s}{2}}u_0\right\|_2 + t^{\frac{1}{2}}[J(u_0) - J(u(t))]^{\frac{1}{2}} \\
&\leq \left\||x|^{-\frac{s}{2}}u_0\right\|_2 + J(u_0)^{\frac{1}{2}}t^{\frac{1}{2}}.
\end{aligned} \tag{4.34}$$

Combining (4.33), we get the following inequality

$$\sqrt{2\varphi(0)e^{\frac{(p-1)\lambda_1}{2}t}} \leq \left\||x|^{-\frac{s}{2}}u_0\right\|_2 + J(u_0)^{\frac{1}{2}}t^{\frac{1}{2}}, \qquad t \geq 0, \tag{4.35}$$

which will result in an error when $t \to \infty$. Hence, $u(t)$ will blow-up at some finite time $T_{\max}$.

Next, we estimate the upper bound of $T_{\max}$. From (4.27) and (4.24), we have

$$\frac{d}{dt}\left(\left\||x|^{-\frac{s}{2}}u\right\|_2^2\right) = -2I(u(t)) > 0, \qquad \forall t \in [0, T_{\max}), \tag{4.36}$$

that is to say

$$\left\||x|^{-\frac{s}{2}}u\right\|_2^2 \text{ is strictly increasing on } [0, T_{\max}). \tag{4.37}$$

Let

$$F(t) := \int_0^t \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2 d\tau + (T_{\max} - t)\left\||x|^{-\frac{s}{2}}u_0\right\|_2^2 + \eta(t+\xi)^2, \qquad t \in [0, T_{\max}), \tag{4.38}$$

where η, ξ are two positive constants which will be specified later.

Then, for any $t \in [0, T_{\max})$, through (2.24), (4.27), (3.16), (4.37) and a series of calculations we can get

$$\begin{cases}
F'(t) = \left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2 - \left\||x|^{-\frac{s}{2}}u_0\right\|_2^2 + 2\eta(t+\xi) \geq 2\eta(t+\xi) > 0, \\
F(0) = T_{\max}\left\||x|^{-\frac{s}{2}}u_0\right\|_2^2 + \eta\xi^2 > 0, \\
F'(0) = 2\eta\xi > 0,
\end{cases} \tag{4.39}$$

and

$$
\begin{aligned}
F''(t) &= -2I(u(t)) + 2\eta \\
&\geq (p-1)\|u(t)\|^2 - 2(p+1)J(u(t)) \\
&\geq (p-1)\lambda_1 \left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2 - 2(p+1)J(u_0) + 2(p+1)\int_0^t \||x|^{-\frac{s}{2}}u_\tau\|_2^2 d\tau \\
&\geq (p-1)\lambda_1 \left\||x|^{-\frac{s}{2}}u_0\right\|_2^2 - 2(p+1)J(u_0) + 2(p+1)\int_0^t \||x|^{-\frac{s}{2}}u_\tau\|_2^2 d\tau \\
&> 0.
\end{aligned}
\tag{4.40}
$$

So

$$
F(t) \geq F(0) > 0, \qquad t \in [0, T_{\max}). \tag{4.41}
$$

Let

$$
a(t) := \left[\int_0^t \||x|^{-\frac{s}{2}}u\|_2^2 d\tau\right]^{\frac{1}{2}}, \quad b(t) := \left[\int_0^t \||x|^{-\frac{s}{2}}u_\tau\|_2^2 d\tau\right]^{\frac{1}{2}}. \tag{4.42}
$$

By Hölder's inequality, we have

$$
\begin{aligned}
&\left[\int_0^t \left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2 d\tau + \eta(t+\xi)^2\right]\left[\int_0^t \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2 d\tau + \eta\right] \\
&\quad - \left[\frac{1}{2}\left(\left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2 - \left\||x|^{-\frac{s}{2}}u_0\right\|_2^2\right) + \eta(t+\xi)^2\right]^2 \\
=&[a^2(t) + \eta(t+\xi)^2][b^2(t) + \eta] - \left[\frac{1}{2}\int_0^t \frac{d}{d\tau}\left\||x|^{-\frac{s}{2}}u\right\|_2^2 d\tau + \eta(t+\xi)^2\right]^2 \\
=&[a^2(t) + \eta(t+\xi)^2][b^2(t) + \eta] - \left[\int_0^t \int_\Omega |x|^{-s}uu_\tau dx d\tau + \eta(t+\xi)^2\right]^2 \\
\geq&[a^2(t) + \eta(t+\xi)^2][b^2(t) + \eta] - \left[\int_0^t \left\||x|^{-\frac{s}{2}}u\right\|_2 \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2 d\tau + \eta(t+\xi)^2\right]^2 \\
\geq&[a^2(t) + \eta(t+\xi)^2][b^2(t) + \eta] - [a(t)b(t) + \eta(t+\xi)]^2 \\
=&[\sqrt{\eta}a(t)]^2 - 2\eta(t+\xi)a(t)b(t) + [\sqrt{\eta}(t+\xi)b(t)]^2 \\
=&[\sqrt{\eta}a(t) - \sqrt{\eta}(t+\xi)b(t)]^2 \\
\geq&0.
\end{aligned}
\tag{4.43}
$$

Combining (4.43) with (4.38), (4.39) and (4.40), we obtain

$$
\begin{aligned}
-(F'(t))^2 &= -4\left[\frac{1}{2}\left(\left\||x|^{-\frac{s}{2}}u\right\|_2^2 - \left\||x|^{-\frac{s}{2}}u_0\right\|_2^2\right) + \eta(t+\xi)^2\right]^2 \\
&= 4\left\{\left[\int_0^t \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2 d\tau + \eta(t+\xi)^2\right]\left[\int_0^t \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2 d\tau + \eta\right]\right. \\
&\quad - \left[\frac{1}{2}\left(\left\||x|^{-\frac{s}{2}}u\right\|_2^2 - \left\||x|^{-\frac{s}{2}}u_0\right\|_2^2\right) + \eta(t+\xi)^2\right]^2 \\
&\quad \left. - \left[F(t) - (T_{\max} - t)\left\||x|^{-\frac{s}{2}}u_0\right\|_2^2\right]\left[\int_0^t \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2 d\tau + \eta\right]\right\} \\
&\geq -4F(t)\left[\int_0^t \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2 d\tau + \eta\right].
\end{aligned}
\tag{4.44}
$$

Then it follows from (4.39), (4.40) and (4.44) that

$$
\begin{aligned}
&F(t)F''(t) - \frac{(p+1)}{2}(F'(t))^2 \\
\geq &F(t)\left[F''(t) - 2(p+1)\left(\int_0^t \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2 d\tau + \eta\right)\right] \\
\geq &F(t)\left[(p-1)\lambda_1 \left\||x|^{-\frac{s}{2}}u_0\right\|_2^2 - 2(p+1)J(u_0) - 2(p+1)\eta\right].
\end{aligned}
\tag{4.45}
$$

Take η small enough, such that

$$
\eta \in \left(0, \frac{\sigma}{2(p+1)}\right], \tag{4.46}
$$

where

$$
\sigma := (p-1)\lambda_1 \left\||x|^{-\frac{s}{2}}u_0\right\|_2^2 - 2(p+1)J(u_0).
$$

Then it is obvious that

$$
FF'' - \frac{(p+1)}{2}F'^2 \geq 0.
$$

Thus, for any $t \in [0, T_{\max})$, we have

$$
\begin{aligned}
\left(F^{-\frac{p-1}{2}}\right)' &= -\frac{p-1}{2}F^{-\frac{p+1}{2}}F' < 0, \\
\left(F^{-\frac{p-1}{2}}\right)'' &= -\frac{p-1}{2}F^{-\frac{p+3}{2}}[FF'' - \frac{p+1}{2}F'^2] \leq 0,
\end{aligned}
\tag{4.47}
$$

which implies that the function $F^{-\frac{p-1}{2}}$ is concave, which will extinct at finite time $T_{\max}$. Furthermore, $T_{\max}$ satisfies

$$
T_{\max} \leq \frac{2F(0)}{(p-1)F'(0)} = \frac{\left\||x|^{-\frac{s}{2}}u_0\right\|_2^2}{(p-1)\eta\xi}T_{\max} + \frac{\xi}{p-1}. \tag{4.48}
$$

Let ξ be large enough such that

$$\xi \in \left(\frac{\left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2}{(p-1)\eta}, \infty \right), \tag{4.49}$$

then by (4.48)

$$T_{\max} \leq \frac{\eta \xi^2}{(p-1)\eta\xi - \left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2}. \tag{4.50}$$

In view of (4.46) and (4.49), we can define

$$\begin{aligned} \Psi &:= \left\{ (\eta, \xi) : \eta \in \left(0, \frac{\sigma}{2(p+1)} \right], \xi \in \left(\frac{\left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2}{(p-1)\eta}, \infty \right) \right\} \\ &= \left\{ (\xi, \eta) : \xi \in \left(\frac{2(p+1) \left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2}{(p-1)\sigma}, \infty \right), \eta \in \left(\frac{\left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2}{(p-1)\xi}, \frac{\sigma}{2(p+1)} \right] \right\}, \end{aligned}$$

and then

$$T_{\max} \leq \inf_{(\eta,\xi)\in\Psi} \frac{\eta \xi^2}{(p-1)\eta\xi - \left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2}.$$

Let $\varrho = \eta\xi$ and

$$f(\xi, \varrho) := \frac{\varrho \xi}{(p-1)\varrho - \left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2}.$$

It is easy to find that $f(\xi, \varrho)$ is decreasing with ϱ. Then,

$$\begin{aligned} T_{\max} &\leq \inf_{\xi \in \left(\frac{2(p+1)\left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2}{(p-1)\sigma}, \infty \right)} f\left(\xi, \frac{\sigma}{2(p+1)} \xi \right) \\ &= \inf_{\xi \in \left(\frac{2(p+1)\left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2}{(p-1)\sigma}, \infty \right)} \frac{\sigma \xi^2}{(p-1)\sigma\xi - 2(p+1) \left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2} \\ &= \left. \frac{\sigma \xi^2}{(p-1)\sigma\xi - 2(p+1) \left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2} \right|_{\xi = \frac{4(p+1)\left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2}{(p-1)\sigma}} \\ &= \frac{8(p+1) \left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2}{(p-1)^2 \sigma} \end{aligned}$$

Hence, by the definition of σ and the above inequality, we have

$$T_{\max} \leq \frac{8(p+1) \left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2}{(p-1)^2 [(p-1)\lambda_1 \left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2 - 2(p+1) J(u_0)]}. \qquad \square$$

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