J. Math. Anal. Appl. 478 (2019) 458–465

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Remark on the blow-up of solutions for the semilinear wave equation with overdamping term

Kenji Nishihara

Waseda University, Tokyo 169-8050, Japan

ARTICLE INFO

Article history:
Received 22 October 2018
Available online 30 May 2019
Submitted by D.M. Ambrose

Keywords:
Semilinear wave equation
Overdamping
Blow-up
Comparison theorem

ABSTRACT

We consider the Cauchy problem for the wave equation with overdamping and semilinear source terms:

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = N(u), \ (t,x) \in \mathbf{R}^+ \times \mathbf{R}^N \\ (u,u_t)(0,x) = (u_0,u_1)(x), \ \ x \in \mathbf{R}^N, \end{cases} \tag{P}$$

with $b(t) = b_0(t+1)^{-\beta}$, $b_0 > 0$, $\beta < -1$ and $N(u) = |u|^{p-1}u$, $p > 1$. Ikeda and Wakasugi [8] have recently showed that, when $|N(u)| \leq C|u|^p$ for any $p > 1$, there is a global-in-time solution to (P) for suitable small data, and that, when $N(u) = \pm|u|^p$, the local-in-time solution blows up within a finite time for suitable large data. To show the blow-up result, their method seems to be not applicable to our semilinear term. Our aim is to show the blow-up of solutions for suitable large data, by the method much different from theirs.

1. Introduction

We consider the Cauchy problem for the semilinear damped wave equation

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = N(u), & (t,x) \in \mathbf{R}^+ \times \mathbf{R}^N \\ (u,u_t)(0,x) = (u_0,u_1)(x), & x \in \mathbf{R}^N, \end{cases} \tag{1.1}$$

with the coefficient of damping

$$b(t) = b_0(t+1)^{-\beta}, \quad (\beta < -1, \ b_0 > 0) \tag{1.2}$$

and the semilinear source term

E-mail address: kenji@waseda.jp.

https://doi.org/10.1016/j.jmaa.2019.05.037

$$N(u) = |u|^{p-1}u, \ p > 1. \tag{1.3}$$

In (1.2), depending on β, the damping $b(t)u_t$ is classified as follows:

$$\begin{array}{lll} (i) & \beta > 1 & \Rightarrow \text{non-effective}, \\ (ii) & -1 \le \beta < 1 \ (\int_0^\infty \frac{dt}{b(t)} = \infty) & \Rightarrow \text{effective}, \\ (iii) & \beta < -1 \ (\int_0^\infty \frac{dt}{b(t)} < \infty) & \Rightarrow \text{overdamping} \end{array}$$

(Wirth [29,30], cf. Yamazaki [31,32].) In (i) "non-effective" means that the damping is weak and the solution to (1.1) behaves as wave-like and scatters as time tends to infinity. In (ii) the solution behaves as $t \to \infty$ like that to the corresponding parabolic equation, which is called diffusion phenomenon. In this paper we treat the overdamping case (iii). Ikeda and Wakasugi [8] recently obtained in this case that, if $|N(u)| \le C|u|^p$ for any $p > 1$, then there is a global-in-time solution u for small data (u_0, u_1), and that, when $N(u) = \pm|u|^p \ (p > 1)$, for suitable large data (u_0, u_1) the local-in-time solution u blows up in a finite time. The blow-up result in [8] is obtained by applying the test function method, and so it seems to be not applicable to our semilinear term (1.3), because $N(u)$ in (1.3) may change the sign.

Thus, our aim in this paper is to show the blow-up of the solution to (1.1) with (1.2)–(1.3) for any $p > 1$ and suitable large data with compact support. To show this we apply the comparison theorem for second order ordinary differential inequalities (Li and Zhou [13]) and essentially use the support of data is compact (cf. Zhou [34]). Our method is much different from that in [8]. Our theorem obtained will be stated precisely in the end of this paper.

We here mention the related results. For semilinear damped wave equations one main interest is to determine the critical exponent p_c in the sense that, if $p > p_c$, then there is a global-in-time solution for small data, and, if $p \le p_c$, then a local-in-time solution blows up in a finite time for any suitable small data. In this sense, it may be said that $p_c = 1$ in overdamping case by the result in [8]. Roughly speaking, we also expect that, when the damping is non-effective, p_c is the Strauss exponent, and, on the other hand, when the damping is effective, p_c is the Fujita exponent or its variant. For these there are many literatures. In case of constant coefficient, see [13,26,33,20,7,17,19,5,6] etc. In the time-dependent case, see [29–32,22,16,1–4] etc., and also [18,9,10,21,24,25,27] etc. in the space-dependent case. The case of both space and time dependent coefficient is interesting, but the blow-up of solutions is not known well. See [28,14,15,11,12] for the small data global existence of solutions. To more detailed references see our recent paper [23].

We end this section by stating the local existence theorem without its proof, because the proof is standard.

Proposition 1.1 *(Local-in-time solution). Let* $(u_0, u_1) \in H^1 \times L^2$ *with* $\operatorname{supp}(u_0, u_1) \subset \{x \in \mathbf{R}^N; \ |x| \le L\} =: B_L (L > 1)$*, and* $1 < p < \infty$ *(*$N = 1, 2$*),* $1 < p \le \frac{N}{N-2} (N \ge 3)$*. Then there exists a solution* $u \in C([0,T); H^1) \cap C^1([0,T); L^2)$ *to (1.1) with (1.2)–(1.3) for some* $T > 0$*, which satisfies*

$$\operatorname{supp} u(t, \cdot) \subset \{x \in \mathbf{R}^N; \ |x| \le t + L\} = B_{t+L}. \tag{1.4}$$

2. Comparison theorem

For the second order ordinary differential inequalities the following comparison theorem holds.

Lemma 2.1 *(Comparison theorem, Li-Zhou [13]). Let the functions* $k, h \in C^2([0,\infty))$ *satisfy*

$$f_0(t)k''(t) + k'(t) \ge g_0(t)k(t)^{1+\alpha}, \quad t \ge 0, \tag{2.1}$$

$$f_0(t)h''(t) + h'(t) \le g_0(t)h(t)^{1+\alpha}, \quad t \ge 0, \tag{2.2}$$

where $\alpha \ge 0$ *and* $f_0, g_0 \in C([0,\infty))$ *are positive. Then, if*

$$k(0) > h(0),\ k'(0) \geq h'(0) \quad or \quad k(0) \geq h(0),\ k'(0) > h'(0), \tag{2.3}$$

then

$$k'(t) \geq h'(t),\ t \geq 0 \tag{2.4}$$

and so

$$k(t) \geq h(t),\ t \geq 0. \tag{2.5}$$

The proof is given in Li and Zhou [13], but for the reader's convenience we state the proof shortly.

Proof. First, instead of (2.3), we can assume

$$k(0) > h(0),\ k'(0) > h'(0), \quad t \geq 0 \tag*{(2.3)$'$}$$

without loss of generality. In fact, if $k'(0) > h'(0)$ and $k(0) = h(0)$, then, for $\delta_0 > 0$ small, $k(t) > h(t),\ k'(t) > h'(t),\ 0 < t \leq \delta_0$ and hence we can take $t = \delta_0$ as the initial time. If $k(0) > h(0)$ and $k'(0) = h'(0)$, then, by (2.1)–(2.2), $k''(0) \geq \frac{g_0(0)}{f_0(0)} k(0)^{1+\alpha} > \frac{g_0(0)}{f_0(0)} h(0)^{1+\alpha} \geq h''(0)$, and hence $k'(t) > h'(t)$ and $k(t) > h(t)$ for $0 < t \leq \delta_0$.

Assume that (2.4) fails, then there is $t^* > 0$ such that

$$k'(t) > h'(t)\ (0 < t < t^*), \quad k'(t^*) = h'(t^*).$$

Hence, for $t < t^*$, $\frac{k'(t)-k'(t^*)}{t-t^*} < \frac{h'(t)-h'(t^*)}{t-t^*}$ and $k''(t^*) \leq h''(t^*)$ as $t \to t^* - 0$. On the other hand, by $k'(t) > h'(t)\,(0 < t < t^*)$ and $k(t^*) > h(t^*)$,

$$f_0(t^*)k''(t^*) + k'(t^*) \geq g_0(t^*)k(t^*)^{1+\alpha} > g_0(t^*)h(t^*)^{1+\alpha} \geq f_0(t^*)h''(t^*) + h'(t^*).$$

Hence $k''(t^*) > h''(t^*)$, which contradicts to $k''(t^*) \leq h''(t^*)$. $\square$

Lemma 2.2 *(Blow-up). Let the function $\Psi(t) \geq 0\,(t \geq 0)$ satisfy*

$$f(t)\Psi''(t) + \Psi'(t) \geq C_0 g(t)\Psi(t)^{1+\alpha}\ (t \geq 0), \quad \Psi(0) > 0,\ \Psi'(0) > 0, \tag{2.6}$$

where $\alpha > 0$, $C_0 > 0$ are constants and positive functions $f(t), g(t)$ satisfy

$$f(t) \leq C_1,\ g(0) = 1,\ g'(t) \leq 0 \text{ and } G_0 := \int_0^\infty g(t)\,dt < \infty \tag{2.7}$$

for some constant $C_1 > 0$. Then, if

$$\min\left(\frac{\alpha\Psi'(0)}{4\Psi(0)}, \frac{\alpha C_0\Psi(0)^\alpha}{1+\sqrt{1+4(1+\alpha/2)C_1C_0\Psi(0)^\alpha}}\right) > G_0^{-1}, \tag{2.8}$$

then $\Psi(t)$ blows up in $(0, T^*]$ for some $T^* > 0$.

Proof. Define the function $\Phi(t)$ by the solution to

$$\Phi'(t) = \eta g(t)\Phi(t)^{1+\frac{\alpha}{2}} \ (t \geq 0), \quad \Phi(0) = \Psi(0). \tag{2.9}$$

Here $\eta > 0$ is suitably defined later. Differentiate

$$\begin{aligned}\Phi''(t) &= \eta g'(t)\Phi(t)^{1+\frac{\alpha}{2}} + \eta(1+\frac{\alpha}{2})g(t)\Phi(t)^{\frac{\alpha}{2}} \cdot \Phi'(t) \\ &\leq \eta^2(1+\frac{\alpha}{2})g(t)\Phi(t)^{1+\alpha} \quad \text{(by (2.9) and (2.7))},\end{aligned}$$

and

$$\begin{aligned}f(t)\Phi''(t) + \Phi'(t) &\leq \eta^2(1+\frac{\alpha}{2})f(t)\Phi(t)^{1+\alpha} + \eta g(t)\Phi(t)^{1+\frac{\alpha}{2}} \\ &\leq (\eta^2(1+\frac{\alpha}{2})C_1 + \frac{\eta}{\Psi(0)^{\alpha/2}})g(t)\Phi(t)^{1+\alpha}.\end{aligned}$$

Choose η as

$$\eta^2(1+\frac{\alpha}{2})C_1 + \frac{\eta}{\Psi(0)^{\alpha/2}} \leq C_0$$

or

$$\eta \leq \frac{2C_0\Psi(0)^{\alpha/2}}{1+\sqrt{1+4(1+\frac{\alpha}{2})C_1C_0\Psi(0)^{\alpha}}}, \tag{2.10}$$

then $\Phi(t)$ satisfies

$$f(t)\Phi''(t) + \Phi'(t) \leq C_0 g(t)\Phi(t)^{1+\alpha} \ (t \geq 0), \quad \Phi(0) = \Psi(0). \tag{2.11}$$

We additionally pose

$$\Phi'(0) < \Psi'(0), \tag{2.12}$$

then, by Lemma 2.1, we have

$$\Phi(t) < \Psi(t) \ (t \geq 0). \tag{2.13}$$

By (2.9), (2.12) means

$$\Phi'(0) = \eta g(0)\Phi(0)^{1+\frac{\alpha}{2}} < \Psi'(0) \quad \text{and} \quad \eta < \frac{\Psi'(0)}{\Psi(0)^{1+\frac{\alpha}{2}}}.$$

Hence η should be taken as

$$\eta \leq \frac{\Psi'(0)}{2\Psi(0)^{1+\frac{\alpha}{2}}}. \tag{2.14}$$

Due to (2.10) and (2.14) we choose η as

$$\eta = \min\left(\frac{\Psi'(0)}{2\Psi(0)}, \frac{2C_0\Psi(0)^{\alpha/2}}{1+\sqrt{1+4(1+\alpha/2)C_1C_0\Psi(0)^{\alpha}}}\right). \tag{2.15}$$

We now solve (2.9) and get

$$\Phi(t) = \left(\frac{1}{\Psi(0)^{\alpha/2}} - \frac{\alpha\eta}{2} \int_0^t g(\tau)\, d\tau \right)^{-\alpha/2}. \tag{2.16}$$

Hence, if

$$\frac{1}{\Psi(0)^{\alpha/2}} < \frac{\alpha\eta}{2} \int_0^\infty g(t)\, dt = \frac{\alpha\eta}{2} G_0, \tag{2.17}$$

then there is some $T^* > 0$ such that $\Phi(t)$ blows up within $(0, T^*]$ and so $\Psi(t)$. The condition (2.17) is written as $G_0^{-1} < \frac{\alpha}{2}\Psi(0)^{\alpha/2}\eta$, that is, if (2.8) holds, then $\Psi(t)$ blows up within $(0, T^*]$. $\square$

3. Blow-up of the solution to (1.1)–(1.3)

Let u be a local-in-time solution obtained in Proposition 1.1. To show the blow-up of the solution, we refer Zhou [34]. First, applying u_t to (1.1) and integrate it over $B_{t+L}(\subset \mathbf{R}^N)$, we have

$$\frac{d}{dt}E(t) := \frac{d}{dt}\left[\frac{1}{2}\int (u_t^2 + |\nabla u|^2)\, dx - \frac{1}{p+1}\int |u|^{p+1} dx \right] = -b(t)\|u_t(t)\|^2 \leq 0.$$

Here and after the domain B_{t+L} of integral is often abbreviated and $\|\cdot\|$ denotes the L^2-norm. Hence

$$E(t) \leq E(0) = \frac{1}{2}(\|u_1\|^2 + \|\nabla u_0\|^2) - \frac{1}{p+1}\int_{B_L} |u_0|^{p+1} dx. \tag{3.1}$$

Here we assume

$$E(0) \leq 0 \quad \text{and so} \quad E(t) \leq 0. \tag{3.2}$$

We now define $\Psi(t)$ by

$$\Psi(t) = \frac{1}{2}\int_{\mathbf{R}^N} |u(t,x)|^2 dx = \frac{1}{2}\int_{B_{t+L}} u(t,x)^2 dx = \frac{1}{2}\|u(t,\cdot)\|^2. \tag{3.3}$$

Then

$$\Psi'(t) = \int (uu_t)(t,x)\, dx \tag{3.4}$$

and

$$\begin{aligned}
\Psi''(t) &= \int u_t(t,x)^2\, dx + \int (uu_{tt})(t,x)\, dx \\
&= -2\left[\frac{1}{2}\int (u_t^2 + |\nabla u|^2)\, dx - \frac{1}{p+1}\int |u|^{p+1} dx \right] \\
&\quad + 2\int u_t^2 dx + \int |\nabla u|^2 dx - \frac{2}{p+1}\int |u|^{p+1} dx + \int uu_{tt} dx.
\end{aligned}$$

By (1.1)

$$\int uu_{tt}dx + \int |\nabla u|^2 dx + b(t)\int uu_t dx = \int |u|^{p+1}dx.$$

Hence, by (3.4)

$$\Psi''(t) \geq 2\int u_t^2\, dx - b(t)\Psi'(t) + \frac{p-1}{p+1}\int |u|^{p+1}dx$$

and

$$\Psi''(t) + b(t)\Psi'(t) \geq \frac{p-1}{p+1}\int\limits_{B_{t+L}} |u|^{p+1}dx. \tag{3.5}$$

Here, using Hölder's inequality with $\frac{1}{(p+1)/2} + \frac{1}{(p+1)/(p-1)} = 1$, we have

$$\int\limits_{B_{t+L}} |u|^2 dx \leq \Big(\int\limits_{B_{t+L}} |u|^{p+1}dx\Big)^{\frac{2}{p+1}}\Big(\int\limits_{B_{t+L}} 1\, dx\Big)^{\frac{p-1}{p+1}} \leq (t+L)^{\frac{(p-1)N}{p+1}}\Big(\int\limits_{B_{t+L}} |u|^{p+1}dx\Big)^{\frac{2}{p+1}}$$

and

$$\int\limits_{B_{t+L}} |u|^{p+1}dx \geq L^{-\frac{(p-1)N}{2}}(1+t)^{-\frac{(p-1)N}{2}}\Psi(t)^{\frac{p+1}{2}}.$$

Hence (3.5) yields

$$b(t)^{-1}\Psi''(t) + \Psi'(t) \geq \frac{p-1}{p+1}b_0^{-1}L^{-\frac{(p-1)N}{2}}(1+t)^{\beta-\frac{(p-1)N}{2}}\Psi(t)^{1+\frac{p-1}{2}}. \tag{3.6}$$

In Lemma 2.2, we naturally take

$$\begin{aligned} &\alpha = \tfrac{p-1}{2},\ f(t) = b(t)^{-1} \leq b_0^{-1} = C_1,\ C_0 = \tfrac{p-1}{p+1}b_0^{-1}L^{-\frac{(p-1)N}{2}}, \\ &g(t) = (1+t)^{\beta-\frac{p-1}{2}},\ g'(t) < 0,\ \textstyle\int_0^\infty g(t)\,dt = (\frac{(p-1)N}{2} - \beta - 1)^{-1} = G_0(>0). \end{aligned} \tag{3.7}$$

Therefore, if $\Psi(0) = \frac{1}{2}\int_{B_L} u_0^2 dx$ and $\Psi'(0) = \int_{B_L} u_0u_1 dx$ fulfill (2.8), then the solution u blows up within $(0, T^*]$.

Thus we have the following theorem.

Theorem 3.1 *(Blow-up). The solution $u \in C([0,T);\, H^1) \cap C^1([0,T);\, L^2)$ to (1.1)–(1.3) obtained in Proposition 1.1 blows up within $(0, T^*]$ for some $T^* > 0$ provided that the data $(u_0, u_1) \in H^1 \times L^2$ satisfies (3.2) and (2.8) with $\Psi(0) = \frac{1}{2}\int_{B_L} u_0^2 dx,\ \Psi'(0) = \int_{B_L} u_0u_1 dx$ and (3.7).*

Remark 3.1. *The set of the data (u_0, u_1) satisfying the conditions in Theorem 3.1 is not empty. In fact, for an example, let $w_0(x)$ be in C^∞ with $\operatorname{supp} w_0 \subset B_L$, and define*

$$u_0 = Kw_0,\ u_1 = K^{1+\gamma}w_0\,(0 < \gamma < \frac{p-1}{2})$$

for $K \gg 1$. Then, since $\Psi(0) = \frac{K^2}{2}\|w_0\|^2,\ \frac{\Psi'(0)}{\Psi(0)} = 2K^\gamma \to \infty$ as $K \to \infty$, and

$$E(0) = \frac{1}{2}(K^{2+2\gamma}\|w_0\|^2 + K^2\|\nabla w_0\|^2) - \frac{K^{p+1}}{p+1}\int_{B_L} |w_0|^{p+1}dx \to -\infty \ (K \to \infty),$$

necessary conditions (2.8) and (3.2) are fulfilled for large $K > 0$.

References

[1] M. D'Abbicco, The threshold of effective damping for semilinear wave equations, Math. Methods Appl. Sci. 38 (2015) 1032–1045.

[2] M. D'Abbicco, S. Lucente, A modified test function method for damped wave equations, Adv. Nonlinear Stud. 13 (2013) 867–892.

[3] M. D'Abbicco, S. Lucente, M. Reissig, Semi-Linear wave equations with effective damping, Chin. Ann. Math. Ser. B 34 (2013) 345–380.

[4] M. D'Abbicco, S. Lucente, M. Reissig, A shift in the Strauss exponent for semilinear wave equations with a not effective damping, J. Differential Equations 259 (2015) 5040–5073.

[5] N. Hayashi, E.I. Kaikina, P.I. Naumkin, Damped wave equation with super critical nonlinearities, Differential Integral Equations 17 (2004) 637–652.

[6] N. Hayashi, E.I. Kaikina, P.I. Naumkin, Damped wave equation in the subcritical case, J. Differential Equations 207 (2004) 161–194.

[7] T. Hosono, T. Ogawa, Large time behavior and L^p-L^q estimate of solutions of 2-dimensional nonlinear damped wave equations, J. Differential Equations 203 (2004) 82–118.

[8] M. Ikeda, Y. Wakasugi, Global well-posedness for the semilinear wave equation with time dependent damping in the overdamping case, Proc. Amer. Math. Soc. (2019), in press.

[9] R. Ikehata, G. Todorova, B. Yordanov, Critical exponent for semilinear wave equations with space-dependent potential, Funkcial. Ekvac. 52 (2009) 411–435.

[10] R. Ikehata, G. Todorova, B. Yordanov, Optimal decay rate of the energy for wave equations with critical potential, J. Math. Soc. Japan 65 (2013) 183–236.

[11] J.S. Kenigson, J.J. Kenigson, Energy decay estimates for the dissipative wave equation with space-time dependent potential, Math. Methods Appl. Sci. 34 (2011) 48–62.

[12] M. Khader, Nonlinear dissipative wave equations with space-time dependent potential, Nonlinear Anal. 74 (2011) 3945–3963.

[13] T.-T. Li, Yi. Zhou, Breakdown of solutions to $\square u + u_t = u^{1+\alpha}$, Discrete Contin. Dyn. Syst. 1 (1995) 503–520.

[14] J. Lin, K. Nishihara, J. Zhai, L^2 estimates of solutions for the damped wave equations with space-time dependent damping term, J. Differential Equations 248 (2010) 403–422.

[15] J. Lin, K. Nishihara, J. Zhai, Decay property of solutions for damped wave equations with space-time dependent damping term, J. Math. Anal. Appl. 374 (2011) 602–614.

[16] J. Lin, K. Nishihara, J. Zhai, Critical exponent for the semilinear wave equation with time-dependent damping, Discrete Contin. Dyn. Syst. 32 (2012) 4307–4320.

[17] P. Marcati, K. Nishihara, The L^p-L^q estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media, J. Differential Equations 191 (2003) 445–469.

[18] K. Mochizuki, Scattering theory for wave equations with dissipative terms, Publ. Res. Inst. Math. Sci. 12 (1976) 383–390.

[19] T. Narazaki, L^p-L^q estimates for damped wave equations and their applications to semi-linear problem, J. Math. Soc. Japan 56 (2004) 585–626.

[20] K. Nishihara, L^p-L^q estimates of solutions to the damped wave equation in 3-dimensional space and their application, Math. Z. 244 (2003) 631–649.

[21] K. Nishihara, Decay properties for the damped wave equation with space dependent potential and absorbed semilinear term, Comm. Partial Differential Equations 35 (2010) 1402–1418.

[22] K. Nishihara, Asymptotic behavior of solutions to the semilinear wave equation with time-dependent damping, Tokyo J. Math. 34 (2011) 327–343.

[23] K. Nishihara, M. Sobajima, Y. Wakasugi, Critical exponent for the semilinear wave equations with a damping in the far field, NoDEA Nonlinear Differential Equations Appl. 25 (2018) 55.

[24] M. Sobajima, Y. Wakasugi, Diffusion phenomena for the wave equation with space-dependent damping in an exterior domain, J. Differential Equations 261 (2016) 5690–5718.

[25] M. Sobajima, Y. Wakasugi, Diffusion phenomena for the wave equation with space-dependent damping term growing at infinity, Adv. Differential Equations 23 (2018) 581–614.

[26] G. Todorova, B. Yordanov, Critical exponent for a nonlinear wave equation with damping, J. Differential Equations 174 (2001) 464–489.

[27] G. Todorova, B. Yordanov, Nonlinear dissipative wave equations with potential, Contemp. Math. 426 (2007) 317–337.

[28] Y. Wakasugi, Small data global existence for the semilinear wave equation with space-time dependent damping, J. Math. Anal. Appl. 393 (2012) 66–79.

[29] J. Wirth, Wave equations with time-dependent dissipation. I. Non-effective dissipation, J. Differential Equations 222 (2006) 487–514.

[30] J. Wirth, Wave equations with time-dependent dissipation. II. Effective dissipation, J. Differential Equations 232 (2007) 74–103.

[31] T. Yamazaki, Asymptotic behavior for abstract wave equations with decaying dissipation, Adv. Differential Equations 11 (2006) 419–456.

[32] T. Yamazaki, Diffusion phenomenon for abstract wave equations with decaying dissipation, Adv. Stud. Pure Math. 47 (2007) 363–381.

[33] Qi. Zhang, A blow-up result for a nonlinear wave equation with damping: the critical case, C. R. Acad. Sci. Paris 333 (2001) 109–114.

[34] Yong Zhou, A blow-up result for a nonlinear wave equation with damping and vanishing initial energy in $\mathbf{R}^N$, Appl. Math. Lett. 18 (2005) 281–286.