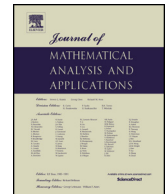




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Ground state solutions of Schrödinger–Poisson systems with variable potential and convolution nonlinearity

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ABSTRACT

In the present paper, we consider the following nonlinear Schrödinger–Poisson system with convolution nonlinearity:

$$\begin{cases} -\Delta u + V(x)u + \phi u = (I_\alpha * F(u))f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\alpha \in (0, 3)$, $I_\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}$ is the Riesz potential, $V \in C(\mathbb{R}^3, [0, \infty))$, $f \in C(\mathbb{R}, \mathbb{R})$ and $F(t) = \int_0^t f(s)ds$ satisfies $\lim_{|t| \rightarrow \infty} F(t)/|t|^\sigma = \infty$ with $\sigma = \min\{2, (6 + \alpha)/4\}$. By using some new analytic techniques and new inequalities, we prove the above system admits a ground state solution under mild assumptions on V and f . In particular, our results cover and improve the existing ones for the Schrödinger–Poisson system which can be considered as the limited problem when $\alpha \rightarrow 0$.

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1. Introduction

This paper is concerned with solitary wave solutions of the following nonlinear Schrödinger system with convolution nonlinearity:

$$\begin{cases} i \frac{\partial \psi}{\partial t} = -\Delta \psi + W(x)\psi + \phi \psi - \left(K(x) * \int_0^{|\psi|} \tilde{f}(s)ds \right) \tilde{f}(|\psi|)\psi, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ -\Delta \phi = |\psi|^2, & x \in \mathbb{R}^3, \end{cases} \quad (1.1)$$

where $\psi : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C}$ is the wave function, $W(x)$ is a real external potential, ϕ represents an internal potential for a nonlocal self-interaction of the wave function, $K(x)$ is the response function which possesses information on the mutual interaction between the particles, and the nonlinear term $f(\psi) := \tilde{f}(|\psi|)\psi$ describes the interaction effect among particles. In many situations the nonlinear interaction can be of

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nonlocal nature. For example, for identical and non-relativistic basic particles (such as bosons or electrons) under the influence of an external potential, the condensate in the mean field regime is governed by (1.1) with Hartree nonlinearity, i.e. $K(x) = 1/|x|$ and $\tilde{f}(|\psi|) = 1$, see [18,23,24]. (1.1) is also used in the description of the Bose–Einstein condensates, in which the nonlocal nonlinearity describes the interaction between the bosons in the condensate, see [14,35]. If the response function is a Riesz potential, i.e. $K(x) = I_\alpha(x)$, where

$$I_\alpha(x) = \frac{\Gamma\left(\frac{3-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) 2^\alpha \pi^{3/2} |x|^{3-\alpha}}, \quad x \in \mathbb{R}^3 \setminus \{0\},$$

$\alpha \in (0, 3)$, then the solitary wave solution of (1.1) leads to the following Schrödinger–Poisson system with convolution nonlinearity:

$$\begin{cases} -\Delta u + V(x)u + \phi u = (I_\alpha * F(u))f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \quad (1.2)$$

where and in the sequel $V(x) = W(x) + E$ and $F(u) = \int_0^u f(t)dt$. In the absence of the internal potential ϕu , (1.2) becomes the following generalized Choquard equation:

$$-\Delta u + V(x)u = (I_\alpha * F(u))f(u), \quad u \in H^1(\mathbb{R}^3). \quad (1.3)$$

When $\alpha = 2$, $V(x) \equiv 1$ and $f(u) = u$, (1.3) is known as the Choquard–Pekar equation or the stationary nonlinear Hartree equation, which introduced at least in 1954, in a work by Pekar [33] describing the quantum mechanics of a polaron at rest, for more details and applications, we refer to [27,31]. For the case where $V(x) \equiv 1$ and $f(u) = |u|^{p-2}u$, (1.3) is known to have a solution if and only if $1 + \alpha/3 < p < 3 + \alpha$ ([30, p. 457], [31, Theorem 1]; see also [22, Lemma 2.7]). As described Moroz and Van Schaftingen in [32], $3 + \alpha$ and $1 + \alpha/3$ are the upper and lower critical exponents, which appear as extensions of exponents 6 and 2 for the corresponding local problem. Later, these results were extended to more general the potential V or the nonlinearity f , see [1,2,12,20,21,29,36].

Inspired by [30–32], we introduce the following basic assumptions on the nonlinearity f :

(F1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, $F(t) \geq 0$ for all $t \in \mathbb{R}$ and there exist constants $C_0 > 0$ and $1 + \alpha/3 < p < 3 + \alpha$ such that

$$|f(t)| \leq C_0 \left(|t|^{\alpha/3} + |t|^{p-1} \right), \quad \forall t \in \mathbb{R};$$

(F2) $F(t) = o(|t|^{1+\alpha/3})$ as $|t| \rightarrow 0$.

Let $\alpha \rightarrow 0$ in (1.2), we can get the following Schrödinger–Poisson system with $g = Ff$:

$$\begin{cases} -\Delta u + V(x)u + \phi u = g(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \quad (1.4)$$

which is also reduced from (1.1) by letting the response function be a Dirac-delta function, i.e. $K(x) = \delta(x)$, and the usual ansatz $\psi(t, x) = e^{-iEt}u(x)$, $E \in \mathbb{R}$. As quoted by Benci and Fortunato in [7], (1.4) works as a model describing solitary waves for the nonlinear stationary Schrödinger equation interacting with the electrostatic field and also in semiconductor theory, nonlinear optics and plasma physics.

The existence, multiplicity and concentration of nontrivial solutions of (1.4) have been the subject of extensive mathematical studies in recent years, for example, [4–6,8,16,17,10,11,34,37,43] and references

therein. When $V = 1$ and $g(u) = |u|^{p-2}u$, by introducing a new manifold that is defined by a condition which is a combination of the Nehari equation and the Pohožaev equality, Ruiz [34] showed that (1.4) admits a positive radial solution if $3 < p < 6$, but does not have a nontrivial solution for $2 < p \leq 3$. In the same assumptions, based on Ruiz' approach in [34], Azzollini and Pomponio [6] obtained the existence of ground state solutions for (1.4) by using a concentration-compactness argument. When $g(u) = |u|^{p-2}u$ and V satisfies the following assumptions:

- (V1) $V \in \mathcal{C}(\mathbb{R}^3, [0, \infty))$ and $V_\infty := \lim_{|y| \rightarrow \infty} V(y) \geq V(x)$ for all $x \in \mathbb{R}^3$;
 (V2') $V(x)$ is weakly differentiable, and satisfies $\nabla V(x) \cdot x \in L^\infty(\mathbb{R}^3) \cup L^{3/2}(\mathbb{R}^3)$, and

$$2V(x) + \nabla V(x) \cdot x \geq 0 \quad \text{a.e. } x \in \mathbb{R}^3,$$

Zhao and Zhao [43] established the existence of ground state solutions for (1.4) by using the Jeanjean's monotonicity trick [25]. In recent paper [38], Tang and Chen introduced some new tricks to generalize and improve the results in [6, 34, 43] to the more general case where V satisfies (V1) and (V2') and g satisfies the following assumptions:

- (G1) $g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, and there exist constants $C > 0$ and $q \in (2, 6)$ such that

$$|g(t)| \leq C(1 + |t|^{q-1}), \quad \forall t \in \mathbb{R};$$

- (G2) $g(t) = o(t)$ as $t \rightarrow 0$;

- (G3) $\lim_{|t| \rightarrow \infty} \frac{G(t)}{|t|^3} = \infty$, where $G(t) = \int_0^t g(s)ds$;

- (G4) $[2g(t)t - 3G(t)]/t^3$ is nondecreasing on $(-\infty, 0) \cup (0, +\infty)$.

Motivated by [6, 9, 34, 37, 38, 43], in the present paper, we shall generalize and improve the results in [38] to (1.2). Compared with (1.3) and (1.4), it is more difficult to deal with (1.2). Indeed, on the one hand, because of the competing effect of $\phi_u u$, the existing methods dealing with Choquard type equations fail for (1.2). On the other hand, due to the appearance of the convolution nonlinearity, the approaches used in [6, 9, 34, 37, 38, 43] cannot be applied directly to obtain the desired solutions for (1.2). These difficulties enforce the implementation of new ideas and techniques. To the best of our knowledge, there seem to be no results for (1.2) on this topic until now.

It is well known that for any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that $-\Delta \phi = u^2$ by using the Lax–Milgram theorem, moreover,

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy = \frac{1}{|x|} * u^2, \quad (1.5)$$

then inserted into the first equation in (1.2), gives

$$-\Delta u + V(x)u + \phi_u(x)u = (I_\alpha * F(u))f(u). \quad (1.6)$$

By the Hardy–Littlewood–Sobolev inequality [28], one has

$$\int_{\mathbb{R}^3} (I_\alpha * h_1)h_2(x)dx \leq C(\alpha)\|h_1\|_{6/(3+\alpha)}\|h_2\|_{6/(3+\alpha)}, \quad \forall h_1, h_2 \in L^{6/(3+\alpha)}. \quad (1.7)$$

In view of (F1), (1.7) and Sobolev embedding theorem [42], one has

$$\begin{aligned}
& \int_{\mathbb{R}^3} (I_\alpha * F(u)) F(u) dx \\
& \leq C(\alpha) \left(\int_{\mathbb{R}^3} |F(u)|^{6/(3+\alpha)} dx \right)^{(3+\alpha)/3} \\
& \leq C_1 \left(\|u\|_2^{2+2\alpha/3} + \|u\|_{6p/(3+\alpha)}^{2p} \right), \quad \forall u \in H^1(\mathbb{R}^3)
\end{aligned} \tag{1.8}$$

and

$$\begin{aligned}
& \int_{\mathbb{R}^3} (I_\alpha * F(u)) f(u) v dx \\
& \leq C(\alpha) \left(\int_{\mathbb{R}^3} |F(u)|^{6/(3+\alpha)} dx \right)^{(3+\alpha)/6} \left(\int_{\mathbb{R}^3} |f(u) v|^{6/(3+\alpha)} dx \right)^{(3+\alpha)/6} \\
& \leq C_2 \left(\|u\|_2^{1+\alpha/3} + \|u\|_{6p/(3+\alpha)}^p \right) \left(\|u\|_2^{\alpha/3} \|v\|_2 + \|u\|_{6p/(3+\alpha)}^{p-1} \|v\|_{6p/(3+\alpha)} \right), \\
& \quad \forall u, v \in H^1(\mathbb{R}^3).
\end{aligned} \tag{1.9}$$

By (1.8) and (1.9), the functional

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x) u^2] dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u)) F(u) dx \tag{1.10}$$

is well defined in $H^1(\mathbb{R}^3)$ and $\Phi \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$. Moreover, for any $u, v \in H^1(\mathbb{R}^3)$,

$$\begin{aligned}
\langle \Phi'(u), v \rangle &= \int_{\mathbb{R}^3} [\nabla u \cdot \nabla v + V(x) uv] dx + \int_{\mathbb{R}^3} \phi_u(x) uv dx \\
&\quad - \int_{\mathbb{R}^3} (I_\alpha * F(u)) f(u) v dx.
\end{aligned} \tag{1.11}$$

Hence, the solutions of (1.2) are critical points of Φ . For the sake of simplicity, in many cases we just say $u \in H^1(\mathbb{R}^3)$, instead of $(u, \phi_u) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$, is a solution of (1.2). A solution is called a ground state solution if its energy is minimal among all nontrivial solutions.

In the first part of this paper, we consider the case when $0 < \alpha < 2$, in addition to (V1), (F1) and (F2), we also introduce the following assumptions:

- (V2) $V \in C^1(\mathbb{R}^3)$, the set $\{x \in \mathbb{R}^3 : |\nabla V(x) \cdot x| \geq \epsilon\}$ has finite Lebesgue measure for every $\epsilon > 0$, and the function $t \mapsto t^2[V(tx) - \nabla V(tx) \cdot (tx)]$ is increasing on $(0, +\infty)$ for every $x \in \mathbb{R}^3$;
(V3) $V \in C^1(\mathbb{R}^3)$, $\nabla V(x) \cdot x \in L^\infty(\mathbb{R}^3)$ and there exists $\varrho > 0$ such that

$$2V(x) + \nabla V(x) \cdot x \geq \varrho, \quad \forall x \in \mathbb{R}^3;$$

- (V3') $V \in C^1(\mathbb{R}^3)$, $\nabla V(x) \cdot x \in L^\infty(\mathbb{R}^3)$, $2V(x) + \nabla V(x) \cdot x \geq 0$ for all $x \in \mathbb{R}^3$ and there exists $\mu > \frac{6+\alpha}{4}$ such that

$$f(t)t - \mu F(t) \geq 0, \quad \forall t \in \mathbb{R};$$

$$(F3) \quad \lim_{|t| \rightarrow \infty} \frac{F(t)}{|t|^{(6+\alpha)/4}} = \infty;$$

$$(F4) \quad [4f(t)t - (3 + \alpha)F(t)]/t|t|^{(2+\alpha)/4} \text{ is nondecreasing on both } (-\infty, 0) \text{ and } (0, +\infty).$$

To state our results, we define the Nehari–Pohožaev manifold as follows:

$$\mathcal{M} := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : J(u) := 2\langle \Phi'(u), u \rangle - \mathcal{P}(u) = 0\}, \quad (1.12)$$

where

$$\begin{aligned} \mathcal{P}(u) = & \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [3V(x) + \nabla V(x) \cdot x] u^2 dx + \frac{5}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 dx \\ & - \frac{3 + \alpha}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u)) F(u) dx, \end{aligned} \quad (1.13)$$

which is associated with the Pohožaev identity $\mathcal{P}(u) = 0$ of (1.2) that can be obtained by the same argument as in [15,32].

We are now in a position to state the main results of this paper in the case when $0 < \alpha < 2$.

Theorem 1.1. *Assume that $0 < \alpha < 2$, V and f satisfy (V1), (V2) and (F1)–(F4). Then problem (1.2) has a ground state solution $\bar{u} \in H^1(\mathbb{R}^3)$ such that*

$$\Phi(\bar{u}) = \inf_{\mathcal{M}} \Phi = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \max_{t > 0} \Phi(t^2 u_t) > 0, \text{ where and in the sequel } u_t(x) := u(tx).$$

Theorem 1.2. *Assume that $0 < \alpha < 2$, V and f satisfy (V1), (V3) and (F1)–(F4). Then problem (1.2) has a positive ground state solution.*

Theorem 1.3. *Assume that $0 < \alpha < 2$, V and f satisfy (V1), (V3') and (F1)–(F4). Then problem (1.2) has a positive ground state solution.*

Remark 1.4. In this paper, (V2) weaken the corresponding condition used in [38, (V4)]. There are indeed functions which satisfy (V1)–(V3), for example

- (i) $V(x) = a - \frac{b}{|x|^{\beta+1}}$ with $a \geq (\beta^2 + \beta + 4)b/6 > 0$, $a > b$ and $\beta > 0$;
- (ii) $V(x) = a - be^{-|x|^\beta}$ with $a \geq b[1 + \beta(\beta + 1)/2e] > 0$ and $\beta > 0$.

In particular, (V1) and (V3) are satisfied by many non-monotonic functions, for example,

- (iii) $V(x) = a - \frac{b \sin^2 |x|^\beta}{1 + |x|^\beta}$ with $a \geq 2b > 0$ and $\beta > 0$.

Inspired of [38,39], we shall prove Theorem 1.1 following this scheme:

- step i). We verify $\mathcal{M} \neq \emptyset$ and establish the minimax characterization of $m := \inf_{\mathcal{M}} \Phi > 0$;
- step ii). We prove that m can be achieved;
- step iii). We show that the minimizer of Φ on \mathcal{M} is a critical point.

Although we mainly follow the procedure of [38], we have to face many new difficulties due to the mutual competing effect between $\phi_u u$ and $(I_\alpha * F(u))f(u)$. More precisely, in step i), we first establish a

key inequality related to $\Phi(u)$, $J(u)$ and $\Phi(t^2 u_t)$ in Lemma 2.4, where some more careful analyses on the convolution nonlinearity are required, see Lemmas 2.1–2.3; then we construct a saddle point structure with respect to the fibre $\{t^2 u_t : t > 0\} \subset H^1(\mathbb{R}^3)$ for $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, see Lemma 2.8, finally based on these constructions we obtain the minimax characterization of m , see Lemma 2.10. In step ii), we first choose a minimizing sequence $\{u_n\}$ of Φ on \mathcal{M} , and show that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, then with the help of the key inequality established in Lemma 2.4 and a concentration-compactness argument, we prove that there exist $\hat{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ and $\hat{t} > 0$ such that $u_n \rightharpoonup \hat{u}$ in $H^1(\mathbb{R}^3)$ up to translations and extraction of a subsequence, and $\hat{t}^2 \hat{u}_{\hat{t}} \in \mathcal{M}$ is a minimizer of $\inf_{\mathcal{M}} \Phi$, see Lemmas 2.13 and 2.14. Step iii) is similar to [38, Lemma 2.10]. We point out that there are no global compactness and any information on $\Phi'(u_n)$ in our arguments.

Motivated by [37,38,43], we use the Jeanjean's monotonicity trick [25] to prove Theorems 1.2 and 1.3. Due to the presence of the convolution nonlinearity, the main difficulty is to construct a bounded (PS) sequence and to prove that the (PS) sequence weakly converges to a critical point of Φ in $H^1(\mathbb{R}^3)$. By using Theorem 1.1 and applying the global compactness lemma and (V1) and (V3) (or (V3')), we can overcome the above difficulty, see Lemma 3.5. During the proofs, a more careful analysis is needed to consider the relationship between the Mountain Pass level for Φ and the least energy of the functional associated the "limit problem" of (1.2), see Lemma 3.4.

In the second part of this paper, we consider the case when $2 \leq \alpha < 3$, besides (V1), (F1) and (F2), we introduce the following assumptions:

$$(F3') \quad \lim_{|t| \rightarrow \infty} \frac{F(t)}{|t|^2} = \infty;$$

$$(F4') \quad f(t)t \geq 0 \text{ for all } t \in \mathbb{R} \text{ and } f(t)/|t| \text{ is nondecreasing on } (-\infty, 0) \cup (0, +\infty).$$

In this case, we define the Nehari manifold

$$\mathcal{N} := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : \langle \Phi'(u), u \rangle = 0\}.$$

Our result is as follows.

Theorem 1.5. Assume that $2 \leq \alpha < 3$, V and f satisfy (V1), (F1), (F2), (F3') and (F4'). Then problem (1.2) has a ground state solution $\bar{u} \in H^1(\mathbb{R}^3)$ such that

$$\Phi(\bar{u}) = \inf_{\mathcal{N}} \Phi = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \max_{t \geq 0} \Phi(tu) > 0.$$

Remark 1.6. Problem (1.4) with $g(u) = F(u)f(u)$ can be considered as a limiting problem of (1.2) when $\alpha \rightarrow 0$. Note (G3) implies that $G(t)/t^3$ is nondecreasing on $(-\infty, 0) \cup (0, +\infty)$. From this fact and (2.3), one can deduce that (G3) is equivalent to (F4) with $\alpha = 0$. In this sense, let $\alpha \rightarrow 0$, our results cover the ones in [6,9,34,37,38,43] which dealt with (1.4).

Throughout the paper we use the following notations:

- $H^1(\mathbb{R}^3)$ denotes the usual Sobolev space equipped with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv) dx, \quad \|u\| = (u, u)^{1/2}, \quad \forall u, v \in H^1(\mathbb{R}^3);$$

- $L^s(\mathbb{R}^3)$ ($1 \leq s < \infty$) denotes the Lebesgue space with the norm $\|u\|_s = (\int_{\mathbb{R}^3} |u|^s dx)^{1/s}$;
- For any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, $u_t(x) := u(tx)$ for $t > 0$;

- For any $x \in \mathbb{R}^3$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^3 : |y - x| < r\}$;
- $S = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \|\nabla u\|_2^2 / \|u\|_6^2$;
- C_1, C_2, \dots denote positive constants possibly different in different places.

The rest of the paper is organized as follows. In Section 2, we study the existence of ground state solutions for (1.2) with $0 < \alpha < 2$ by using the Nehari–Pohožaev manifold, and give the proof of Theorem 1.1. In Section 3, based on the Jeanjean’s monotonicity trick, we show the existence of ground state solutions for (1.2) with $0 < \alpha < 2$, and complete the proofs of Theorems 1.2 and 1.3. In Section 4, we consider the existence of ground state solutions for (1.2) with $2 \leq \alpha < 3$ by using the Nehari manifold, and give the proof of Theorem 1.3.

2. Proof of Theorem 1.1

In Sections 2 and 3, we always assume $0 < \alpha < 2$. By (1.4) and (1.11), we have

$$\begin{aligned} J(u) &= \frac{3}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [V(x) - \nabla V(x) \cdot x] u^2 dx + \frac{3}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 dx \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u)) [4f(u)u - (3 + \alpha)F(u)] dx. \end{aligned} \quad (2.1)$$

First, we establish some key inequalities.

Lemma 2.1. Assume that (F1) and (F4) hold. Then for all $s > 0$ and $t \in \mathbb{R}$,

$$g(s, t) := \frac{3}{s^{(3+\alpha)/2}} F(s^2 t) + \left(1 - s^{3/2}\right) [4f(t)t - (3 + \alpha)F(t)] - 3F(t) \geq 0. \quad (2.2)$$

Proof. It is evident that $g(s, 0) \geq 0$ for $s > 0$. For $t \neq 0$, it follows from (F4) that

$$\begin{aligned} \frac{d}{ds} g(s, t) &= \frac{3s^{1/2}|t|^{(6+\alpha)/4}}{2} \left[\frac{4f(s^2 t)s^2 t - (3 + \alpha)F(s^2 t)}{|s^2 t|^{(6+\alpha)/4}} - \frac{4f(t)t - (3 + \alpha)F(t)}{|t|^{(6+\alpha)/4}} \right] \\ &\quad \begin{cases} \geq 0, & s \geq 1, \\ \leq 0, & 0 < s < 1, \end{cases} \end{aligned}$$

which implies that $g(s, t) \geq g(1, t) = 0$ for all $s > 0$ and $t \in (-\infty, 0) \cup (0, +\infty)$. \square

Lemma 2.2. Assume that (F1) and (F4) hold. Then

$$\frac{F(t)}{t|t|^{(2+\alpha)/4}} \text{ is nondecreasing on both } (-\infty, 0) \text{ and } (0, +\infty). \quad (2.3)$$

Proof. By (F1) and (2.2), one has

$$\lim_{s \rightarrow 0} g(s, t) = 4f(t)t - (6 + \alpha)F(t) \geq 0, \quad \forall t \in \mathbb{R}. \quad (2.4)$$

Note that

$$\frac{d}{dt} \left(\frac{F(t)}{t|t|^{(2+\alpha)/4}} \right) = \frac{1}{4|t|^{(10+\alpha)/4}} [4f(t)t - (6 + \alpha)F(t)]. \quad (2.5)$$

Thus, the conclusion follows from (2.4) and (2.5). \square

Lemma 2.3. Assume that (F1) and (F4) hold. Then

$$\begin{aligned} h(t, u) &:= \int_{\mathbb{R}^3} \left\{ \frac{3}{t^{3+\alpha}} (I_\alpha * F(t^2 u)) F(t^2 u) + (1 - t^3) (I_\alpha * F(u)) \right. \\ &\quad \left. [4f(u)u - (3 + \alpha)F(u)] - 3(I_\alpha * F(u))F(u) \right\} dx \\ &\geq 0, \quad \forall t > 0, u \in H^1(\mathbb{R}^3). \end{aligned} \quad (2.6)$$

Proof. Note that (F1) and (2.3) imply

$$I_\alpha * \left(\frac{F(t^2 u)}{|t|^{(6+\alpha)/2}} \right) - I_\alpha * F(u) \begin{cases} \geq 0, & t \geq 1, \\ \leq 0, & 0 < t < 1. \end{cases} \quad (2.7)$$

By (F1), (F4) and (2.7), we have

$$\begin{aligned} \frac{d}{dt} h(t, u) &= \int_{\mathbb{R}^3} \left\{ \frac{12}{t^{3+\alpha}} (I_\alpha * F(t^2 u)) f(t^2 u) t u - \frac{3(3 + \alpha)}{t^{4+\alpha}} (I_\alpha * F(t^2 u)) F(t^2 u) \right. \\ &\quad \left. - 3t^2 (I_\alpha * F(u)) [4f(u)u - (3 + \alpha)F(u)] \right\} \\ &= 3t^2 \int_{\mathbb{R}^3} |u|^{(6+\alpha)/4} \left\{ \left(I_\alpha * \frac{F(t^2 u)}{t^{(6+\alpha)/2}} \right) \frac{4f(t^2 u)t^2 u - (3 + \alpha)F(t^2 u)}{|t^2 u|^{(6+\alpha)/4}} \right. \\ &\quad \left. - (I_\alpha * F(u)) \frac{4f(u)u - (3 + \alpha)F(u)}{|u|^{(6+\alpha)/4}} \right\} dx \\ &\quad \begin{cases} \geq 0, & t \geq 1, \\ \leq 0, & 0 < t < 1, \end{cases} \end{aligned} \quad (2.8)$$

which implies $h(t, u) \geq h(1, u) = 0$ for $t > 0$ and $u \in H^1(\mathbb{R}^3)$. This shows that (2.6) holds. \square

Define

$$\beta(x, t) := 3 [V(x) - tV(t^{-1}x)] - (1 - t^3)[V(x) - \nabla V(x) \cdot x], \quad \forall x \in \mathbb{R}^3, t > 0. \quad (2.9)$$

It is easy to check that (V2) implies

$$\beta(x, t) > 0, \quad \forall x \in \mathbb{R}^3, t \in (0, 1) \cup (1, \infty). \quad (2.10)$$

Lemma 2.4. Assume that (V1), (V2), (F1) and (F4) hold. Then

$$\Phi(u) \geq \Phi(t^2 u_t) + \frac{1 - t^3}{3} J(u) + \frac{1}{6} \int_{\mathbb{R}^3} \beta(x, t) u^2 dx, \quad \forall u \in H^1(\mathbb{R}^3), t > 0. \quad (2.11)$$

Proof. For $u \in H^1(\mathbb{R}^3)$ and $t > 0$, one has

$$\begin{aligned} \Phi(t^2 u_t) &= \frac{t^3}{2} \|\nabla u\|_2^2 + \frac{t}{2} \int_{\mathbb{R}^3} V(t^{-1}x) u^2 dx + \frac{t^3}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 dx \\ &\quad - \frac{1}{2t^{3+\alpha}} \int_{\mathbb{R}^3} (I_\alpha * F(t^2 u)) F(t^2 u) dx. \end{aligned} \quad (2.12)$$

Thus, by (1.10), (2.1), (2.6) and (2.12), one has

$$\begin{aligned} &\Phi(u) - \Phi(t^2 u_t) \\ &= \frac{1-t^3}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [V(x) - tV(t^{-1}x)] u^2 dx + \frac{1-t^3}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} \left[\frac{1}{t^{3+\alpha}} (I_\alpha * F(t^2 u)) F(t^2 u) - (I_\alpha * F(u)) F(u) \right] dx \\ &= \frac{1-t^3}{3} J(u) + \frac{1}{6} \int_{\mathbb{R}^3} \{ 3 [V(x) - tV(t^{-1}x)] - (1-t^3) [V(x) - \nabla V(x) \cdot x] \} u^2 dx \\ &\quad + \frac{1}{6} \int_{\mathbb{R}^3} \left\{ \frac{3}{t^{3+\alpha}} (I_\alpha * F(t^2 u)) F(t^2 u) + (1-t^3) (I_\alpha * F(u)) [4f(u)u - (3+\alpha)F(u)] \right. \\ &\quad \left. - 3(I_\alpha * F(u)) F(u) \right\} dx \\ &\geq \frac{1-t^3}{3} J(u) + \frac{1}{6} \int_{\mathbb{R}^3} \beta(x, t) u^2 dx. \end{aligned}$$

This shows that (2.11) holds. \square

Remark that (2.11) with $t \rightarrow 0$ implies

$$\Phi(u) \geq \frac{1}{3} J(u) + \frac{1}{6} \int_{\mathbb{R}^3} [2V(x) + \nabla V(x) \cdot x] u^2 dx, \quad \forall u \in H^1(\mathbb{R}^3). \quad (2.13)$$

To overcome the lack of compactness of the Sobolev spaces embeddings in \mathbb{R}^3 , we define the following energy functional

$$\Phi^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u|^2 + V_\infty u^2] dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u)) F(u) dx. \quad (2.14)$$

Corresponding to (1.12) and (2.1), we define

$$\mathcal{M}^\infty := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : J^\infty(u) = 0\} \quad (2.15)$$

and

$$J^\infty(u) := \frac{3}{2} \|\nabla u\|_2^2 + \frac{V_\infty}{2} \|u\|_2^2 + \frac{3}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u)) [4f(u)u - (3 + \alpha)F(u)] dx. \quad (2.16)$$

From Lemma 2.4, we have the following two corollaries.

Corollary 2.5. Assume that (F1) and (F4) hold. Then

$$\Phi^\infty(u) \geq \Phi^\infty(t^2 u_t) + \frac{1-t^3}{3} J^\infty(u) + \frac{(1-t)^2(2+t)V_\infty}{6} \|u\|_2^2, \\ \forall u \in H^1(\mathbb{R}^3), \quad t \geq 0. \quad (2.17)$$

Corollary 2.6. Assume that (V1), (V2), (F1) and (F4) hold. Then for $u \in \mathcal{M}$

$$\Phi(u) = \max_{t \geq 0} \Phi(t^2 u_t).$$

Lemma 2.7. Assume that (V1) and (V2) hold. Then there exist $\varrho_1, \varrho_2 > 0$ such that

$$2V(x) + \nabla V(x) \cdot x \geq \varrho_1, \quad \forall x \in \mathbb{R}^3, \quad (2.18)$$

$$V(x) - \nabla V(x) \cdot x \geq \varrho_2, \quad \forall x \in \mathbb{R}^3. \quad (2.19)$$

Proof. Note that (2.10) implies

$$2V(x) + \nabla V(x) \cdot x > 3tV(t^{-1}x) + t^3 \nabla V(x) \cdot x - t^3 V(x), \quad \forall x \in \mathbb{R}^3, \quad t \in (0, 1) \cup (1, \infty) \quad (2.20)$$

and

$$|\nabla V(x) \cdot x| \leq V_1, \quad \forall x \in \mathbb{R}^3 \quad (2.21)$$

for some constant $V_1 > 0$. By (V1), there exists $R_0 > 0$ such that

$$V(x) \geq \frac{V_\infty}{3}, \quad \forall |x| \geq R_0. \quad (2.22)$$

For any $x_0 \in \mathbb{R}^3 \setminus \{0\}$, by (2.22), one has

$$V(t^{-1}x_0) \geq \frac{V_\infty}{3}, \quad \forall 0 < t \leq \frac{|x_0|}{R_0}. \quad (2.23)$$

Let $t_0 = \min\{|x_0|/R_0, \sqrt{V_\infty/3(V_\infty + V_1)}\}$. Then it follows from (2.20), (2.21) and (2.23) that

$$2V(x_0) + \nabla V(x_0) \cdot x_0 > 3t_0 V(t_0^{-1}x_0) + t_0^3 \nabla V(x_0) \cdot x_0 - t_0^3 V(x_0) \\ \geq t_0 [V_\infty - t_0^2(V_1 + V_\infty)] > 0. \quad (2.24)$$

Moreover, by (V1), (V2) and (2.20), one has $V(0) > 0$. This, together with (2.24), implies

$$2V(x) + \nabla V(x) \cdot x > 0, \quad \forall x \in \mathbb{R}^3. \quad (2.25)$$

From (V1), (V2) and (2.25), we can deduce that there exists $V_0 \in (0, V_\infty)$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^3$. Otherwise, there exists $x_1 \in B_{R_0}(0)$ such that $V(x_1) = 0$ and $\nabla V(x_1) = 0$ due to $V \in \mathcal{C}^1(\mathbb{R}^3, [0, \infty))$ and (2.22), which contradicts to (2.25). Choose $t_1 = \sqrt{V_0/(V_1 + V_\infty)} \in (0, 1)$, it follows from (2.20) and (2.21) that

$$2V(x) + \nabla V(x) \cdot x > 3t_1 V_0 - t_1^3 V_1 - t_1^3 V_\infty = 2V_0 \sqrt{\frac{V_0}{V_1 + V_\infty}} := \varrho_1, \quad \forall x \in \mathbb{R}^3. \quad (2.26)$$

Note that (2.10) implies

$$\begin{aligned} V(x) - \nabla V(x) \cdot x &> \frac{3[tV(t^{-1}x) - V(x)]}{t^3 - 1} \\ &\geq \frac{3[tV_0 - V_\infty]}{t^3 - 1}, \quad \forall x \in \mathbb{R}^3, t > 1. \end{aligned} \quad (2.27)$$

Choose $t_2 = (V_\infty + 1)/V_0 > 1$, (2.27) implies

$$V(x) - \nabla V(x) \cdot x > \frac{3V_0^3}{(V_\infty + V_0)^3 - V_0^3} := \varrho_2, \quad \forall x \in \mathbb{R}^3. \quad \square$$

Lemma 2.8. Assume that (V1), (V2), (F1), (F3) and (F4) hold. Then for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u^2 u_{t_u} \in \mathcal{M}$.

Proof. Let $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ be fixed and define a function $\zeta(t) := \Phi(t^2 u_t)$ on $(0, \infty)$. Clearly, by (2.1) and (2.12), we have

$$\begin{aligned} \zeta'(t) = 0 &\Leftrightarrow \frac{1}{2} \int_{\mathbb{R}^3} \{3t^3 |\nabla u|^2 + t[V(t^{-1}x) - \nabla V(t^{-1}x) \cdot (t^{-1}x)]u^2\} dx \\ &\quad + \frac{3t^3}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 dx - \frac{1}{2t^{3+\alpha}} \int_{\mathbb{R}^3} (I_\alpha * F(t^2 u)) \\ &\quad [4f(t^2 u)t^2 u - (3 + \alpha)F(t^2 u)] dx = 0 \\ &\Leftrightarrow J(t^2 u_t) = 0 \quad \Leftrightarrow \quad t^2 u_t \in \mathcal{M}. \end{aligned}$$

By (V1), (F1) and (F3), we have $\lim_{t \rightarrow 0^+} \zeta(t) = 0$, $\zeta(t) > 0$ for $t > 0$ small and $\zeta(t) < 0$ for t large. Therefore $\max_{t \in (0, \infty)} \zeta(t)$ is achieved at $t_0 = t_u > 0$ so that $\zeta'(t_0) = 0$ and $t_0^2 u_{t_0} \in \mathcal{M}$.

Next we claim that t_u is unique for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. In fact, for any given $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, let $t_1, t_2 > 0$ such that $\zeta'(t_1) = \zeta'(t_2) = 0$. Then $J(t_1^2 u_{t_1}) = J(t_2^2 u_{t_2}) = 0$. Jointly with (2.11), we have

$$\begin{aligned} \Phi(t_1^2 u_{t_1}) &\geq \Phi(t_2^2 u_{t_2}) + \frac{t_1^3 - t_2^3}{3t_1^3} J(t_1^2 u_{t_1}) + \frac{t_1}{6} \int_{\mathbb{R}^3} \beta(x, t_2/t_1) u^2 dx \\ &= \Phi(t_2^2 u_{t_2}) + \frac{t_1}{6} \int_{\mathbb{R}^3} \beta(x, t_2/t_1) u^2 dx \end{aligned} \quad (2.28)$$

and

$$\begin{aligned}
\Phi(t_2^2 u_{t_2}) &\geq \Phi(t_1^2 u_{t_1}) + \frac{t_2^3 - t_1^3}{3t_2^3} J(t_2^2 u_{t_2}) + \frac{t_2}{6} \int_{\mathbb{R}^3} \beta(x, t_1/t_2) u^2 dx \\
&= \Phi(t_1^2 u_{t_1}) + \frac{t_2}{6} \int_{\mathbb{R}^3} \beta(x, t_1/t_2) u^2 dx.
\end{aligned} \tag{2.29}$$

Combining (2.10), (2.28) and (2.29), we have $t_1 = t_2$. Therefore, $t_u > 0$ is unique for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. \square

Corollary 2.9. Assume that (F1), (F3) and (F4) hold. Then for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u^2 u_{t_u} \in \mathcal{M}^\infty$.

Combining Corollary 2.6 with Lemma 2.8, we can obtain the following minimax characterization.

Lemma 2.10. Assume that (V1), (V2), (F1), (F3) and (F4) hold. Then

$$m = \inf_{\mathcal{M}} \Phi = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \max_{t > 0} \Phi(t^2 u_t).$$

Lemma 2.11. Assume that (V1), (V2), (F1), (F3) and (F4) hold. Then

- (i) there exists $\rho > 0$ such that $\|u\| \geq \rho$, $\forall u \in \mathcal{M}$;
- (ii) $m = \inf_{\mathcal{M}} \Phi > 0$.

Proof. (i). Since $J(u) = 0$, $\forall u \in \mathcal{M}$, by (1.8), (1.9), (2.1), (2.19) and the Sobolev embedding theorem, one has

$$\begin{aligned}
\frac{\min\{3, \varrho_2\}}{2} \|u\|^2 &\leq \frac{3}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [V(x) - \nabla V(x) \cdot x] u^2 dx + \frac{3}{4} \int_{\mathbb{R}^3} \phi_u(x) u^2 dx \\
&= \frac{1}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u)) [4f(u)u - (3 + \alpha)F(u)] dx \\
&\leq C_1 \left(\|u\|^{2+2\alpha/3} + \|u\|^{2p} \right),
\end{aligned}$$

which implies

$$\|u\| \geq \rho := \min \left\{ 1, \left(\frac{\min\{3, \varrho_2\}}{4C_1} \right)^{3/2\alpha} \right\}, \quad \forall u \in \mathcal{M}. \tag{2.30}$$

(ii). Let $\{u_n\} \subset \mathcal{M}$ be such that $\Phi(u_n) \rightarrow m$. There are two possible cases: 1) $\inf_{n \in \mathbb{N}} \|u_n\|_2 > 0$ and 2) $\inf_{n \in \mathbb{N}} \|u_n\|_2 = 0$.

Case 1). $\inf_{n \in \mathbb{N}} \|u_n\|_2 := \rho_1 > 0$. In this case, by (2.13) and (2.18), one has

$$m + o(1) = \Phi(u_n) = \Phi(u_n) - \frac{1}{3} J(u_n) \geq \frac{\varrho_1}{6} \rho_1^2. \tag{2.31}$$

Case 2). $\inf_{n \in \mathbb{N}} \|u_n\|_2 = 0$. By (2.30), passing to a subsequence, we have

$$\|u_n\|_2 \rightarrow 0, \quad \|\nabla u_n\|_2 \geq \frac{1}{2} \rho. \tag{2.32}$$

Note that (F1) implies that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(t)| \leq C_\varepsilon |t|^{1+\alpha/3} + \varepsilon |t|^{3+\alpha}, \quad \forall t \in \mathbb{R}. \quad (2.33)$$

By (1.7), (2.33) and the Sobolev embedding inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^3} (I_\alpha * F(u)) F(u) dx &\leq C_2 \|u\|_2^{2+2\alpha/3} + \frac{1}{2} S^{3+\alpha} \|u\|_6^{6+2\alpha} \\ &\leq C_2 \|u\|_2^{2+2\alpha/3} + \frac{1}{2} \|\nabla u\|_2^{6+2\alpha}, \quad \forall u \in H^1(\mathbb{R}^3). \end{aligned} \quad (2.34)$$

Let $t_n = \|\nabla u_n\|_2^{-2/3}$, then (2.32) implies that $\{t_n\}$ is bounded. Since $J(u_n) = 0$, it follows from (2.11), (2.12), (2.32) and (2.34) that

$$\begin{aligned} m + o(1) &= \Phi(u_n) \geq \Phi(t_n^2(u_n)_{t_n}) \\ &= \frac{t_n^3}{2} \|\nabla u_n\|_2^2 + \frac{t_n}{2} \int_{\mathbb{R}^3} V(t_n^{-1}x) u_n^2 dx + \frac{t_n^3}{4} \int_{\mathbb{R}^3} \phi_{u_n}(x) u_n^2 dx \\ &\quad - \frac{1}{2t_n^{3+\alpha}} \int_{\mathbb{R}^3} (I_\alpha * F(t_n^2 u_n)) F(t_n^2 u_n) dx \\ &\geq \frac{1}{2} t_n^3 \|\nabla u_n\|_2^2 - \frac{C_2}{2} (t_n \|u_n\|_2^2)^{1+\alpha/3} - \frac{1}{4} (t_n^3 \|\nabla u_n\|_2^2)^{3+\alpha} \\ &= \frac{1}{4} t_n^3 \|\nabla u_n\|_2^2 \left[2 - (t_n^3 \|\nabla u_n\|_2^2)^{2+\alpha} \right] + o(1) = \frac{1}{4} + o(1). \end{aligned}$$

Cases 1) and 2) show that $m = \inf_{\mathcal{M}} \Phi > 0$. \square

By combining [38, Lemmas 2.7 and 2.8], [3, Lemma 5.1], [19, Lemma 2.2], [43, Lemma 2.2]) and [42], we can obtain the following Brezis–Lieb type Lemma.

Lemma 2.12. Assume that (V1), (V2), (F1) and (F2) hold. If $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$, then along a subsequence

$$\Phi(u_n) = \Phi(\bar{u}) + \Phi(u_n - \bar{u}) + o(1), \quad J(u_n) = J(\bar{u}) + J(u_n - \bar{u}) + o(1) \quad (2.35)$$

$$\Phi'(u_n) = \Phi'(\bar{u}) + \Phi'(u_n - \bar{u}) + o(1), \quad (2.36)$$

$$\langle \Phi'(u_n), u_n \rangle = \langle \Phi'(\bar{u}), \bar{u} \rangle + \langle \Phi'(u_n - \bar{u}), u_n - \bar{u} \rangle + o(1). \quad (2.37)$$

Lemma 2.13. Assume that (V1), (V2) and (F1)–(F4) hold. Then $m^\infty := \inf_{\mathcal{M}^\infty} \Phi^\infty \geq m$.

Proof. In view of Lemma 2.4 and Corollary 2.6, we have $\mathcal{M}^\infty \neq \emptyset$. Arguing indirectly, we assume that $m > m^\infty$. Let $\varepsilon := m - m^\infty$. Then there exists u_ε^∞ such that

$$u_\varepsilon^\infty \in \mathcal{M}^\infty \quad \text{and} \quad m^\infty + \frac{\varepsilon}{2} > \Phi^\infty(u_\varepsilon^\infty) \quad (2.38)$$

In view of Lemma 2.8, there exists $t_\varepsilon > 0$ such that $t_\varepsilon^2(u_\varepsilon^\infty)_{t_\varepsilon} \in \mathcal{M}$. Thus, it follows from (V1), (1.10), (2.14), (2.17) and (2.38) that

$$m^\infty + \frac{\varepsilon}{2} > \Phi^\infty(u_\varepsilon^\infty) \geq \Phi^\infty(t_\varepsilon^2(u_\varepsilon^\infty)_{t_\varepsilon}) \geq \Phi(t_\varepsilon^2(u_\varepsilon^\infty)_{t_\varepsilon}) \geq m.$$

This contradiction shows that $m^\infty \geq m$. \square

Lemma 2.14. Assume that (V1), (V2) and (F1)–(F4) hold. Then m is achieved.

Proof. In view of Lemmas 2.8 and 2.11, we have $\mathcal{M} \neq \emptyset$ and $m > 0$. Let $\{u_n\} \subset \mathcal{M}$ be such that $\Phi(u_n) \rightarrow m$. Since $J(u_n) = 0$, then it follows from (2.13) and (2.18) that

$$\begin{aligned} m + o(1) &= \Phi(u_n) = \Phi(u_n) - \frac{1}{3}J(u_n) \\ &\geq \frac{1}{6} \int_{\mathbb{R}^3} [2V(x) + \nabla V(x) \cdot x] u_n^2 dx \geq \frac{\theta_1}{6} \|u_n\|_2^2. \end{aligned} \quad (2.39)$$

This shows that $\{\|u_n\|_2\}$ is bounded. Next, we prove that $\{\|\nabla u_n\|_2\}$ is also bounded. Arguing by contradiction, suppose that $\|\nabla u_n\|_2 \rightarrow \infty$. By (1.7), (2.33) and the Sobolev embedding inequality, for $u \in H^1(\mathbb{R}^3)$ one has

$$\begin{aligned} \int_{\mathbb{R}^3} (I_\alpha * F(u)) F(u) dx &\leq C_3 \|u\|_2^{2+2\alpha/3} + \frac{1}{2(8m)^{2+\alpha}} S^{3+\alpha} \|u\|_6^{6+2\alpha} \\ &\leq C_3 \|u\|_2^{2+2\alpha/3} + \frac{1}{2(8m)^{2+\alpha}} \|\nabla u\|_2^{6+2\alpha}. \end{aligned} \quad (2.40)$$

Let $t_n = (8m/\|\nabla u_n\|_2^2)^{1/3}$. Since $J(u_n) = 0$, it follows from (1.10), (2.11) and (2.40) that

$$\begin{aligned} m + o(1) &= \Phi(u_n) \geq \Phi(t_n^2(u_n)_{t_n}) \\ &= \frac{t_n^3}{2} \|\nabla u_n\|_2^2 + \frac{t_n}{2} \int_{\mathbb{R}^3} V(t_n^{-1}x) u_n^2 dx + \frac{t_n^3}{4} \int_{\mathbb{R}^3} \phi_{u_n}(x) u_n^2 dx \\ &\quad - \frac{1}{2t_n^{3+\alpha}} \int_{\mathbb{R}^3} (I_\alpha * F(t_n^2 u_n)) F(t_n^2 u_n) dx \\ &\geq \frac{1}{2} t_n^3 \|\nabla u_n\|_2^2 - \frac{C_3}{2} (t_n \|u_n\|_2^2)^{1+\alpha/3} - \frac{1}{4(8m)^{2+\alpha}} (t_n^3 \|\nabla u_n\|_2^2)^{3+\alpha} \\ &= \frac{1}{2} t_n^3 \|\nabla u_n\|_2^2 \left[1 - \frac{1}{2} \left(\frac{t_n^3 \|\nabla u_n\|_2^2}{8m} \right)^{2+\alpha} \right] + o(1) \\ &= 2m + o(1). \end{aligned} \quad (2.41)$$

This contradiction shows that $\{\|\nabla u_n\|_2\}$ is also bounded, and so $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Passing to a subsequence, we have $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$. Then $u_n \rightarrow \bar{u}$ in $L_{\text{loc}}^s(\mathbb{R}^3)$ for $2 \leq s < 6$ and $u_n \rightarrow \bar{u}$ a.e. in \mathbb{R}^3 . There are two possible cases: i). $\bar{u} = 0$ and ii). $\bar{u} \neq 0$.

Case i). $\bar{u} = 0$, i.e. $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. Then $u_n \rightarrow 0$ in $L_{\text{loc}}^s(\mathbb{R}^3)$ for $2 \leq s < 2^*$ and $u_n \rightarrow 0$ a.e. in \mathbb{R}^3 . Using (V1) and (V2), it is easy to show that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} [V_\infty - V(x)] u_n^2 dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \nabla V(x) \cdot x u_n^2 dx = 0. \quad (2.42)$$

From (1.10), (2.1), (2.14), (2.16) and (2.42), one can get

$$\Phi^\infty(u_n) \rightarrow m, \quad J^\infty(u_n) \rightarrow 0. \quad (2.43)$$

Note that (F1) and (F2) imply that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(t)| \leq \varepsilon |t|^{1+\alpha/3} + C_\varepsilon |t|^p, \quad \forall t \in \mathbb{R}. \quad (2.44)$$

By (1.7), (2.1), (2.19), (2.44) and Lemma 2.11 (i), one has

$$\begin{aligned} \frac{\min\{3, \varrho_2\}}{2} \rho^2 &\leq \frac{1}{2} \int_{\mathbb{R}^3} (3|\nabla u_n|^2 + [V(x) - \nabla V(x) \cdot x] u_n^2) dx + \frac{3}{4} \int_{\mathbb{R}^3} \phi_{u_n}(x) u_n^2 dx \\ &= \frac{1}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u_n)) [4f(u_n) u_n - (3 + \alpha) F(u_n)] dx \\ &\leq C_4 \left(\varepsilon \|u_n\|_2^{1+\alpha/3} + C_\varepsilon \|u_n\|_{6p/(3+\alpha)}^p \right) \\ &\quad \left(\|u_n\|_2^{1+\alpha/3} + \|u_n\|_{6p/(3+\alpha)}^p \right). \end{aligned} \quad (2.45)$$

Using (2.45) and Lions' concentration compactness principle [42, Lemma 1.21], we can prove that there exist $\delta > 0$ and $y_n \in \mathbb{R}^3$ such that $\int_{B_1(y_n)} |u_n|^2 dx > \delta$. Let $\hat{u}_n(x) = u_n(x + y_n)$. Then we have $\|\hat{u}_n\| = \|u_n\|$ and

$$J^\infty(\hat{u}_n) = o(1), \quad \Phi^\infty(\hat{u}_n) \rightarrow m, \quad \int_{B_1(0)} |\hat{u}_n|^2 dx > \delta. \quad (2.46)$$

Therefore, there exists $\hat{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that, passing to a subsequence,

$$\begin{cases} \hat{u}_n \rightharpoonup \hat{u}, & \text{in } H^1(\mathbb{R}^3); \\ \hat{u}_n \rightharpoonup \hat{u}, & \text{in } L_{\text{loc}}^s(\mathbb{R}^3), \quad \forall s \in [1, 6); \\ \hat{u}_n \rightarrow \hat{u}, & \text{a.e. on } \mathbb{R}^3. \end{cases} \quad (2.47)$$

Let $w_n = \hat{u}_n - \hat{u}$. Then (2.47) and Lemma 2.12 yield

$$\Phi^\infty(\hat{u}_n) = \Phi^\infty(\hat{u}) + \Phi^\infty(w_n) + o(1), \quad J^\infty(\hat{u}_n) = J^\infty(\hat{u}) + J^\infty(w_n) + o(1). \quad (2.48)$$

We define the functional $\Psi^\infty : H^1(\mathbb{R}^3) \rightarrow \mathbb{R}$ for $u \in H^1(\mathbb{R}^3)$ by

$$\Psi^\infty(u) = \Phi^\infty(u) - \frac{1}{3} J^\infty(u) = \frac{V_\infty}{3} \|u\|_2^2 + \frac{1}{6} \int_{\mathbb{R}^3} (I_\alpha * F(u)) [4f(u)u - (6 + \alpha)F(u)] dx. \quad (2.49)$$

From (2.14), (2.16), (2.46), (2.48) and (2.49), one has

$$\Psi^\infty(w_n) = m - \Psi^\infty(\hat{u}) + o(1), \quad J^\infty(w_n) = -J^\infty(\hat{u}) + o(1). \quad (2.50)$$

If there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, then we have

$$\Phi^\infty(\hat{u}) = m, \quad J^\infty(\hat{u}) = 0. \quad (2.51)$$

Next, we assume that $w_n \neq 0$. We claim that $J^\infty(\hat{u}) \leq 0$. Otherwise, if $J^\infty(\hat{u}) > 0$, then (2.50) implies $J^\infty(w_n) < 0$ for large n . In view of Corollary 2.9, there exists $t_n > 0$ such that $t_n^2(w_n)_{t_n} \in \mathcal{M}^\infty$ for large n . From (2.14), (2.16), (2.17), (2.50) and Lemma 2.13, we obtain

$$\begin{aligned}
 m - \Psi^\infty(\hat{u}) + o(1) &= \Psi^\infty(w_n) = \Phi^\infty(w_n) - \frac{1}{3}J^\infty(w_n) \\
 &\geq \Phi^\infty(t_n^2(w_n)_{t_n}) - \frac{t_n^3}{3}J^\infty(w_n) + \frac{(1-t_n)^2(2+t_n)V_\infty}{6}\|w_n\|_2^2 \\
 &\geq m^\infty - \frac{t_n^3}{3}J^\infty(w_n) + \frac{(1-t_n)^2(2+t_n)V_\infty}{6}\|w_n\|_2^2 \\
 &\geq m,
 \end{aligned}$$

which is a contradiction due to $\Psi^\infty(\hat{u}) > 0$. This shows that $J^\infty(\hat{u}) \leq 0$. In view of Lemma 2.4, there exists $t_\infty > 0$ such that $t_\infty^2 \hat{u}_{t_\infty} \in \mathcal{M}^\infty$. By (2.4), (2.14), (2.16), (2.17), (2.43), (2.46), (2.49), Fatou's lemma and Lemma 2.13, we have

$$\begin{aligned}
 m &= \lim_{n \rightarrow \infty} \left[\Phi^\infty(\hat{u}_n) - \frac{1}{3}J^\infty(\hat{u}_n) \right] \\
 &= \lim_{n \rightarrow \infty} \Psi(\hat{u}_n) \geq \Psi(\hat{u}) = \Phi^\infty(\hat{u}) - \frac{1}{3}J^\infty(\hat{u}) \\
 &\geq \Phi^\infty(t_\infty^2 \hat{u}_{t_\infty}) - \frac{t_\infty^3}{3}J^\infty(\hat{u}) + \frac{(1-t_\infty)^2(2+t_\infty)V_\infty}{6}\|\hat{u}\|_2^2 \\
 &\geq m^\infty - \frac{t_\infty^3}{3}J^\infty(\hat{u}) + \frac{(1-t_\infty)^2(2+t_\infty)V_\infty}{6}\|\hat{u}\|_2^2 \geq m,
 \end{aligned}$$

which implies (2.51) holds also. In view of Lemma 2.8, there exists $\hat{t} > 0$ such that $\hat{t}^2 \hat{u}_{\hat{t}} \in \mathcal{M}$, moreover, it follows from (V1), (1.10), (2.14), (2.51) and Corollary 2.5 that

$$m \leq \Phi(\hat{t}^2 \hat{u}_{\hat{t}}) \leq \Phi^\infty(\hat{t}^2 \hat{u}_{\hat{t}}) \leq \Phi^\infty(\hat{u}) = m.$$

This shows that m is achieved at $\hat{t}^2 \hat{u}_{\hat{t}} \in \mathcal{M}$.

Case ii). $\bar{u} \neq 0$. In this case, analogous to the proof of (2.51), by using Φ and J instead of Φ^∞ and J^∞ , we can deduce that $\Phi(\bar{u}) = m$ and $J(\bar{u}) = 0$. \square

In the same way as [13] or [38], we can obtain the following lemma.

Lemma 2.15. Assume that (V1), (V2) and (F1)–(F4) hold. If $\bar{u} \in \mathcal{M}$ and $\Phi(\bar{u}) = m$, then \bar{u} is a critical point of Φ .

Proof of Theorem 1.1. In view of Lemmas 2.14 and 2.15, there exists $\bar{u} \in \mathcal{M}$ such that

$$\Phi(\bar{u}) = m = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \max_{t > 0} \Phi(t^2 u_t), \quad \Phi'(\bar{u}) = 0.$$

This shows that \bar{u} is a ground state solution of (1.2) such that $\Phi(\bar{u}) = m = \inf_{\mathcal{M}} \Phi$. \square

3. Proofs of Theorems 1.2 and 1.3

Since we are looking for positive solutions to (1.2), without loss of generality, we suppose that $f(t) = 0$ for $t < 0$ in this section.

To use the Jeanjean's monotonicity trick [25, Theorem 1.1], for $\lambda \in [1/2, 1]$ we introduce two families of C^1 -functionals on $H^1(\mathbb{R}^3)$ defined by

$$\Phi_\lambda(u) := \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 dx - \frac{\lambda}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u))F(u) dx, \quad (3.1)$$

$$\Phi_\lambda^\infty(u) := \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V_\infty u^2) dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 dx - \frac{\lambda}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u))F(u) dx. \quad (3.2)$$

In the same way as [15,32], we can obtain the following lemma.

Lemma 3.1. Assume that (V1), (V3) (or (V3')) and (F1) hold. Let u be a critical point of Φ_λ in $H^1(\mathbb{R}^3)$, then we have the following Pohožaev type identity

$$\begin{aligned} \mathcal{P}_\lambda(u) := & \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [3V(x) + \nabla V(x) \cdot x] u^2 dx + \frac{5}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 dx \\ & - \frac{3+\alpha}{2} \lambda \int_{\mathbb{R}^3} (I_\alpha * F(u))F(u) dx = 0. \end{aligned} \quad (3.3)$$

We set $J_\lambda(u) := 2\langle \Phi'_\lambda(u), u \rangle - \mathcal{P}_\lambda(u)$, then for $\lambda \in [1/2, 1]$

$$\begin{aligned} J_\lambda(u) = & \frac{3}{2} \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} [V(x) - \nabla V(x) \cdot x] u^2 dx + \frac{3}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 dx \\ & - \frac{\lambda}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u)) [4f(u)u - (3+\alpha)F(u)] dx. \end{aligned} \quad (3.4)$$

Correspondingly, for $\lambda \in [1/2, 1]$ we also let

$$\begin{aligned} J_\lambda^\infty(u) = & \frac{3}{2} \|\nabla u\|_2^2 + \frac{V_\infty}{2} \|u\|_2^2 + \frac{3}{4} \int_{\mathbb{R}^3} \phi_u(x)u^2 dx \\ & - \frac{\lambda}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u)) [4f(u)u - (3+\alpha)F(u)] dx. \end{aligned} \quad (3.5)$$

Set

$$\mathcal{M}_\lambda^\infty := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : J_\lambda^\infty(u) = 0\}, \quad m_\lambda^\infty := \inf_{\mathcal{M}_\lambda^\infty} \Phi_\lambda^\infty.$$

By Corollary 2.5, we have the following lemma.

Lemma 3.2. Assume that (F1), (F3) and (F4) hold. Then

$$\Phi_\lambda^\infty(u) \geq \Phi_\lambda^\infty(t^2 u_t) + \frac{1-t^3}{3} J_\lambda^\infty(u) + \frac{(1-t)^2(2+t)V_\infty}{6} \|u\|_2^2, \quad \forall u \in H^1(\mathbb{R}^3), \quad t > 0. \quad (3.6)$$

Since $f(t) = 0$ for $t < 0$, from Theorem 1.1, the strong maximum principle and standard arguments, we can deduce that Φ_1^∞ has a minimizer $u_1^\infty > 0$ on \mathcal{M}_1^∞ , i.e.

$$u_1^\infty \in \mathcal{M}_1^\infty, \quad (\Phi_1^\infty)'(u_1^\infty) = 0 \quad \text{and} \quad m_1^\infty = \Phi_1^\infty(u_1^\infty). \quad (3.7)$$

Lemma 3.3. Under assumptions of Theorem 1.2 or 1.3, we have

- (i) there exists $T > 0$ independent of λ such that $\Phi_\lambda(T^2(u_1^\infty)_T) < 0$ for all $\lambda \in [1/2, 1]$;
(ii) there exists a positive constant κ_0 independent of λ such that for all $\lambda \in [1/2, 1]$,

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\lambda(\gamma(t)) \geq \kappa_0 > \max\{\Phi_\lambda(0), \Phi_\lambda(T^2(u_1^\infty)_T)\},$$

where

$$\Gamma = \{\gamma \in \mathcal{C}([0, 1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = T^2(u_1^\infty)_T\};$$

- (iii) c_λ and m_λ^∞ are non-increasing on $\lambda \in [1/2, 1]$.

The proof of Lemma 3.3 is standard, so we omit it.

Lemma 3.4. Under assumptions of Theorem 1.2 or 1.3, there exists $\bar{\lambda} \in [1/2, 1)$ such that $c_\lambda < m_\lambda^\infty$ for $\lambda \in (\bar{\lambda}, 1]$.

Proof. It is easy to see that $\Phi_\lambda(t^2(u_1^\infty)_t)$ is continuous on $t \in (0, \infty)$. Hence for any $\lambda \in [1/2, 1)$, we can choose $t_\lambda \in (0, T)$ such that $\Phi_\lambda(t_\lambda^2(u_1^\infty)_{t_\lambda}) = \max_{t \in [0, T]} \Phi_\lambda(t^2(u_1^\infty)_t)$. Let $\beta_0 = \inf_{\lambda \in [1/2, 1]} t_\lambda$. If $\beta_0 = 0$, then there exists a sequence $\{\lambda_n\} \subset [1/2, 1]$ such that $\lambda_n \rightarrow \lambda_0 \in [1/2, 1]$ and $t_{\lambda_n} \rightarrow 0$, and so by (3.1) and Lemma 3.3 (iii), one has

$$0 < c_1 \leq c_{\lambda_n} \leq \Phi_{\lambda_n}(t_{\lambda_n}^2(u_1^\infty)_{t_{\lambda_n}}) = o(1).$$

This contradiction shows $\beta_0 > 0$. Thus $0 < \beta_0 \leq t_\lambda < T$ for all $\lambda \in [1/2, 1]$. Let

$$\bar{\lambda} := \max \left\{ \frac{1}{2}, 1 - \frac{\beta_0^{4+\alpha} \min_{\beta_0 \leq s \leq T} \int_{\mathbb{R}^3} [V_\infty - V(s^{-1}x)] |u_1^\infty|^2 dx}{\int_{\mathbb{R}^3} (I_\alpha * F(T^2 u_1^\infty)) F(T^2 u_1^\infty) dx} \right\}. \quad (3.8)$$

Then $1/2 \leq \bar{\lambda} < 1$. From (3.1), (3.2), (3.6), (3.8) and Lemma 3.3 (iii), we derive

$$\begin{aligned} m_\lambda^\infty &\geq m_1^\infty = \Phi_1^\infty(u_1^\infty) \geq \Phi_1^\infty(t_\lambda^2(u_1^\infty)_{t_\lambda}) \\ &= \Phi_\lambda(t_\lambda^2(u_1^\infty)_{t_\lambda}) - \frac{1-\lambda}{2t_\lambda^{3+\alpha}} \int_{\mathbb{R}^3} (I_\alpha * F(t_\lambda^2 u_1^\infty)) F(t_\lambda^2 u_1^\infty) dx \\ &\quad + \frac{t_\lambda}{2} \int_{\mathbb{R}^3} [V_\infty - V(t_\lambda^{-1}x)] |u_1^\infty|^2 dx \\ &\geq c_\lambda - \frac{1-\lambda}{2\beta_0^{3+\alpha}} \int_{\mathbb{R}^3} (I_\alpha * F(T^2 u_1^\infty)) F(T^2 u_1^\infty) dx \\ &\quad + \frac{\beta_0}{2} \min_{\beta_0 \leq s \leq T} \int_{\mathbb{R}^3} [V_\infty - V(s^{-1}x)] |u_1^\infty|^2 dx \\ &> c_\lambda, \quad \forall \lambda \in (\bar{\lambda}, 1]. \quad \square \end{aligned}$$

Lemma 3.5. Under assumptions of Theorem 1.2 or 1.3, for almost every $\lambda \in (\bar{\lambda}, 1]$, there exists $u_\lambda \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that

$$\Phi'_\lambda(u_\lambda) = 0, \quad \Phi_\lambda(u_\lambda) = c_\lambda. \quad (3.9)$$

Proof. In view of the Jeanjean's monotonicity trick [25, Theorem 1.1] and Lemma 3.3, for almost every $\lambda \in [1/2, 1]$, there exists a bounded sequence $\{u_n(\lambda)\} \subset H^1(\mathbb{R}^3)$ (for simplicity, we denote it by $\{u_n\}$ instead of $\{u_n(\lambda)\}$ such that

$$\Phi_\lambda(u_n) \rightarrow c_\lambda > 0, \quad \|\Phi'_\lambda(u_n)\| \rightarrow 0. \quad (3.10)$$

Similar to the proof of [26, Lemma 5.1], using Lemma 2.12, we can deduce that there exist $u_\lambda \in H^1(\mathbb{R}^3)$, an integer $l \in \mathbb{N} \cup \{0\}$, a sequence $\{y_n^k\} \subset \mathbb{R}^3$ and $w^k \in H^1(\mathbb{R}^3)$ for $1 \leq k \leq l$ such that $u_n \rightharpoonup u_\lambda$ in $H^1(\mathbb{R}^3)$, $\Phi'_\lambda(u_\lambda) = 0$, $(\Phi_\lambda^\infty)'(w^k) = 0$ and $\Phi_\lambda^\infty(w^k) \geq m_\lambda^\infty$ for $1 \leq k \leq l$,

$$\left\| u_n - u_\lambda - \sum_{k=1}^l w^k(\cdot + y_n^k) \right\| \rightarrow 0 \quad \text{and} \quad \Phi_\lambda(u_n) \rightarrow \Phi_\lambda(u_\lambda) + \sum_{i=1}^l \Phi_\lambda^\infty(w^i). \quad (3.11)$$

Since $\Phi'_\lambda(u_\lambda) = 0$, then $J_\lambda(u_\lambda) = 0$. It follows from (3.1) and (3.4) that

$$\begin{aligned} \Phi_\lambda(u_\lambda) &= \Phi_\lambda(u_\lambda) - \frac{1}{3} J_\lambda(u_\lambda) \\ &= \frac{1}{6} \int_{\mathbb{R}^3} [2V(x) + \nabla V(x) \cdot x] u_\lambda^2 dx \\ &= + \frac{\lambda}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u_\lambda)) [4f(u_\lambda)u_\lambda - (6 + \alpha)F(u_\lambda)] dx \geq 0. \end{aligned} \quad (3.12)$$

If $l \neq 0$, then

$$c_\lambda = \lim_{n \rightarrow \infty} \Phi_\lambda(u_n) = \Phi_\lambda(u_\lambda) + \sum_{i=1}^l \Phi_\lambda^\infty(w^i) \geq m_\lambda^\infty, \quad \forall \lambda \in (\bar{\lambda}, 1],$$

which is a contradiction by Lemma 3.4. Thus $l = 0$, and (3.11) implies that $u_n \rightarrow u_\lambda$ in $H^1(\mathbb{R}^3)$ and $\Phi_\lambda(u_\lambda) = c_\lambda$ for almost every $\lambda \in (\bar{\lambda}, 1]$. \square

Lemma 3.6. Under assumptions of Theorem 1.2 or 1.3, there exists $\bar{u} > 0$ such that

$$\Phi'(\bar{u}) = 0, \quad \Phi(\bar{u}) = c_1 > 0. \quad (3.13)$$

Proof. Under assumptions of Theorem 1.2 or 1.3, in view of Lemma 3.5, there exist two sequences of $\{\lambda_n\} \subset (\bar{\lambda}, 1]$ and $\{u_{\lambda_n}\} \subset H^1(\mathbb{R}^3)$, denoted it by $\{u_n\}$, such that

$$\lambda_n \rightarrow 1, \quad \Phi'_{\lambda_n}(u_n) = 0, \quad \Phi_{\lambda_n}(u_n) = c_{\lambda_n}. \quad (3.14)$$

By (3.1), (3.4), (3.14) and Lemma 3.3 (iii), one has

$$\begin{aligned} c_{1/2} &\geq c_{\lambda_n} = \Phi_{\lambda_n}(u_n) - \frac{1}{3} J_{\lambda_n}(u_n) \\ &= \frac{1}{6} \int_{\mathbb{R}^3} [2V(x) + \nabla V(x) \cdot x] u_n^2 dx \\ &\quad + \frac{\lambda_n}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u_n)) [4f(u_n)u_n - (6 + \alpha)F(u_n)] dx. \end{aligned} \quad (3.15)$$

To prove the boundedness of $\{\|u_n\|\}$, we distinguish two cases: 1). (V3) holds; 2). (V3') holds.

Case 1). (V3) holds. In this case, by (2.4) and (3.15), one has

$$c_{1/2} \geq \frac{\rho}{6} \|u_n\|_2^2, \quad (3.16)$$

which implies that $\{\|u_n\|_2\}$ is bounded. Next, we prove that $\{\|\nabla u_n\|_2\}$ is also bounded. Arguing by contradiction, suppose that $\|\nabla u_n\|_2 \rightarrow \infty$. By (V1), (V3), (3.16) and Lemma 3.3 (iii), one has

$$c_{\lambda_n} + \int_{\mathbb{R}^3} [V_\infty - V(x) + |\nabla V(x) \cdot x|] u_n^2 dx \leq M_0 \quad (3.17)$$

for some constant $M_0 > 0$. Let $t_n = \min \{1, 2(M_0/\|\nabla u_n\|_2^2)^{1/3}\}$, then $t_n \rightarrow 0$. Thus, it follows from (3.1), (3.2), (3.4), (3.5) and (3.17) that

$$\begin{aligned} \Phi_{\lambda_n}^\infty(t_n^2(u_n)_{t_n}) &\leq \Phi_{\lambda_n}^\infty(u_n) - \frac{1-t_n^3}{3} J_{\lambda_n}^\infty(u_n) \\ &= \Phi_{\lambda_n}(u_n) + \frac{1}{2} \int_{\mathbb{R}^3} [V_\infty - V(x)] u_n^2 dx \\ &\quad - \frac{1-t_n^3}{3} \left[J_{\lambda_n}(u_n) + \frac{1}{2} \int_{\mathbb{R}^3} [V_\infty - V(x) + \nabla V(x) \cdot x] u_n^2 dx \right] \\ &\leq c_{\lambda_n} + \int_{\mathbb{R}^3} [V_\infty - V(x) + |\nabla V(x) \cdot x|] u_n^2 dx \leq M_0. \end{aligned} \quad (3.18)$$

Analogous to the proof of (2.41), we can deduce a contradiction by using (3.18). Hence, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ under assumptions of Theorem 1.2.

Case 2). (V3') holds. In this case, (V3') and (3.15) imply

$$\begin{aligned} c_{1/2} &\geq \frac{\lambda_n}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u_n)) [4f(u_n)u_n - (6+\alpha)F(u_n)] dx \\ &\geq \left(\mu - \frac{6+\alpha}{4} \right) \int_{\mathbb{R}^3} (I_\alpha * F(u_n)) F(u_n) dx. \end{aligned} \quad (3.19)$$

Then it follows from (V1), (3.1) and (3.19) that

$$\begin{aligned} \frac{\gamma_0}{2} \|u_n\|^2 &\leq \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u_n|^2 + V(x)u_n^2] dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx \\ &= \frac{\lambda_n}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u_n)) F(u_n) dx \leq C_4, \end{aligned} \quad (3.20)$$

where γ_0 is a positive constant. Hence, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ under assumptions of Theorem 1.3.

Similar to the proof of Lemma 3.5, there exists $\bar{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that (3.13) holds. Moreover, by the strong maximum principle and standard argument, we can conclude that $\bar{u} > 0$. \square

Proofs of Theorems 1.2 and 1.3. Let

$$\mathcal{K} := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : \Phi'(u) = 0\}, \quad \hat{m} := \inf_{u \in \mathcal{K}} \Phi(u).$$

Then Lemma 3.6 shows that $\mathcal{K} \neq \emptyset$ and $\hat{m} \leq c_1$. For any $u \in \mathcal{K}$, (2.1), (3.4) and Lemma 3.1 imply $J(u) = J_1(u) = 2\langle \Phi'(u), u \rangle - \mathcal{P}(u) = 0$. As in (3.12), we have $\Phi(u) = \Phi_1(u) \geq 0$ for any $u \in \mathcal{K}$, and so $\hat{m} \geq 0$. Let $\{u_n\} \subset \mathcal{K}$ such that $\Phi'(u_n) = 0$ and $\Phi(u_n) \rightarrow \hat{m}$. In view of Lemma 3.4, $\hat{m} \leq c_1 < m_1^\infty$. Similar to the proof of Lemma 3.6, we can deduce that there exists $\hat{u} > 0$ such that $\Phi'(\hat{u}) = 0$ and $\Phi(\hat{u}) = \hat{m}$. This shows that $\hat{u} \in H^1(\mathbb{R}^3)$ is a positive ground state solution of (1.2). \square

4. Proof of Theorem 1.5

In this section, we always assume that $2 \leq \alpha < 3$. Inspired by [40,41], we prove Theorem 1.5.

Lemma 4.1. Assume that (V1), (F1) and (F4') hold. Then for all $u \in H^1(\mathbb{R}^3)$ and $t \geq 0$,

$$\Phi(u) \geq \Phi(tu) + \frac{1-t^4}{4} \langle \Phi'(u), u \rangle + \frac{(1-t^2)^2}{4} \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2] dx. \quad (4.1)$$

Proof. From (F1) and (F4'), we can deduce

$$F(t) \geq 0, \quad \forall t \in \mathbb{R}, \quad \frac{1}{2}f(t)t - F(t) \geq 0, \quad \forall t \in \mathbb{R}, \quad (4.2)$$

$$\frac{F(t)}{t|t|} \text{ is nondecreasing on } (-\infty, 0) \cup (0, +\infty). \quad (4.3)$$

Next, we claim

$$\begin{aligned} h_0(t, u) &:= \int_{\mathbb{R}^3} \left[(I_\alpha * F(tu)) F(tu) - (I_\alpha * F(u)) F(u) + \frac{1-t^4}{2} (I_\alpha * F(u)) f(u)u \right] dx \\ &\geq 0, \quad \forall t \geq 0, u \in H^1(\mathbb{R}^3). \end{aligned} \quad (4.4)$$

Clearly, (4.2) implies that $h_0(0, u) \geq 0$ for $u \in H^1(\mathbb{R}^3)$. For $t \neq 0$, by (F4'), (4.2) and (4.3), we have

$$\begin{aligned} \frac{d}{dt} h_0(t, u) &= 2t^3 \int_{\mathbb{R}^3} u^2 \left\{ \left(I_\alpha * \frac{F(tu)}{t^2} \right) \frac{f(tu)}{tu} - (I_\alpha * F(u)) \frac{f(u)}{u} \right\} dx \\ &\begin{cases} \geq 0, & t \geq 1, \\ \leq 0, & 0 < t < 1, \end{cases} \end{aligned}$$

which implies $h_0(t, u) \geq h_0(1, u) = 0$ for $t > 0$ and $u \in H^1(\mathbb{R}^3)$. This shows that (4.4) holds. Then it follows from (1.10), (1.11) and (4.4) that

$$\begin{aligned} \Phi(u) - \Phi(tu) &= \frac{1-t^4}{4} \langle \Phi'(u), u \rangle + \frac{(1-t^2)^2}{4} \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2] dx + \frac{1}{2} h_0(t, u) \\ &\geq \frac{1-t^4}{4} \langle \Phi'(u), u \rangle + \frac{(1-t^2)^2}{4} \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2] dx \end{aligned}$$

for $t \geq 0$ and $u \in H^1(\mathbb{R}^3)$. This shows that (4.1) holds. \square

From Lemma 4.1, we have the following corollary.

Corollary 4.2. Assume that (V1), (F1) and (F4') hold. Then $\Phi(u) = \max_{t \geq 0} \Phi(tu)$ for $u \in \mathcal{N}$.

Lemma 4.3. Assume that (V1), (F1), (F3') and (F4') hold. Then for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$.

Proof. Let $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ be fixed and define a function $\zeta_0(t) := \langle \Phi'(tu), tu \rangle$ on $(0, \infty)$. Using (F1), (F3') and (1.11), it is easy to verify that $\zeta_0(0) = 0$, $\zeta_0(t) > 0$ for $t > 0$ small and $\zeta_0(t) < 0$ for t large. Therefore, there exist $t_0 = t_u > 0$ so that $\zeta_0(t_0) = 0$ and $t_u u \in \mathcal{N}$. We claim that t_u is unique for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. In fact, for any given $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, let $t_1, t_2 > 0$ such that $\zeta_0(t_1) = \zeta_0(t_2) = 0$. Jointly with (4.1), we have

$$\Phi(t_1 u) \geq \Phi(t_2 u) + \frac{(t_1^2 - t_2^2)^2}{4t_1^2} \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2] dx, \quad (4.5)$$

$$\Phi(t_2 u) \geq \Phi(t_1 u) + \frac{(t_2^2 - t_1^2)^2}{4t_2^2} \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2] dx. \quad (4.6)$$

(4.5) and (4.6) imply $t_1 = t_2$. Hence, $t_u > 0$ is unique for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. \square

Combining Lemma 4.1, Corollary 4.2 and Lemma 4.3, we can obtain the following lemma.

Lemma 4.4. Assume that (V1), (F1), (F3') and (F4') hold. Then

$$c = \inf_{\mathcal{N}} \Phi = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \max_{t \geq 0} \Phi(tu) > 0.$$

By a similar argument as [9, Lemmas 2.5, 2.7], we can obtain the following two lemmas.

Lemma 4.5. Assume that (V1), (F1), (F3') and (F4') hold. Then there exist a constant $c_* \in (0, c]$ and a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ satisfying

$$\Phi(u_n) \rightarrow c_*, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \rightarrow 0. \quad (4.7)$$

Lemma 4.6. Assume that (V1), (F1), (F3') and (F4') hold. If $\bar{u} \in \mathcal{N}$ and $\Phi(\bar{u}) = c$, then \bar{u} is a critical point of Φ .

Proof of Theorem 1.5. In view of Lemma 4.5, there exists $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that (4.7) holds. By Lemma 4.1, one has

$$c_* + o(1) = \Phi(u_n) - \frac{1}{4} \langle \Phi'(u_n), u_n \rangle \geq \frac{1}{4} \int_{\mathbb{R}^3} [|\nabla u_n|^2 + V(x)u_n^2] dx,$$

which implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Passing to a subsequence, we have $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$ and $u_n \rightarrow \bar{u}$ a.e. on \mathbb{R}^3 . There are two possible cases: i). $\bar{u} = 0$; ii). $\bar{u} \neq 0$.

Case i). $\bar{u} = 0$. Then $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, and so $u_n \rightarrow 0$ in $L_{loc}^s(\mathbb{R}^3)$, $2 \leq s < 6$ and $u_n \rightarrow 0$ a.e. on \mathbb{R}^3 . Since

$$\int_{\mathbb{R}^3} [V_\infty - V(x)]u_n^2 dx = o(1), \quad \int_{\mathbb{R}^3} [V_\infty - V(x)]u_n v dx = o(1), \quad \forall v \in H^1(\mathbb{R}^3),$$

it follows from (1.10), (1.11) and (4.7) that

$$\Phi^\infty(u_n) \rightarrow c_*, \quad \|(\Phi^\infty)'(u_n)\|(1 + \|u_n\|) \rightarrow 0. \quad (4.8)$$

By (1.7), (2.14), (2.16), (2.44) and (4.8), one has

$$\begin{aligned} 0 &< c_* + o(1) = \Phi^\infty(u_n) - \frac{1}{2} \langle (\Phi^\infty)'(u_n), u_n \rangle \\ &= -\frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n}(x) u_n^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} (I_\alpha * F(u_n)) [f(u_n) u_n - F(u_n)] dx \\ &\leq C_5 \left(\varepsilon \|u_n\|_2^{1+\alpha/3} + C_\varepsilon \|u_n\|_{6p/(3+\alpha)}^p \right) \left(\|u_n\|_2^{1+\alpha/3} + \|u_n\|_{6p/(3+\alpha)}^p \right). \end{aligned} \quad (4.9)$$

Using (4.8) and Lions' concentration compactness principle [42, Lemma 1.21], we can prove that there exist $\delta > 0$ and $y_n \in \mathbb{R}^3$ such that $\int_{B_1(y_n)} |u_n|^2 dx > \delta$. Let $\tilde{u}_n(x) = u_n(x + y_n)$. Then we have

$$\Phi^\infty(\tilde{u}_n) \rightarrow c_* \in (0, c], \quad \|(\Phi^\infty)'(\tilde{u}_n)\|(1 + \|\tilde{u}_n\|) \rightarrow 0, \quad (4.10)$$

and there exists $\tilde{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that, up to a subsequence, $\tilde{u}_n \rightharpoonup \tilde{u}$ in $H^1(\mathbb{R}^3)$, $\tilde{u}_n \rightarrow \tilde{u}$ in $L_{\text{loc}}^s(\mathbb{R}^3)$, $2 \leq s < 6$ and $\tilde{u}_n \rightarrow \tilde{u}$ a.e. on \mathbb{R}^3 . Using (4.10), a standard argument shows that $(\Phi^\infty)'(\tilde{u}) = 0$. Thus, by (4.2), (4.10) and Fatou's lemma, we have

$$\begin{aligned} c &\geq c_* = \lim_{n \rightarrow \infty} \left[\Phi^\infty(\tilde{u}_n) - \frac{1}{4} \langle (\Phi^\infty)'(\tilde{u}_n), \tilde{u}_n \rangle \right] \\ &= \lim_{n \rightarrow \infty} \left\{ \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla \tilde{u}_n|^2 + V_\infty \tilde{u}_n^2) dx \right. \\ &\quad \left. + \frac{1}{2} \int_{\mathbb{R}^3} (I_\alpha * F(\tilde{u}_n)) \left[\frac{1}{2} f(\tilde{u}_n) \tilde{u}_n - F(\tilde{u}_n) \right] dx \right\} \\ &\geq \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla \tilde{u}|^2 + V_\infty \tilde{u}^2) dx + \frac{1}{2} \int_{\mathbb{R}^3} (I_\alpha * F(\tilde{u})) \left[\frac{1}{2} f(\tilde{u}) \tilde{u} - F(\tilde{u}) \right] dx \\ &= \Phi^\infty(\tilde{u}) - \frac{1}{4} \langle (\Phi^\infty)'(\tilde{u}), \tilde{u} \rangle = \Phi^\infty(\tilde{u}). \end{aligned} \quad (4.11)$$

Since $\tilde{u} \neq 0$, in view of Lemma 4.3, there exists $\tilde{t} = t_{\tilde{u}} > 0$ such that $\tilde{t}\tilde{u} \in \mathcal{N}$, and so $\Phi(\tilde{t}\tilde{u}) \geq c$. On the other hand, since $(\Phi^\infty)'(\tilde{u}) = 0$, similar to (4.1), we have

$$\begin{aligned} \Phi^\infty(\tilde{u}) &\geq \Phi^\infty(\tilde{t}\tilde{u}) + \frac{1 - \tilde{t}^4}{4} \langle (\Phi^\infty)'(\tilde{u}), \tilde{u} \rangle + \frac{(1 - \tilde{t}^2)^2}{4} \int_{\mathbb{R}^3} (|\nabla \tilde{u}|^2 + V_\infty \tilde{u}^2) dx \\ &\geq \Phi^\infty(\tilde{t}\tilde{u}) = \Phi(\tilde{t}\tilde{u}) + \frac{\tilde{t}^2}{2} \int_{\mathbb{R}^3} [V_\infty - V(x)] \tilde{u}^2 dx \\ &\geq \Phi(\tilde{t}\tilde{u}) \geq c, \end{aligned}$$

which, together with (4.11), implies $\Phi(\tilde{t}\tilde{u}) = c$. Let $u_0 = \tilde{t}\tilde{u}$. Then $u_0 \in \mathcal{N}$ and $\Phi(u_0) = c$. In view of Lemma 4.6, we have $\Phi'(u_0) = 0$. This shows that $u_0 \in H^1(\mathbb{R}^3)$ is a ground state solution of (1.2) such that $\Phi(u_0) = \inf_{\mathcal{N}} \Phi > 0$.

Case ii). $\bar{u} \neq 0$. A standard argument shows that $\Phi'(\bar{u}) = 0$ and $\Phi(\bar{u}) \geq c$. Moreover, we can deduce from (4.2), (4.7) and Fatou's lemma that

$$c \geq c_* = \lim_{n \rightarrow \infty} \left[\Phi(u_n) - \frac{1}{4} \langle \Phi'(u_n), u_n \rangle \right] \geq \Phi(\bar{u}) - \frac{1}{4} \langle \Phi'(\bar{u}), \bar{u} \rangle = \Phi(\bar{u}).$$

Hence, $\bar{u} \in H^1(\mathbb{R}^3)$ is a ground state solution of (1.2) such that $\Phi(\bar{u}) = \inf_{\mathcal{N}} \Phi > 0$. \square

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