



# G-outer inverse of Banach spaces operators

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## ABSTRACT

The aim of this paper is to present a new generalized inverse, which is called a G-outer inverse, of operators between two Banach spaces as a generalization of G-Drazin inverses of square matrices and Banach space operators. Several characterizations of this new inverse are given. Based on the G-outer inverse, we define and characterize the G-outer partial order, extending the G-Drazin partial order to operators between two Banach spaces. We also relate this new partial order to the minus partial order. In the end, we study an outer binary relation which is an equivalence relation on the corresponding set.

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## 1. Introduction

Let  $X$ ,  $Y$ ,  $Z$  and  $W$  be arbitrary Banach spaces in this article. The set of all bounded linear operators from  $X$  to  $Y$  will be denoted by  $\mathcal{B}(X, Y)$ . In particular, we state  $\mathcal{B}(X) = \mathcal{B}(X, X)$ . For  $A \in \mathcal{B}(X, Y)$ , the notations  $N(A)$  and  $R(A)$  represent the null space and the range of  $A$ , respectively.

Generalized inverses have been extensively applied in many scientific areas, such as statistics, engineering, prediction theory, control system analysis, curve fitting, image restoration and many areas of applied mathematics. An operator  $A \in \mathcal{B}(X, Y)$  is relatively regular if there exists some  $B \in \mathcal{B}(Y, X)$  such that  $ABA = A$ . The operator  $B$  is called an inner inverse (or generalized inverse or  $g$ -inverse) of  $A$  and it is not unique. By  $A\{1\}$  we denote the set of all inner inverses of  $A$ . Recall that  $A \in \mathcal{B}(X, Y)$  is relatively regular if and only if  $N(A)$  and  $R(A)$  are closed and complemented subspaces of  $X$  and  $Y$ , respectively [3]. In the case that  $X$  and  $Y$  are Hilbert spaces,  $A$  is relatively regular if and only if  $R(A)$  is closed [3].

The well-known types of relations defined over a set are equivalence relations (i.e. reflexive, symmetric and transitive binary relations) and partial orders (i.e. reflexive, anti-symmetric and transitive binary relations), whose usefulness is indisputable in the whole Mathematics. Pre-orders are more general relations than equivalence relations and partial orders, because they are reflexive and transitive binary relations.

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Using generalized inverses, many pre-orders and partial orders were introduced and investigated [5,6,12–14, 21].

Let  $A \in \mathcal{B}(X, Y)$  be relatively regular and  $B \in \mathcal{B}(X, Y)$ . Then  $A$  is said to be below  $B$  under the minus partial order (denoted by  $A \leq^- B$ ) if there exists  $A^- \in A\{1\}$  such that  $AA^- = BA^-$  and  $A^-A = A^-B$  [8].

Let  $A \in \mathcal{B}(X, Y)$ . If there exists some operator  $0 \neq B \in \mathcal{B}(Y, X)$  satisfying  $BAB = B$ , then  $B$  is called an outer inverse of  $A$ . Notice that there exists an outer inverse of  $A$  if and only if  $A \neq 0$  [3]. We use  $A\{2\}$  to denote the set of all outer inverses of  $A$ . The set of  $\{2\}$ -inverses have been used in the iterative methods for solving the nonlinear equations [1] as well as in statistics [7]. In particular, outer inverses show great influence in stable approximations of ill-posed problems and in linear and nonlinear problems involving rank-deficient generalized inverses [18].

**Lemma 1.1.** [3,10] *Let  $A \in \mathcal{B}(X, Y) \setminus \{0\}$  and let  $T$  and  $S$  be subspaces of  $X$  and  $Y$ , respectively. Then the following statements are equivalent:*

- (i) *there exists  $0 \neq B \in \mathcal{B}(Y, X)$  such that  $BAB = B$ ,  $R(B) = T$  and  $N(B) = S$ ;*
- (ii)  *$T$  and  $S$  are closed and complemented subspaces of  $X$  and  $Y$ , respectively,  $A(T)$  is closed,  $A(T) \oplus S = Y$  and the reduction  $A|_T : T \rightarrow A(T)$  is invertible.*

If (i) or (ii) is satisfied, then the operator  $B$  in the part (i) is unique and denoted by  $A_{T,S}^{(2)}$ .

In the case that the outer inverse  $A_{T,S}^{(2)}$  exists, we have

$$X = T \oplus N(A_{T,S}^{(2)}A) \quad \text{and} \quad Y = A(T) \oplus S.$$

The matrix form of  $A$  is given by:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} T \\ N(A_{T,S}^{(2)}A) \end{bmatrix} \longrightarrow \begin{bmatrix} A(T) \\ S \end{bmatrix}, \quad (1)$$

where  $A_1 \in \mathcal{B}(T, A(T))$  is invertible. Therefore,

$$A_{T,S}^{(2)} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A(T) \\ S \end{bmatrix} \longrightarrow \begin{bmatrix} T \\ N(A_{T,S}^{(2)}A) \end{bmatrix}. \quad (2)$$

Let  $T$  and  $S$  be closed subsets of  $X$  and  $Y$ , respectively. We use  $\mathcal{B}(X, Y)_{T,S}$  to denote the set of all  $A \in \mathcal{B}(X, Y)$  such that  $A_{T,S}^{(2)}$  exists. A number of iterative methods are developed for computing various classes of outer inverses with prescribed range and null space [4,19,22,23]. Also, denote by  $\mathcal{B}(X, Y)_{T,S}^- = \{A \in \mathcal{B}(X, Y)_{T,S} : A \text{ is relatively regular}\}$ .

One of many special kinds of outer inverses is the generalized Drazin inverse introduced by Koliha [9]. An operator  $A \in \mathcal{B}(X)$  is generalized Drazin invertible if there exists a unique  $B \in \mathcal{B}(X)$  such that

$$AB = BA, \quad BAB = B \quad \text{and} \quad A - ABA \text{ is quasinilpotent.}$$

The generalized Drazin inverse  $B$  of  $A$  will be denoted by  $A^d$  [9]. When  $A - ABA$  is nilpotent in the above definition,  $A^d = A^D$  is the Drazin inverse of  $A$ . The symbol  $\mathcal{B}(X)^d$  denotes the set of all generalized Drazin invertible operators of  $\mathcal{B}(X)$ . The group inverse is a particular case of Drazin inverse for which the condition  $A - ABA$  is nilpotent is replaced with  $A = ABA$ . By  $A^\#$  will be denoted the group inverse of  $A$ . For some recent results related to the (generalized) Drazin see [11,15].

Let  $A, B \in \mathcal{B}(X)^d$ . The operator  $A$  is below to  $B$  under the generalized Drazin pre-order ( $A \leq^d B$ ) if  $A^d A = A^d B$  and  $AA^d = BA^d$  [17].

Wang and Liu [20] introduced a G-Drazin inverse of a square matrix. Let  $\mathbb{C}^{n \times n}$  be the set of  $n \times n$  complex matrices,  $A \in \mathbb{C}^{n \times n}$  and  $k = \text{ind}(A)$ , where  $\text{ind}(A)$  is the index of  $A$ . A matrix  $C \in \mathbb{C}^{n \times n}$  is a G-Drazin inverse of  $A$  if

$$ACA = A, \quad A^{k+1}C = A^k \quad \text{and} \quad CA^{k+1} = A^k.$$

Note that, in general, the G-Drazin inverse is not unique. The G-Drazin partial order was studied in [20] too. Some interesting results related to the extension of G-Drazin inverses for rectangular matrices using a weight matrix can be found in [2].

The definition of a G-Drazin inverse of a Banach space operator was given in [16] extending the results from [20] to more general settings. Let  $A \in \mathcal{B}(X)$  be generalized Drazin invertible such that  $A$  is relatively regular. An operator  $C \in \mathcal{B}(X)$  is a G-Drazin inverse of  $A$  if the following equalities hold:

$$ACA = A \quad \text{and} \quad A^d AC = CA^d A.$$

Observe that this definition and the above definition of G-Drazin inverses for square matrices are equivalent in complex matrix case [16]. The G-Drazin partial order  $\leq^{GD}$  for operators on Banach spaces was characterized in [16]. Recall that, for  $B \in \mathcal{B}(X)$  and  $A \in \mathcal{B}(X)^d$  such that  $A$  is relatively regular,  $A \leq^{GD} B$  if and only if  $A \leq^- B$  and  $A \leq^d B$ .

Motivated by recent research about G-Drazin inverse, the main contribution of this manuscript is to present a new generalized inverse which extends the notation of G-Drazin inverse. Precisely, we use an outer inverse instead of the generalized Drazin inverse in the definition of the G-Drazin inverse for an operator between two Banach spaces to define a G-outer inverse. Thus, the G-Drazin inverses given in [20] and [16] are special cases of G-outer inverse, which means that we proposed a wider class of generalized inverses. The second research stream is investigation of the G-outer relation introduced by means of the G-outer inverse. We show that this new relation is a partial order on the corresponding set which generalizes the G-Drazin partial order.

This paper is organized as follows. Section 2 introduces and characterizes the G-outer inverse of an operator between two Banach spaces. Section 3 defines and characterizes the G-outer partial order on  $\mathcal{B}(X, Y)_{T, S}^-$ . This relation is based on the G-outer inverse and extends the G-Drazin partial orders given in [20] and [16]. In addition, some connections of this new partial order and the minus order are presented. In Section 4, we consider an outer relation and prove that it is an equivalence relation on the set  $\mathcal{B}(X, Y)_{T, S}$ .

## 2. G-outer inverses

As a generalization of the G-Drazin inverse for a square matrix, we introduce the G-outer inverse of an operator between two Banach spaces.

**Definition 2.1.** Let  $A \in \mathcal{B}(X, Y)_{T, S}^-$ . An operator  $C \in \mathcal{B}(Y, X)$  is a G-outer  $(T, S)$ -inverse of  $A$  if the following equalities hold:

$$ACA = A \quad \text{and} \quad A_{T, S}^{(2)} AC = CAA_{T, S}^{(2)}.$$

Notice that, if  $X = Y$  and  $A_{T, S}^{(2)} = A^d$ , then the G-outer  $(T, S)$ -inverse of  $A$  becomes the G-Drazin inverse of  $A$  defined in [16, Definition 4.3].

In the complex matrix case, we do not need the assumption that  $A$  is relatively regular in the definition of a G-outer  $(T, S)$ -inverse of  $A$ , because it is satisfied.

**Example 2.1.** Let  $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{2 \times 2}$ ,  $T = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{C} \right\} = \mathbb{C} \times \{0\}$  and  $S = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} : y \in \mathbb{C} \right\} = \{0\} \times \mathbb{C}$ . Then  $A_{T,S}^{(2)} = A$  and, for arbitrary  $c \in \mathbb{C}$ ,  $C = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$  is a G-outer  $(T, S)$ -inverse of  $A$ . Thus, a G-outer inverse is not uniquely determined.

Several equivalent conditions for an operator to be a G-outer inverse are studied in the following result. Observe that Theorem 2.1 extends [16, Corollary 4.3] concerning characterizations of the G-Drazin inverse. Also, by part (iv) of Theorem 2.1, we have that a G-outer inverse of an operator  $A \in \mathcal{B}(X, Y)_{T,S}^-$  is not unique. We denote by  $A\{GO, T, S\}$  the set of all G-outer  $(T, S)$ -inverses of  $A$ . Remark that  $A\{GO, T, S\} \subseteq A\{1\}$ .

**Theorem 2.1.** Let  $A \in \mathcal{B}(X, Y)_{T,S}^-$ . For  $C \in \mathcal{B}(Y, X)$ , the following statements are equivalent:

- (i)  $C \in A\{GO, T, S\}$ ;
- (ii)  $ACA = A$  and  $A_{T,S}^{(2)}AC = A_{T,S}^{(2)} = CAA_{T,S}^{(2)}$ ;
- (iii)  $ACA = A$ ,  $AA_{T,S}^{(2)}AC = AA_{T,S}^{(2)}$  and  $CAA_{T,S}^{(2)}A = A_{T,S}^{(2)}A$ ;
- (iv) there exist topological direct sums  $X = T \oplus N(A_{T,S}^{(2)}A)$  and  $Y = A(T) \oplus S$  such that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & C_2 \end{bmatrix},$$

where  $A_1 \in \mathcal{B}(T, A(T))$  is invertible,  $A_2 \in \mathcal{B}(N(A_{T,S}^{(2)}A), S)$  is relatively regular and  $C_2 \in A_2\{1\}$ .

**Proof.** (i)  $\Rightarrow$  (ii): Since  $ACA = A$ , then  $A_{T,S}^{(2)}ACAA_{T,S}^{(2)} = A_{T,S}^{(2)}AA_{T,S}^{(2)} = A_{T,S}^{(2)}$ . Furthermore, the assumption  $A_{T,S}^{(2)}AC = CAA_{T,S}^{(2)}$  gives

$$A_{T,S}^{(2)} = (A_{T,S}^{(2)}AC)AA_{T,S}^{(2)} = CAA_{T,S}^{(2)}AA_{T,S}^{(2)} = CAA_{T,S}^{(2)}$$

and similarly  $A_{T,S}^{(2)} = A_{T,S}^{(2)}AC$ .

(ii)  $\Rightarrow$  (iii): If we multiply the condition  $A_{T,S}^{(2)} = A_{T,S}^{(2)}AC$  by  $A$  from the left side, we obtain  $AA_{T,S}^{(2)} = AA_{T,S}^{(2)}AC$ . Multiplying  $A_{T,S}^{(2)} = CAA_{T,S}^{(2)}$  by  $A$  from the right side, we observe that  $A_{T,S}^{(2)}A = CAA_{T,S}^{(2)}A$ .

(iii)  $\Rightarrow$  (i): Using the hypothesis  $AA_{T,S}^{(2)} = AA_{T,S}^{(2)}AC$ , we get

$$A_{T,S}^{(2)} = A_{T,S}^{(2)}(AA_{T,S}^{(2)}) = A_{T,S}^{(2)}AA_{T,S}^{(2)}AC = A_{T,S}^{(2)}AC.$$

In the same way,  $A_{T,S}^{(2)}A = CAA_{T,S}^{(2)}A$  yields  $A_{T,S}^{(2)} = CAA_{T,S}^{(2)}$ . Hence,  $A_{T,S}^{(2)}AC = A_{T,S}^{(2)} = CAA_{T,S}^{(2)}$  and  $C \in A\{GO, T, S\}$ .

(i)  $\Leftrightarrow$  (iv): It is well-known that there exist topological direct sums  $X = T \oplus N(A_{T,S}^{(2)}A)$  and  $Y = A(T) \oplus S$  such that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

where  $A_1 \in \mathcal{B}(T, A(T))$  is invertible and

$$A_{T,S}^{(2)} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Let

$$C = \begin{bmatrix} C_1 & C_3 \\ C_4 & C_2 \end{bmatrix} : \begin{bmatrix} A(T) \\ S \end{bmatrix} \longrightarrow \begin{bmatrix} T \\ N(A_{T,S}^{(2)}A) \end{bmatrix}.$$

Then, by

$$A_{T,S}^{(2)}AC = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} C = \begin{bmatrix} C_1 & C_3 \\ 0 & 0 \end{bmatrix}$$

and

$$CAA_{T,S}^{(2)} = C \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ C_4 & 0 \end{bmatrix},$$

we deduce that  $A_{T,S}^{(2)}AC = CAA_{T,S}^{(2)}$  is equivalent to  $C_3 = 0$  and  $C_4 = 0$ . Now, because  $A_1$  is invertible,

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = A = ACA = \begin{bmatrix} A_1C_1A_1 & 0 \\ 0 & A_2C_2A_2 \end{bmatrix}$$

if and only if  $A_1 = A_1C_1A_1$  and  $A_2 = A_2C_2A_2$  if and only if  $C_1 = A_1^{-1}$ ,  $A_2$  is relatively regular and  $C_2 \in A_2\{1\}$ . Thus, (i) is equivalent to (iv).  $\square$

It is interesting to consider the special case when  $T$  and  $S$  are the range and null space of some operators. If  $A_{R(D),N(E)}^{(2)}$  exists for some operators  $D$  and  $E$ , we obtain the next characterizations of a G-outer inverse.

**Theorem 2.2.** *Let  $D \in \mathcal{B}(Z, X)$  and  $E \in \mathcal{B}(Y, W)$  such that  $R(D)$  and  $N(E)$  are closed subspaces of  $X$  and  $Y$ , respectively, and let  $A \in \mathcal{B}(X, Y)_{R(D),N(E)}^-$ . For  $C \in \mathcal{B}(Y, X)$ , the following statements are equivalent:*

- (i)  $C \in A\{GO, R(D), N(E)\}$ ;
- (ii)  $ACA = A$ ,  $EAC = E$  and  $CAD = D$ ;
- (iii)  $ACA = A$ ,  $N(AC) \subseteq N(E)$  and  $R(D) \subseteq R(CA)$ .

**Proof.** (i)  $\Rightarrow$  (ii): If  $C \in A\{GO, R(D), N(E)\}$ , then, by Theorem 2.1(iii),  $ACA = A$ ,  $AA_{R(D),N(E)}^{(2)}AC = AA_{R(D),N(E)}^{(2)}$  and  $CAA_{R(D),N(E)}^{(2)}A = A_{R(D),N(E)}^{(2)}A$ . Because

$$R(I - AA_{R(D),N(E)}^{(2)}) = N(AA_{R(D),N(E)}^{(2)}) = N(A_{R(D),N(E)}^{(2)}) = N(E)$$

and

$$N(I - A_{R(D),N(E)}^{(2)}A) = R(A_{R(D),N(E)}^{(2)}A) = R(A_{R(D),N(E)}^{(2)}) = R(D),$$

we have  $E = EAA_{R(D),N(E)}^{(2)}$  and  $D = A_{R(D),N(E)}^{(2)}AD$ . Thus,

$$E = E(AA_{R(D),N(E)}^{(2)}) = (EAA_{R(D),N(E)}^{(2)})AC = EAC$$

and

$$D = (A_{R(D),N(E)}^{(2)} A)D = CA(A_{R(D),N(E)}^{(2)} AD) = CAD.$$

(ii)  $\Rightarrow$  (iii): This implication is obvious.

(iii)  $\Rightarrow$  (i): Using  $ACA = A$ ,  $N(AC) \subseteq N(E)$  and  $R(D) \subseteq R(CA)$ , we obtain

$$R(I - AC) = N(AC) \subseteq N(E) = N(A_{R(D),N(E)}^{(2)})$$

and

$$R(A_{R(D),N(E)}^{(2)}) = R(D) \subseteq R(CA) = N(I - CA),$$

which give  $A_{R(D),N(E)}^{(2)} = A_{R(D),N(E)}^{(2)} AC$  and  $A_{R(D),N(E)}^{(2)} = CA A_{R(D),N(E)}^{(2)}$ . Hence, by Theorem 2.1, (i) holds.  $\square$

We can characterize the G-outer invertible operators by means of idempotents.

**Theorem 2.3.** *Let  $A \in \mathcal{B}(X, Y)_{T,S}$ . The following statements are equivalent:*

- (i) *A is relatively regular and  $A\{GO, T, S\} \neq \emptyset$ ;*
- (ii) *there exist idempotents  $P \in \mathcal{B}(Y)$  and  $Q \in \mathcal{B}(X)$  such that*

$$R(P) = R(A), \quad N(Q) = N(A) \quad \text{and} \quad A_{T,S}^{(2)} P = Q A_{T,S}^{(2)}.$$

*In addition, for arbitrary  $A^- \in A\{1\}$ ,  $QA^-P \in A\{GO, T, S\}$ , that is,*

$$Q \cdot A\{1\} \cdot P \subseteq A\{GO, T, S\}.$$

**Proof.** (i)  $\Rightarrow$  (ii): For  $C \in A\{GO, T, S\}$ , set  $P = AC$  and  $Q = CA$ . Since  $C \in A\{1\}$ , we have  $P = P^2$ ,  $Q = Q^2$ ,  $R(P) = R(A)$  and  $N(Q) = N(A)$ . We observe that

$$A_{T,S}^{(2)} P = A_{T,S}^{(2)} AC = CAA_{T,S}^{(2)} = QA_{T,S}^{(2)}.$$

(ii)  $\Rightarrow$  (i): The hypothesis  $R(P) = R(A)$  implies  $A = PA$  and  $P = AU$ , for some  $U \in \mathcal{B}(Y, X)$ . Hence,  $A = PA = AUA$  and  $A$  is relatively regular. Let  $A^- \in A\{1\}$  and  $C = QA^-P$ . Notice that  $P = AU = AA^-(AU) = AA^-P$ . From  $N(Q) = N(A)$ , we get  $R(I - Q) = N(A)$  and  $N(Q) = N(A^-A) = R(I - A^-A)$  which give  $A = AQ$  and  $Q = QA^-A$ . Therefore,

$$ACA = (AQ)A^-(PA) = AA^-A = A.$$

Using  $A_{T,S}^{(2)} P = QA_{T,S}^{(2)}$ , we obtain

$$\begin{aligned} A_{T,S}^{(2)} AC &= A_{T,S}^{(2)} (AQ)A^-P = A_{T,S}^{(2)} (AA^-P) = A_{T,S}^{(2)} P = QA_{T,S}^{(2)} \\ &= QA^-AA_{T,S}^{(2)} = (QA^-P)AA_{T,S}^{(2)} = CAA_{T,S}^{(2)}. \end{aligned}$$

So, we deduce that  $C \in A\{GO, T, S\}$ .  $\square$

For two arbitrary G-outer  $(T, S)$ -inverses  $C_1$  and  $C_2$  of  $A$ , we now verify that  $C_1AC_2$  is a G-outer  $(T, S)$ -inverse of  $A$ .

**Theorem 2.4.** *Let  $A \in \mathcal{B}(X, Y)_{T,S}^-$ . Then*

$$A\{GO, T, S\} \cdot A \cdot A\{GO, T, S\} \subseteq A\{GO, T, S\}.$$

**Proof.** Suppose that  $C_1, C_2 \in A\{GO, T, S\}$  and  $C = C_1AC_2$ . Firstly, we get  $ACA = (AC_1A)C_2A = AC_2A = A$ . Further, by Theorem 2.1(ii),

$$A_{T,S}^{(2)}AC = (A_{T,S}^{(2)}AC_1)AC_2 = A_{T,S}^{(2)}AC_2 = A_{T,S}^{(2)}$$

and

$$CAA_{T,S}^{(2)} = C_1A(C_2AA_{T,S}^{(2)}) = C_1AA_{T,S}^{(2)} = A_{T,S}^{(2)},$$

which imply  $A_{T,S}^{(2)}AC = A_{T,S}^{(2)} = CAA_{T,S}^{(2)}$ . Therefore,  $C \in A\{GO, T, S\}$ .  $\square$

### 3. G-outer partial order

In this section, we start with the definition of a new binary relation on  $\mathcal{B}(X, Y)$ , extending the definitions of the G-Drazin relations given in [20] for complex square matrices and in [16] for bounded linear operators on a Banach space.

**Definition 3.1.** Let  $A \in \mathcal{B}(X, Y)_{T,S}^-$  and  $B \in \mathcal{B}(X, Y)$ . Then we say that  $A$  is below to  $B$  under the G-outer  $(T, S)$ -relation (denoted by  $A \leq^{GO, T, S} B$ ) if there exist  $C_1, C_2 \in A\{GO, T, S\}$  such that

$$AC_1 = BC_1 \quad \text{and} \quad C_2A = C_2B.$$

We present some necessary and sufficient conditions for  $A \leq^{GO, T, S} B$  to be satisfied.

**Theorem 3.1.** *Let  $A \in \mathcal{B}(X, Y)_{T,S}^-$  and  $B \in \mathcal{B}(X, Y)$ . Then the following statements are equivalent:*

- (i)  $A \leq^{GO, T, S} B$ ;
- (ii) *there exist  $C \in A\{GO, T, S\}$  such that*

$$AC = BC \quad \text{and} \quad CA = CB;$$

- (iii) *there exist topological direct sums  $X = T \oplus N(A_{T,S}^{(2)}A)$  and  $Y = A(T) \oplus S$  such that*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

- where  $A_1 \in \mathcal{B}(T, A(T))$  is invertible,  $A_2 \in \mathcal{B}(N(A_{T,S}^{(2)}A), S)$  is relatively regular and  $A_2 \leq^- B_2$ ;
- (iv) *there exists  $C \in A\{GO, T, S\}$  such that*

$$ACB = A = BCA;$$

(v) there exist idempotents  $P \in \mathcal{B}(Y)$  and  $Q \in \mathcal{B}(X)$  such that

$$R(P) = R(A), \quad N(Q) = N(A), \quad A_{T,S}^{(2)}P = QA_{T,S}^{(2)} \quad \text{and} \quad PB = A = BQ.$$

In addition,  $B_{T,S}^{(2)}$  exists and, if  $B$  is relatively regular, then  $F \in B\{GO, T, S\}$  if and only if

$$F = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & F_2 \end{bmatrix},$$

where  $F_2 \in B_2\{1\}$ .

**Proof.** (i)  $\Rightarrow$  (ii): Since  $A \leq^{GO, T, S} B$ , there exist  $C_1, C_2 \in A\{GO, T, S\}$  such that  $AC_1 = BC_1$  and  $C_2A = C_2B$ . Set  $C = C_1AC_2$ . By Theorem 2.4, we have that  $C \in A\{GO, T, S\}$ . Also,

$$AC = (AC_1)AC_2 = B(C_1AC_2) = BC$$

and analogously  $CA = CB$ .

(ii)  $\Rightarrow$  (iii): Let  $C \in A\{GO, T, S\}$  such that  $AC = BC$  and  $CA = CB$ . Applying Theorem 2.1(iv), there exist topological direct sums  $X = T \oplus N(A_{T,S}^{(2)}A)$  and  $Y = A(T) \oplus S$  such that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & C_2 \end{bmatrix},$$

where  $A_1 \in \mathcal{B}(T, A(T))$  is invertible,  $A_2 \in \mathcal{B}(N(A_{T,S}^{(2)}A), S)$  is relatively regular and  $C_2 \in A_2\{1\}$ . Assume that

$$B = \begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix} : \begin{bmatrix} T \\ N(A_{T,S}^{(2)}A) \end{bmatrix} \longrightarrow \begin{bmatrix} A(T) \\ S \end{bmatrix}.$$

Using

$$\begin{bmatrix} I & 0 \\ 0 & A_2C_2 \end{bmatrix} = AC = BC = \begin{bmatrix} B_1A_1^{-1} & B_3C_2 \\ B_4A_1^{-1} & B_2C_2 \end{bmatrix},$$

we observe that  $B_1 = A_1$ ,  $B_4 = 0$  and  $A_2C_2 = B_2C_2$ . Now, by

$$\begin{bmatrix} I & 0 \\ 0 & C_2A_2 \end{bmatrix} = CA = CB = \begin{bmatrix} I & A_1^{-1}B_3 \\ 0 & C_2B_2 \end{bmatrix},$$

$B_3 = 0$  and  $C_2A_2 = C_2B_2$ . Thus,  $A_2 \leq^- B_2$ .

(iii)  $\Rightarrow$  (i): The assumption  $A_2 \leq^- B_2$  implies that there exists  $C_2 \in A_2\{1\}$  such that  $A_2C_2 = B_2C_2$  and  $C_2A_2 = C_2B_2$ . Set

$$C = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & C_2 \end{bmatrix}.$$

The part (iv) of Theorem 2.1 gives  $C \in A\{GO, T, S\}$ . We can easily check that  $AC = BC$  and  $CA = CB$ . Hence,  $A \leq^{GO, T, S} B$ .



(ii)  $\Rightarrow$  (iv): If  $C \in A\{GO, T, S\}$  such that  $AC = BC$  and  $CA = CB$ , then  $ACB = ACA = A$  and  $BCA = ACA = A$ .

(iv)  $\Rightarrow$  (iii): Suppose that  $C \in A\{GO, T, S\}$  such that  $ACB = A = BCA$ . Applying Theorem 2.1, there exist topological direct sums  $X = T \oplus N(A_{T,S}^{(2)}A)$  and  $Y = A(T) \oplus S$  such that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & C_2 \end{bmatrix},$$

where  $A_1 \in \mathcal{B}(T, A(T))$  is invertible,  $A_2 \in \mathcal{B}(N(A_{T,S}^{(2)}A), S)$  is relatively regular and  $C_2 \in A_2\{1\}$ . Set

$$B = \begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix}.$$

By

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = A = ACB = \begin{bmatrix} B_1 & B_3 \\ A_2 C_2 B_4 & A_2 C_2 B_2 \end{bmatrix},$$

notice that  $B_1 = A_1$ ,  $B_3 = 0$  and  $A_2 C_2 B_2 = A_2$ . From

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = A = BCA = \begin{bmatrix} A_1 & 0 \\ B_4 & B_2 C_2 A_2 \end{bmatrix},$$

we get  $B_4 = 0$  and  $B_2 C_2 A_2 = A_2$ . Let  $C'_2 = C_2 A_2 C_2$ . Then  $C'_2 \in A_2\{1\}$ ,  $A_2 C'_2 = A_2 C_2 = B_2 C_2 A_2 C_2 = B_2 C'_2$  and  $C'_2 A_2 = C_2 A_2 = C_2 A_2 C_2 B_2 = C'_2 B_2$ , which give  $A_2 \leq^- B_2$ .

(ii)  $\Rightarrow$  (v): Because there exist  $C \in A\{GO, T, S\}$  such that  $AC = BC$  and  $CA = CB$ , for  $P = AC$  and  $Q = CA$ , we get  $R(P) = R(A)$ ,  $N(Q) = N(A)$  and  $A_{T,S}^{(2)}P = Q A_{T,S}^{(2)}$  as in the proof of part (i)  $\Rightarrow$  (ii) of Theorem 2.3. Furthermore,

$$A = A(CA) = ACB = PB \quad \text{and} \quad A = (AC)A = BCA = BQ.$$

(v)  $\Rightarrow$  (ii): Assume that there exist idempotents  $P \in \mathcal{B}(Y)$  and  $Q \in \mathcal{B}(X)$  such that  $R(P) = R(A)$ ,  $N(Q) = N(A)$ ,  $A_{T,S}^{(2)}P = Q A_{T,S}^{(2)}$  and  $PB = A = BQ$ . If  $C = QA^-P$ , for  $A^- \in (A)\{1\}$ , by Theorem 2.3, we have that  $C \in A\{GO, T, S\}$ . Also,

$$BC = (BQ)A^-P = AA^-P = A(QA^-P) = AC$$

and similarly  $CB = CA$ .

Suppose that  $F \in B\{GO, T, S\}$  and

$$F = \begin{bmatrix} F_1 & F_3 \\ F_4 & F_2 \end{bmatrix} : \begin{bmatrix} A(T) \\ S \end{bmatrix} \longrightarrow \begin{bmatrix} T \\ N(A_{T,S}^{(2)}A) \end{bmatrix}.$$

The equality  $B = BFB$  can be written as

$$\begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix} = \begin{bmatrix} A_1 F_1 A_1 & A_1 F_3 B_2 \\ B_2 F_4 A_1 & B_2 F_2 B_2 \end{bmatrix}.$$

So,  $F_1 = A_1^{-1}$  and  $B_2 = B_2 F_2 B_2$ . Notice that, by Lemma 1.1,  $B_{T,S}^{(2)}$  exists and

$$B_{T,S}^{(2)} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Then  $B_{T,S}^{(2)} B F = F B B_{T,S}^{(2)}$  implies  $F_3 = 0$  and  $F_4 = 0$ .

For

$$F = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & F_2 \end{bmatrix},$$

where  $F_2 \in B_2\{1\}$ , we easily show that  $F \in B\{GO, T, S\}$ .  $\square$

In the theory of partial orders, one fundamental problem is to characterize the partial order in terms of the other partial orders with constraints. By Theorem 3.1 and the fact  $A\{GO, T, S\} \subseteq (WAW)\{1\}$ , note that  $A \leq^{GO, T, S} B$  implies  $A \leq^- B$ .

**Corollary 3.1.** *Let  $A, B \in \mathcal{B}(X, Y)_{T,S}^-$ . If  $A \leq^{GO, T, S} B$ , then*

$$B\{GO, T, S\} \subseteq A\{GO, T, S\}.$$

**Proof.** The hypothesis  $A \leq^{GO, T, S} B$  implies that  $A$  and  $B$  can be represented as in part (iii) of Theorem 3.1.

By  $A_2 \leq^- B_2$ , there exists  $C_2 \in A_2\{1\}$  such that  $A_2 C_2 = B_2 C_2$  and  $C_2 A_2 = C_2 B_2$ . Since  $B = \begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix}$  is relatively regular, we deduce that  $B_2$  is relatively regular too. If  $U \in B_2\{1\}$ , then

$$\begin{aligned} A_2 U A_2 &= A_2 (C_2 A_2) U (A_2 C_2) A_2 = A_2 C_2 (B_2 U B_2) C_2 A_2 \\ &= A_2 (C_2 B_2) C_2 A_2 = A_2 C_2 A_2 C_2 A_2 \\ &= A_2. \end{aligned}$$

Thus, we conclude that  $B_2\{1\} \subseteq A_2\{1\}$ .

Let  $F \in B\{GO, T, S\}$ . By Theorem 3.1,

$$F = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & F_2 \end{bmatrix},$$

where  $F_2 \in B_2\{1\}$ . Therefore,  $F_2 \in B_2\{1\} \subseteq A_2\{1\}$ . Using Theorem 2.1, we obtain that  $F \in A\{GO, T, S\}$ .  $\square$

Now, we show that the G-outer relation is a partial order on the set  $\mathcal{B}(X, Y)_{T,S}^-$ .

**Theorem 3.2.** *The G-outer relation is a partial order on the set  $\mathcal{B}(X, Y)_{T,S}^-$ .*

**Proof.** It is clear that  $\leq^{GO, T, S}$  is reflexive.

To check that  $\leq^{GO, T, S}$  is antisymmetric, let  $A, B \in \mathcal{B}(X, Y)_{T,S}^-$  such that  $A \leq^{GO, T, S} B$  and  $B \leq^{GO, T, S} A$ . Then  $A \leq^- B$  and  $B \leq^- A$ . Because  $\leq^-$  is antisymmetric, we conclude that  $A = B$ .

In order to verify that  $\leq^{GO, T, S}$  is transitive, assume that  $A, B, E \in \mathcal{B}(X, Y)_{T,S}^-$ ,  $A \leq^{GO, T, S} B$  and  $B \leq^{GO, T, S} E$ . The operators  $A$  and  $B$  can be represented as in Theorem 3.1(iii). Also, there exists  $F \in B\{GO, T, S\}$  such that  $BF = EF$ ,  $FB = FE$  and

$$F = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & F_2 \end{bmatrix},$$

where  $F_2 \in B_2\{1\}$ . Set

$$E = \begin{bmatrix} E_1 & E_3 \\ E_4 & E_2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} I & 0 \\ 0 & B_2F_2 \end{bmatrix} = BF = EF = \begin{bmatrix} E_1A_1^{-1} & E_3F_2 \\ E_4A_1^{-1} & E_2F_2 \end{bmatrix}$$

gives  $E_1 = A_1$ ,  $E_4 = 0$  and  $B_2F_2 = E_2F_2$ . From

$$\begin{bmatrix} I & 0 \\ 0 & F_2B_2 \end{bmatrix} = FB = FE = \begin{bmatrix} I & A_1^{-1}E_3 \\ F_2E_4 & F_2E_2 \end{bmatrix},$$

we obtain  $E_3 = 0$  and  $F_2B_2 = F_2E_2$ . Hence,  $B_2 \leq^- E_2$ . Since the relation  $\leq^-$  is transitive, we deduce that  $A_2 \leq^- E_2$ . Applying Theorem 3.1, we have that  $A \leq^{GO,T,S} E$ .  $\square$

It is well-known that  $A \leq^{GO,T,S} B$  implies  $A \leq^- B$ . We present additional conditions such that the converse is also true.

**Theorem 3.3.** *Let  $A \in \mathcal{B}(X, Y)_{T,S}^-$  and  $B \in \mathcal{B}(X, Y)$ . Then the following statements are equivalent:*

- (i)  $A \leq^{GO,T,S} B$ ;
- (ii)  $A \leq^- B$ ,  $A_{T,S}^{(2)}B = A_{T,S}^{(2)}A$  and  $BA_{T,S}^{(2)} = AA_{T,S}^{(2)}$ ;
- (iii)  $A \leq^- B$ ,  $N(A_{T,S}^{(2)}A) \subseteq N(A_{T,S}^{(2)}B)$  and  $R(BA_{T,S}^{(2)}) \subseteq R(AA_{T,S}^{(2)})$ .

**Proof.** (i)  $\Rightarrow$  (ii): By Theorem 3.1, there exist topological direct sums  $X = T \oplus N(A_{T,S}^{(2)}A)$  and  $Y = A(T) \oplus S$  such that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

where  $A_1 \in \mathcal{B}(T, A(T))$  is invertible,  $A_2 \in \mathcal{B}(N(A_{T,S}^{(2)}A), S)$  is relatively regular and  $A_2 \leq^- B_2$ . There exists  $C_2 \in A_2\{1\}$  such that  $A_2C_2 = B_2C_2$  and  $C_2A_2 = C_2B_2$ . Suppose that

$$C = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & C_2 \end{bmatrix}.$$

We can easily check that  $ACA = A$ ,  $AC = BC$  and  $CA = CB$ . So,  $A \leq^- B$ . We also have

$$A_{T,S}^{(2)}B = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = A_{T,S}^{(2)}A \quad \text{and} \quad BA_{T,S}^{(2)} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = AA_{T,S}^{(2)}.$$

(ii)  $\Rightarrow$  (iii): It is obvious.

(iii)  $\Rightarrow$  (i): Assume that  $A$  and  $A_{T,S}^{(2)}$  are given by (1) and (2), respectively, and

$$B = \begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix} : \begin{bmatrix} T \\ N(A_{T,S}^{(2)}A) \end{bmatrix} \longrightarrow \begin{bmatrix} A(T) \\ S \end{bmatrix}.$$

Since

$$A_{T,S}^{(2)}B = \begin{bmatrix} A_1^{-1}B_1 & A_1^{-1}B_3 \\ 0 & 0 \end{bmatrix}, \quad A_{T,S}^{(2)}A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

and  $N(A_{T,S}^{(2)}A) \subseteq N(A_{T,S}^{(2)}B)$ , we get  $B_3 = 0$ . From  $R(BA_{T,S}^{(2)}) \subseteq R(AA_{T,S}^{(2)})$ ,

$$BA_{T,S}^{(2)} = \begin{bmatrix} B_1A_1^{-1} & 0 \\ B_4A_1^{-1} & 0 \end{bmatrix} \quad \text{and} \quad AA_{T,S}^{(2)} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

we observe that  $B_4 = 0$ .

By the condition  $A \leq^- B$ , there exists  $C \in A\{1\}$  such that  $AC = BC$  and  $CA = CB$ . If

$$C = \begin{bmatrix} C_1 & C_3 \\ C_4 & C_2 \end{bmatrix} : \begin{bmatrix} A(T) \\ S \end{bmatrix} \longrightarrow \begin{bmatrix} T \\ N(A_{T,S}^{(2)}A) \end{bmatrix},$$

the equality  $ACA = A$  gives  $C_1 = A_1^{-1}$  and  $A_2C_2A_2 = A_2$ . Using  $AC = BC$ , we obtain  $B_1 = A_1$  and  $A_2C_2 = B_2C_2$ . By  $CA = CB$ , we have  $C_2A_2 = C_2B_2$  and so  $A_2 \leq^- B_2$ . Applying Theorem 3.1, note that  $A \leq^{GO,T,S} B$ .  $\square$

For  $T = R(D)$  and  $S = N(E)$ , where  $D \in \mathcal{B}(Z, X)$  and  $E \in \mathcal{B}(Y, W)$ , we can characterize the relation  $A \leq^{GO,R(D),N(E)} B$  by Theorem 3.3.

**Corollary 3.2.** *Let  $D \in \mathcal{B}(Z, X)$  and  $E \in \mathcal{B}(Y, W)$  such that  $R(D)$  and  $N(E)$  are closed subspaces of  $X$  and  $Y$ , respectively, and let  $A \in \mathcal{B}(X, Y)_{R(D),N(E)}^-$ . For  $C \in \mathcal{B}(Y, X)$ , the following statements are equivalent:*

- (i)  $A \leq^{GO,R(D),N(E)} B$ ;
- (ii)  $A \leq^- B$ ,  $EB = EA$  and  $BD = AD$ .

**Proof.** (i)  $\Rightarrow$  (ii): According to Theorem 3.3,  $A \leq^{GO,R(D),N(E)} B$  is equivalent to  $A \leq^- B$ ,  $A_{R(D),N(E)}^{(2)}B = A_{R(D),N(E)}^{(2)}A$  and  $BA_{R(D),N(E)}^{(2)} = AA_{R(D),N(E)}^{(2)}$ . As in the proof of Theorem 2.2, we obtain that  $E = EAA_{R(D),N(E)}^{(2)}$  and  $D = A_{R(D),N(E)}^{(2)}AD$ . Therefore,

$$EB = EA(A_{R(D),N(E)}^{(2)}B) = (EAA_{R(D),N(E)}^{(2)})A = EA$$

and

$$BD = (BA_{R(D),N(E)}^{(2)})AD = A(A_{R(D),N(E)}^{(2)}AD) = AD.$$

(ii)  $\Rightarrow$  (i): Because  $EB = EA$ , then  $E(B - A) = 0$  gives  $R(B - A) \subseteq N(E) = N(A_{R(D),N(E)}^{(2)})$  and so  $A_{R(D),N(E)}^{(2)}B = A_{R(D),N(E)}^{(2)}A$ . Since  $R(A_{R(D),N(E)}^{(2)}) = R(D)$ , we have that  $A_{R(D),N(E)}^{(2)} = DV$ , for some  $V \in \mathcal{B}(Y, Z)$ . Hence, by  $BD = AD$ ,

$$BA_{R(D),N(E)}^{(2)} = (BD)V = A(DV) = AA_{R(D),N(E)}^{(2)}.$$

Using Theorem 3.3, we conclude that  $A \leq^{GO,R(D),N(E)} B$ .  $\square$

Recall that if  $A, B \in \mathcal{B}(X, Y)$  are relatively regular, then  $A \leq^- B$  if and only if  $B - A \leq^- B$ .

**Theorem 3.4.** Let  $A \in \mathcal{B}(X, Y)_{T,S}^-$  and  $B \in \mathcal{B}(X, Y)^-$  such that  $A \leq^{GO,T,S} B$ . If  $A$  and  $B$  are represented as in Theorem 3.1(iii), then

- (i)  $B - A \leq^- B$ ;
- (ii)  $B_2 - A_2 \leq^- B_2$ .

**Proof.** By Theorem 3.1 and Theorem 2.2, notice that  $A, B, A_2$  and  $B_2$  are relatively regular,  $A \leq^- B$  and  $A_2 \leq^- B_2$ . Hence,  $B - A \leq^- B$  and  $B_2 - A_2 \leq^- B_2$ .  $\square$

#### 4. Outer relation

Motivated by the equalities of Theorem 3.3(ii), we create the outer  $(T, S)$ -relation in terms of the outer inverse with the range  $T$  and null space  $S$ .

**Definition 4.1.** Let  $A \in \mathcal{B}(X, Y)_{T,S}$  and  $B \in \mathcal{B}(X, Y)$ . Then we say that  $A$  is below to  $B$  under the outer  $(T, S)$ -relation (denoted by  $A \leq^{T,S} B$ ) if

$$A_{T,S}^{(2)}B = A_{T,S}^{(2)}A \quad \text{and} \quad BA_{T,S}^{(2)} = AA_{T,S}^{(2)}.$$

It is interesting to note that  $A \leq^{T,S} B$  implies  $A_{T,S}^{(2)} \in B\{2\}$ .

Using idempotents, we can obtain an equivalent condition for the existence of  $A_{T,S}^{(2)}$ .

**Lemma 4.1.** Let  $A \in \mathcal{B}(X, Y) \setminus \{0\}$  and let  $T$  and  $S$  be subspaces of  $X$  and  $Y$ , respectively. Then the following statements are equivalent:

- (i)  $A_{T,S}^{(2)}$  exists;
- (ii) there exist idempotents  $P \in \mathcal{B}(Y)$  and  $Q \in \mathcal{B}(X)$  such that

$$R(P) = A(T), \quad N(P) = S, \quad R(Q) = T \quad \text{and} \quad N(A) \subseteq N(Q).$$

**Proof.** (i)  $\Rightarrow$  (ii): For  $P = AA_{T,S}^{(2)}$  and  $Q = A_{T,S}^{(2)}A$ , we easily verify that (ii) holds.

(ii)  $\Rightarrow$  (i): Notice that  $X = R(Q) \oplus N(Q) = T \oplus N(Q)$  and  $Y = R(P) \oplus N(P) = A(T) \oplus S$ . Since  $N(A) \subseteq N(Q)$ , then  $A|_T : T \rightarrow A(T)$  is invertible and, by Lemma 1.1,  $A_{T,S}^{(2)}$  exists.  $\square$

As Theorem 3.1, we prove some characterizations of the outer  $(T, S)$ -relation.

**Theorem 4.1.** Let  $A \in \mathcal{B}(X, Y)_{T,S}$  and  $B \in \mathcal{B}(X, Y)$ . Then the following statements are equivalent:

- (i)  $A \leq^{T,S} B$ ;
- (ii)  $AA_{T,S}^{(2)}A = AA_{T,S}^{(2)}B = BA_{T,S}^{(2)}A$ ;

(iii) there exist topological direct sums  $X = T \oplus N(A_{T,S}^{(2)}A)$  and  $Y = A(T) \oplus S$  such that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

where  $A_1 \in \mathcal{B}(T, A(T))$  is invertible.

In addition,  $B_{T,S}^{(2)}$  exists and

$$B_{T,S}^{(2)} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \quad (3)$$

We observe that the  $(T, S)$ -relation is not a partial order, but it is an equivalence relation on the set  $\mathcal{B}(X, Y)_{T,S}$ .

**Theorem 4.2.** *The outer  $(T, S)$ -relation is an equivalence relation on the set  $\mathcal{B}(X, Y)_{T,S}$ .*

**Proof.** Firstly, note that the relation  $\leq^{T,S}$  is reflexive.

In order to prove that  $\leq^{T,S}$  is symmetric, suppose that  $A, B \in \mathcal{B}(X, Y)_{T,S}$  such that  $A \leq^{T,S} B$ . Then  $A$  and  $B$  can be represented as in Theorem 4.1(ii). Using (3), we can verify that  $B_{T,S}^{(2)}B = B_{T,S}^{(2)}A$  and  $BB_{T,S}^{(2)} = AB_{T,S}^{(2)}$ . So,  $B \leq^{T,S} A$ .

To check that  $\leq^{T,S}$  is transitive, let  $A, B, E \in \mathcal{B}(X, Y)_{T,S}$ ,  $A \leq^{T,S} B$  and  $B \leq^{T,S} E$ . We have representations of operators  $A$  and  $B$  as in Theorem 4.1(ii). The equalities  $B_{T,S}^{(2)}B = B_{T,S}^{(2)}E$  and  $BB_{T,S}^{(2)} = EB_{T,S}^{(2)}$  give

$$E = \begin{bmatrix} A_1 & 0 \\ 0 & E_2 \end{bmatrix}.$$

Therefore, we get  $A_{T,S}^{(2)}A = A_{T,S}^{(2)}E$  and  $AA_{T,S}^{(2)} = EA_{T,S}^{(2)}$  which yields  $A \leq^{T,S} E$ .  $\square$

Applying Theorem 3.3, we obtain the next result.

**Corollary 4.1.** *Let  $A \in \mathcal{B}(X, Y)_{T,S}^-$  and  $B \in \mathcal{B}(X, Y)$ . Then the following statements are equivalent:*

- (i)  $A \leq^{GO,T,S} B$ ;
- (ii)  $A \leq^- B$  and  $A \leq^{T,S} B$ .

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