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Center conditions of a particular polynomial differential system with a nilpotent singularity

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ABSTRACT

In this work we study the center conditions of a particular polynomial differential system with a nilpotent singularity using a new proposed algorithm. This problem was initially studied in [19] where some center conditions were found using the Cherkas' method and its full characterization was established as an open problem.

1. Introduction and statement main results

The case when a nilpotent singularity of a vector field is a center was theoretically characterized in [4,11]. In fact, any nilpotent system with a center is orbitally equivalent to a time-reversible, more specifically the phase portrait is symmetric respect to the y-axis after a change in the time and state variables, see also [1,13,18]. However, there are few families of nilpotent vector fields where the center conditions are known, see for instance [5–7]. Indeed the centers of the more simple semi-quasi-homogeneous systems, that is, systems that are sum of two quasi-homogeneous vector fields are not yet completely classified.

In [12] was studied the analytically integrable centers of the semi-quasi-homogeneous nilpotent vector fields of the type $(1, k)$ with $k \geq 2$

$$\dot{x} = -y + a_1 xy + a_2 x^{k+1}, \quad \dot{y} = x^{2k-1} + b_1 y^2 + b_2 x^k y + b_3 x^{2k}. \tag{1.1}$$

We recall that not all the nilpotent centers are locally analytic integrable, see [4,13,17]. In [19] the center conditions of system (1.1) were studied via Cherkas' method, see [14,17,20]. The authors distinguished two

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cases, k even and k odd, that we now write as the following differential systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x^{4q+3} \end{pmatrix} + \begin{pmatrix} a_1xy + a_2x^{2q+3} \\ b_1y^2 + b_2x^{2q+2}y + b_3x^{4q+4} \end{pmatrix}, \tag{1.2}$$

with $q \in \mathbb{N} \cup \{0\}$, and

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x^{4q+1} \end{pmatrix} + \begin{pmatrix} a_1xy + a_2x^{2q+2} \\ b_1y^2 + b_2x^{2q+1}y + b_3x^{4q+2} \end{pmatrix}, \tag{1.3}$$

with $q \in \mathbb{N}$. We remark that the center problem for these nilpotent families is as important as the nondegenerate center problem of the semi-homogeneous vector field formed by a linear component of type $(-y, x)$ and a quadratic component or a homogeneous cubic component, studied in [10,24] respectively. The family (1.2) is completely solved in [19], that is, all the centers are known. However, the classification for the family (1.3) is still open.

In this work we develop an orbital reversibility algorithm to obtain the orbital reversible obstructions to have a center for any vector field with a nilpotent singularity. We complete the center conditions for the family (1.3) for $q = 1$. In addition, a new center case is obtained for $q \geq 2$, and it is conjectured that all the centers given in this work, for the case $q \geq 2$, are the unique ones.

Notice that if $\mathbf{F}$ is the vector field associated to system (1.3) then $\mathbf{F} = \mathbf{F}_{2q} + \mathbf{F}_{2q+1}$ is a sum of two quasi-homogeneous vector fields of type $\mathbf{t} = (1, 2q + 1)$ and degree $2q$ and $2q + 1$, respectively. The center problem for the above family is the simplest unsolved problem with arbitrary q. We recall that the analytic integrability and the center problem for system (1.3) has been studied but not solved in [12] and later in [19]. Next result completes the analytical integrability problem for system (1.3).

Theorem 1.1. *System (1.3) for $q \geq 1$ is analytically integrable if, and only if, one of the following conditions is verified:*

a) $a_2 = b_2 = 0$.
b) $a_1 + 2b_1 = b_2 + 2(q + 1)a_2 = 0$.

Next result gives center conditions of system (1.3) for $q \geq 2$.

Theorem 1.2. *The origin of system (1.3) for $q \geq 2$ is a center if one of the following conditions is satisfied:*

(a) $a_2 = b_2 = 0$.
(b) $a_1 + 2b_1 = b_2 + 2(q + 1)a_2 = 0$.
(c) $a_1 = b_1 = b_3 = 0$.
(d) $a_2 = b_1 + a_1 = b_3 - a_1 = 0$.
(e) $a_2 = b_1 - 2(q + 1)a_1 = b_3 + 2a_1 = 0$.
(f) $a_2 = 2b_1 - (2q + 1)a_1 = 2b_3 + a_1 = 0$.
(g) $b_3 = b_2 - (2q + 1)a_2 = b_1 - (2q + 1)a_1 = 0$.

The cases **(a)**, **(b)**, **(c)**, **(d)**, **(e)**, **(f)** correspond to the cases **(i)**, **(v)**, **(vi)**, **(ii)**, **(iii)**, **(iv)** of [19], respectively. However the case **(g)** is a new case not found in the previous work [19].

We have shown that, for $2 \leq q \leq 100$, the only centers of system (1.3) are those described in Theorem 1.2, so for these values of q these cases are necessary and sufficient conditions. We conjecture that the same statement happens for all $q > 100$.

The following result characterizes the center conditions of system (1.3) for $q = 1$.

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Theorem 1.3. *The origin of system (1.3) for $q=1$ is a center if, and only if, one of the following conditions is satisfied: Cases* **(a)-(g)** *for $q=1$ of Theorem 1.2 and*

(h) $b_2-35a_2=3b_1-5a_1=9b_3+4a_1=405a_2^2-a_1^2=0$.

Case **(h)** is a new case only for $q=1$. Moreover, it is also a center case not found in [19].

2. Preliminary results

Before of showing our results, we recall the following concepts and definitions. Given $\mathbf{t}=(t_1,t_2)$ with t_1 and t_2 natural numbers without common factors, a function f of two variables is quasi-homogeneous of type $\mathbf{t}$ and degree k if $f(\varepsilon^{t_1}x,\varepsilon^{t_2}y)=\varepsilon^k f(x,y)$. The vector space of quasi-homogeneous polynomials of type $\mathbf{t}$ and degree k will be denoted by $\mathcal{P}_k^{\mathbf{t}}$. A vector field $\mathbf{F}=(P,Q)^T$ is quasi-homogeneous of type $\mathbf{t}$ and degree k if $P\in\mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $Q\in\mathcal{P}_{k+t_2}^{\mathbf{t}}$. We will denote $\mathcal{Q}_k^{\mathbf{t}}$ the vector space of the quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree k.

Any vector field can be expanded into quasi-homogeneous terms of type $\mathbf{t}$ of successive degrees. Thus, the vector field $\mathbf{F}$ can be written in the form $\mathbf{F}=\mathbf{F}_r+\mathbf{F}_{r+1}+\cdots$ for some $r\in\mathbb{Z}$, where $\mathbf{F}_j=(P_{j+t_1},Q_{j+t_2})^T\in\mathcal{Q}_j^{\mathbf{t}}$ and $\mathbf{F}_r\not\equiv\mathbf{0}$. If we select the type $\mathbf{t}=(1,1)$, we are using in fact the Taylor expansion, but in general, each term in the above expansion involves monomials with different degrees.

We will denote by $\mathbf{D}_0=(t_1x,t_2y)^T\in\mathcal{Q}_0^{\mathbf{t}}$ (a dissipative quasi-homogeneous vector field) and by $\mathbf{X}_h=(-\partial h/\partial y,\partial h/\partial x)^T$ (the Hamiltonian vector field associated to the polynomial h). If $h\in\mathcal{P}_{r+|\mathbf{t}|}^{\mathbf{t}}$ then $\mathbf{X}_h\in\mathcal{Q}_r^{\mathbf{t}}$ where $|\mathbf{t}|=t_1+t_2$. Moreover, it is proved that every $\mathbf{F}_j\in\mathcal{Q}_j^{\mathbf{t}}$ can be expressed as

$$\mathbf{F}_j=\mathbf{X}_{h_j}+\mu_j\mathbf{D}_0 \tag{2.4}$$

with $h_j=(\mathbf{D}_0\wedge\mathbf{F}_j)/(j+|\mathbf{t}|)$ and $\mu_j=\operatorname{div}(\mathbf{F}_j)/(j+|\mathbf{t}|)$, where $\mathbf{D}_0\wedge\mathbf{F}_j:=t_1xQ_{j+t_2}-t_2yP_{j+t_1}\in\mathcal{P}_{j+|\mathbf{t}|}^{\mathbf{t}}$ and $\operatorname{div}(\mathbf{F}_j)\in\mathcal{P}_j^{\mathbf{t}}$ is the divergence of $\mathbf{F}_j$, see [3].

We define the vector spaces.

$$\begin{aligned}
\widetilde{\mathcal{Q}}_k^{\mathbf{t}}&=\left\{(P,Q)^T\in\mathcal{Q}_k^{\mathbf{t}}:P(-x,y)=P(x,y),Q(-x,y)=-Q(x,y)\right\},\\
\overline{\mathcal{Q}}_k^{\mathbf{t}}&=\left\{(P,Q)^T\in\mathcal{Q}_k^{\mathbf{t}}:P(x,y)=-P(x,y),Q(-x,y)=Q(x,y)\right\},\\
\widetilde{\mathcal{P}}_k^{\mathbf{t}}&=\left\{\mu\in\mathcal{P}_k^{\mathbf{t}}:\mu(-x,y)=-\mu(x,y)\right\},\\
\overline{\mathcal{P}}_k^{\mathbf{t}}&=\left\{\mu\in\mathcal{P}_k^{\mathbf{t}}:\mu(-x,y)=\mu(x,y)\right\}
\end{aligned}$$

$\widetilde{\mathcal{Q}}_k^{\mathbf{t}}$, $\overline{\mathcal{Q}}_k^{\mathbf{t}}$ are the vector field in $\mathcal{Q}_k^{\mathbf{t}}$ such that the differential systems associated to these vector field are invariant to $(x,y,t)\to(-x,y,-t)$, or invariant to $(x,y,t)\to(-x,y,t)$, respectively. It is verified that $\mathcal{Q}_k^{\mathbf{t}}=\widetilde{\mathcal{Q}}_k^{\mathbf{t}}\bigoplus\overline{\mathcal{Q}}_k^{\mathbf{t}}$ and $\mathcal{P}_k^{\mathbf{t}}=\widetilde{\mathcal{P}}_k^{\mathbf{t}}\bigoplus\overline{\mathcal{P}}_k^{\mathbf{t}}$.

The vector field transformed of $\mathbf{F}$ by means of the change of variables with generators (spatial $\mathbf{U}$ and temporal μ) is given by (see [15])

$$\mathbf{U}_{**}(\mu\mathbf{F}):=\mu\mathbf{F}+[\mu\mathbf{F},\mathbf{U}]+\tfrac{1}{2!}\left[[\mu\mathbf{F},\mathbf{U}],\mathbf{U}\right]+\tfrac{1}{3!}\left[\left[[\mu\mathbf{F},\mathbf{U}],\mathbf{U}\right],\mathbf{U}\right]+\cdots.$$

Next result provides a simplified normal form which uses reduced changes of variables and time reparametrization that are convenient for calculating necessary conditions of centers for systems with first quasi-homogeneous component of the type $\mathbf{F}_{2q}=(y,-x^{4q+1})^T$, $q\in\mathbb{N}$.

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Proposition 2.4. *Let* $\mathbf{F}=\sum_{j\geq 2q}\mathbf{F}_j$ *with* $\mathbf{F}_j\in\mathcal{Q}_j^{(1,2q+1)}$, $\mathbf{F}_{2q}=(y,-x^{4q+1})^T\in\widetilde{\mathcal{Q}}_{2q}^{(1,2q+1)}$ *then it is possible to choose adequately* $\widetilde{\mathbf{U}}=\sum_{j\geq 1}\widetilde{\mathbf{U}}_j$ *with* $\widetilde{\mathbf{U}}_j\in\widetilde{\mathcal{Q}}_j^{\mathbf{t}}$ *and* $\tilde{\mu}=1+\sum_{j\geq 1}\tilde{\mu}_j$, *with* $\tilde{\mu}_j\in\widetilde{\mathcal{P}}_j^{\mathbf{t}}$, $\mathbf{t}=(1,2q+1)$, *such that*

$$\widetilde{\mathbf{U}}_{**}\left(\tilde{\mu}\mathbf{F}\right) \quad = \quad \mathbf{F}_{2q}+\sum_{i\geq 0}\left(\widetilde{\mathbf{G}}_{2(q+i)+1}+\left(\widetilde{\mathbf{G}}_{2(q+i+1)}+\alpha_{2(q+i+1)}x^{2(q+i+1)}\mathbf{D}_0\right)\right), \tag{2.5}$$

where $\mathbf{D}_0=(x,(2q+1)y)^T$ *and* $\widetilde{\mathbf{G}}_j\in\widetilde{\mathcal{Q}}_j^{\mathbf{t}}$.

Proof. By [8, Proposition 12-15] a formally orbital equivalent normal form of system $\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x})$ is $\dot{\mathbf{x}}=\mathbf{G}(\mathbf{x}):=\mathbf{F}_{2q}+\sum_{i>2q}\mathbf{G}_i$, with $\mathbf{G}_i\in\mathrm{Cor}\left(\mathcal{L}_i\right)$ (complementary space to the range of $\mathcal{L}_i$), being

$$\begin{array}{rclcrcl}
\mathcal{L}_i:\mathcal{Q}_{i-2q}^{\mathbf{t}}\times\mathrm{Cor}\left(\ell_{i-2q}\right) & \rightarrow & \mathcal{Q}_i^{\mathbf{t}} & \qquad & \ell_i:\mathcal{P}_{i-2q}^{\mathbf{t}} & \rightarrow & \mathcal{P}_i^{\mathbf{t}}\\
(\mathbf{U}_{i-2q},\mu_{i-2q}) & & -\left[\mathbf{F}_{2q},\mathbf{U}_{i-2q}\right]-\mu_{i-2q}\mathbf{F}2q & & \mu_{i-2q} & & \nabla\mu_{i-2q}\cdot\mathbf{F}2q
\end{array}$$

Moreover $\mathbf{G}=\mathbf{F}_{2q}+\sum_{i>2q}\nu_i\mathbf{D}_0$ with $\nu_i\in\mathrm{Cor}\left(\ell_i\right)$ (complementary space to the range of ℓ_i). Taking into account that $\mathbf{F}_{2q}=\mathbf{X}_h$ with $h=-\frac{1}{2}y^2-\frac{1}{4q+2}x^{4q+2}$, we have that $\nu_i=\beta_j^{(l)}x^jh^l$ if $j\neq 4q+1$ where $j=i-l(4q+2)$ and $l=\left\lfloor\frac{i}{2(2q+1)}\right\rfloor$ ($\lfloor\cdot\rfloor$ is floor function), and $\nu_i=0$ if $j=4q+1$. It is easy to prove that another subspace complementary to the range of ℓ_i is given by $\nu_i=\alpha_ix^i$ if $\mathrm{mod}(i,4q+2)\neq 4q+1$ and $\nu_i=0$ otherwise.

It is possible to choose a complementary subspace to the range of ℓ_k for all k, $\mathrm{Cor}\left(\ell_k\right)$ of the form $\mathrm{Cor}\left(\ell_k\right)=\widetilde{\mathrm{Cor}\left(\ell_k\right)}\bigoplus\overline{\mathrm{Cor}\left(\ell_k\right)}$, where $\widetilde{\mathrm{Cor}\left(\ell_k\right)}=\mathrm{Cor}\left(\ell_k\right)\cap\widetilde{\mathcal{P}}_k^{\mathbf{t}}$ and $\overline{\mathrm{Cor}\left(\ell_k\right)}=\mathrm{Cor}\left(\ell_k\right)\cap\overline{\mathcal{P}}_k^{\mathbf{t}}$. Therefore $\mathcal{Q}_{i-2q}^{\mathbf{t}}\times\mathrm{Cor}\left(\ell_{i-2q}\right)=\widetilde{\mathcal{Q}}_{i-2q}^{\mathbf{t}}\times\widetilde{\mathrm{Cor}\left(\ell_{i-2q}\right)}\bigoplus\overline{\mathcal{Q}}_{i-2q}^{\mathbf{t}}\times\overline{\mathrm{Cor}\left(\ell_{i-2q}\right)}$.

With this choice we consider now the operator $\mathcal{L}_i:\widetilde{\mathcal{Q}}_{i-2q}^{\mathbf{t}}\times\widetilde{\mathrm{Cor}\left(\ell_{i-2q}\right)}\bigoplus\bar{\mathcal{Q}}_{i-2q}^{\mathbf{t}}\times\overline{\mathrm{Cor}\left(\ell_{i-2q}\right)}\rightarrow\bar{\mathcal{Q}}_i^{\mathbf{t}}\bigoplus\widetilde{\mathcal{Q}}_i^{\mathbf{t}}$ whose matrix can be expressed as

$$\left(\begin{array}{c|c||c}
\bullet & \mathbf{0} & \overline{\mathcal{Q}}_i^{\mathbf{t}}\\
\hline
\mathbf{0} & \bullet & \widetilde{\mathcal{Q}}_i^{\mathbf{t}}\\
\hline
\widetilde{\mathcal{Q}}_{i-2q}^{\mathbf{t}}\times\widetilde{\mathrm{Cor}\left(\ell_{i-2q}\right)} & \overline{\mathcal{Q}}_{i-2q}^{\mathbf{t}}\times\overline{\mathrm{Cor}\left(\ell_{i-2q}\right)} &
\end{array}\right)$$

where $\bullet$ means a non-null submatrix, since $\mathbf{F}_{2q}\in\widetilde{\mathcal{Q}}_{2q}^{\mathbf{t}}$ and hence the following properties are verified: $\left[\mathbf{F}_{2q},\widetilde{\mathbf{U}}_{i-2q}\right]\in\overline{\mathcal{Q}}_i^{\mathbf{t}}$, $\left[\mathbf{F}_{2q},\overline{\mathbf{U}}_{i-2q}\right]\in\widetilde{\mathcal{Q}}_i^{\mathbf{t}}$ and if $\widetilde{\mu}_{i-2q}\in\widetilde{\mathrm{Cor}\left(\ell_{i-2q}\right)}$, $\overline{\mu}_{i-2q}\in\overline{\mathrm{Cor}\left(\ell_{i-2q}\right)}$, then $\widetilde{\mu}_{2-2q}\mathbf{F}_{2q}\in\overline{\mathcal{Q}}_i^{\mathbf{t}}$ and $\overline{\mu}_{2-2q}\mathbf{F}_{2q}\in\widetilde{\mathcal{Q}}_i^{\mathbf{t}}$.

Therefore if we define the reduced operator

$$\begin{array}{rcl}
\bar{\mathcal{L}}_i:\widetilde{\mathcal{Q}}_{i-2q}^{\mathbf{t}}\times\widetilde{\mathrm{Cor}\left(\ell_{i-2q}\right)} & \rightarrow & \bar{\mathcal{Q}}_i^{\mathbf{t}}\\
(\tilde{\mathbf{U}}_{i-2q},\tilde{\mu}_{i-2q}) & & -\left[\mathbf{F}_{2q},\tilde{\mathbf{U}}_{i-2q}\right]-\tilde{\mu}_{i-2q}\mathbf{F}2q
\end{array}$$

the matrix of the operator $\bar{\mathcal{L}}_i$ is a submatrix of the matrix of $\mathcal{L}_i$ and the formally orbital equivalent normal form using reduced spatial generators $\widetilde{\mathbf{U}}_k\in\widetilde{\mathcal{Q}}_k^{\mathbf{t}}$ and reduced temporal generators $\widetilde{\mu}_k\in\widetilde{\mathrm{Cor}\left(\ell_k\right)}$ is given by

$$\widetilde{\mathbf{U}}_{**}\left(\widetilde{\mu}\mathbf{F}\right) \quad = \quad \mathbf{F}_{2q}+\sum_{i\geq 0}\left(\widetilde{\mathbf{G}}_{2(q+i)+1}+\left(\widetilde{\mathbf{G}}_{2(q+i+1)}+\alpha_{2(q+i+1)}x^{2(q+i+1)}\mathbf{D}_0\right)\right)$$

and the result follows. $\square$

Next result characterizes the centers of system $\dot{\mathbf{x}}=\mathbf{F}_{2q}(\mathbf{x})+\cdots$.

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Theorem 2.5. *The origin of system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ where $\mathbf{F} = \sum_{j\geq 2q} \mathbf{F}_j$ with $\mathbf{F}_j \in \mathcal{Q}_j^{(1,2q+1)}$, $\mathbf{F}_{2q} = (y, -x^{4q+1})^T$, is a center if, and only if, the coefficients $\alpha_{2(q+i+1)}$ of equation (2.5) are null for all $i \geq 0$.*

Proof. By Proposition 2.4 the origin of system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is a center if, and only if, the origin of system $\dot{\mathbf{x}} = \widetilde{\mathbf{U}}_{**}(\widetilde{\mu}\mathbf{F})(\mathbf{x})$ defined in (2.5) is a center. The sufficiency is trivial. We prove the necessity by reduction to absurdity. Assume that not all the coefficients $\alpha_{2(q+i+1)}$ be null and take $N = \min\left\{i \in \mathbb{N} : \alpha_{2(q+i+1)} \neq 0\right\}$, that is, $\alpha_{2(q+i+1)} = 0$ for all $i < N$ and $\alpha_{2(q+N+1)} \neq 0$. Let $\mathbf{F}$ be the vector field associated to the system (2.5) and $\mathbf{G} = \mathbf{F}_{2q} + \sum_{i\geq 0}\sum_{j=1}^{2} \widetilde{\mathbf{G}}_{2q+2i+j}$ then $\mathbf{F} = \mathbf{G} + \nu\mathbf{D}_0$ with $\nu = \sum_{i\geq N} \alpha_{2(q+i+1)} x^{2(q+i+1)}$. Taking the vector field $\mathbf{G}$ as center, considering the decomposition (2.4) we have $\widetilde{\mathbf{G}}_{2q+2i+j} = \mathbf{X}_{h_{i,j}} + \lambda_{i,j}\mathbf{D}_0$ $j = 1, 2$, where $h_{i,j} \in \mathcal{P}^{\mathbf{t}}_{4q+2i+j+2}$ and $\lambda_{i,j} \in \mathcal{P}^{\mathbf{t}}_{2q+2i+j}$. Taking into account that $\mathbf{F}_{2q} = \mathbf{X}_h$ with $h = -\frac{1}{2}y^2 - \frac{1}{4q+2}x^{4q+2}$, we get:

$$\begin{aligned}\mathbf{G}\cdot\mathbf{F}^{\perp} = \mathbf{G}\wedge\nu\mathbf{D}_0 &= \nu\left(\mathbf{F}_{2q} + \sum_{i\geq 0}\sum_{j=1}^{2}\tilde{\mathbf{G}}_{2q+2i+j}\right)\wedge\mathbf{D}_0 = \nu\left(\mathbf{X}_h + \sum_{i\geq 0}\sum_{j=1}^{2}\mathbf{X}_{h_{i,j}} + \lambda_{i,j}\mathbf{D}_0\right)\wedge\mathbf{D}_0 \\ &= x^{2(q+N+1)}\left(\alpha_{2(q+N+1)} + \sum_{l>0}\alpha_{2(q+N+1+l)}x^{2l}\right)\left((4q+2)h + \sum_{i\geq 0}\sum_{j=1}^{2}(4q+2i+j)h_{i,j}\right)\end{aligned}$$

In this way for all (x, y) in a neighborhood of the origin with $x \neq 0$, we have that $\mathrm{sig}\left(\mathbf{G}\cdot\mathbf{F}^{\perp}\right) = \mathrm{sig}(\alpha_{2(q+N+1)})$ and for $x = 0$ we get $\mathbf{G}\cdot\mathbf{F}^{\perp} = 0$ therefore all orbits of $\mathbf{F}$ leave from any closed orbits of the center $\mathbf{G}$ if $\alpha_{2(q+N+1)} > 0$ or all orbit of $\mathbf{F}$ enter from any closed orbit of the center $\mathbf{G}$ if $\alpha_{2(q+N+1)} < 0$. Therefore the origin of system (1.3) is not a center. □

These constants $\alpha_{2(q+i+1)}$ give the center conditions, their vanish is a necessary condition to have a center. Therefore, Theorem 2.5 provides an effective algorithm for computing center conditions in the particular case of system (1.3) and for nilpotent systems in general. Notice that the algorithm is efficient because only use change of variables necessary to achieve a reduced normal form. This fact allows to calculate more Lyapunov constants than the classical algorithm that includes all change of variables, as it is shown in the following proofs. Moreover, the constants $\alpha_{2(q+i+1)}$ that appear in (2.5) give the conditions of orbital reversibility, that is, system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ of Theorem 2.5 is orbital reversible if, and only if, $\alpha_{2(q+i+1)} = 0$ for all $i \geq 0$ in the equation (2.5), see [1,11].

3. Proof of results

3.1. Proof of Theorem 1.1

If system (1.3) is analytically integrable from [9, Theorem 18] a first integral must be of the form $I = \frac{y^2}{2} + \frac{x^{4q+2}}{4q+2} + \cdots$. Then using [12, Theorem 2] is obtained the result.

3.2. Sufficient conditions for Theorem 1.2

In case **(a)** the system is R_y-reversible, and as the origin is monodromic, the origin is a center. In the case **(b)** system (1.3) is Hamiltonian with $\mathbf{F} = \mathbf{X}_g$, where g is given in (3.7). Therefore is integrable and as the origin is monodromic, it is a center. The case **(c)** is R_x-reversible. For the cases **(d)-(g)**, in order to prove the sufficiency, we first apply to system (1.3) the change of variables $x = u$, $y = v(1 + a_1 u)^{b_1/a_1}$ and the system becomes

$$\begin{aligned}\dot{u} &= v(1+a_1u)^{b_1/a_1+1} + a_2u^{2q+2},\\ \dot{v} &= \frac{-u^{4q+1}+b_3u^{4q+2}}{(1+a_1u)^{b_1/a_1}} + \frac{b_2+(b_2a_1-b_1a_2)u}{1+a_1u}u^{2q+1}v.\end{aligned} \tag{3.6}$$

In the case **(d)** system (3.6) is given by

$$\dot{u} = v, \qquad \dot{v} = -u^{4q+1} + b_2u^{2q+1}v + a_1^2u^{4q+3},$$

which is R_u-reversible. In the case **(e)** system (3.6) is

$$\dot{u} = v(1+a_1u)^{2q+3}, \qquad \dot{v} = -\frac{1+2a_1u}{(1+a_1u)^{2q+2}}u^{4q+1} + b_2u^{2q+1}v$$

Applying the change $Y = v$, $X = u/(1+a_1u)$, and $dt = (1-a_1X)^{2q+1}dT$ we obtain the system

$$X' = Y, \qquad Y' = -X^{4q+1} + 2(q+1)a_1X^{2q+1}Y - \sigma a_1^2X^{4q+3},$$

which is R_X-reversible. In the case **(f)** system (3.6) is

$$\dot{u} = v(1+a_1u)^{(2q+3)/2}, \qquad \dot{v} = \frac{-u^{4q+1}(1+\frac{a_1}{2}u)}{(1+a_1u)^{(2q+1)/2}} + b_2u^{2q+1}v.$$

Applying the change $Y = v$, $X = u/\sqrt{1+a_1u}$, $dt = dT/((1+\frac{a_1}{2}u)(1-a_1u)^q)$ and taking into account that $1+\frac{a_1^2}{4}X^2 = (1+\frac{a_1}{2}u)^2/(1+a_1u)$, we obtain the system

$$X' = Y, \qquad Y' = -X^{4q+1} + b_2\frac{X^{2q+1}}{\sqrt{1+\frac{a_1^2}{4}X^2}}Y,$$

which is R_X-reversible. In the case **(g)** system (3.6) takes the form

$$\dot{u} = v(1+a_1u)^{2q+2} + a_2u, \qquad \dot{v} = \frac{-u^{4q+1}}{(1+a_1u)^{2q+1}} + \frac{(2q+1)a_2}{1+a_1u}u^{2q+1}v.$$

Applying the change $Y = v$, $X = u/(1+a_1u)$, $dt = (1-a_1X)^{2q}dT$ we obtain the system

$$X' = Y + a_2X^{2q+2}, \qquad Y' = -X^{4q+1} + (2q+1)a_2X^{2q+1}Y,$$

which is R_X-reversible and this completes the proof.

3.3. Necessary conditions for Theorem 1.2

We compute the first orbital reversibility constants or the first center condition $\alpha_{2(q+i+1)} = 0$ that appear in (2.5). This first constant is

$$\alpha_{2q+2} = \big(2qa_1 - 3b_1 - (2q+3)b_3\big)\big(b_2 + 2(q+1)a_2\big) + (4q+3)a_2(a_1+2b_1), \;\; q \geq 1$$

The next constants are too large to include them here but the reader can compute them using the method derived from Theorem 2.5, see also [2]. From the form of α_{2q+2} we have the following four possibilities:

(1) $b_2 + 2(q+1)a_2 = 0$;

(2) $b_2 + 2(q+1)a_2 \neq 0$ and $a_1 + 2b_1 = 0$;
(3) $(b_2 + 2(q+1)a_2)(a_1 + 2b_1) \neq 0$ and $a_2 = 0$;
(4) $(b_2 + 2(q+1)a_2)(a_1 + 2b_1)a_2 \neq 0$.

Case **(1)**. If $b_2 + 2(q+1)a_2 = 0$ then we have $\alpha_{2q+2} = (4q+3)a_2(a_1 + 2b_1)$, for $q \geq 1$ which gives rise to the following possibilities:

(i) If $a_2 = 0$ then we obtain $b_2 = 0$ and in this case system (1.3) is R_y-reversible. This situation corresponds to case **(a)**.
(ii) If $a_1 + 2b_1 = 0$ system (1.3) is a Hamiltonian system with $\mathbf{F} = \mathbf{X}_g$, where

$$g(x,y) = -\frac{1}{2}y^2 - \frac{1}{2(2q+1)}x^{4q+2} + b_1 xy^2 - a_2 x^{2q+2} y + \frac{b_3}{4q+3}x^{4q+3}. \tag{3.7}$$

This case corresponds to case **(b)**.

Case **(2)**. If $b_2 + 2(q+1)a_2 \neq 0$ and $a_1 + 2b_1 = 0$ then we obtain $\alpha_{2q+2} = (4q+3)b_1 + (2q+3)b_3$. From the vanishing of this constant we can isolate $b_3 = -(4q+3)b_1/(2q+3)$. The next orbitally reversible constant or center condition is

$$\alpha_{2q+4} = b_1\big(-16q(4q+5)(q+1)b_1^2 + 3(2q+5)(2q+3)^2 a_2\left[(2q+3)a_2 - b_2\right]\big), \quad q \geq 1,$$

which gives rise to the following possibilities:

(i) If $b_1 = 0$ we obtain $a_1 = b_3 = 0$. In this case the system is R_x-reversible and corresponds to case **(c)**.
(ii) If $b_1 \neq 0$ from the vanishing of α_{2q+4} we have

$$b_1^2 = \frac{3(2q+5)(2q+3)^2 a_2\left[(2q+3)a_2 - b_2\right]}{16q(4q+5)(q+1)}$$

with $a_2\left[(2q+3)a_2 - b_2\right] \neq 0$, otherwise $b_1 = 0$. The next constant is

$$\alpha_{2q+6} = (2q+5)(2q+3)(16q^2 + 150q + 161)a_2 - 2(16q^3 + 130q^2 + 251q + 140)b_2, \quad q \geq 1.$$

The vanishing of this constant provides the value of b_2 given by

$$b_2 = \frac{(2q+5)(2q+3)(16q^2 + 150q + 161)}{2(16q^3 + 130q^2 + 251q + 140)}a_2$$

and the next one is

$$\alpha_{2q+8} = b_1 a_2\left[(32(16q^3 + 130q^2 + 251q + 140))(q+1)qb_1^2 + 45a_2^2(2q+7)(2q+5)(2q+3)^3\right].$$

Hence the unique possibility to vanish α_{2q+8} taking into account that $q \in \mathbb{N}$ is $a_2 = 0$ and in this case $b_2 = 0$, which gives the already considered case **(a)**.

Case **(3)**. The condition $a_2 = 0$ with $(b_2 + 2(q+1)a_2)(a_1 + 2b_1) \neq 0$ is equivalent to $a_2 = 0$ and $b_2(a_1 + 2b_1) \neq 0$. If $a_2 = 0$ we obtain that $\alpha_{2q+2} = \left[2qa_1 - 3b_1 - (2q+3)b_3\right]b_2$. From its vanishing we have $b_3 = (2qa_1 - 3b_1)/(2q+3)$ and the next constant is $\alpha_{2q+4} = (b_1 + a_1)(b_1 - 2(q+1)a_1)(2b_1 - (2q+1)a_1)$, which gives rise to the following possibilities:

(i) If $b_1 + a_1 = 0$, then we obtain $b_3 = a_1$ and this corresponds to the case **(d)**.
(ii) If $b_1 - 2(q+1)a_1 = 0$ we have $b_3 = -2a_1$ and this situation corresponds to case **(e)**.
(iii) If $2b_1 - (2q+1)a_1 = 0$ then $b_3 = -a_1/2$ and this corresponds to the case **(f)**.

Case **(4)**. In the last case $(b_2 + 2(q+1)a_2)(a_1 + 2b_1)a_2 \neq 0$ we apply the scalar change $x = x_1/a_2$, $y = x_2/a_2^{2q+1}$, and $dt = dT/a_2^{2q}$ and system (1.3) takes the form

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1^{4q+1} \end{pmatrix} + \begin{pmatrix} A_1 x_1 x_2 + x_1^{2q+2} \\ B_1 x_2^2 + B_2 x_1^{2q+1} x_2 + B_3 x_1^{4q+2} \end{pmatrix},$$

where $' = d/dT$, $A_1 = a_1/a_2$, $B_1 = b_1/a_2$, $B_2 = b_2/a_2$ and $B_3 = b_3/a_2$. In this case the first nonzero constant is

$$\alpha_{2q+2} = \big((2q+3)B_3 + B_1 - (2q+1)A_1\big)\big(B_2 + 2(q+1)\big) + \big(B_2 - (2q+1)\big)\big(A_1 + 2B_1\big).$$

We remark that $B_2 + 2(q+1) \neq 0$ since otherwise implies $b_2 + 2(q+1)a_2 = 0$ against the hypothesis of the case. Hence we can isolate from α_{2q+2} the parameter B_3 and we obtain

$$B_3 = \frac{B_1(-3B_2 + 2q) + A_1(3 + 2(4 + B_2)q + 4q^2)}{(3 + 2q)(B_2 + 2(q+1))}.$$

We recall that $3 + 2q \neq 0$ because $q \in \mathbb{N}$. Substituting this value of B_3 in the next constants we obtain a rational functions in the parameters A_1, B_1, B_2, q where the denominators are different from zero. So we take the polynomial numerators of these constants that we rename of the same form. We have not been able to finish the problem using computer algebra system Singular [23] to find the decomposition in prime ideals of the ideal generate by the constants. Hence we use the resultant method, see for instance [5]. In fact any component of the center variety must be a zero of the resultants between the initial orbital reversible constants and the resultants of the resulting successive polynomials of the resultants. Hence we first compute the following resultant between α_{2q+4} and α_{2q+j}, for $j = 8, 10, 12$ respect to the parameter A_1, and these resultants are

$$\begin{aligned}
\text{Resultant}[\alpha_{2q+4}, \alpha_{2q+8}, A_1] &= B_1^5 q^3 (3+2q)^{15} (B_2 - (2q+1))(B_2 + 2(q+1))^{15} (5+4q)^3 \mathcal{P}_1(B_1, B_2, q), \\
\text{Resultant}[\alpha_{2q+4}, \alpha_{2q+10}, A_1] &= B_1^5 q^3 (3+2q)^{21} (B_2 - (2q+1))(B_2 + 2(q+1))^{21} (5+4q)^6 \mathcal{P}_2(B_1, B_2, q), \\
\text{Resultant}[\alpha_{2q+4}, \alpha_{2q+12}, A_1] &= B_1^5 q^3 (3+2q)^{27} (B_2 - (2q+1))(B_2 + 2(q+1))^{27} (5+4q)^9 \mathcal{P}_3(B_1, B_2, q),
\end{aligned}$$

where $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ are polynomials in the variables B_1, B_2 and q. If we recall that $q \in \mathbb{N}$ we have three cases to study. The first one is $B_1 = 0$, the second is $B_2 = (2q+1)$ and the third is $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}_3 = 0$.

a) If $B_1 = 0$ we have that

$$\alpha_{2q+4} = A_1 \mathcal{Q}_1(A_1, B_2, q), \quad \alpha_{2q+8} = A_1 \mathcal{Q}_2(A_1, B_2, q) \quad \text{and} \quad \alpha_{2q+10} = A_1 \mathcal{Q}_3(A_1, B_2, q),$$

where $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are polynomials in A_1, B_2 and q. The case $A_1 = 0$ implies $a_1 = b_1 = 0$ against the hypothesis of the case because $(a_1 + 2b_1)a_2 \neq 0$. Now we do the following resultants

$$\begin{aligned}
\text{Resultant}[\mathcal{Q}_1, \mathcal{Q}_2, A_1] &= B_2^4 q^2 (B_2 - 2q - 3)^2 (3+2q)^{10} (B_2 + 2(q+1))^8 (5+4q)^2 \mathcal{R}_1(B_2, q), \\
\text{Resultant}[\mathcal{Q}_1, \mathcal{Q}_3, A_1] &= B_2^4 q^2 (B_2 - 2q - 3)^2 (3+2q)^{14} (B_2 + 2(q+1))^{12} (5+4q)^4 \mathcal{R}_2(B_2, q),
\end{aligned}$$

which gives the following possibilities:

i) If $B_2 = 0$ we have $\alpha_{2q+4} = 2A_1^3(1+2q)(3+2q)^4(5+2q)$ which never vanishes.

ii) If $B_2 = 2q+3$ we have $\alpha_{2q+4} = 4A_1^3 q(3+2q)^3(5+4q)(5+8q)$ which also never vanishes.
iii) If $\mathcal{R}_1 = \mathcal{R}_2 = 0$ we made the Resultant$[\mathcal{R}_1, \mathcal{R}_2, B_2]$ which is a polynomial in q without any natural root.

b) If $B_2 = 2q+1$ we have that $\alpha_{2q+4} = (B_1 - A_1(2q+1))\mathcal{S}_1(A_1, B_1, q)$ and

$$\alpha_{2q+8} = (B_1 - A_1(2q+1))\mathcal{S}_2(A_1, B_1, q) \quad \text{and} \quad \alpha_{2q+10} = (B_1 - A_1(2q+1))\mathcal{S}_3(A_1, B_1, q).$$

If $B_1 - A_1(2q+1) = 0$ this implies $B_3 = 0$ and then $b_3 = b_1 - (2q+1)a_1 = b_2 - (2q+1)a_2 = 0$ which corresponds to the case **(g)**. The remaining case is $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}_3 = 0$. The resultant between these polynomials gives

$$\begin{aligned}\text{Resultant}[\mathcal{S}_1, \mathcal{S}_2, A_1] &= q^2(3+2q)^8(3+4q)^8(5+4q)^2\mathcal{T}_1(B_2, q),\\ \text{Resultant}[\mathcal{S}_1, \mathcal{S}_3, A_1] &= q^2(3+2q)^12(3+4q)^12(5+4q)^4\mathcal{T}_2(B_2, q),\end{aligned}$$

and the resultant Resultant$[\mathcal{T}_1, \mathcal{T}_2, B_1]$ gives a polynomial in q without any natural root.

c) The last case is $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}_3 = 0$ with $B_1 \neq 0$. Next we do the following resultants:

$$\begin{aligned}\text{Resultant}[\mathcal{P}_1, \mathcal{P}_2, B_1] = \;& (2B_2 - (4q+1))^2(B_2 - (2q+5))^2(B_2 - (2q+3))^{26}(B_2 - 2q+1)^2\\ & (1+q)^6(3+2q)^{12}(B_2+2q+5)^2(5+4q)^{10}(7+4q)^{10}(B_2+6q+9)^6\bar{\mathcal{P}}_1(B_2, q),\\ \text{Resultant}[\mathcal{P}_1, \mathcal{P}_3, B_1] = \;& (2B_2 - (4q+1))^2(B_2 - (2q+5))^2(B_2 - (2q+3))^{26}(B_2 - 2q+1)^2\\ & (1+q)^6(3+2q)^{12}(B_2+2q+5)^2(5+4q)^{10}(7+4q)^{10}(B_2+6q+9)^6\bar{\mathcal{P}}_2(B_2, q),\end{aligned}$$

with

$$\bar{\mathcal{P}}_1(B_2, q) = \bar{\mathcal{P}}_{11}^2(B_2, q)\bar{\mathcal{P}}_{12}^2(B_2, q), \qquad \bar{\mathcal{P}}_2(B_2, q) = \bar{\mathcal{P}}_{21}^2(B_2, q)\bar{\mathcal{P}}_{22}^2(B_2, q),$$

where $\bar{\mathcal{P}}_{11}(B_2, q)$ is of degree 38 in q and $\bar{\mathcal{P}}_{12}(B_2, q)$ is of degree 325 in q and $\bar{\mathcal{P}}_{21}(B_2, q)$ is of degree 74 in q and $\bar{\mathcal{P}}_{22}(B_2, q)$ is of degree 457 in q. Then we have the following particular cases:

i) If $B_2 = (4q+1)/2$ the Resultant$[\mathcal{P}_1/B_1^2, \mathcal{P}_2/B_1^2, B_1]$ has not any natural root for q.
ii) If $B_2 = (2q+5)$ the Resultant$[\mathcal{P}_1/B_1^2, \mathcal{P}_2/B_1^2, B_1]$ has not any natural root for q.
iii) If $B_2 = (2q+3)$ the Resultant$[\mathcal{P}_1/B_1^6, \mathcal{P}_2/B_1^6, B_1]$ has not any natural root for q.
iv) If $B_2 = (2q-1)$ the Resultant$[\mathcal{P}_1/B_1^2, \mathcal{P}_2/B_1^2, B_1]$ has not any natural root for q.
v) If $B_2 = -(2q+5)$ the Resultant$[\mathcal{P}_1, \mathcal{P}_2, B_1] = 0$, due to $\mathcal{P}_1 = (4+B_1^2+2q)\mathcal{M}_1(B_1, q)$ and $\mathcal{P}_2 = (4+B_1^2+2q)\mathcal{M}_2(B_1, q)$. However $4+B_1^2+2q \neq 0$ because q must be a natural number. Moreover the resultant Resultant$[\mathcal{M}_1, \mathcal{M}_2, B_1]$ has not any natural root for q.
vi) If $B_2 = -(6q+9)$ the Resultant$[\mathcal{P}_1, \mathcal{P}_2, B_1] = 0$, due to $\mathcal{P}_1 = (27+B_1^2+54q+36q^2+8q^3)\mathcal{N}_1(B_1, q)$ and $\mathcal{P}_2 = (27+B_1^2+54q+36q^2+8q^3)\mathcal{N}_2(B_1, q)$. However $27+B_1^2+54q+36q^2+8q^3 \neq 0$ because q must be a natural number. Moreover the resultant Resultant$[\mathcal{N}_1, \mathcal{N}_2, B_1]$ has not any natural root for q.
vii) The last case is $\bar{\mathcal{P}}_1 = \bar{\mathcal{P}}_2 = 0$. In this case we have to do the different resultant between the polynomials that define $\bar{\mathcal{P}}_1$ and $\bar{\mathcal{P}}_2$. The resultant between $\mathcal{P}_{11}(B_2, q)$ and $\mathcal{P}_{21}(B_2, q)$ has not any natural root for q except the case $q = 1$ that we study later. However it lacks to make the computations of the resultants

$$\text{Resultant}[\mathcal{P}_{11}, \mathcal{P}_{22}, B_1], \quad \text{Resultant}[\mathcal{P}_{12}, \mathcal{P}_{21}, B_1], \quad \text{Resultant}[\mathcal{P}_{12}, \mathcal{P}_{22}, B_1],$$

to see if there exist any natural root for q except the case $q = 1$.

In short we have not been able to finish the problem neither using computer algebra system Singular [23] to find the decomposition in prime ideals of the ideal generate by the constants nor using the resultant method, see [5]. We recall that this second method is based in the fact that any component of the center variety must be a zero of the resultants between the initial orbital reversible constants and of the resultants of the resulting successive polynomials. Doing this resultant process we are not able to finish the process with our computational facilities in order to arrive to a polynomial in q that must not have roots in $\mathbb{N}$ except for $q = 1$. However for a fixed value of $2 \le q \le 100$ we always obtain the cases given in the statement of the theorem obtaining the decomposition of the ideal using the computer algebra system Singular.

3.4. Proof of Theorem 1.3

Necessity. Reasoning as in section about necessary conditions of the Theorem 1.2 we get the cases **(a)-(g)**. Only left the study for $(b_2 + 2(q+1)a_2)(b_2 - (2q+1)a_2)(a_1 + 2b_1)a_2 \neq 0$ but now with $q = 1$. In this case using the computer algebra system Singular we can obtain the decomposition of the ideal generated by the orbital reversible constants and we obtain the new case **(h)** and this case completes the proof of the necessary conditions.

Sufficiency. Sufficient conditions for the cases **(a)-(g)** are given in the proof of Theorem 1.2. We only need to prove the sufficient conditions for the case **(h)**. The conditions of case (**h**) are $b_2 = 35a_2$, $b_1 = 5a_1/3$, $b_3 = -4a_1/9$ and $a_1 = \pm 9\sqrt{5}a_2$. Here we only analyze the case of the positive root but proof for the negative one is similar. Under these conditions system (3.6) takes the form

$$\dot{x} = a_2x^4 + y + 9\sqrt{5}\, a_2xy, \qquad \dot{y} = -x^5 - 4\sqrt{5}a_2x^6 + 35a_2x^3y + 15\sqrt{5}\, a_2y^2. \tag{3.8}$$

Now we apply the change of variables $x = -u/(1 + a_1u)$, $y = -v/(1 + a_1u)^3$ and the scaling of time $dt = dT/(1 + 9\sqrt{5}a_2u)^3$ where $a_1 = 9\sqrt{5}a_2$, the previous system becomes

$$\dot{u} = (1 + 9\sqrt{5}a_2u)(-a_2u^4 + v), \qquad \dot{v} = -u^5 - 5\sqrt{5}a_2u^6 - 35a_2u^3v - 27\sqrt{5}a_2^2u^4v + 12\sqrt{5}a_2v^2. \tag{3.9}$$

Now we do the change $Y = \dot{u}$ from where $v = (a_2u^4 + 9\sqrt{5}a_2^2u^5 + Y)/(1 + 9\sqrt{5}a_2u)$ and system (3.9) is transformed to

$$\dot{u} = Y, \qquad \dot{Y} = p_0(u) + p_1(u)Y + p_2(u)Y^2, \tag{3.10}$$

where $p_0(u) = -u^5(1 + a_2u(14\sqrt{5} + 5a_2u(52 + 3a_2u(22\sqrt{5} + 45a_2u))))$, $p_1(u) = -39a_2u^3(1 + \sqrt{5}a_2u)$ and $p_2(u) = (21\sqrt{5}a_2)/(1 + 9\sqrt{5}a_2u)$. Systems of the form (3.10) are called Cherkas system, see for instance [21,25]. These Cherkas systems can be transformed to Lienard systems by the change $y_1 = Y\psi = Ye^{\int_0^u p_2 ds}$ that transforms system (3.10) into the Liénard system

$$\dot{u} = y_1, \qquad \dot{y_1} = p_0(u)\psi^2 + p_1(u)\psi y_1 = g(u) + f(u)y_1 \tag{3.11}$$

where ψ is in this case $\psi = 1/(1 + 9\sqrt{5}a_2u)^{7/3}$. Doing the indicated change we obtain that

$$g(u) = \frac{u^5(\sqrt{5} + 5a_2u)^2(1 + 3\sqrt{5}a_2u)}{5(1 + 9\sqrt{5}a_2u)^{11/3}}, \qquad f(u) = \frac{39a_2u^3(1 + 10\sqrt{5}a_2u + 45a_2^2u^2)}{(1 + 9\sqrt{5}a_2u)^{10/3}}.$$

Now we compute the primitives of these functions $G(u) = \int_0^u g(s)ds$ and $F(u) = \int_0^u f(s)ds$ in order to apply the following theorem for Liénard systems, see [16] and also [22].

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Theorem 3.6. *Liénard system $\dot{x} = y$, $\dot{y} = g(x) + f(x)y$ has a center at the origin if, and only if, there exists a function $z(x)$ satisfying $F(x) = F(z)$, $G(x) = G(z)$ with $z(0) = 0$ and $z'(0) < 0$.*

This solution $z(x)$ must correspond to a common factor between $F(x) - F(z)$ and $G(x) - G(z)$ other that $x - z$. Thus we have the following corollary for the polynomial case.

Corollary 3.7. *If the Liénard system $\dot{x} = y$, $\dot{y} = g(x) + f(x)y$ with f and g polynomial has a center at the origin, then it is necessary that the resultant of*

$$\frac{F(x) - F(z)}{x - z} \quad \text{and} \quad \frac{G(x) - G(z)}{x - z}$$

with respect to x or z vanishes. This condition is sufficient if the common factor of the two polynomials vanishes at $x = z = 0$.

For system (3.11) we have that $G(u) = G_1(u) - G_1(0) = G_1(u) - 1/(7290000a_2^6)$ with

$$G_1(u) = \frac{(\sqrt{5} + 15a_2u)^2(1 + 18\sqrt{5}a_2u + 315a_2^2u^2 - 900\sqrt{5}a_2^3u^3 + 6075a_2^4u^4 + 20250\sqrt{5}a_2^5u^5 + 50625a_2^6u^6)}{36450000\, a_2^6(1 + 9\sqrt{5}a_2u)^{8/3}},$$

and $F(u) = F_1(u) - F_1(0) = F_1(u) - 13/(5400a_2^3)$ where

$$F_1(u) = \frac{13(\sqrt{5} + 15a_2u)(-\sqrt{5} - 45a_2u + 45\sqrt{5}a_2^2u^2 + 225a_2^3u^3)}{27000\, a_2^3(1 + 9\sqrt{5}a_2u)^{4/3}}.$$

We are going to apply Corollary 3.7 to the rational functions

$$\frac{F_1(x) - F_1(z)}{x - z} \quad \text{and} \quad \frac{G_1(x) - G_1(z)}{x - z}$$

Note that $F(x) - F(z) = F_1(x) - F_1(z)$ and $G(x) - G(z) = G_1(x) - G_1(z)$. We can do this because the denominators of F_1 and G_1 are unity elements. In order to do that we compute the resultant of the numerators of

$$\frac{F_1(u)^3 - F_1(z)^3}{u - z} \quad \text{and} \quad \frac{G_1(u)^3 - G_1(z)^3}{u - z} \tag{3.12}$$

with respect to u or z that vanishes. Since $F_1^3(x) - F_1^3(z) = (F_1(x) - F_1(z))(F_1^2(x) + F_1(x)F_1(z) + F_1^2(z))$ and the same for $G_1^3(x) - G_1^3(z)$, and the factors $(F_1^2(x) + F_1(x)F_1(z) + F_1^2(z))$, $(G_1^2(x) + G_1(x)G_1(z) + G_1^2(z))$ do not introduce any factor of the form $z + u + \cdots$. Moreover the numerators the functions (3.12) have a common factor given by

$$f(u, z) = u + \sqrt{5}a_2u^2 + z + 10\sqrt{5}a_2uz + 45a_2^2u^2z + \sqrt{5}a_2z^2 + 45a_2^2uz^2.$$

Applying the Implicit function theorem we see that $f(u, z) = 0$ has a solution of the form $z = -u + \cdots$. A solution that satisfies $z(0) = 0$ and $z'(0) < 0$. In fact the first terms are

$$z(u) = -u + \frac{32}{4}\sqrt{5}a_2u^2 - 320a_2^2u^3 + 2600\sqrt{5}a_2^3u^4 + \cdots .$$

Hence applying Theorem 3.6 system (3.11) has a center at the origin and consequently the original system (3.8) also.

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