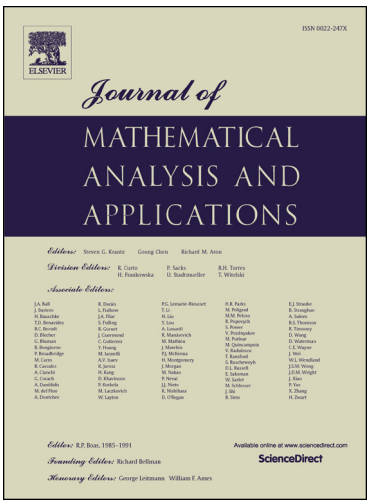
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Rotational surfaces of constant astigmatism in space forms

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Abstract

A surface in a Riemannian space is called surface of constant astigmatism if the difference between the principal radii of curvature at each point is a constant function. In this paper we give a classification of all rotational surfaces of constant astigmatism in space forms. We also prove that generating curves of such surfaces are critical points of a variational problem for a curvature energy. Using the description of these curves, we locally construct all rotational surfaces of constant astigmatism as the associated binormal evolution surfaces from the generating curves.

Keywords: surface of constant astigmatism, critical curve, spherical rotational surface, binormal evolution surface
2010 MSC: 34C05, 37K25, 53A10

1. Introduction

A surface in Euclidean space $\mathbb{R}^3$ is said to be a surface of constant astigmatism if the difference between the principal radii of curvature $\rho_2 - \rho_1$ at each point is a constant function. From

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Journal Pre-proof

the physical viewpoint, this difference $\rho_2 - \rho_1$ measures the amplitude of astigmatism and consequently, a surface where this difference is constant has constant amplitude of astigmatism in the normal directions ([8, 23]).

Surfaces of constant astigmatism were studied in early works of Bianchi and Ribaucour proving that their focal surfaces have constant negative Gaussian curvature [4, 5, 13, 21]. Moreover, the Gauss equation of these surfaces is related with the sine-Gordon equation. Therefore, recently there has been an increasing interest in studying the Gauss equation of these surfaces using the theory of integrable systems: [3, 9, 10, 11, 12, 18, 20].

In this paper, we extend the notion of surfaces of constant astigmatism to 3-dimensional Riemannian spaces. Let $(\mathbb{M}^3, \langle , \rangle)$ be a 3-dimensional Riemannian space with Levi-Civita connection $\tilde{\nabla}$. Let S be an immersed (orientable) surface in $\mathbb{M}^3$ endowed with the induced metric and let η be a given unit vector field normal to S. If $v, w \in T_pS$ are two tangent vectors at a point $p \in S$, the shape operator A_p at p is a self-adjoint endomorphism of T_pS defined as $\langle A_p(v), w\rangle = -\langle \tilde{\nabla}_v \eta, w\rangle$. Then, the two *principal curvatures* $\kappa_1(p)$ and $\kappa_2(p)$ of S at p are defined as the eigenvalues of A_p ([6]).

Definition 1.1. *Let S be a surface in a Riemannian space $\mathbb{M}^3$ with non-zero principal curvatures κ_1 and κ_2 at every point. We say that S is a surface of constant astigmatism if there is a constant $c \in \mathbb{R}$ such that*

$$\frac{1}{\kappa_1(p)} - \frac{1}{\kappa_2(p)} = c \tag{1}$$

holds for any $p \in S$.

The inverse of κ_i, $\rho_i(p) = 1/\kappa_i(p)$, is called the *principal radius of curvature* at p. Notice that if $c = 0$ in (1), then S is totally umbilical. From now, we will discard this trivial case and we will assume that $c \neq 0$.

The surfaces of constant astigmatism are a type of Weingarten surfaces. Recall that a Weingarten surface is a surface where a relation between the principal curvatures is satisfied. In particular, the constant astigmatism equation (1) is a simple linear relation between the radii of curvature. The class of Weingarten surfaces is a topic of great interest for geometers, especially after the works of Hopf, Chern and Hartman, among others in the 1950s. The authors have completely described all rotational surfaces in Euclidean space satisfying a linear relation between the principal curvatures, $a\kappa_1 + b\kappa_2 = c$, where $a, b, c \in \mathbb{R}$ ([17]).

In order to find examples of surfaces of constant astigmatism, it is natural to impose some symmetric conditions on the surface, and it is clear that rotational surfaces are a remarkable

Journal Pre-proof

class of symmetric surfaces. Rotational surfaces in $\mathbb{R}^3$ of constant astigmatism were described by Lilienthal in 1887 proving that the profile curves are the involutes of the tractrix curve, which generates the pseudosphere. A picture of the generating curves of the Lilienthal's surfaces is shown in [3, Fig. 1] and in figure 3 of the present paper. Recently, Baran and Marvan have proved that a surface of constant astigmatism is the involute of a surface of constant negative Gaussian curvature ([3]). On the other hand, in [19] it has been proved that tractrices are critical curves for a total curvature type variational problem. More generally, the same paper proves that profile curves of any rotational surface with constant negative Gaussian curvature are critical curves for the total curvature type energy

$$\mathcal{F}(\gamma) = \int_\gamma \sqrt{\kappa^2 + a^2} \tag{2}$$

for some constant $a \neq 0$. The interest on these curves lies on the following property: their lifts to the space $\mathbb{R}^2 \times \mathbb{S}^1$ endowed with a suitable sub-Riemannian structure are believed to be used by the brain to complete contours of pictures that are missing to the eye vision. See [19] and references therein for more details. Definitively, for this reason, the visual curve completion problem and the surfaces of constant astigmatism are geometrically related.

Until now, surfaces of constant astigmatism have only been studied in Euclidean space. One aim of this paper is to extend the study of these surfaces to space forms, that is, beyond the Euclidean space, to the sphere and the hyperbolic space. We will classify all rotational surfaces of constant astigmatism in space forms, giving a full description of them (in the hyperbolic space we only consider rotational surfaces of spherical type). The second objective consists on proving that generating curves of the rotational examples are solutions of a variational problem associated to a curvature energy involving the curvature κ and the constant c in (1). This energy is measured in a suitable space of curves that deforms the initial curve.

This paper is organized as follows. In Section 2 we define an energy functional $\mathbf{\Theta}_\mu$ for curves in 2-space forms and we calculate its first variation formula. In Section 3 we prove that critical curves of $\mathbf{\Theta}_\mu$ are the generating curves of rotational surfaces of constant astigmatism in space forms. In Section 4 we will prove the converse process by evolving critical curves of $\mathbf{\Theta}_\mu$ under their associated binormal flow. This method gives a way of constructing all rotational surfaces of constant astigmatism in space forms. Once we have characterized the generating curves, we proceed to its classification. Firstly, in Section 5 we give some geometric properties of critical curves of $\mathbf{\Theta}_\mu$ and we associate to them a system of ODE where we analyze the phase plane and its singular points. Finally, Section 6 is devoted to classify and describe all extremal curves of $\mathbf{\Theta}_\mu$ in each space form: see Theorems 6.1, 6.3, 6.4 and 6.5 for precise statements. In particular, in Euclidean space we revisit Lilienthal's surfaces obtaining a full description of all cases in terms of an integration constant.

Journal Pre-proof

2. The curvature energy

Let $\mathbb{M}^n(\rho)$ denote a n-dimensional Riemannian space form of constant sectional curvature ρ, that is, the Euclidean space $\mathbb{R}^n$ ($\rho = 0$), the round sphere $\mathbb{S}^n(\rho)$ ($\rho > 0$) and the hyperbolic space $\mathbb{H}^n(\rho)$ ($\rho < 0$). We denote by $\widetilde{\nabla}$ the Levi-Civita connection on $\mathbb{M}^n(\rho)$. For $n = 3$, let $\gamma(s)$ be a non-geodesic curve in $\mathbb{M}^3(\rho)$ parametrized by arc-length and let $T(s) = \gamma'(s)$ represent the *unit tangent vector field* of γ. Since γ is a non-geodesic curve, $\widetilde{\nabla}_T T(s) \neq 0$ and the standard *Frenet frame* is well defined along $\gamma(s)$. We denote it by $\{T, N, B\}$, where N and B are the *unit normal vector field* and *unit binormal vector field*, respectively. Then, the Frenet equations are

$$\begin{aligned} \widetilde{\nabla}_T T(s) &= \kappa(s)N(s) \\ \widetilde{\nabla}_T N(s) &= -\kappa(s)T(s) + \tau(s)B(s) \\ \widetilde{\nabla}_T B(s) &= -\tau(s)N(s), \end{aligned} \tag{3}$$

where κ and τ are the *curvature* and the *torsion* of γ, respectively. Notice that if $\tau = 0$, the curve is included in a totally geodesic surface which is identified with a 2-space form $\mathbb{M}^2(\rho)$. In such a case, we say that γ is a *planar curve*. Conversely, a curve in $\mathbb{M}^2(\rho)$ can be viewed as a curve in $\mathbb{M}^3(\rho)$ with zero torsion, embedding $\mathbb{M}^2(\rho)$ in $\mathbb{M}^3(\rho)$ as a totally geodesic surface. In particular, the binormal vector $B = T \times N$ in this case is defined to be constant.

Let $\gamma : [0, L] \to \mathbb{M}^2(\rho)$ be a curve parametrized by the arc-length parameter s. For any $\mu \in \mathbb{R}$, we define the following *curvature energy functional*

$$\mathbf{\Theta}_\mu(\gamma) = \int_\gamma \kappa \, \mathrm{e}^{\mu/\kappa} = \int_0^L \kappa(s) \mathrm{e}^{\mu/\kappa(s)} ds, \tag{4}$$

where μ is called the *energy index*. If $\mu = 0$, the curvature energy $\mathbf{\Theta}_0$ is just the functional $\mathcal{F}$ in (2) for $a = 0$ and it coincides with the total curvature of γ. From now on, we discard this case assuming that $\mu \neq 0$. In this setting, let Γ be a variation of γ by non-geodesic curves, that is, for sufficiently small $\varepsilon > 0$, $\Gamma = \Gamma(s, t) : [0, L] \times (-\varepsilon, \varepsilon) \to \mathbb{M}^2(\rho)$ is a smooth map with $\Gamma(s, 0) = \gamma(s)$ such that the curves $\{\Gamma(\text{-}, t) : t \in (-\varepsilon, \varepsilon)\}$ are not geodesics. Let $W = W(s) = \frac{\partial \Gamma}{\partial t}(s, 0)$ be the variational vector field. We write $V = V(s, t) = \frac{\partial \Gamma}{\partial s}(s, t)$, $W = W(s, t) = \frac{\partial \Gamma}{\partial t}(s, t)$, $v = v(s, t) = |V(s, t)|$, and so on. For the next computations, we follow [1] and [14]. By using the Frenet equations (3), the variations of v and κ in the direction of W are

$$\begin{aligned} W(v) &= v\langle \widetilde{\nabla}_T W, T\rangle, \\ W(\kappa) &= \langle \widetilde{\nabla}_T^2 W, N\rangle - 2\kappa\langle \widetilde{\nabla}_T W, T\rangle + \rho\langle W, N\rangle. \end{aligned}$$

Journal Pre-proof

After a standard computation involving integration by parts and the above expressions of $W(v)$ and $W(\kappa)$, the *first variation formula* of $\mathbf{\Theta}_\mu$ is

$$\frac{d}{dt}\Big|_{t=0}\mathbf{\Theta}_\mu(\Gamma(\text{-},t)) = \int_0^L \langle \mathcal{E}(\gamma), W\rangle ds + \mathcal{B}\,[W,\gamma]_0^L,$$

where

$$\mathcal{E}(\gamma) = \widetilde{\nabla}_T\mathcal{J} - R(T,\mathcal{K})T = \widetilde{\nabla}_T\mathcal{J} + \rho\mathcal{K},$$
$$\mathcal{B}\,[W,\gamma]_0^L = \Big[\langle \mathcal{K}, \widetilde{\nabla}_T W\rangle - \langle \mathcal{J}, W\rangle\Big]_0^L,$$

and

$$\mathcal{K}(\gamma) = \Big(1-\frac{\mu}{\kappa}\Big)\,\mathrm{e}^{\mu/\kappa}N, \tag{5}$$
$$\mathcal{J}(\gamma) = \widetilde{\nabla}_T\mathcal{K} + (\kappa-2\mu)\,\mathrm{e}^{\mu/\kappa}T. \tag{6}$$

Throughout this paper, a curve γ is called a *critical curve* or *extremal curve* if $\mathcal{E}(\gamma)=0$ for any variation Γ of γ. Notice that this is an abuse of notation for the sake of simplicity because, in general, criticality depends on the boundary conditions, as it is clear by the boundary term $\mathcal{B}\,[W,\gamma]_0^L$ in the first variation formula.

Now, using the first and second Frenet equations in (3), a straightforward computation shows that $\mathcal{E}(\gamma)$ has no component in T. Moreover, $\mathcal{E}(\gamma)=0$ is equivalent to the vanishing of the component in N, obtaining the Euler-Lagrange equation for $\mathbf{\Theta}_\mu$.

Proposition 2.1. *Let $\gamma=\gamma(s)$ be a non-geodesic curve in $\mathbb{M}^2(\rho)$. Then γ is a critical curve of the curvature energy $\mathbf{\Theta}_\mu$ if and only if*

$$\frac{d^2}{ds^2}\left(\Big(1-\frac{\mu}{\kappa}\Big)\,\mathrm{e}^{\mu/\kappa}\right) + \left(\rho\Big(1-\frac{\mu}{\kappa}\Big) - \mu\kappa\right)\mathrm{e}^{\mu/\kappa} = 0. \tag{7}$$

We start with the particular case that the curvature of γ is constant.

Corollary 2.2. *The only (non-geodesic) critical curves of $\mathbf{\Theta}_\mu$ with constant curvature κ_0 in $\mathbb{M}^2(\rho)$ are the following:*

1. *Case $\mathbb{R}^2$. There are not critical curves.*
2. *Case $\mathbb{S}^2(\rho)$. In this case $\mu\in\big(0,\sqrt{\rho}/2\big]$. Moreover, if $\mu<\sqrt{\rho}/2$, then there are two solutions, which correspond with two parallels; and, if $\mu=\sqrt{\rho}/2$, then there is only one solution, a circle with curvature $\kappa_0=\sqrt{\rho}$.*

Journal Pre-proof

3. *Case $\mathbb{H}^2(\rho)$. There are always two critical curves with constant curvature, namely, a circle and a hypercycle.*

Proof. From equation (7), if $\kappa = \kappa_0$ is constant then

$$\kappa = \kappa_0 = \frac{\rho \pm \sqrt{\rho^2 - 4\mu^2\rho}}{2\mu}. \tag{8}$$

In particular, there are restrictions on the values of μ in relation with the constant ρ, namely, $\rho^2 \geq 4\mu^2\rho$.

Firstly, it is clear that if $\rho = 0$, then (8) implies that $\kappa_0 = 0$, which is not possible.

If $\rho > 0$, then $\mu \leq \sqrt{\rho}/2$. When $\mu < \sqrt{\rho}/2$, there are two solutions that correspond with two parallels of $\mathbb{S}^2(\rho)$. On the other hand, if $\mu = \sqrt{\rho}/2$, there is only one solution that corresponds with a circle of curvature $\kappa_0 = \sqrt{\rho}$.

Finally, if $\rho < 0$, the condition $\rho^2 > 4\mu^2\rho$ is always satisfied, obtaining two critical curves with constant curvature. From (8), the curvature corresponding with the plus sign satisfies that $\kappa_0^2 < -\rho$ and the critical curve is a hypercycle. Similarly, for the minus sign, $\kappa_0^2 > -\rho$ and the curve is a circle. □

We finish this section obtaining a first integral of the Euler-Lagrange equation (7). Observe that from (7), and after a change of orientation on γ if necessary, we can assume $\mu > 0$. Set $b = \mu/\kappa$, then equation (7) becomes

$$(b^2)'' + b'(b^2)' - 2\left(\rho\,(1-b) - \frac{\mu^2}{b}\right) = 0. \tag{9}$$

Define the function

$$f(s) = \mathrm{e}^{2b}\left(b^2\left(b'\right)^2 + \rho\,(1-b)^2 + \mu^2\right).$$

The derivative of f is

$$\begin{aligned} f'(s) &= \mathrm{e}^{2b}\left(2\mu^2 b' - \rho(1-b)\left(b^2\right)' + 2b^2\left(b'\right)^3 + \frac{1}{2}\left(b^2\right)'\left(b^2\right)''\right) \\ &= 2\mu b'\mathrm{e}^{2b}\left(\mu - b\kappa\right) = 0, \end{aligned}$$

where for the second identity we have used (7). It follows that f is a constant function $f(s) = d$, $d \in \mathbb{R}$. From the definition of f and b, the derivative with respect to the arc-length

Journal Pre-proof

parameter κ_s of κ satisfies

$$\kappa_s^2 = \frac{\kappa^4}{\mu^4}\left(d\kappa^2 \mathrm{e}^{-2\mu/\kappa} - \mu^2\kappa^2 - \rho\,(\kappa-\mu)^2\right). \tag{10}$$

We turn back to equation (7). Let us introduce the notation

$$x = \mathrm{e}^{\mu/\kappa}, \quad y = x_s. \tag{11}$$

Then (7) can be written as

$$\frac{d^2}{ds^2}\left((1-\log x)\,x\right) + \left(\rho\,(1-\log x) - \frac{\mu^2}{\log x}\right)x = 0.$$

After some manipulations, this equation is equivalent to the system of ordinary differential equations

$$\begin{cases} x' &= y \\ y' &= \dfrac{-\mu^2x^2 - \rho x^2\log^2 x + \rho x^2\log x - y^2\log x}{x\log^2 x}. \end{cases} \tag{12}$$

Moreover, using the first integral of the Euler-Lagrange equation (10), we obtain in the following result an implicit equation that describes the orbits of the ODE system (12).

Proposition 2.3. *If $\gamma \subset \mathbb{M}^2(\rho)$ is a critical curve of the energy $\mathbf{\Theta}_\mu$, then there is a constant $d \in \mathbb{R}$ such that γ in the coordinates introduced in* (11) *satisfies*

$$F(x,y) := y^2\log^2 x + \mu^2x^2 + \rho\,(1-\log x)^2\,x^2 = d, \quad x \in \mathbb{R}^+ \setminus \{1\}. \tag{13}$$

In particular, the constant d is positive if $\rho \geq 0$.

3. A variational characterization of profile curves

A surface S in $\mathbb{M}^3(\rho)$ is called *rotationally symmetric*, or simply a *rotational surface*, if S is invariant under the action of a one-parameter group of rotations $\{\phi_t : t \in \mathbb{R}\}$ of $\mathbb{M}^3(\rho)$. The group leaves a geodesic pointwise fixed. This geodesic is called the *rotation axis*. In $\mathbb{R}^3$ and $\mathbb{S}^3(\rho)$, any one-parameter group of rotations is isomorphic to $\mathbb{S}^1$ (and, as a consequence, the orbits are circles), but in the hyperbolic space $\mathbb{H}^3(\rho)$ there are three different types of rotations ([7]). Our interest in $\mathbb{H}^3(\rho)$ focuses only for those groups of rotations that are isomorphic to $\mathbb{S}^1$, which are called *spherical rotations*.

Journal Pre-proof

A rotational surface is generated by a planar curve γ called the *profile curve* of S. Since γ is a planar curve, we can assume that it lies fully in a totally geodesic surface of $\mathbb{M}^3(\rho)$ identified with a 2-space form $\mathbb{M}^2(\rho)$. We prove that the profile curve γ of a rotational surface of constant astigmatism has a variational characterization.

Theorem 3.1. *Let S be a rotational surface in $\mathbb{M}^3(\rho)$ satisfying the constant astigmatism equation* (1)*. Then, locally, the profile curve γ of S satisfies the Euler-Lagrange equation* (7) *for the curvature energy*

$$\mathbf{\Theta}_\mu(\gamma) = \int_\gamma \kappa\, \mathrm{e}^{\mu/\kappa}\, ds, \tag{14}$$

where $\mu = 1/c$ and κ is the curvature of γ.

Proof. Let ξ be the Killing vector field which is the infinitesimal generator of the rotations that leave S invariant. Since the result is local, let us take Fermi geodesic coordinates $X : U \subset \mathbb{R}^2 \to S$ on S,

$$X(s,t) = \phi_t\left(\gamma(s)\right), \quad \xi = \frac{\partial}{\partial t}, \tag{15}$$

where s measures the arc-length along the geodesics orthogonal to ξ and $\gamma(s) = X(s,0)$. From (15), the curve γ and all its copies $\gamma_t(s) = \phi_t(\gamma(s))$ are arc-length parametrized planar geodesics of S and they are orthogonal to ξ. Moreover, γ and all γ_t are not geodesics in $\mathbb{M}^3(\rho)$ from (1). Therefore, we can denote by $\{T(s,t), N(s,t), B(s,t)\}$ and $\kappa(s,t)$ the Frenet frame and the curvature of γ_t, respectively.

On the other hand, following [1, Sec. 3], the Gauss-Codazzi equations of S in $\mathbb{M}^3(\rho)$ have a simple local expression, namely,

$$\frac{\partial}{\partial s}\left(\frac{1}{\kappa}\left(G_{ss} + G(\kappa^2 + \rho)\right)\right) - \kappa_s G = 0, \tag{16}$$

where G is the length of the Killing vector field ξ, that is, $G(s)^2 = \langle X_t, X_t\rangle$. Furthermore, not only $G(s)$, but also all the involved functions depend only on s. The principal curvatures of S are $\kappa_1(s) = h_{22}(s)$, the second coefficient of the second fundamental form, and $\kappa_2(s) = -\kappa(s)$. Since

$$h_{22} = \frac{1}{\kappa}\left(\frac{G_{ss}}{G} + \rho\right), \tag{17}$$

relation (1) becomes

$$(c\kappa - 1)\, G_{ss} = G\left(\kappa^2 - \rho\,(c\kappa - 1)\right). \tag{18}$$

Journal Pre-proof

First, we study the particular case that κ is a constant function $\kappa(s) = \kappa_0$. From (1), $\kappa_0 \neq 1/c$ and combining (16) and (18), we deduce that $G(s)$ is a positive constant. Hence, S is a flat isoparametric surface. Using once more (18), we deduce that

$$\kappa_0^2 = \rho\,(c\kappa_0 - 1)\,.$$

From equation (8), γ is a critical curve with constant curvature for $\mathbf{\Theta}_\mu$ with $\mu = 1/c$, and the theorem is proved in this case.

Suppose now that the curvature κ of γ is not constant. From the Inverse Function Theorem, we can suppose that s is locally a function of κ. Set $G(\kappa) = \dot{P}(\kappa)$, where the upper dot denotes the derivative with respect to κ. We integrate once equation (16), obtaining

$$\dot{P}_{ss} + \dot{P}\left(\kappa^2 + \rho\right) - \kappa\,(P + \lambda) = 0, \tag{19}$$

for a constant $\lambda \in \mathbb{R}$. With this notation, equation (18) can be now expressed as

$$\dot{P}_{ss} - \dot{P}\left(\frac{\kappa^2}{c\kappa - 1} - \rho\right) = 0,$$

and if we substitute in this equation the value of $\dot{P}_{ss}$ from (19), we find

$$\frac{c\kappa^3}{c\kappa - 1}\dot{P} - \kappa P - \lambda\kappa = 0.$$

A direct integration of this equation gives

$$P(\kappa) = \kappa\,\mathrm{e}^{\mu/\kappa} - \lambda, \tag{20}$$

where $\mu = 1/c$. For this function $P(\kappa)$, equation (19) is just the Euler-Lagrange equation (7), concluding that γ is a critical curve of $\mathbf{\Theta}_\mu$. This proves the result. □

4. A geometric description of the rotational surfaces

In this section we will prove the converse of Theorem 3.1 by evolving extremal curves under their associated binormal flow. This will describe in Theorem 4.3 below a way of constructing all rotational surfaces of constant astigmatism in space forms.

First, we observe that critical curves of $\mathbf{\Theta}_\mu$ have a distinguished vector field defined along them. Let $\gamma = \gamma(s)$ be a curve in $\mathbb{M}^2(\rho)$ critical for $\mathbf{\Theta}_\mu$. A vector field W along γ, which

Journal Pre-proof

infinitesimally preserves unit speed parametrization, is called a *Killing vector field along* γ in the sense of Langer-Singer ([14]), if γ evolves in the direction of W without changing shape, only its position. In other words, if the equations

$$W(v)(s,0) = W(\kappa)(s,0) = 0,$$

hold for any variational vector field of γ having W as variation vector field, where $v = |\gamma'|$.

We prove that critical curves of $\mathbf{\Theta}_\mu$ have a naturally associated Killing vector field defined along γ. Indeed, we define the vector field $\mathcal{I}$ along γ by

$$\mathcal{I} = T \times \mathcal{K} = \left(1 - \frac{\mu}{\kappa}\right) \mathrm{e}^{\mu/\kappa} B, \tag{21}$$

where $\mathcal{K}$ is defined in (5). Here $B = T \times N$ is the (constant) binormal of γ viewing $\mathbb{M}^2(\rho)$ inside $\mathbb{M}^3(\rho)$ and γ as a planar curve ($\tau = 0$). The following result is proved in [1, Prop. 3].

Proposition 4.1. *If $\gamma \subset \mathbb{M}^2(\rho)$ is a critical curve of $\mathbf{\Theta}_\mu$, then the vector field along γ*

$$\mathcal{I} = \left(1 - \frac{\mu}{\kappa}\right) \mathrm{e}^{\mu/\kappa} B$$

is a Killing vector field along γ.

From this result and using a similar argument to [14], we uniquely extend $\mathcal{I}$ to a Killing vector field in the whole ambient space $\mathbb{M}^3(\rho)$. We denote it by $\mathcal{I}$ again. Since $\mathbb{M}^3(\rho)$ is complete, we consider the one-parameter group of isometries $\{\phi_t : t \in \mathbb{R}\}$ determined by the flow of $\mathcal{I}$, and define the surface $S_\gamma = \{\phi_t(\gamma(s)) : s \in I = [0, L], t \in \mathbb{R}\}$ obtained as the evolution of γ under the $\mathcal{I}$-flow. Observe that S_γ is an $\mathcal{I}$-invariant surface which is foliated by congruent copies of γ, namely, $\gamma_t(s) = \phi_t(\gamma(s))$.

Since $X(s,t) = \phi_t(\gamma(s))$ is a parametrization of S_γ, and ϕ_t are isometries of $\mathbb{M}^3(\rho)$,

$$X_t(s,t) = \left(1 - \frac{\mu}{\kappa}\right) \mathrm{e}^{\mu/\kappa} B(s,t),$$

where κ is the curvature of γ and $B(s,t)$ is the unit binormal vector of $\gamma_t(s)$. Thus S_γ is a *binormal evolution surface* with velocity $V(s) = \langle X_t, X_t \rangle^{1/2} = \langle \mathcal{I}, \mathcal{I} \rangle^{1/2}$.

Now, since all the filaments of S_γ satisfy $\tau = 0$, the fibers of S_γ have constant curvature and zero torsion in $\mathbb{M}^3(\rho)$, as proved in [1, Prop. 3]. On one hand, if the curvature $\kappa(s,t)$ of all filaments is constant, then S_γ is a flat isoparametric surface. What is more, since S_γ is not a flat totally umbilical surface, then S_γ is a right circular cylinder ([22]). On the other hand, if $\kappa(s,t)$ is non-constant, then the curvature of γ is not constant and S_γ is a rotational surface, as it can be proved adapting the computations of [2, Prop. 5.3]. We summarize this result.

Journal Pre-proof

Proposition 4.2. *Let $\gamma \subset \mathbb{M}^2(\rho)$ be a critical curve of the energy $\mathbf{\Theta}_\mu$. Let S_γ be a binormal evolution surface in $\mathbb{M}^3(\rho)$ such that all filaments have zero torsion and non-constant curvature. Then S_γ is a rotational surface.*

We are in conditions to prove the converse of Theorem 3.1, which provides a method of constructing rotational surfaces of constant astigmatism in $\mathbb{M}^3(\rho)$.

Theorem 4.3. *Let $\gamma \subset \mathbb{M}^2(\rho)$ be a critical curve of the energy $\mathbf{\Theta}_\mu$ with non-constant curvature. Then S_γ is a rotational surface of $\mathbb{M}^3(\rho)$ of constant astigmatism for the value $c = 1/\mu$ in* (1).

Proof. We know that the square of the length of the Killing vector field $\mathcal{I}$ is

$$V(s)^2 = \langle \mathcal{I}, \mathcal{I} \rangle = \frac{(\kappa - \mu)^2}{\kappa^2} \mathrm{e}^{2\mu/\kappa}. \tag{22}$$

Since the evolution is done by the isometries ϕ_t, the curve γ and all its congruent copies $\phi_t(\gamma)$ are planar critical curves of $\mathbf{\Theta}_\mu$ and Proposition 4.2 implies that S_γ is a rotational surface. Finally, any γ_t satisfies the Euler-Lagrange equation (7) which is, using (22), equivalent to

$$(\kappa - \mu)\, V_{ss} = \left(\mu\kappa^2 - \rho\,(\kappa - \mu)\right) V.$$

The principal curvatures of S_γ are $\kappa_1 = h_{22}(s)$ (see equation (17), where now $G(s) = V(s)$) and $\kappa_2 = -\kappa(s)$. Thus, we deduce, using above equation that

$$\frac{1}{\kappa_1} - \frac{1}{\kappa_2} = \frac{1}{h_{22}(s)} + \frac{1}{\kappa(s)} = \frac{\kappa(s) - \mu}{\mu\kappa(s)} + \frac{1}{\kappa(s)} = \frac{1}{\mu}\,.$$

This proves that S_γ is a surface of constant astigmatism in $\mathbb{M}^3(\rho)$ with $c = 1/\mu$ in (1). □

Theorem 4.3 and Theorem 3.1 locally characterize all surfaces of constant astigmatism in $\mathbb{M}^3(\rho)$ as binormal evolution surfaces generated by extremal planar curves of $\mathbf{\Theta}_\mu$. Thus we conclude that these surfaces must be locally either totally umbilical, right circular cylinders constructed over an extremal curve with constant curvature (Corollary 2.2) or binormal evolution surfaces generated by extremal planar curves of $\mathbf{\Theta}_\mu$.

Remark 4.4. *In the sphere $\mathbb{S}^3(\rho)$, the constant astigmatism flat isoparametric rotational surfaces (right circular cylinders) are Hopf tori given by the product*

$$\mathbb{S}^1\left(\frac{\sqrt{2}\mu}{m}\right) \times \mathbb{S}^1\left(\frac{\sqrt{m^2 - 2\mu^2\rho}}{\sqrt{\rho}\, m}\right),$$

where $m^2 = |\rho^2 \pm \sqrt{\rho^2 - 4\mu^2\rho}|$. The particular case $\mu = \sqrt{\rho}/2$ is the Clifford torus, which is a minimal surface in $\mathbb{S}^3(\rho)$.

Journal Pre-proof

5. Analysis of the extremal curves

In this section we study some geometric properties of the critical curves of the curvature energy $\mathbf{\Theta}_\mu$, obtaining a useful expression of the parametrization of these curves in terms of its curvature.

Consider the 2-space form $\mathbb{M}^2(\rho)$ viewed as a subset of the affine space $\mathbb{R}^3$ with canonical coordinates (x_1, x_2, x_3), where the Euclidean plane $\mathbb{R}^2$ is $\mathbb{R}^2 \times \{0\}$, the sphere $\mathbb{S}^2(\rho)$ is $x_1^2 + x_2^2 + x_3^2 = 1/\rho$ and the hyperbolic space $\mathbb{H}^2(\rho)$ is $x_1^2 + x_2^2 - x_3^2 = 1/\rho$, $x_3 > 0$. Here $\mathbb{R}^2$ and $\mathbb{S}^2(\rho)$ are endowed with the Euclidean metric of $\mathbb{R}^3$ and $\mathbb{H}^2(\rho)$ with the induced one from the Lorentzian metric $dx_1^2 + dx_2^2 - dx_3^3$.

As a special case, we now prove that in the hyperbolic space, the case $d > 0$ in (13) corresponds with rotational surfaces of spherical type. In fact, the three possible signs of the constant d in (13) indicate the three types of rotational surfaces in the hyperbolic space. Following the notation of Section 4 and from [1, equation (47)], the curvature κ_δ of the orbits of S_γ satisfies

$$\kappa_\delta^2 = \frac{\dot{P}_s^2}{\dot{P}^2} + h_{22}^2.$$

By combining (10), (17) and (20), we find

$$\kappa_\delta^2 = \frac{d\kappa^2 \mathrm{e}^{-2\mu/\kappa}}{(\mu - \kappa)^2} - \rho.$$

This implies that the orbits are circles ($\kappa_\delta^2 > -\rho$) if and only if $d > 0$, which means that the rotational surface is of spherical type.

From now, we suppose that the constant d in (13) is positive: recall that d is always positive if $\rho \geq 0$. Define the function $\phi : \mathbb{R}^2 \to \mathbb{M}^2(\rho) \subset \mathbb{R}^3$ by

$$\phi(u,v) = \begin{cases} \dfrac{1}{\sqrt{d}}\,(u, dv, 0) & \text{if} \quad \rho = 0 \\ \dfrac{1}{\sqrt{d}}\left(u, \sqrt{d - \rho u^2}\sin(\sqrt{\rho d}\,v), \sqrt{d - \rho u^2}\cos(\sqrt{\rho d}\,v)\right) & \text{if} \quad \rho > 0 \\ \dfrac{1}{\sqrt{d}}\left(u, \sqrt{d - \rho u^2}\sinh(\sqrt{-\rho d}\,v), \sqrt{d - \rho u^2}\cosh(\sqrt{-\rho d}\,v)\right) & \text{if} \quad \rho < 0. \end{cases} \tag{23}$$

As it turns out, up to rigid motions, it is always possible to find a coordinate system in $\mathbb{M}^2(\rho) \subset \mathbb{R}^3$ such that the critical curves of any curvature energy satisfy that their first component is a multiple of $\dot{P}(\kappa)$. From this, we deduce the following parametrization of a critical curve of $\mathbf{\Theta}_\mu$.

Journal Pre-proof

Proposition 5.1. *Let γ be any critical curve of $\mathbf{\Theta}_\mu$ with non-constant curvature $\kappa(s)$. Then, up to rigid motions, γ can be parametrized as*

$$\gamma(s) = \phi\left((1-\log x)x, \psi(s)\right), \quad x = \mathrm{e}^{\mu/\kappa}, \tag{24}$$

where ϕ is defined in (23) *and the function $\psi(s)$ comes from the parametrization by arc-length.*

Furthermore, without loss of generality, we can assume that the arc-length parameter s is chosen so that $x(0) = x_0$, hence

$$\psi(s) = -\mu \int_0^s \frac{x(t)}{d - \rho\left(1-\log x(t)\right)^2 x^2(t)}\, dt. \tag{25}$$

Notice that the denominator cannot vanish because if $d = \rho(1-\log x)^2 x^2$ at some point, we find from (13) that $y^2 \log^2 x + \mu^2 x^2 = 0$, which is not possible.

The parametrization (24) allows us to obtain some geometric properties concerning the symmetries and cuts with respect to some coordinate axis of critical curves. In this direction we first prove that critical curves are symmetric about a fixed axis.

Proposition 5.2. *Up to a rigid motion, any critical curve of $\mathbf{\Theta}_\mu$ in $\mathbb{M}^2(\rho)$ is symmetric with respect to the geodesic $\alpha = \Pi_{13} \cap \mathbb{M}^2(\rho)$, where Π_{13} is the plane of equation $x_2 = 0$.*

Proof. If $\gamma(s)$ is a critical curve of $\mathbf{\Theta}_\mu$, a change of variables in (25) gives $\psi(-s) = -\psi(s)$. From the parametrization (24), the second coordinate of $\gamma(-s)$ has the opposite sign to the corresponding one of $\gamma(s)$ and the first and third coordinates coincide. This finishes the proof. □

We now study those points where extremal curves of $\mathbf{\Theta}_\mu$ cut or tend to cut the other coordinate axis.

Proposition 5.3. *In $\mathbb{M}^2(\rho)$, consider the geodesic $\beta = \Pi_{23} \cap \mathbb{M}^2(\rho)$ where Π_{23} is the plane of equation $x_1 = 0$. Then, up to a rigid motion, any critical curve γ of $\mathbf{\Theta}_\mu$ may meet β at two types of points, namely:*

1. *Regular points. The curve γ cuts the geodesic β at points satisfying $x = \mathrm{e}$.*
2. *Singular points. If x tends to zero, then γ tends to meet β.*

Journal Pre-proof

Proof. From the parametrization (24), γ meets the geodesic β whenever its first component vanishes, that is, when $(1-\log x)\,x = 0$. If $x = \mathrm{e}$, then the intersection occurs at a regular point of γ. If $x = 0$ then the point is outside of the parametrization of γ, although it may be added in order to obtain complete curves. In this case, if x tends to 0, then γ tends to β. □

We now prove a result about the behavior of the critical curves by change of orientation and by dilations.

Proposition 5.4. *For each fixed $d > 0$, the critical curves of $\mathbf{\Theta}_\mu$ and $\mathbf{\Theta}_{-\mu}$ coincide up to change of orientation. In the Euclidean plane $\mathbb{R}^2$, any dilation of ratio $\lambda > 0$ of a critical curve of $\mathbf{\Theta}_\mu$ for $d > 0$ is a critical curve of $\mathbf{\Theta}_{\mu/\lambda}$ for d/λ^2.*

Proof. Let $\gamma = \gamma(s)$ be a critical curve of $\mathbf{\Theta}_\mu$. We write (13) as $F(x, y; \mu) = d$ to indicate the dependence of F on the energy index μ. We reverse the orientation on γ, say, $\widetilde{\gamma}(s) = \gamma(-s)$. If $\widetilde{\kappa}$ is the curvature of $\widetilde{\gamma}$, then $\widetilde{\kappa}(s) = -\kappa(-s)$ and

$$\begin{aligned} x(s;\mu) &= \widetilde{x}(-s;-\mu) \\ y(s;\mu) &= -\widetilde{y}(-s;-\mu). \end{aligned}$$

Using above equations, we see that

$$F(x, y; \mu) = F(\widetilde{x}, \widetilde{y}; -\mu) = d,$$

proving that critical curves for μ and for $-\mu$ are the same (with opposite orientation).

Now, in the Euclidean plane, if we apply a dilation of ratio $\lambda > 0$ to γ, say $\widetilde{\gamma}(\lambda s) = \lambda\gamma(s)$, then its curvature $\widetilde{\kappa}$ satisfies $\kappa(s) = \lambda\widetilde{\kappa}(\lambda s)$. In this case, we obtain

$$\begin{aligned} x(s;\mu) &= \widetilde{x}(\lambda s;\mu/\lambda) \\ y(s;\mu) &= \lambda\widetilde{y}(\lambda s;\mu/\lambda). \end{aligned}$$

Since $\rho = 0$ for the Euclidean plane, it follows that

$$F(x, y; \mu) = \lambda^2 F(\widetilde{x}, \widetilde{y}; \mu/\lambda) = d$$

Thus critical curves of $\mathbf{\Theta}_\mu$ are also critical for $\mathbf{\Theta}_{\mu/\lambda}$ for the constant of integration $d = \lambda^2\widetilde{d}$. □

Observe that for dilations, the result does not hold if $\rho \neq 0$. Indeed, after applying the corresponding relations between x, y, $\widetilde{x}$ and $\widetilde{y}$ as above, the sectional curvature ρ of $\mathbb{M}^2(\rho)$ should also be deformed to ρ/λ, which is not possible since it must be fixed from the beginning.

Journal Pre-proof

Consider now the expression (25) of $\psi(s)$. We make a change of variable $y = x_s$ and by using (13), we obtain

$$\psi(x) = \mu \int_x^{x_0} \frac{r \log r}{\left(d - \rho\,(1 - \log r)^2\, r^2\right) \sqrt{d - \mu^2 r^2 - \rho\,(1 - \log r)^2\, r^2}}\, dr. \tag{26}$$

The function $\psi(x)$ in (26) (or $\psi(s)$ in (25)) provides information of the critical curve γ thanks to (24). For example, since $d > \rho\,(1 - \log x)^2\, x^2$, then $\psi'(x) > 0$ if $x \in (0, 1)$ and $\psi'(x) < 0$ if $x > 1$. This implies that $\psi(x)$ monotonically increases if $x \in (0, 1)$ and $\psi(x)$ decreases if $x > 1$. For fixed μ, set $\psi(x) = \psi(x; d)$ to indicate the dependence on d. Consider the value $\lim_{x\to 0} \psi(x; d)$ as a function depending on the parameter d and let d_* be the only value of d such that $\lim_{x\to 0} \psi(x; d) = 0$.

In order to study the ODE system (12), we analyze the corresponding phase plane. Let $Q(x, y)$ be the tangent vector field

$$Q(x, y) = \left(y, \frac{-\mu^2 x^2 - \rho x^2 \log x(\log x - 1) - y^2 \log x}{x \log^2 x}\right),$$

which is defined in $(\mathbb{R}^+ \setminus \{1\}) \times \mathbb{R}$. The singular points of Q are the points of the form $(x, 0)$, where x satisfies

$$\mu^2 = \rho \log x\,(1 - \log x)\,. \tag{27}$$

In particular, if $\rho = 0$, there are not singular points (figure 1, left). Therefore, from now on assume that $\rho \neq 0$. From (27) and if $\rho^2 - 4\mu^2\rho \geq 0$, we find

$$\log x = \frac{\rho \pm \sqrt{\rho^2 - 4\mu^2\rho}}{2\rho}. \tag{28}$$

Thus there are two singular points $P_+ = (x_+, 0)$ and $P_- = (x_-, 0)$, which may coincide, and correspond with the choices $+$ and $-$ in (28), respectively. After some computations, at these singular points we have

$$\begin{pmatrix} \dfrac{\partial Q_1}{\partial x} & \dfrac{\partial Q_1}{\partial y} \\ \dfrac{\partial Q_2}{\partial x} & \dfrac{\partial Q_2}{\partial y} \end{pmatrix} (x_\pm, 0) = \begin{pmatrix} 0 & 1 \\ \mp\dfrac{\sqrt{\rho^2 - 4\mu^2\rho}}{\log^2 x} & 0 \end{pmatrix}. \tag{29}$$

The analysis of the type of singular points depends on the eigenvalues of (29), which are

$$\lambda_1 = -\sqrt{2\mu^2 - \rho \log(x)}, \quad \lambda_2 = \sqrt{2\mu^2 - \rho \log(x)}. \tag{30}$$

Journal Pre-proof

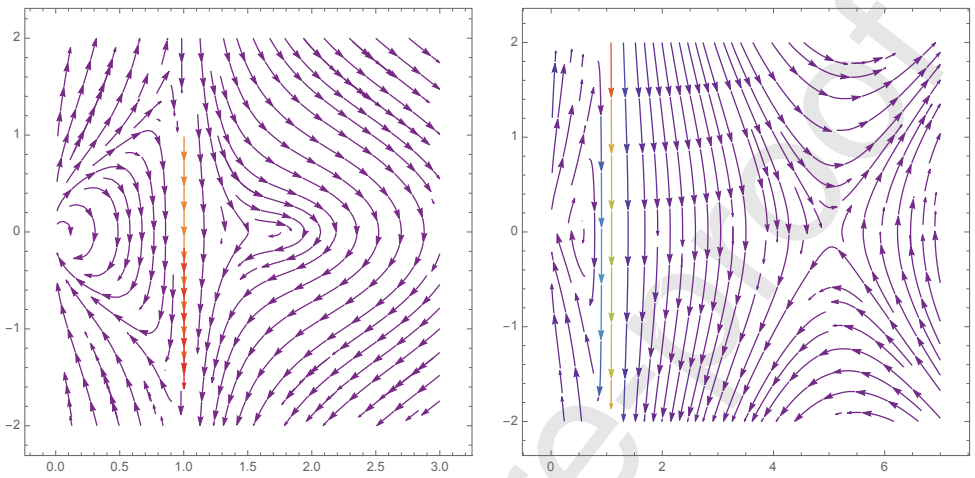


Figure 1: (Left) Phase plane in the Euclidean plane $\mathbb{R}^2$. (Right) Phase plane in the hyperbolic plane $\mathbb{H}^2(\rho)$: here $\rho = -1$, $\mu = -1$ and the singular points are $P_+ = (0.54, 0)$ (centre) and $P_- = (5.04, 0)$ (saddle point)

We first study the case $\rho > 0$. From (28), there are not singular points when $\rho < 4\mu^2$. If $\rho = 4\mu^2$, there is a unique singular point $P_+ = P_- = P = (\sqrt{\mathrm{e}}, 0)$, where the matrix (29) is not diagonalizable and the eigenvalue is 0, so P is a degenerate point. Finally, if $\rho > 4\mu^2$, $P_+ \neq P_-$, with $x_- < x_+$. From (30), the eigenvalues for P_+ are two distinct pure imaginary complex numbers and the eigenvalues for P_- are two distinct real numbers with different sign. Thus P_+ is a centre and P_- is an unstable saddle point. See the phase plane in figure 2.

Let us see now the case $\rho < 0$. From (28) and since $\rho^2 - 4\mu^2\rho$ is always positive, then there are always two singular points, $P_+ = (x_+, 0)$ and $P_- = (x_-, 0)$ where now $x_+ < 1 < x_-$. The eigenvalues corresponding for P_+ are two opposite pure imaginary complex numbers, so P_+ is a centre, whereas for P_- the eigenvalues are two real numbers with opposite sign, hence P_- is an unstable saddle point: see the phase plane in figure 1, right.

We finish this section by analyzing the existence of closed critical curves. Notice that the problem of existence of closed critical curves is a difficult matter. Here we establish what is the relation between the value d in (13) and the energy index μ of $\mathbf{\Theta}_\mu$.

Proposition 5.5. *Let $d > 0$. Then there are no closed critical curves of $\mathbf{\Theta}_\mu$ in $\mathbb{R}^2$ and $\mathbb{H}^2(\rho)$. In $\mathbb{S}^2(\rho)$, if γ is a closed critical curve of $\mathbf{\Theta}_\mu$, then the number*

$$\mu\sqrt{\rho d}\int_0^{\varrho} \frac{x(t)}{d - \rho(1 - \log x(t))^2 x^2(t)}\, dt \tag{31}$$

Journal Pre-proof

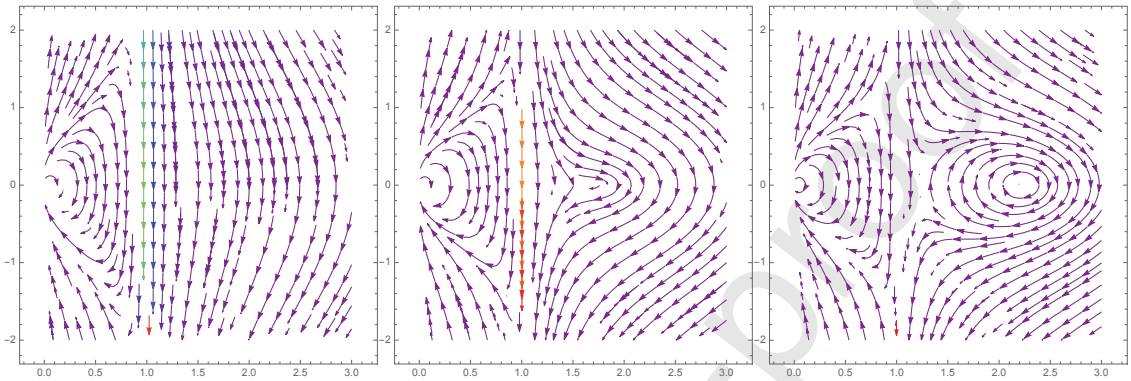


Figure 2: Phase plane in the sphere $\mathbb{S}^2(\rho)$. (Left) Case $\rho < 4\mu^2$: there are not singular points. (Middle) Case $\rho = 4\mu^2$: the point $P = (\sqrt{e}, 0)$ is a degenerate singular point. (Right) Case $\rho > 4\mu^2$: here $\rho = 1$ and $\mu = 0.4$ and the singular points are $P_- = (1.22, 0)$, which is a saddle point, and $P_+ = (2.22, 0)$, which is a centre

is a rational multiple of 2π *for some* $d \in (\rho x_+^2 \log x_-, \rho x_-^2 \log x_+)$*, where* ϱ *is the period of the curvature* κ*.*

Proof. Since closed curves have periodic curvatures, we begin by studying if equation (10) admits periodic solutions, that is, closed orbits in the phase plane (12). Closed orbits appear only in local neighborhoods of centre points, and consequently, we can just restrict ourselves to the case $\rho \neq 0$ and $\rho^2 - 4\mu^2\rho > 0$: this excludes the ambient space $\mathbb{R}^2$.

Let γ be a closed critical curve in $\mathbb{H}^2(\rho)$. If $\varrho > 0$ is the period of the curvature of γ, $\kappa(s)$, from (25) we have

$$\int_0^{\varrho} \frac{x(t)}{d - \rho\left(1 - \log x(t)\right)^2 x^2(t)}\, dt = 0,$$

which is impossible because the denominator is positive, proving the result in $\mathbb{H}^2(\rho)$.

Suppose now that the ambient space is $\mathbb{S}^2(\rho)$ with $\rho > 4\mu^2$. In order to obtain the values of the constant of integration d for which there exist periodic solutions, notice that orbits in this case cut the x-axis in either one or three points, in the latter case, we obtain the closed ones. Indeed, the function $F(x, 0) = \mu^2 x^2 + \rho\left(1 - \log x\right)^2 x^2$ increases in the interval $(0, x_-)$, decreases as x moves from x_- to x_+ and, finally, increases if $x > x_+$. We deduce that $F(x, 0)$ reaches a local maximum at x_- and a local minimum at x_+. Therefore, there are exactly three cuts if and only if $d = F(x, 0)$ for $x \in (x_-, x_+)$. Since the function $F(x, 0)$ is decreasing

Journal Pre-proof

in the interval (x_-, x_+), a periodic solution appears if and only if

$$F(x_+, 0) = \rho x_+^2 \log x_- < d < \rho x_-^2 \log x_+ = F(x_-, 0).$$

Once more from (24) and (25), if γ is a closed curve, then the number $\sqrt{\rho} d\psi(\varrho)$ is a rational multiple of 2π, obtaining (31). □

6. Classification of extremal curves

In this section we give the classification of generating curves of rotational surfaces of constant astigmatism in space forms. To this end, in each of the three following subsections we will study each space form $\mathbb{M}^3(\rho)$. Recall that in Euclidean space $\mathbb{R}^3$, rotational surfaces of constant astigmatism were studied by Lilienthal proving that the profile curves are involutes of the tractrix ([16]). Besides revisiting the class of Lilienthal's surfaces, we will provide a description of all possible shapes for profile curves in the three space forms.

We know from Section 5 that profile curves are parametrized by (24), and we analyzed the phase plane (12) in each 2-space form. From Propositions 5.2 and 5.3, an extremal curve $\gamma \subset \mathbb{M}^2(\rho)$ is symmetric about the geodesic $\alpha = \Pi_{13} \cap \mathbb{M}^2(\rho)$ and, furthermore, γ may meet the geodesic $\beta = \Pi_{23} \cap \mathbb{M}^2(\rho)$ at two points, possibly singular. It turns out that this geodesic β is the rotation axis of the rotational surface of constant astigmatism.

A first set of shapes are the curves in $\mathbb{M}^2(\rho)$ which share similar properties to profile curves of Lilienthal's surfaces and which we consequently call *Lilienthal's type curves*. In Euclidean space, these curves are parametrized by a real constant since they are involutes of the tractrix. In terms of this constant, these curves were studied in [3] where a gallery of graphics was also provided. Our contribution in the analysis of these curves is that, in terms of the constant of integration d in (13), we describe how they meet thee rotation axis β and that we obtain information about the number of singularities.

Lilienthal's type curves in three space forms are the following:

1. *Arch type curves*. Concave graphs of a function defined in a bounded interval of the geodesic β. This function has a maximum where the curve meets the symmetry axis α.
2. *Fishtail type curves*. Non simple curves with one intersection point on the symmetry axis. They have exactly two peaks and a local minimum between the two peaks on the symmetry axis.
3. *Deltoid type curves*. Simple curves having the shape of a deltoid and with two peaks and one vertex at the intersection point between the geodesics α and β.

Journal Pre-proof

4. *Bridge type curves.* Simple curves having the shape of a bridge. The towers bend away from each other and finish at exactly two peaks. Moreover, the cable joining the towers reach a local minimum on the symmetry axis.

We now turn to those extremal curves that do not belong to Lilienthal's type family. These shapes appear only in $\mathbb{S}^2(\rho)$ for $\rho > 4\mu^2$ and in $\mathbb{H}^2(\rho)$: see figures 5 and 7, respectively. We distinguish between the cases of $\mathbb{S}^2(\rho)$ and $\mathbb{H}^2(\rho)$. For each case, we introduce the next terminology:

1. Curves in $\mathbb{S}^2(\rho)$.
   (a) *Anti-deltoid type curves.* Non simple curves having three vertices, where one of the vertices is located at the intersection point between α and β. The opposite segment to this vertex gives one turn around the north pole before closing.
   (b) *Anti-fishtail type curves.* Non simple curves having the fishtail type shape, but between the two peaks the curve turns around the north pole before closing.
   (c) *Anti-arch type curves.* Non simple curves having the arc type shape but before closing, the curve turns around the north pole in the segment between the two peaks .
   (d) *Anti-bridge type curves.* Simple curves having the shape of a bridge where the cable goes outside towers. Towers bend towards each other finishing exactly with two peaks and the cable gives more than half turn around the north pole.
   (e) *Cross type curves.* Non simple curves having some intersection points in the symmetry axis α. They have two peaks. After $x = 1$, the curves give as many turns around the north pole as needed before meeting the symmetry peak so they may have more than one self-intersection points.
   (f) *Braid type curves.* Complete curves with periodic curvature that roll up around a circle giving turns around the pole of the parametrization. When the curvature is not constant, the curve may close up if the integral in (31) is a multiple of 2π.
2. Curves in $\mathbb{H}^2(\rho)$.
   (a) *Hypercycle type curves.* Concave graphs of a function defined on the entire geodesic β and going further from it. This function has a minimum precisely where the curve meets the symmetry axis α. Hypercycles are included here as limit cases.
   (b) *Anchor type curves.* Simple curves having two disjoint symmetric components with respect to the geodesic α. Each component begins at the geodesic β and cross it one more time before they tend to α. It is the only one where x does not have critical points.

Journal Pre-proof

6.1. The Euclidean plane $\mathbb{R}^2$

After a dilation (Proposition 5.4), we assume that the index energy is $\mu = 1$ and thus, extremal curves are parametrized by the constant of integration d in (13). As we have announced, in terms of this constant d, we give an exhaustive description of the number of peaks of the curve and its intersection with the rotation axis.

Theorem 6.1. *Critical curves of* $\mathbf{\Theta}_1$ *in* $\mathbb{R}^2$ *form a one-parameter family of curves depending on the constant of integration* $d > 0$ *(see figure 3).*

1. *Case* $d \leq 1$. *The curves are of arch type.*
2. *Case* $d \in \left(1, \mathrm{e}^2/4\right)$. *The curves are of fishtail type.*
3. *Case* $d = \mathrm{e}^2/4$. *The curves are of deltoid type.*
4. *Case* $d > \mathrm{e}^2/4$. *The curves are of bridge type. If* $d \in (\mathrm{e}^2/4, \mathrm{e}^2)$, *the minimum of the cable is located in the positive part of the* α *axis; if* $d = \mathrm{e}^2$, *the point* $(0, 0)$ *represents this minimum, and; if* $d > \mathrm{e}^2$, *this minimum has negative component: in this case, the curves cut twice more times the* β *axis making an angle*

$$\theta_d = \arccos\left(-\frac{\mathrm{e}}{\sqrt{d}}\right)$$

that varies from π *to* $\pi/2$ *as* d *increases.*

Proof. From (24) and (26), a critical curve γ of $\mathbf{\Theta}_1$ is parametrized by

$$\gamma(x) = \frac{1}{\sqrt{d}}\left((1 - \log x)x, \int \frac{x \log x}{\sqrt{d - x^2}}\, dx\right).$$

The integral can be computed by quadratures obtaining

$$\gamma(x) = \frac{1}{\sqrt{d}}\left((1 - \log x)x, \sqrt{d - x^2}(\log x - 1) + \sqrt{d} \log\left(\frac{\sqrt{d} + \sqrt{d - x^2}}{x}\right)\right). \tag{32}$$

From (13), the function $F(x, 0) = x^2$ is increasing for any $x > 0$. This implies that the orbit $F(x, y) = d$ intersects the x-axis at $x_0 = \sqrt{d}$, that is, at the point of the orbit where the maximum value of x is reached.

We describe the type of curve only for values of x in the interval $(0, x_0)$, because the other half is obtained by symmetry (Proposition 5.2).

Journal Pre-proof

1. Case $x_0 = \sqrt{d} \leq 1$. The function $\psi(x)$ monotonically increases until ψ reaches its maximum at x_0, namely, $\psi(x_0) = 0$. At the same time, the function $(1 - \log x)x$ also increases from $x = 0$ to $x = x_0$. This implies that γ is of arch type.
2. Case $x_0 = \sqrt{d} \in (1, \mathrm{e}/2)$. The curve γ is defined by two parts. If $x \in (0, 1)$, the function $\psi(x)$ increases from a negative number ($\lim_{x\to 0} \psi(x) < 0$) to a positive one ($\lim_{x\to 1} \psi(x) > 0$). Therefore, there is a point where γ intersects the geodesic α and, from the symmetry, it represents a self-intersection point. This self-intersection point occurs far from the geodesic β because $\lim_{x\to 0} \psi(x) < 0$. On the other hand, the second part of γ corresponds with $x > 1$, where the function $\psi(x)$ decreases from $\lim_{x\to 1} \psi(x) > 0$ until the value $\psi(x_0) = 0$. The curve is of fishtail type.
3. Case $x_0 = \sqrt{d} = \mathrm{e}/2$. The behavior of γ is as in previous case but now the self-intersection point appears when $x \to 0$, that is, precisely at the intersection point with β. Thus the curve is of deltoid type.
4. Case $x_0 = \sqrt{d} > \mathrm{e}/2$. The curve is again defined in two parts as in item 2. There are no self-intersection points because $\lim_{x\to 0} \psi(x) > 0$ and $\psi(x)$ increases in the interval $(0, 1)$. For $x > 1$, the function $\psi(x)$ decreases until $\psi(x_0) = 0$. Now the value $(1 - \log x_0)x_0$ may be positive, zero or negative depending if $x_0 < \mathrm{e}$, $x_0 = \mathrm{e}$ or $x_0 > \mathrm{e}$, respectively. In each case, we obtain bridge type curves of the three different possible cases. Using $y = x_s$ and equation equation (13) for $\rho = 0$, the tangent vector field T of γ can be computed in terms of the arc-length parameter obtaining

$$T(s) = -\frac{1}{\sqrt{d}}\,(y \log x, x) = -\frac{1}{\sqrt{d}}\left(\sqrt{d - x^2}, x\right).$$

The angle $\theta(s)$ between $T(s)$ and any parallel curve to β satisfies $\cos\theta(s) = -x(s)/\sqrt{d}$, in particular, regular points representing the intersection of γ with the axis β appear when $x = \mathrm{e}$.

□

It is known that parallel curves to a given Lilienthal's curve are also Lilienthal's curves. This can be proved again by using that Lilienthal's curves are critical curves of $\mathbf{\Theta}_\mu$.

Proposition 6.2. *Let γ be a critical curve of $\mathbf{\Theta}_\mu$ in $\mathbb{R}^2$ for $d > 0$. Then, any parallel curve to γ of parameter $r > 0$ is a critical curve for $\mathbf{\Theta}_\mu$ for $e^{-2r\mu}d$.*

Proof. Consider a parallel curve to γ, $\widetilde{\gamma}(u(s)) = \gamma(s) + rN(s)$ where u denotes the arc-length parameter of $\widetilde{\gamma}$. Then, the curvature of $\widetilde{\gamma}$, $\widetilde{\kappa}$, can be computed in terms of the curvature of γ,

Journal Pre-proof

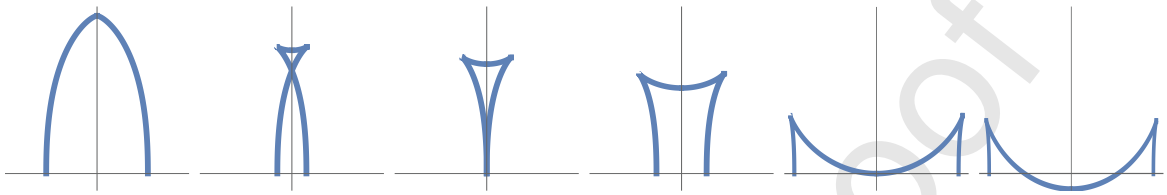


Figure 3: Critical curves of $\mathbf{\Theta}_1$ in the Euclidean plane depending on the constant $d > 0$. The geodesic β is represented by the horizontal axis, whereas α is the vertical one. From left to right: arch type ($d = 1$), fishtail type ($d = 1.55$), deltoid type ($d = \mathrm{e}^2/4$) and the three different possible cases of bridge type ($d = 2.5$, $d = \mathrm{e}^2$ and $d = 9$, respectively)

κ, obtaining

$$\widetilde{\kappa}(u(s)) = \frac{\kappa(s)}{1 - r\kappa(s)}\,.$$

Therefore, using it, we have

$$x(s;\mu) = e^{r\mu}\widetilde{x}(u(s);\mu)\,.$$

Moreover, since the arc-length parameter of $\widetilde{\gamma}$, u, and s are related by

$$u'(s) = 1 - r\kappa(s) = \frac{\log \widetilde{x}(u(s);\mu)}{\log x(s;\mu)}\,,$$

we obtain

$$y(s;\mu) = e^{r\mu}\widetilde{y}(u(s);\mu)u'(s) = e^{r\mu}\frac{\log \widetilde{x}(u(s);\mu)}{\log x(s;\mu)}\widetilde{y}(u(s);\mu)\,.$$

Finally, from above expressions of $x(s;\mu)$ and $y(s;\mu)$, and the definition of $F(x,y;\mu)$ in (13), we conclude that

$$F(x,y;\mu) = e^{2r\mu}F(\widetilde{x},\widetilde{y};\mu) = d\,,$$

proving the statement. □

6.2. The sphere $\mathbb{S}^2(\rho)$

Consider the 2-sphere $\mathbb{S}^2(\rho)$. From Proposition 5.4, it is enough to consider that the energy index μ is positive. In the following two theorems we sum up the classification of the critical curves of $\mathbf{\Theta}_\mu$ depending on the relation between ρ, μ and the value d_* defined after equation (26). The first result discusses the case $\rho \leq 4\mu^2$, where all shapes belong to Lilienthal's type family: see figure 4.

Theorem 6.3. (*Case $\rho \leq 4\mu^2$*) *Critical curves of $\mathbf{\Theta}_\mu$ in $\mathbb{S}^2(\rho)$ for $\rho \leq 4\mu^2$ represent a one-parameter family depending on the constant of integration $d > 0$ in* (13).

Journal Pre-proof

1. *Case $d \leq \rho + \mu^2$. The curves are of arch type.*
2. *Case $d \in \left(\rho + \mu^2, d_*\right)$. The curves are of fishtail type.*
3. *Case $d = d_*$. The curves are of deltoid type.*
4. *Case $d > d_*$. The curves are of bridge type. If $d \in \left(d_*, \mu^2 \mathrm{e}^2\right)$, the minimum of the cable is located in the upper halfsphere; if $d = \mu^2 \mathrm{e}^2$, the minimum is the point where the equator β intersects the symmetry axis; and, finally, if $d > \mu^2 \mathrm{e}^2$, the minimum is located in the lower halfsphere, what means that the curves meet twice the equator β.*

Proof. Let $\gamma \subset \mathbb{S}^2(\rho)$ be a critical curve of $\mathbf{\Theta}_\mu$ for $\mu > 0$ such that $\rho \leq 4\mu^2$. We know from Section 5 that if $\rho < 4\mu^2$ there are no singular points and if $\rho = 4\mu^2$, there is only one degenerate singular point P. In this cases, we have $F(x, 0) = x^2(\mu^2 + \rho(1 - \log x)^2)$ and the derivative of $F(x, 0)$ with respect to x is $2x(\mu^2 - \rho \log x(1 - \log x)) > 0$ since $\rho \leq 4\mu^2$. In particular, $F(x, 0)$ increases from 0 to $+\infty$. Thus, for a given $d > 0$, the orbit $F(x, y) = d$ always intersects the x-axis of the phase plane at exactly one point x_0, which is the maximum value of x in that orbit.

As in the Euclidean case, we are going to argue only for values $x \in (0, x_0)$, since the other half part of γ is completed by symmetry. We have the following cases:

1. Case $x_0 \leq 1$. Then $d \leq \rho + \mu^2$ and $\psi(x)$ increases until $x = x_0$, where $\psi(x_0) = 0$ and γ meets the symmetry axis. Since the function $(1 - \log x)x$ also increases, then γ is a curve of arch type.
2. Case $x_0 > 1$. Thus $d > \rho + \mu^2$ and there appear three subcases:
   (a) Subcase $d \in \left(\rho + \mu^2, d_*\right)$. Then $\lim_{x \to 0} \psi(x) < 0$ and $\psi(x)$ increases until $x = 1$. Since $\lim_{x \to 1} \psi(x) > 0$, there exists a self-intersection point for some $x \in (0, 1)$. For $x > 1$, the function $\psi(x)$ decreases until reaching the value x_0, where $\psi(x_0) = 0$. Since the function $(1 - \log x)x$ increases if $x \in (0, 1)$ and decreases for $x \in (1, x_0)$, it gives rise to critical curves of fishtail type.
   (b) Subcase $d = d_*$. From the definition of d_*, we have that $\lim_{x \to 0} \psi(x) = 0$ and arguing as above we conclude that γ is a curve of deltoid type.
   (c) Subcase $d > d_*$. Then $\lim_{x \to 0} \psi(x) > 0$ and $\psi(x)$ increases for $x \in (0, 1)$. For these values of x, the function $(1 - \log x)x$ also increases. However, in the interval $(1, x_0)$, the functions $\psi(x)$ and $(1 - \log x)x$ decrease. Exactly, at $x = x_0$, the function $\psi(x)$ vanishes which means that γ meets the geodesic α. Moreover, if $d < \mu^2 \mathrm{e}^2$, this contact point is located in the upper halfsphere; if $d = \mu^2 \mathrm{e}^2$, $\gamma(x_0)$ is the intersection point between α and β, and; for $d > \mu^2 \mathrm{e}^2$, the point $\gamma(x_0)$ is located in the lower halfsphere. Thus there appear the three different possibilities of bridge type curves.

Journal Pre-proof

□

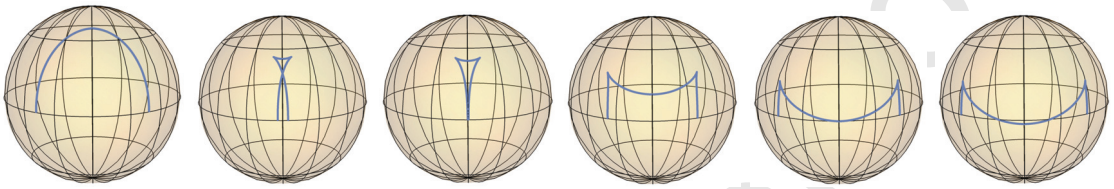


Figure 4: Critical curves of $\mathbf{\Theta}_\mu$ in $\mathbb{S}^2(\rho)$ for $\rho \leq 4\mu^2$. Here $\rho = \mu = 1$. From left to right: arch type ($d = 1.3$), fishtail type ($d = 2.5$), deltoid type ($d = d_* \simeq 2.64$) and the three different cases of bridge type ($d = 4.5$, $d = \mu^2 \mathrm{e}^2$ and $d = 8$, respectively)

We now study the case $\rho > 4\mu^2$, which has not a counterpart in Euclidean plane, being some of these shapes of non-Lilienthal's type curves: see figure 5.

Theorem 6.4. (*Case* $\rho > 4\mu^2$) *Critical curves of* $\mathbf{\Theta}_\mu$ *in* $\mathbb{S}^2(\rho)$ *for* $\rho > 4\mu^2$ *are of arch type, fishtail type, deltoid type, bridge type, cross type, anti-deltoid type, anti-fishtail type, anti-arch type and anti-bridge type. Moreover, we also have braid type curves whenever* $d \in \left(\rho x_+^2 \log x_-, \rho x_-^2 \log x_+\right)$.

Proof. Assume that γ is a critical curve of $\mathbf{\Theta}_\mu$ in $\mathbb{S}^2(\rho)$ for $\rho > 4\mu^2 > 0$. From Section 5, there are two singular points $P_+ = (x_+, 0)$ and $P_- = (x_-, 0)$ with $x_- < x_+$, representing a centre and a saddle point, respectively. Arguing as in the beginning of the proof of Theorem 6.3, the function $F(x, 0)$ increases if $x \in (0, x_-)$, then decreases if $x \in (x_-, x_+)$ and finally increases if $x > x_+$. Therefore, for each $d > 0$, the associated orbit $F(x, y) = d$ meets either once or thrice the x-axis (see Proposition 5.5 for details).

Suppose that the orbit $F(x, y) = d$ has only one intersection point with the x-axis at $x = x_0$. As before, we observe that x_0 represents the maximum value of x in that orbit, which happens for any $d \in \left(0, \rho x_+^2 \log x_-\right) \cup \left(\rho x_-^2 \log x_+, \infty\right)$. In this setting we have the same type of critical curves of Theorem 6.3, that is, arch type, fishtail type, deltoid type and bridge type curves. Moreover, whenever d is close enough to $\rho x_-^2 \log x_+$, some different kind of critical curves may also appear for some prescribed values of μ. We consider that $|\lim_{x\to 0} \psi(x)|$ is bigger than $\pi/4$. This means that the critical curve γ begins and ends in the geodesic β at a distance bigger than a half round, what is more, the value $|\lim_{x\to 0} \psi(x)|$ grows as d decreases. Then, if the points where γ is not defined (the vertices appearing with $x = 1$) are far from each other we have, from a similar argument as in Theorem 6.3, anti-bridge type curves. Now, if the points corresponding with $x = 1$ touch themselves we are dealing with anti-arch type curves. Then, for smaller values of d, the endpoints of γ are close enough (after one turn in

Journal Pre-proof

the sphere) so that the towers have a self-intersection point, that is, γ is of anti-fishtail type. Following with this argument, there may exist a value of d where the towers meet precisely at the intersection point between α and β. This case is of anti-deltoid type. Finally, for smaller values of d the distance between the endpoints grows (linearly measured) and the critical curve may give as many turns around the north pole as needed. These curves have at least one self-intersection point in the symmetry axis α so they are of cross type.

The other possibility is that $d \in \left(\rho x_+^2 \log x_-, \rho x_-^2 \log x_+\right)$ so the orbit has three points of intersection with the x-axis. If x_0 denotes the first one and, whenever $x < x_0$ we can argue as in the proof of Theorem 6.3 obtaining that this part of γ has been previously described. However, for these values of d we also have another two intersection points which appear around the critical point x_+. Moreover, since P_+ is a centre, we have that the associated orbits are closed (Proposition 5.5). Thus, the curve γ has periodic curvature and γ goes around the north pole being a curve of braid type. In general, the curve γ is not closed because it is necessary that the integral in (31) is a multiple of 2π. □

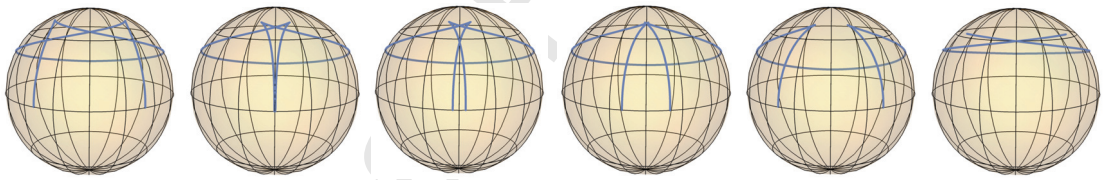


Figure 5: Non-Lilienthal's type critical curves of $\mathbf{\Theta}_\mu$ in $\mathbb{S}^2(\rho)$ for $\rho > 4\mu^2$. Here $\rho = 1$ and $\mu = 0.45$. From left to right: cross type ($d = 1.29$), anti-deltoid type ($d \simeq 1.32$), anti-fishtail type ($d = 1.33$), anti-arch type ($d \simeq 1.35$), anti-bridge type ($d = 1.4$) and one period of the curvature of a braid type ($d = 1.23$)

6.3. The hyperbolic plane $\mathbb{H}^2(\rho)$

We classify the critical curves for $\mathbf{\Theta}_\mu$ in $\mathbb{H}^2(\rho)$ when the rotational surfaces are of spherical type, which correspond with the constant d in (13) being positive. We know from Section 5 that in the hyperbolic plane $\mathbb{H}^2(\rho)$ there are two singular points $P_+ = (x_+, 0)$ and $P_- = (x_-, 0)$ in the phase plane, $x_+ < 1 < x_-$, being a centre and an unstable saddle point, respectively: see figure 1, right. The classification result is the following: see figures 6 and 7.

Theorem 6.5. *Let $d > 0$. Critical curves of $\mathbf{\Theta}_\mu$ in $\mathbb{H}^2(\rho)$ for $d \in \left(0, \rho x_-^2 \log x_+\right)$ are arch type curves, fishtail type curves, deltoid type curves, bridge type curves and hypercycle type curves. Moreover, whenever $d > \rho x_-^2 \log x_+$ we also have anchor type curves.*

Journal Pre-proof

Proof. Let γ be a critical curve of $\mathbf{\Theta}_\mu$ in $\mathbb{H}^2(\rho)$ with $d > 0$. Since $\rho < 0$, the function $F(x, 0)$ decreases while $x < x_+$. The point $P_+ = (x_+, 0)$ is a centre and, thus $F(x, 0)$ reaches a local minimum at $x = x_+$. On the other hand, in the interval (x_+, x_-) the function $F(x, 0)$ increases from the number $F(x_+, 0) < 0$ to the $F(x_-, 0) = \rho x_-^2 \log x_+ > 0$, hence $F(x, 0)$ reaches its local maximum at x_+. Finally, the function $F(x, 0)$ is decreasing for $x > x_+$.

Therefore, the orbit $F(x, y) = d$ meets the x-axis if and only if $d \leq \rho x_-^2 \log x_+$ and, in this case, the orbit intersects the x-axis at exactly two points. The first intersection point corresponds with a point $x = x_0$, and $x_0 < x_-$, while the second one appears for some $x = x_0'$, and $x_0' > x_-$. Since P_- is a saddle point, the orbit $F(x, y) = d$ has two connected components, each of them corresponding with one intersection point. For the connected component corresponding with a point $x_0 < x_-$, we can argue as in $\mathbb{R}^2$ and $\mathbb{S}^2(\rho)$ obtaining curves of arch type, of fishtail type, of deltoid type and of bridge type (Lilienthal's type curves).

Consider now the connected component corresponding with the cut $x_0' > x_-$. For this case, we have that the functions $\psi(x)$ and $(1-\log x)x$ always decrease. Moreover, $\lim_{x\to\infty}(1-\log x)x = -\infty$, which means that γ is defined on the entire geodesic β. Therefore γ is a curve of hypercycle type.

Finally, we consider that $d > \rho x_-^2 \log x_+$. In this case, the associated orbits do not meet the x-axis and are defined for all $x > 0$ and $x \neq 1$. As $x \to 0$, the critical curve tends to meet β. Thus in the interval $x \in (0, 1)$, the functions $\psi(x)$ and $(1 - \log x)x$ increase. The point where $x = 1$ represents a peak and γ is not defined there. After that, both functions decrease. Since γ is defined in $(0, 1) \cup (1, \infty)$, there are points where γ intersects β (corresponding with $x = \mathrm{e}$, see Proposition 5.3). Moreover, γ never intersects the symmetry axis α since $\psi(x) > 0$ for all possible values of x. However, $\lim_{x\to\infty} \psi(x) = 0$, that is, γ tends to the symmetry axis but it has two disjoint parts.. Thus, we get anchor type curves. □

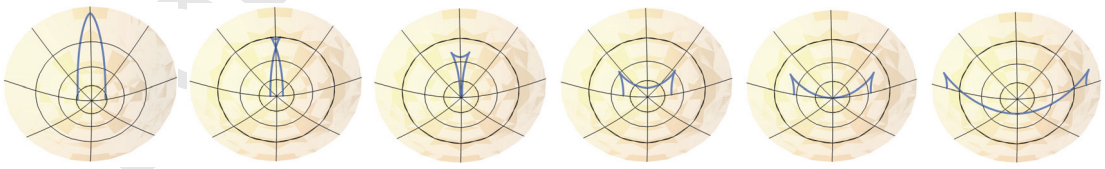


Figure 6: Lilienthal's type critical curves of $\mathbf{\Theta}_\mu$ in $\mathbb{H}^2(\rho)$ depending on the constant $d > 0$. Here $\rho = -1$ and $\mu = 1.25$. From left to right: arch type ($d = 0.35$), fishtail type ($d = 1.5$), deltoid type ($d = d_* \simeq 2.5$) and bridge type curves of the three possible cases ($d = 8$, $d = \mu^2\mathrm{e}^2$ and $d = 20$, respectively). Here we look $\mathbb{H}^2(\rho)$ as the upper sheet of the hyperboloid model viewed from the top of the x_3-axis

Remark 6.6. *Observe that for a given triple (ρ, μ, d) there may exist more than one critical curve of $\mathbf{\Theta}_\mu$ in $\mathbb{M}^2(\rho)$ for $d > 0$ (see, for instance, figures 6 and 7 for $d = 20$). Indeed, the*

Journal Pre-proof

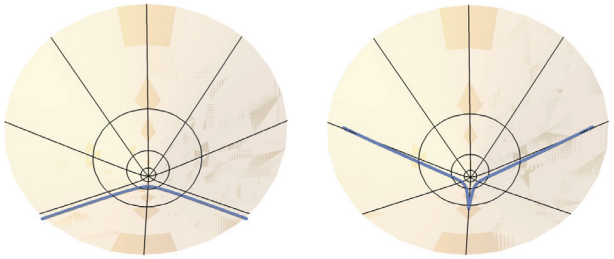


Figure 7: Non-Lilienthal's type critical curves of $\mathbf{\Theta}_\mu$ in $\mathbb{H}^2(\rho)$. The first one is an hypercycle type curve for $d < \rho x_-^2 \log x_+$ (here, $\rho = -1$, $\mu = 1.25$ and $d = 20$); while, the second one corresponds with $d > \rho x_-^2 \log x_+$ and is an anchor type curve (here, $\rho = -1$, $\mu = 1.25$ and $d = 36$)

number of critical curves is related with the number of connected components of the associated orbit in the phase space $F(x, y) = d$, (13). *Therefore, we have that in* $\mathbb{R}^2$ *and in* $\mathbb{S}^2(\rho)$ *for* $\rho \leq 4\mu^2$ *there exists only one critical curve for* $d > 0$. *On the other hand, in* $\mathbb{S}^2(\rho)$ *for* $\rho > 4\mu^2$ *there are exactly two if* $d \in \left(\rho x_+^2 \log x_-, \rho x_-^2 \log x_+\right)$*; otherwise, there exists only one. Similarly, in* $\mathbb{H}^2(\rho)$*, there is only one, but for* $d < \rho x_-^2 \log x_+$. *For these values there are two.*

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