



Uniform regularity of the weak solution to higher-order Navier-Stokes-Cahn-Hilliard systems [☆]



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ABSTRACT

Higher-order Navier-Stokes-Cahn-Hilliard system is a noted interface system that describes the evolution of two immiscible incompressible fluids. Regularity of the solutions to this system is a primary issue from a mathematical point of view. The purpose of this paper is to improve the known results regarding the regularity with respect to time of weak solutions to two-dimensional higher-order Navier-Stokes-Cahn-Hilliard system. Namely, (H^1, H^k) -regularity of the first and second derivatives of the weak solutions to two-dimensional higher-order Navier-Stokes-Cahn-Hilliard system is obtained with the methods of energy estimates.

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1. Introduction

Interfacial dynamics have a wide range of applications in problems of materials science involving the mixture of different fluids, solids, or gasses. Navier-Stokes-Cahn-Hilliard system is a well-known model describing topological transitions in binary fluid flows [1,8,10–12,16]. In this paper, we consider higher-order Navier-Stokes-Cahn-Hilliard system for the flow of two homogeneous and incompressible fluids in a 2-dimensional bounded domain. This model consists of convective higher-order Cahn-Hilliard equations coupled with two-dimensional incompressible Navier-Stokes equations governing fluid velocity. Endowing the system with suitable boundary and initial conditions, we investigate the following higher-order Navier-Stokes-Cahn-Hilliard system

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$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla p - w \nabla \varphi = h, & \text{in } \Omega_T = \Omega \times (0, T), \\ \operatorname{div} u = 0, & \text{in } \Omega_T, \\ \varphi_t + u \cdot \nabla \varphi - \Delta w = 0, & \text{in } \Omega_T, \\ w = P(-\Delta)\varphi + f(\varphi), & \text{in } \Omega_T, \\ u = 0, & \text{on } \Gamma \times (0, T), \\ \partial_\nu \varphi = \partial_\nu (\Delta \varphi) = \cdots = \partial_\nu (\Delta^k \varphi) = 0, & \text{on } \Gamma \times (0, T), \\ u(x, 0) = u_0(x), \quad \varphi(x, 0) = \varphi_0(x), & \text{in } \Omega, \end{cases} \quad (1.1)$$

where, $P(s) = \sum_{i=1}^k a_i s^i$, $k \geq 2$ and $a_k > 0$, Ω is a 2-dimensional regular bounded domain with unit normal vector ν to the boundary $\Gamma = \partial\Omega$, and h is an external volume force. The higher-order Cahn-Hilliard system follows from the free energy equation

$$\Psi = \int_{\Omega} \frac{1}{2} \sum_{i=1}^k a_i |(-\Delta)^{\frac{i}{2}} \varphi|^2 + F(\varphi) dx,$$

where, $F(z) = \int_0^z f(s) ds$.

The model (1.1) contains the Cahn-Hilliard system coupled with two-dimensional incompressible Navier-Stokes system. In last decades, there are many works on Cahn-Hilliard system and Navier-Stokes system. Cherfil, Miranville and Peng obtained the well-posedness and the global attractor of the higher-order Cahn-Hilliard system, in addition to the higher-order (in space) anisotropic generalized Cahn-Hilliard system [4] and [5,6]. The regularity of the solutions to two-dimensional Navier-Stokes system was proven in H^k space in [24], and the regularity of the solutions to three-dimensional Cahn-Hilliard system was obtained in H^k space [19]. Furthermore, Navier-Stokes-Cahn-Hilliard systems have been investigated by many researchers [2,3,13,14,17,23]. The existence and uniqueness of the weak solution to the higher-order Navier-Stokes-Cahn-Hilliard system were proven, and the existence of exponential attractor of corresponding dynamical system was also proven [3]. The numerical results of the two-dimensional Navier-Stokes-Cahn-Hilliard system in a velocity-pressure phase field with chemical potential formulation were presented with the method of Galerkin-based Isogeometric Analysis (IGA) in [14]. In [2], the global regularity of the strong solutions was considered for the two-dimensional Navier-Stokes-Cahn-Hilliard system with a mixed partial viscosity and mobility, and the global existence and uniqueness of the classical solution were demonstrated. H^1 -regularity and H^2 -regularity with respect to time of the weak solutions to the two-dimensional Navier-Stokes/Cahn-Hilliard system with the method of energy estimates were obtained in [13]. In [17], Mininni, Miranville and Romanelli obtained the existence and uniqueness of the solution as well as the global attractor of the higher-order Cahn-Hilliard system with dynamic boundary conditions. Furthermore, long-time behavior of the solutions to the Cahn-Hilliard-Navier-Stokes system with moving contact lines was taken into consideration, and the existence of the global attractor was obtained in this system in [23].

In this paper, we investigate uniform regularity of the weak solutions to the higher-order Navier-Stokes-Cahn-Hilliard system in a two-dimensional domain. In the theory of partial differential equations, higher-order regularity of the weak solutions is an important topic. The importance of higher-order regularity of the solutions lies in the following two aspects. On the one hand, if higher-order regularity of the solutions is obtained, weak solutions are upgraded to strong solutions or even classical solutions. On the other hand, if the solutions of many equations have enough regularity in Sobolev spaces to be approximated to higher order in L^p by piecewise polynomial splines defined on uniform grids, the higher-order approximation by splines is considered. That is to say the higher-order regularity of the solutions plays an important role in the numerical methods for solving partial differential equations. In the past decades, the higher-order regularity of the weak solutions to some equations has been discussed [7,9,15,19,21,22,24].

The remainder of this paper is organized as follows. In Section 2, we present a few basic spaces, some special symbols, several interpolation inequalities, and preliminary conclusions. In Section 3, the theorems about the regularity estimates of (u_t, φ_t) to higher-order Navier-Stokes-Cahn-Hilliard system are presented. In Section 4, we present the theorems about the regularity of (u_{tt}, φ_{tt}) to higher-order Navier-Stokes-Cahn-Hilliard system.

2. Preliminaries

For two-dimensional higher-order Navier-Stokes-Cahn-Hilliard equations (1.1), we introduce several basic spaces as follows

$$\mathcal{H}^k(\Omega) = \{v \in H^k(\Omega), \frac{\partial v}{\partial \nu} = \frac{\partial(\Delta v)}{\partial \nu} = \dots = \frac{\partial \Delta^{\lfloor \frac{k-2}{2} \rfloor} v}{\partial \nu} = 0 \text{ on } \partial\Omega\}, \quad k \in \mathbb{N}, \quad k \geq 2,$$

$$L_{\text{div}}^2(\Omega, R^2) = \{v \in L^2(\Omega, R^2) \mid \operatorname{div} v = 0, \quad v_n|_{\partial\Omega} = 0\},$$

$$V_0 = \{v \in H_0^1(\Omega, R^2) \mid \operatorname{div} v = 0\},$$

where, the norm in V_0 is defined as $\|v\|_{V_0}^2 = \|\nabla v\|^2$, $[a]$ is the greatest integer less or equal to a , and we denote the Leray projector by $\mathbf{P}: L^2(\Omega, R^2) \rightarrow L_{\text{div}}^2(\Omega)$.

The norm in $L^2(\Omega, R^2)$ is denoted by $\|\cdot\|$ and the double parentheses $((\cdot, \cdot))$ represents the $L^2(\Omega, R^2)$ -inner product as well as $H^1(\Omega)$ - $H^{-1}(\Omega)$ -dual product. Furthermore, the $\mathcal{H}^k(\Omega)$ -norm is expressed by the $\|\cdot\|_{\mathcal{H}^k}$, for $k = 1, 2, 3, \dots$.

By Hodge decomposition, we obtain $((\nabla p, v)) = 0$ for any $v \in V_0$. Then, we obtain following variational formulation of the problem (1.1)

$$\begin{cases} ((u_t, v)) + ((u \cdot \nabla)u, v)) + ((\nabla u, \nabla v)) - ((w \nabla \varphi, v)) = ((h, v)), \\ ((\varphi_t, \psi)) + ((u \cdot \nabla \varphi, \psi)) + ((\nabla w, \nabla \psi)) = 0, \\ \sum_{i=1}^k a_i (((-\Delta)^{\frac{i}{2}} \varphi, (-\Delta)^{\frac{i}{2}} \xi)) + ((f(\varphi), \xi)) - ((w, \xi)) = 0, \end{cases} \quad (2.1)$$

where, $v \in V_0$, $\psi \in H^1(\Omega)$, $\xi \in H^k(\Omega)$ and $u(0) = u_0$, $\varphi(0) = \varphi_0$.

Definition 2.1. If $(u_0, \varphi_0) \in L_{\text{div}}^2(\Omega, R^2) \times \mathcal{H}^k(\Omega)$ and $T > 0$, and

$$u \in L^\infty(0, T; L_{\text{div}}^2(\Omega, R^2)) \cap L^2(0, T; V_0), \quad \varphi \in L^\infty(0, T; \mathcal{H}^k(\Omega)) \cap H^1(0, T; H^{-1})$$

satisfying the variational formulation (2.1), we say that (u, φ) is a weak solution to higher-order Navier-Stokes-Cahn-Hilliard system (1.1).

Assume that the nonlinear term $f(\cdot)$ in (1.1) satisfies

$$f(s) \in C^4(R), \quad f(0) = 0, \quad |f'(s)|, |f''(s)|, |f^{(3)}(s)|, |f^{(4)}(s)| \leq c_0, \quad c_0 > 0. \quad (2.2)$$

Lemma 2.2. [3] For fixed $T > 0$, given $(u_0, \varphi_0) \in L_{\text{div}}^2(\Omega, R^2) \times \mathcal{H}^k(\Omega)$, there exists (u, φ) satisfying (2.1), and the following conclusions hold

$$u \in L^\infty(0, T; V_0(\Omega)), \quad \varphi \in L^\infty(0, T; \mathcal{H}^{2k}(\Omega)).$$

Additionally, we have

$$\int_t^{t+r} \|u_t\|_{L^2}^2 + \|\varphi_t\|_{H^k}^2 ds \leq C(r), \quad w \in L^\infty(0, T; H^1(\Omega)), \quad \forall t \geq 2r, \quad r > 0.$$

Lemma 2.3. [21] (Uniform Gronwall's Lemma) Let g, h, y be three positive locally integrable functions on $[t_0, +\infty]$, such that y' is locally integrable on $[t_0, +\infty]$ and satisfies

$$\frac{dy}{dt} \leq gy + h, \quad \int_t^{t+r} g(s)ds \leq a_1, \quad \int_t^{t+r} h(s)ds \leq a_2, \quad \int_t^{t+r} y(s)ds \leq a_3, \quad \text{for } t \geq t_0,$$

where, r, a_1, a_2 , and a_3 are positive constants. Then,

$$y(t+r) \leq \left(\frac{a_3}{r} + a_2\right)\exp(a_1), \quad \forall t \geq t_0.$$

Lemma 2.4. [18] (Gagliardo-Nirenberg-Sobolev inequality) Let u belong to L^q in R^n and its derivatives of order m , $D^m u$, belong to L^r , $1 \leq q, r \leq \infty$. For the derivatives $D^j u$, $0 \leq j < m$, the following inequalities hold

$$|D^j u|_{L^p} \leq C |D^m u|_{L^r}^a |u|_{L^q}^{1-a},$$

where, $\frac{1}{p} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q}$, for all a in the interval $\frac{j}{m} \leq a \leq 1$, and the constant C depending only on n, m, j, q, r and a .

Thus, we have several interpolation inequalities which are used throughout this paper. For any $k \in \mathbf{N}$, $k \geq 2$, $\forall i = 1, 2, \dots, k-1$, there exists $c(i) > 0$ such that

$$\begin{cases} \|(-\Delta)^i v\|_{L^2} \leq c(i) \|(-\Delta)^k v\|_{L^2}^{\frac{i}{k}} \|v\|_{L^2}^{1-\frac{i}{k}}, \quad \forall v \in H^{2k}(\Omega), \\ \|(-\Delta)^{\frac{i-1}{2}} v\|_{L^2} \leq c(i) \|(-\Delta)^{\frac{k-1}{2}} v\|_{L^2}^{\frac{i-1}{k-1}} \|v\|_{L^2}^{1-\frac{i-1}{k-1}}, \quad \forall v \in H^k(\Omega), \\ \|(-\Delta)^{\frac{i+1}{2}} v\|_{L^2} \leq c(i) \|(-\Delta)^{\frac{k+1}{2}} v\|_{L^2}^{\frac{i+1}{k+1}} \|v\|_{L^2}^{1-\frac{i+1}{k+1}}, \quad \forall v \in H^{k+1}(\Omega), \\ \|(-\Delta)^{\frac{i+k}{2}} v\|_{L^2} \leq c(i) \|(-\Delta)^k v\|_{L^2}^{\frac{i+k}{2k}} \|v\|_{L^2}^{1-\frac{i+k}{2k}}, \quad \forall v \in H^{2k}(\Omega), \\ \|(-\Delta)^{i-1} v\|_{L^2} \leq c(i) \|(-\Delta)^k v\|_{L^2}^{\frac{i-1}{k}} \|v\|_{L^2}^{1-\frac{i-1}{k}}, \quad \forall v \in H^{2k}(\Omega), \end{cases} \quad (2.3)$$

besides,

$$\begin{cases} \|v\|_{L^4} \leq C \|v\|_{L^2}^{\frac{1}{2}} \|v\|_{H^1}^{\frac{1}{2}}, \quad \forall v \in H^1(\Omega), \\ \|v\|_{L^\infty} \leq C \|v\|_{L^2}^{\frac{1}{2}} \|v\|_{H^2}^{\frac{1}{2}}, \quad \forall v \in H^2(\Omega). \end{cases} \quad (2.4)$$

Lemma 2.5. [20] Assume $u \in L^\infty(0, T; V_0(\Omega))$ and $a(u, v, w)$ be the trilinear form

$$a(u, v, w) = \int_{\Omega} (u \cdot \nabla) v \cdot w dx, \quad \forall v \in V_0.$$

Then, we have $a(u, v, w) = -a(u, w, v)$ and $a(u, v, v) = 0$.

3. (H^1, H^k) -Regularity of (u_t, φ_t)

To improve regularity of the first derivative of the weak solution to higher-order Navier-Stokes-Cahn-Hilliard system, we differentiate the Eq. (2.1) with respect to t . So, we have

$$\begin{cases} ((u_{tt}, v)) + ((u_t \cdot \nabla u + u \cdot \nabla u_t, v)) + ((\nabla u_t, \nabla v)) - ((w_t \nabla \varphi + w \nabla \varphi_t, v)) = ((h_t, v)), \\ ((\varphi_{tt}, \psi)) + ((u_t \cdot \nabla \varphi + u \cdot \nabla \varphi_t, \psi)) + ((\nabla w_t, \nabla \psi)) = 0, \\ \sum_{i=1}^k a_i (((-\Delta)^{\frac{i}{2}} \varphi_t, (-\Delta)^{\frac{i}{2}} \xi)) + ((f'(\varphi) \varphi_t, \xi)) - ((w_t, \xi)) = 0. \end{cases} \quad (3.1)$$

Theorem 3.1. Assuming $h \in L^\infty(0, T; L^2(\Omega))$ and $h_t \in L^\infty(0, T; L^2(\Omega))$, we have $u_t \in L^\infty(0, T; V_0(\Omega))$. Furthermore, the weak solution (u, φ) satisfies

$$\begin{aligned} & \|u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + \|\varphi_t\|_{L^2}^2 + \|(-\Delta)^{\frac{k-1}{2}} \varphi_t\|_{L^2}^2 \leq C(r), \\ & \int_t^{t+r} \|\Delta w\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \|(-\Delta)^k \varphi_t\|_{L^2}^2 + \|w_t\|_{L^2}^2 + \|\Delta u_t\|_{L^2}^2 ds \leq C(r), \end{aligned}$$

for $\forall t \geq 2r, r > 0$.

Proof. Assuming $v = -\Delta u$ in (2.1)₁, $\psi = -\Delta w$ in (2.1)₂, we have

$$\begin{cases} \frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + ((w \nabla \varphi, \Delta u)) = (((u \cdot \nabla) u, \Delta u)) + ((h, -\Delta u)), \\ \|\Delta w\|_{L^2}^2 = ((\varphi_t, \Delta w)) + ((u \cdot \nabla \varphi, \Delta w)). \end{cases} \quad (3.2)$$

For (3.2)₁, we have the following inequality

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \leq |((w \nabla \varphi, \Delta u))| + |(((u \cdot \nabla) u, \Delta u))| + |((h, -\Delta u))|. \quad (3.3)$$

Then, we utilize (2.3) and Young's inequality with ϵ to obtain

$$\begin{aligned} & |((w \nabla \varphi, \Delta u))| + |(((u \cdot \nabla) u, \Delta u))| + |((h, -\Delta u))| \\ & \leq C \|w\|_{L^4} \|\nabla \varphi\|_{L^4} \|\Delta u\|_{L^2} + C \|\Delta u\|_{L^2} \|u\|_{L^\infty} \|\nabla u\|_{L^2} + \|\Delta u\|_{L^2} \|h\|_{L^2} \\ & \leq C \|w\|_{L^2}^{\frac{1}{2}} \|\nabla w\|_{L^2}^{\frac{1}{2}} \|\nabla \varphi\|_{L^2}^{\frac{1}{2}} \|\Delta \varphi\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2} + C \|u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{3}{2}} \|\nabla u\|_{L^2} + \epsilon_1 \|\Delta u\|_{L^2}^2 + C_{\epsilon_1} \|h\|_{L^2}^2 \\ & \leq \epsilon_1 \|\Delta u\|_{L^2}^2 + \epsilon_2 \|\Delta u\|_{L^2}^2 + C_{\epsilon_2} \|w\|_{L^2} \|\nabla w\|_{L^2} \|\nabla \varphi\|_{L^2} \|\Delta \varphi\|_{L^2} + \epsilon_3 \|\Delta u\|_{L^2}^2 \\ & \quad + C_{\epsilon_3} \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^4 + C_{\epsilon_1} \|h\|_{L^2}^2. \end{aligned}$$

Let $\epsilon_1 = \epsilon_2 = \epsilon_3 = \frac{1}{4}$ in above estimate and combine with (3.3). With the result of Lemma 2.2, it follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \frac{1}{4} \|\Delta u\|_{L^2}^2 \\ & \leq C \|w\|_{L^2} \|\nabla w\|_{L^2} \|\nabla \varphi\|_{L^2} \|\Delta \varphi\|_{L^2} + C \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^4 + C \|h\|_{L^2}^2 \leq C + C \|h\|_{L^2}^2. \end{aligned} \quad (3.4)$$

Applying Lemma 2.3, we obtain the following results

$$\|\nabla u\|_{L^2}^2 \leq C(r), \quad \int_t^{t+r} \|\Delta u\|_{L^2}^2 ds \leq C(r), \quad \forall t \geq 2r, r > 0. \quad (3.5)$$

For (3.2)₂, we utilize (2.3) and Young's inequality with ϵ to obtain the following estimate

$$\begin{aligned} & |((\varphi_t, \Delta w))| + |((u \cdot \nabla \varphi, \Delta w))| \\ & \leq \|\varphi_t\|_{L^2} \|\Delta w\|_{L^2} + \|u\|_{L^4} \|\nabla \varphi\|_{L^4} \|\Delta w\|_{L^2} \\ & \leq \epsilon_4 \|\Delta w\|_{L^2}^2 + C_{\epsilon_4} \|\varphi_t\|_{L^2}^2 + C \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla \varphi\|_{L^2}^{\frac{1}{2}} \|\Delta \varphi\|_{L^2}^{\frac{1}{2}} \|\Delta w\|_{L^2} \\ & \leq \epsilon_4 \|\Delta w\|_{L^2}^2 + C_{\epsilon_4} \|\varphi_t\|_{L^2}^2 + \epsilon_5 \|\Delta w\|_{L^2}^2 + C_{\epsilon_5} \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \varphi\|_{L^2} \|\Delta \varphi\|_{L^2}. \end{aligned}$$

Let $\epsilon_4 = \epsilon_5 = \frac{1}{4}$ in above inequality. Then, combining (2.3)₂ and Lemma 2.2, we obtain

$$\frac{1}{2} \|\Delta w\|_{L^2}^2 \leq C \|\varphi_t\|_{L^2}^2 + C \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \varphi\|_{L^2} \|\Delta \varphi\|_{L^2} \leq C \|\varphi_t\|_{L^2}^2 + C. \quad (3.6)$$

We integrate (3.6) from t to $t+r$ to obtain

$$\int_t^{t+r} \|\Delta w\|_{L^2}^2 ds \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \quad (3.7)$$

Assume $\psi = \varphi_t$ in (3.1)₂ and $\xi = -\Delta \varphi_t$ in (3.1)₃. From Lemma 2.5, we obtain the following equalities

$$\begin{cases} \frac{1}{2} \frac{d}{dt} \|\varphi_t\|^2 + a(u_t, \varphi, \varphi_t) + ((\nabla w_t, \nabla \varphi_t)) = 0, \\ \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+1}{2}} \varphi_t\|_{L^2}^2 + ((f'(\varphi) \varphi_t, -\Delta \varphi_t)) + ((w_t, \Delta \varphi_t)) = 0. \end{cases} \quad (3.8)$$

Summing the equalities in (3.8) yields

$$\frac{1}{2} \frac{d}{dt} \|\varphi_t\|^2 + a(u_t, \varphi, \varphi_t) + \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+1}{2}} \varphi_t\|_{L^2}^2 = ((f'(\varphi) \varphi_t, \Delta \varphi_t)). \quad (3.9)$$

Then, we have the inequality

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\varphi_t\|^2 + a_k \|(-\Delta)^{\frac{k+1}{2}} \varphi_t\|_{L^2}^2 \\ & \leq |a(u_t, \varphi, \varphi_t)| + \sum_{i=1}^{k-1} |a_i| \|(-\Delta)^{\frac{i+1}{2}} \varphi_t\|_{L^2}^2 + |((f'(\varphi) \varphi_t, \Delta \varphi_t))|. \end{aligned} \quad (3.10)$$

By Lemma 2.4 and Young's inequality with ϵ , we obtain the following estimates for (3.10).

The estimate of the first term of the right side in (3.10) is as follows

$$\begin{aligned} |a(u_t, \varphi, \varphi_t)| &= \left| \int_{\Omega} (u_t \cdot \nabla) \varphi \cdot \varphi_t dx \right| \\ &\leq C \|u_t\|_{L^2} \|\nabla \varphi\|_{L^4} \|\varphi_t\|_{L^4} \\ &\leq C \|\nabla \varphi\|_{L^4} (\|u_t\|_{L^2}^2 + \|\varphi_t\|_{L^2} \|\nabla \varphi_t\|_{L^2}) \\ &\leq C \|\Delta \varphi\|_{L^2} (\|u_t\|_{L^2}^2 + \|\varphi_t\|_{L^2}^2 + \|\nabla \varphi_t\|_{L^2}^2). \end{aligned}$$

The estimate of the second term of the right side in (3.10) is as follows

$$\begin{aligned}
\sum_{i=1}^{k-1} a_i \|(-\Delta)^{\frac{i+1}{2}} \varphi_t\|_{L^2}^2 &\leq \sum_{i=1}^{k-1} |a_i| \|(-\Delta)^{\frac{k+1}{2}} \varphi_t\|_{L^2}^{\frac{2(i+1)}{k+1}} \|\varphi_t\|_{L^2}^{2-\frac{2(i+1)}{k+1}} \\
&\leq \epsilon \|(-\Delta)^{\frac{k+1}{2}} \varphi_t\|_{L^2}^{\frac{2(i+1)}{k+1} \cdot \frac{k+1}{i+1}} + C_\epsilon \|\varphi_t\|_{L^2}^{(2-\frac{2(i+1)}{k+1}) \cdot \frac{k+1}{k-i}} \\
&\leq \varepsilon \|(-\Delta)^{\frac{k+1}{2}} \varphi_t\|_{L^2}^2 + C_\epsilon \|\varphi_t\|_{L^2}^2.
\end{aligned}$$

The estimate of the third term of the right side in (3.10) is as follows

$$|((f'(\varphi)\varphi_t, \Delta\varphi_t))| = \left| \int_{\Omega} f'(\varphi)\varphi_t \Delta\varphi_t dx \right| \leq \left| \int_{\Omega} |f'(\varphi)|\varphi_t \Delta\varphi_t dx \right| \leq c_0 \|\nabla\varphi_t\|_{L^2}^2.$$

Let $\varepsilon = \frac{a_k}{2}$ in the estimate of the second term of the right side in (3.10). Then, combining Lemma 2.2 and the estimates above, we obtain

$$\begin{aligned}
&\frac{d}{dt} \|\varphi_t\|^2 + a_k \|(-\Delta)^{\frac{k+1}{2}} \varphi_t\|_{L^2}^2 \\
&\leq C \|\Delta\varphi\|_{L^2} (\|u_t\|_{L^2}^2 + \|\varphi_t\|_{L^2}^2 + \|\nabla\varphi_t\|_{L^2}^2) + 2c_0 \|\nabla\varphi_t\|_{L^2}^2 + C \|\varphi_t\|_{L^2}^2 \\
&\leq C (\|u_t\|_{L^2}^2 + \|\varphi_t\|_{L^2}^2 + \|\nabla\varphi_t\|_{L^2}^2).
\end{aligned} \tag{3.11}$$

Combining uniform Gronwall's lemma, Lemma 2.2, and (3.11), we obtain

$$\|\varphi_t\|_{L^2}^2 \leq C(r), \quad \int_t^{t+r} \|(-\Delta)^{\frac{k+1}{2}} \varphi_t\|_{L^2}^2 ds \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \tag{3.12}$$

Let $\psi = (-\Delta)^{k-1} \varphi_t$ in (3.1)₂ and $\xi = (-\Delta)^k \varphi_t$ in (3.1)₃. Then, we obtain

$$\begin{cases} \frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\frac{k-1}{2}} \varphi_t\|_{L^2}^2 + ((u_t \cdot \nabla\varphi + u \cdot \nabla\varphi_t, (-\Delta)^{k-1} \varphi_t)) + ((w_t, (-\Delta)^k \varphi_t)) = 0, \\ \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+k}{2}} \varphi_t\|_{L^2}^2 + ((f'(\varphi)\varphi_t, (-\Delta)^k \varphi_t)) = ((w_t, (-\Delta)^k \varphi_t)). \end{cases} \tag{3.13}$$

Summing the two equations in (3.13) yields

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\frac{k-1}{2}} \varphi_t\|_{L^2}^2 + ((u_t \cdot \nabla\varphi + u \cdot \nabla\varphi_t, (-\Delta)^{k-1} \varphi_t)) \\
&+ \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+k}{2}} \varphi_t\|_{L^2}^2 + ((f'(\varphi)\varphi_t, (-\Delta)^k \varphi_t)) = 0.
\end{aligned}$$

Then, we have the following inequality

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\frac{k-1}{2}} \varphi_t\|_{L^2}^2 + a_k \|(-\Delta)^k \varphi_t\|_{L^2}^2 \\
&\leq |((u_t \cdot \nabla\varphi + u \cdot \nabla\varphi_t, (-\Delta)^{k-1} \varphi_t))| + \sum_{i=1}^{k-1} |a_i| \|(-\Delta)^{\frac{i+k}{2}} \varphi_t\|_{L^2}^2 + |((f'(\varphi)\varphi_t, (-\Delta)^k \varphi_t))|.
\end{aligned} \tag{3.14}$$

By Lemma 2.4 and Young's inequality with ϵ , we obtain the following estimates for (3.14).

The estimate of the first term of the right side in (3.14) is as follows

$$\begin{aligned}
& |((u_t \cdot \nabla \varphi + u \cdot \nabla \varphi_t, (-\Delta)^{k-1} \varphi_t))| \\
& \leq C \|(-\Delta)^{k-1} \varphi_t\|_{L^2} (\|u_t\|_{L^2} \|\nabla \varphi\|_{L^\infty} + \|u\|_{L^2} \|\nabla \varphi_t\|_{L^\infty}) \\
& \leq C \|(-\Delta)^k \varphi_t\|_{L^2}^{\frac{k-1}{k}} \|\varphi_t\|_{L^2}^{\frac{1}{k}} (\|u_t\|_{L^2} \|\nabla \varphi\|_{L^2}^{\frac{1}{2}} \|\varphi\|_{H^3}^{\frac{1}{2}} + \|u\|_{L^2} \|\nabla \varphi_t\|_{L^2}^{\frac{1}{2}} \|\varphi_t\|_{H^3}^{\frac{1}{2}}) \\
& \leq \frac{1}{2} \|(-\Delta)^k \varphi_t\|_{L^2}^{\frac{2(k-1)}{k}} \|\varphi_t\|_{L^2}^{\frac{2}{k}} + C (\|u_t\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} + \|u\|_{L^2}^2 \|\nabla \varphi_t\|_{L^2} \|\varphi_t\|_{H^3}) \\
& \leq \epsilon_1 \|(-\Delta)^k \varphi_t\|_{L^2}^{\frac{2(k-1)}{k} \cdot \frac{k}{k-1}} + C_{\epsilon_1} \|\varphi_t\|_{L^2}^{\frac{2}{k} \cdot k} + C (\|u_t\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} + \|u\|_{L^2}^2 \|\nabla \varphi_t\|_{L^2} \|\varphi_t\|_{H^3}) \\
& \leq \epsilon_1 \|(-\Delta)^k \varphi_t\|_{L^2}^2 + C_{\epsilon_1} \|\varphi_t\|_{L^2}^2 + C \|u_t\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} + C \|u\|_{L^2}^2 (\|\nabla \varphi_t\|_{L^2}^2 + \|\varphi_t\|_{H^3}^2).
\end{aligned}$$

The estimate of the second term of the right side in (3.14) is as follows

$$\begin{aligned}
\sum_{i=1}^{k-1} |a_i| \|(-\Delta)^{\frac{i+k}{2}} \varphi_t\|_{L^2}^2 & \leq C \sum_{i=1}^{k-1} |a_i| \|(-\Delta)^k \varphi_t\|_{L^2}^{\frac{i+k}{k}} \|\varphi_t\|_{L^2}^{2-\frac{i+k}{k}} \\
& \leq \epsilon_2 \|(-\Delta)^k \varphi_t\|_{L^2}^{\frac{i+k}{k} \cdot \frac{2k}{i+k}} + C_{\epsilon_2} \|\varphi_t\|_{L^2}^{(2-\frac{i+k}{k}) \cdot \frac{2k}{k-i}} \\
& \leq \epsilon_2 \|(-\Delta)^k \varphi_t\|_{L^2}^2 + C_{\epsilon_2} \|\varphi_t\|_{L^2}^2.
\end{aligned}$$

The estimate of the third term of the right side in (3.14) is as follows

$$|((f'(\varphi) \varphi_t, (-\Delta)^k \varphi_t))| \leq c_0 \left| \int_{\Omega} \varphi_t (-\Delta)^k \varphi_t dx \right| \leq c_0 \|(-\Delta)^{\frac{k}{2}} \varphi_t\|_{L^2}^2.$$

Assume $\epsilon_1 = \epsilon_2 = \frac{ak}{4}$ in above inequalities. Then, we combine Lemma 2.2 and (3.12) to obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\frac{k-1}{2}} \varphi_t\|_{L^2}^2 + \frac{ak}{2} \|(-\Delta)^k \varphi_t\|_{L^2}^2 \\
& \leq C \|u_t\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} + C \|u\|_{L^2}^2 (\|\nabla \varphi_t\|_{L^2}^2 + \|\nabla \Delta \varphi_t\|_{L^2}^2) + C \|\varphi_t\|_{L^2}^2 + c_0 \|(-\Delta)^{\frac{k}{2}} \varphi_t\|_{L^2}^2 \\
& \leq C \|u_t\|_{L^2}^2 + C (\|\nabla \varphi_t\|_{L^2}^2 + \|\nabla \Delta \varphi_t\|_{L^2}^2) + c_0 \|(-\Delta)^{\frac{k}{2}} \varphi_t\|_{L^2}^2 + C.
\end{aligned}$$

Applying uniform Gronwall's lemma and (3.12), we obtain

$$\|(-\Delta)^{\frac{k-1}{2}} \varphi_t\|_{L^2}^2 \leq C(r), \quad \int_t^{t+r} \|(-\Delta)^k \varphi_t\|_{L^2}^2 ds \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \quad (3.15)$$

By interpolation inequalities in Lemma 2.4 and Young's inequality with ϵ , we obtain

$$\begin{aligned}
C \sum_{i=1}^{k-1} |a_i| \|(-\Delta)^i \varphi_t\|_{L^2}^2 & \leq C \sum_{i=1}^{k-1} |a_i| \|(-\Delta)^k \varphi_t\|_{L^2}^{\frac{2i}{k}} \|\varphi_t\|_{L^2}^{2-\frac{2i}{k}} \\
& \leq \epsilon_1 \|(-\Delta)^k \varphi_t\|_{L^2}^{\frac{2i}{k} \cdot \frac{k}{i}} + C_{\epsilon_1} \|\varphi_t\|_{L^2}^{(2-\frac{2i}{k}) \cdot \frac{k}{k-i}} \\
& \leq \epsilon_1 \|(-\Delta)^k \varphi_t\|_{L^2}^2 + C_{\epsilon_1} \|\varphi_t\|_{L^2}^2.
\end{aligned} \quad (3.16)$$

Assuming $\xi = -w_t$ in (3.1)₃, we obtain

$$\|w_t\|_{L^2}^2 = \sum_{i=1}^k a_i(((-\Delta)^{\frac{i}{2}} \varphi_t, (-\Delta)^{\frac{i}{2}} w_t)) + ((f'(\varphi) \varphi_t, w_t)).$$

Utilizing (3.16), the estimate of $\|w_t\|_{L^2}^2$ is as follows

$$\begin{aligned} \|w_t\|_{L^2}^2 &= ((f'(\varphi) \varphi_t, w_t)) + \sum_{i=1}^k a_i(((-\Delta)^{\frac{i}{2}} \varphi_t, (-\Delta)^{\frac{i}{2}} w_t)) \\ &\leq c_0 \|\varphi_t\|_{L^2} \|w_t\|_{L^2} + \sum_{i=1}^k a_i(((-\Delta)^i \varphi_t, w_t)) \\ &\leq \epsilon_2 \|w_t\|_{L^2}^2 + C_{\epsilon_2} \|\varphi_t\|_{L^2}^2 + \sum_{i=1}^k a_i \|(-\Delta)^i \varphi_t\|_{L^2} \|w_t\|_{L^2} \\ &\leq \epsilon_2 \|w_t\|_{L^2}^2 + C_{\epsilon_2} \|\varphi_t\|_{L^2}^2 + \epsilon_3 \|w_t\|_{L^2}^2 + C_{\epsilon_3} \sum_{i=1}^k a_i \|(-\Delta)^i \varphi_t\|_{L^2}^2 \\ &\leq \epsilon_2 \|w_t\|_{L^2}^2 + C_{\epsilon_2} \|\varphi_t\|_{L^2}^2 + \epsilon_3 \|w_t\|_{L^2}^2 + C_{\epsilon_3} \sum_{i=1}^{k-1} a_i \|(-\Delta)^i \varphi_t\|_{L^2}^2 + C_{\epsilon_3} a_k \|(-\Delta)^k \varphi_t\|_{L^2}^2 \\ &\leq \epsilon_2 \|w_t\|_{L^2}^2 + C_{\epsilon_2} \|\varphi_t\|_{L^2}^2 + \epsilon_3 \|w_t\|_{L^2}^2 + (C_{\epsilon_1, \epsilon_3} a_k + \epsilon_1) \|(-\Delta)^k \varphi_t\|_{L^2}^2 + C_{\epsilon_1, \epsilon_3} \|\varphi_t\|_{L^2}^2. \end{aligned}$$

Assuming $\epsilon_1 = 1$ and $\epsilon_2 = \epsilon_3 = \frac{1}{4}$ in above estimate, from (3.15), we have

$$\|w_t\|_{L^2}^2 \leq (C + 1) \|(-\Delta)^k \varphi_t\|_{L^2}^2 + C \|\varphi_t\|_{L^2}^2.$$

Integrating from t to $t + r$, we obtain

$$\int_t^{t+r} \|w_t\|_{L^2}^2 ds \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \quad (3.17)$$

Assume $v = u_t$ in (3.1)₁ yields

$$\frac{1}{2} \frac{d}{dt} \|u_t\|_{L^2}^2 + a(u_t, u, u_t) + \|\nabla u_t\|_{L^2}^2 = ((w_t \nabla \varphi + w \nabla \varphi_t, u_t)) + (h_t, u_t). \quad (3.18)$$

Thus, we have

$$\frac{1}{2} \frac{d}{dt} \|u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \leq |a(u_t, u, u_t)| + |((w_t \nabla \varphi + w \nabla \varphi_t, u_t))| + (h_t, u_t). \quad (3.19)$$

By Lemma 2.4 and Young's inequality with ϵ , we have the following estimates for (3.19) with the method of energy estimates.

The estimate of the first term of the right side in (3.19) is as follows

$$|a(u_t, u, u_t)| \leq \|u_t\|_{L^4}^2 \|\nabla u\|_{L^2} \leq \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^2} \leq \epsilon_1 \|\nabla u_t\|_{L^2}^2 + C_{\epsilon_1} \|\nabla u\|_{L^2}^2 \|u_t\|_{L^2}^2.$$

The estimate of the second term of the right side in (3.19) is as follows

$$\begin{aligned}
& |((w_t \nabla \varphi + w \nabla \varphi_t, u_t))| \\
& \leq C \|u_t\|_{L^4} \|\nabla \varphi\|_{L^4} \|w_t\|_{L^2} + C \|u_t\|_{L^2} \|w\|_{L^4} \|\nabla \varphi_t\|_{L^4} \\
& \leq C \|u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{1}{2}} \|\nabla \varphi\|_{L^2}^{\frac{1}{2}} \|\Delta \varphi\|_{L^2}^{\frac{1}{2}} \|w_t\|_{L^2} + C \|u_t\|_{L^2} \|w\|_{L^2}^{\frac{1}{2}} \|\nabla w\|_{L^2}^{\frac{1}{2}} \|\nabla \varphi_t\|_{L^2}^{\frac{1}{2}} \|\Delta \varphi_t\|_{L^2}^{\frac{1}{2}} \\
& \leq C \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} + C \|\nabla \varphi\|_{L^2} \|\Delta \varphi\|_{L^2} \|w_t\|_{L^2}^2 + C \|u_t\|_{L^2}^2 \|w\|_{L^2} \|\nabla w\|_{L^2} + C \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2} \\
& \leq \epsilon_2 \|\nabla u_t\|_{L^2}^2 + C_{\epsilon_2} \|u_t\|_{L^2}^2 + C \|\nabla \varphi\|_{L^2} \|\Delta \varphi\|_{L^2} \|w_t\|_{L^2}^2 + C \|w\|_{L^2} \|\nabla w\|_{L^2} \|u_t\|_{L^2}^2 \\
& \quad + C (\|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2).
\end{aligned}$$

The estimate of the third term of the right side in (3.19) is as follows

$$((h_t, u_t)) \leq \frac{1}{2} \|h_t\|_{L^2}^2 + \frac{1}{2} \|u_t\|_{L^2}^2.$$

Let $\epsilon_1 = \epsilon_2 = \frac{1}{4}$ in above estimates. Then combine (3.15), (3.17) and Lemma 2.2 to yield

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|u_t\|_{L^2}^2 + \frac{1}{2} \|\nabla u_t\|_{L^2}^2 \\
& \leq C \|\nabla u\|_{L^2}^4 \|u_t\|_{L^2}^2 + C \|u_t\|_{L^2}^2 + C \|\nabla \varphi\|_{L^2} \|\Delta \varphi\|_{L^2} \|w_t\|_{L^2}^2 \\
& \quad + C \|w\|_{L^2} \|\nabla w\|_{L^2} \|u_t\|_{L^2}^2 + C (\|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2) + \frac{1}{2} \|h_t\|_{L^2}^2 + \frac{1}{2} \|u_t\|_{L^2}^2 \\
& \leq C \|u_t\|_{L^2}^2 + C \|w_t\|_{L^2}^2 + C (\|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2) + \frac{1}{2} \|h_t\|_{L^2}^2.
\end{aligned} \tag{3.20}$$

Applying uniform Gronwall's lemma and utilizing (3.15), (3.17), we obtain

$$\|u_t\|_{L^2}^2 \leq C(r), \quad \int_t^{t+r} \|\nabla u_t\|_{L^2}^2 ds \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \tag{3.21}$$

Let $v = -\Delta u_t$ in (3.1)₁. Then, we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla u_t\|_{L^2}^2 + ((u_t \cdot \nabla u + u \cdot \nabla u_t, -\Delta u_t)) + \|\Delta u_t\|_{L^2}^2 \\
& = ((w_t \nabla \varphi + w \nabla \varphi_t, -\Delta u_t)) + ((h_t, -\Delta u_t)).
\end{aligned} \tag{3.22}$$

Thus,

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla u_t\|_{L^2}^2 + \|\Delta u_t\|_{L^2}^2 \leq |((u_t \cdot \nabla u + u \cdot \nabla u_t, \Delta u_t))| \\
& \quad + |((w_t \nabla \varphi + w \nabla \varphi_t, -\Delta u_t))| + |((h_t, -\Delta u_t))|.
\end{aligned} \tag{3.23}$$

By Lemma 2.4 and Young's inequality with ϵ , we have the following estimates for (3.23) with the method of energy estimates.

The estimate of the first term of the right side in (3.23) is as follows

$$\begin{aligned}
& |((u_t \cdot \nabla u + u \cdot \nabla u_t, \Delta u_t))| \\
& \leq C \|\Delta u_t\|_{L^2} \|u_t\|_{L^4} \|\nabla u\|_{L^4} + C \|\Delta u_t\|_{L^2} \|u\|_{L^4} \|\nabla u_t\|_{L^4} \\
& \leq C \|\Delta u_t\|_{L^2} \|u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}} + \|\Delta u_t\|_{L^2}^{\frac{3}{2}} \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned} &\leq \epsilon_1 \|\Delta u_t\|_{L^2}^2 + C_{\epsilon_1} \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} + \epsilon_2 \|\Delta u_t\|_{L^2}^2 + C_{\epsilon_2} \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \|\nabla u_t\|_{L^2}^2 \\ &\leq \epsilon_1 \|\Delta u_t\|_{L^2}^2 + C_{\epsilon_1} \|u_t\|_{L^2} \|\nabla u\|_{L^2} (\|\nabla u_t\|_{L^2}^2 + \|\Delta u\|_{L^2}^2) + \epsilon_2 \|\Delta u_t\|_{L^2}^2 + C_{\epsilon_2} \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \|\nabla u_t\|_{L^2}^2. \end{aligned}$$

The estimate of the second term of the right side in (3.23) is as follows

$$\begin{aligned} &|((w_t \nabla \varphi + w \nabla \varphi_t, -\Delta u_t))| \\ &\leq C \|w_t\|_{L^2} \|\Delta u_t\|_{L^2} \|\nabla \varphi\|_{L^\infty} + C \|w\|_{L^4} \|\nabla \varphi_t\|_{L^4} \|\Delta u_t\|_{L^2} \\ &\leq C \|w_t\|_{L^2} \|\Delta u_t\|_{L^2} \|\nabla \varphi\|_{L^2}^{\frac{1}{2}} \|\varphi\|_{H^3}^{\frac{1}{2}} + C \|w\|_{L^2}^{\frac{1}{2}} \|\nabla w\|_{L^2}^{\frac{1}{2}} \|\nabla \varphi_t\|_{L^2}^{\frac{1}{2}} \|\Delta \varphi_t\|_{L^2}^{\frac{1}{2}} \|\Delta u_t\|_{L^2} \\ &\leq \epsilon_3 \|\Delta u_t\|_{L^2}^2 + C_{\epsilon_3} \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} \|w_t\|_{L^2}^2 + \epsilon_4 \|\Delta u_t\|_{L^2}^2 + C_{\epsilon_4} \|w\|_{L^2} \|\nabla w\|_{L^2} \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2} \\ &\leq \epsilon_3 \|\Delta u_t\|_{L^2}^2 + C_{\epsilon_3} \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} \|w_t\|_{L^2}^2 + \epsilon_4 \|\Delta u_t\|_{L^2}^2 + C_{\epsilon_4} \|w\|_{L^2} \|\nabla w\|_{L^2} (\|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2). \end{aligned}$$

The estimate of the third term of the right side in (3.23) is as follows

$$|((h_t, -\Delta u_t))| \leq \epsilon_5 \|\Delta u_t\|_{L^2}^2 + C_{\epsilon_5} \|h_t\|_{L^2}^2.$$

Let $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = \frac{1}{8}$ in above estimates. Then, we combine (3.12) and Lemma 2.2 to yield

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\nabla u_t\|_{L^2}^2 + \frac{3}{8} \|\Delta u_t\|_{L^2}^2 \\ &\leq C \|u_t\|_{L^2} \|\nabla u\|_{L^2} (\|\nabla u_t\|_{L^2}^2 + \|\Delta u\|_{L^2}^2) + C \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \|\nabla u_t\|_{L^2}^2 \\ &\quad + C \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} \|w_t\|_{L^2}^2 + C \|w\|_{L^2} \|\nabla w\|_{L^2} (\|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2) + C \|h_t\|_{L^2}^2 \\ &\leq C \|\Delta u\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^2 + C \|w_t\|_{L^2}^2 + C (\|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2) + C \|h_t\|_{L^2}^2. \end{aligned} \quad (3.24)$$

Applying uniform Gronwall's lemma and combining (3.5), (3.15), (3.17), and (3.21), we obtain

$$\|\nabla u_t\|_{L^2}^2 \leq C(r), \quad \int_t^{t+r} \|\Delta u_t\|_{L^2}^2 ds \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \quad (3.25)$$

Therefore, from (3.5), (3.7), (3.12), (3.15), (3.17), (3.21), and (3.25), the proof of Theorem 3.1 is completed. \square

Theorem 3.2. Assuming $h \in L^\infty(0, T; L^2(\Omega))$ and $h_t \in L^\infty(0, T; L^2(\Omega))$, we have $\varphi_t \in L^\infty(0, T; \mathcal{H}^k(\Omega))$. Furthermore, the weak solution (u, φ) satisfies

$$\begin{aligned} &\|(-\Delta)^{\frac{k}{2}} \varphi_t\|_{L^2}^2 \leq C(r), \\ &\int_t^{t+r} \|u_{tt}\|_{L^2}^2 + \|\varphi_{tt}\|_{L^2}^2 + \|\Delta w_t\|_{L^2}^2 ds \leq C(r), \end{aligned}$$

for $\forall t \geq 2r, \quad r > 0$.

Proof. Assume $v = u_{tt}$ in (3.1)₁, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla u_t\|_{L^2}^2 + \|u_{tt}\|_{L^2}^2 + ((u_t \cdot \nabla u + u \cdot \nabla u_t, u_{tt})) = ((w_t \nabla \varphi + w \nabla \varphi_t, u_{tt})) + ((h_t, u_{tt})). \quad (3.26)$$

So, we have

$$\frac{1}{2} \frac{d}{dt} \|\nabla u_t\|_{L^2}^2 + \|u_{tt}\|_{L^2}^2 \leq |((u_t \cdot \nabla u + u \cdot \nabla u_t, u_{tt}))| + |((w_t \nabla \varphi + w \nabla \varphi_t, u_{tt}))| + ((h_t, u_{tt})). \quad (3.27)$$

By (2.3) and Young's inequality with ϵ , we have the following estimates for (3.27) on the basis of Lemma 2.2.

The estimate of the first term of the right side in (3.27) is as follows

$$\begin{aligned} & |((u_t \cdot \nabla u + u \cdot \nabla u_t, u_{tt}))| \\ & \leq C \|u_{tt}\|_{L^2} \|u_t\|_{L^4} \|\nabla u\|_{L^4} + C \|u_{tt}\|_{L^2} \|u\|_{L^4} \|\nabla u_t\|_{L^4} \\ & \leq C \|u_{tt}\|_{L^2} \|u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}} + C \|u_{tt}\|_{L^2} \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{1}{2}} \|\Delta u_t\|_{L^2}^{\frac{1}{2}} \\ & \leq \epsilon_1 \|u_{tt}\|_{L^2}^2 + C_{\epsilon_1} \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} + \epsilon_2 \|u_{tt}\|_{L^2}^2 + C_{\epsilon_2} \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} \|\Delta u_t\|_{L^2} \\ & \leq \epsilon_1 \|u_{tt}\|_{L^2}^2 + C_{\epsilon_1} \|u_t\|_{L^2}^2 \|\nabla u_t\|_{L^2}^2 + C_{\epsilon_1} \|\nabla u\|_{L^2}^2 \|\Delta u\|_{L^2}^2 \\ & \quad + \epsilon_2 \|u_{tt}\|_{L^2}^2 + C_{\epsilon_2} \|\Delta u_t\|_{L^2}^2 + C_{\epsilon_2} \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \|\nabla u_t\|_{L^2}^2. \end{aligned}$$

The estimate of the second term of the right side in (3.27) is as follows

$$\begin{aligned} & |((w_t \nabla \varphi + w \nabla \varphi_t, u_{tt}))| \\ & \leq C \|w_t\|_{L^2} \|u_{tt}\|_{L^2} \|\nabla \varphi\|_{L^\infty} + C \|u_{tt}\|_{L^2} \|w\|_{L^4} \|\nabla \varphi_t\|_{L^4} \\ & \leq C \|w_t\|_{L^2} \|u_{tt}\|_{L^2} \|\nabla \varphi\|_{L^2}^{\frac{1}{2}} \|\varphi\|_{H^3}^{\frac{1}{2}} + C \|u_{tt}\|_{L^2} \|w\|_{L^2}^{\frac{1}{2}} \|\nabla w\|_{L^2}^{\frac{1}{2}} \|\nabla \varphi_t\|_{L^2}^{\frac{1}{2}} \|\Delta \varphi_t\|_{L^2}^{\frac{1}{2}} \\ & \leq \epsilon_3 \|u_{tt}\|_{L^2}^2 + C_{\epsilon_3} \|w_t\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} + \epsilon_4 \|u_{tt}\|_{L^2}^2 + C_{\epsilon_4} \|w\|_{L^2} \|\nabla w\|_{L^2} \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2} \\ & \leq \epsilon_3 \|u_{tt}\|_{L^2}^2 + C_{\epsilon_3} \|w_t\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} + \epsilon_4 \|u_{tt}\|_{L^2}^2 + C_{\epsilon_4} \|w\|_{L^2} \|\nabla w\|_{L^2} (\|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2). \end{aligned}$$

The estimate of the third term of the right side in (3.27) is as follows

$$((h_t, u_{tt})) \leq \epsilon_5 \|u_{tt}\|_{L^2}^2 + C_{\epsilon_5} \|h_t\|_{L^2}^2.$$

Assume $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = \frac{1}{8}$ in above estimates. We combine Theorem 3.1 and Lemma 2.2 to obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla u_t\|_{L^2}^2 + \frac{3}{8} \|u_{tt}\|_{L^2}^2 \\ & \leq C \|u_t\|_{L^2}^2 \|\nabla u_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\Delta u\|_{L^2}^2 + C \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \|\nabla u_t\|_{L^2}^2 \\ & \quad + C \|\Delta u_t\|_{L^2}^2 + C \|w_t\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} + C \|w\|_{L^2} \|\nabla w\|_{L^2} (\|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2) + C \|h_t\|_{L^2}^2 \\ & \leq C \|\Delta u\|_{L^2}^2 + \|\Delta u_t\|_{L^2}^2 + C \|w_t\|_{L^2}^2 + C (\|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2) + C \|h_t\|_{L^2}^2 + C. \end{aligned} \quad (3.28)$$

Apply uniform Gronwall's lemma and utilize Theorem 3.1 and Lemma 2.2 to obtain

$$\|\nabla u_t\|_{L^2}^2 \leq C(r), \quad \int_t^{t+r} \|u_{tt}\|_{L^2}^2 ds \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \quad (3.29)$$

Let $\psi = \varphi_{tt}$ in (3.1)₂, $\xi = -\Delta \varphi_{tt}$ in (3.1)₃. Then, we obtain

$$\begin{cases} \|\varphi_{tt}\|_{L^2}^2 + ((u_t \cdot \nabla \varphi + u \cdot \nabla \varphi_t, \varphi_{tt})) + ((\nabla w_t, \nabla \varphi_{tt})) = 0, \\ \frac{1}{2} \frac{d}{dt} \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+1}{2}} \varphi_t\|_{L^2}^2 + ((w_t, \Delta \varphi_{tt})) = ((f'(\varphi) \varphi_t, \Delta \varphi_{tt})). \end{cases} \quad (3.30)$$

Summing the two equations in (3.30) yields

$$\frac{1}{2} \frac{d}{dt} \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+1}{2}} \varphi_t\|_{L^2}^2 + \|\varphi_{tt}\|_{L^2}^2 + ((u_t \cdot \nabla \varphi + u \cdot \nabla \varphi_t, \varphi_{tt})) = ((f'(\varphi) \varphi_t, \Delta \varphi_{tt})). \quad (3.31)$$

Then,

$$\frac{1}{2} \frac{d}{dt} \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+1}{2}} \varphi_t\|_{L^2}^2 + \|\varphi_{tt}\|_{L^2}^2 \leq |((u_t \cdot \nabla \varphi + u \cdot \nabla \varphi_t, \varphi_{tt}))| + |((f'(\varphi) \varphi_t, \Delta \varphi_{tt}))|. \quad (3.32)$$

Owing to Lemma 2.2, Lemma 2.4, Young's inequality with ϵ , and Theorem 3.1, we easily obtain the following estimates for (3.32).

The estimate of the first term of the right side in (3.32) is as follows

$$\begin{aligned} & |((u_t \cdot \nabla \varphi + u \cdot \nabla \varphi_t, \varphi_{tt}))| \\ & \leq C \|\varphi_{tt}\|_{L^2} (\|u_t\|_{L^4} \|\nabla \varphi\|_{L^4} + \|u\|_{L^4} \|\nabla \varphi_t\|_{L^4}) \\ & \leq C \|\varphi_{tt}\|_{L^2} (\|u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{1}{2}} \|\nabla \varphi\|_{L^2}^{\frac{1}{2}} \|\Delta \varphi\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla \varphi_t\|_{L^2}^{\frac{1}{2}} \|\Delta \varphi_t\|_{L^2}^{\frac{1}{2}}) \\ & \leq \epsilon_1 \|\varphi_{tt}\|_{L^2}^2 + C_{\epsilon_1} (\|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla \varphi\|_{L^2} \|\Delta \varphi\|_{L^2} + \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2}) \\ & \leq \epsilon_1 \|\varphi_{tt}\|_{L^2}^2 + C_{\epsilon_1} (\|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla \varphi\|_{L^2} \|\Delta \varphi\|_{L^2} + \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2). \end{aligned}$$

The estimate of the second term of the right side in (3.32) is as follows

$$\begin{aligned} & |((f'(\varphi) \varphi_t, \Delta \varphi_{tt}))| \\ & \leq |((\nabla(f'(\varphi) \nabla \varphi_t + f''(\varphi) \nabla \varphi \varphi_t), \varphi_{tt}))| \\ & \leq |((f'(\varphi) \Delta \varphi_t + 2f''(\varphi) \nabla \varphi \nabla \varphi_t + f''(\varphi) \Delta \varphi \varphi_t + f'''(\varphi) |\Delta \varphi|^2 \varphi_t, \varphi_{tt}))| \\ & \leq C \|\varphi_{tt}\|_{L^2} (c_0 \|\Delta \varphi_t\|_{L^2} + 2c_0 \|\nabla \varphi_t\|_{L^2} \|\nabla \varphi\|_{L^\infty} + c_0 \|\varphi_t\|_{L^2} \|\Delta \varphi\|_{L^\infty} + c_0 \|\Delta \varphi\|_{L^4}^2 \|\varphi_t\|_{L^\infty}) \\ & \leq C \|\varphi_{tt}\|_{L^2} (\|\Delta \varphi_t\|_{L^2} + \|\nabla \varphi_t\|_{L^2} \|\nabla \varphi\|_{L^2}^{\frac{1}{2}} \|\varphi\|_{H^3}^{\frac{1}{2}} + \|\varphi_t\|_{L^2} \|\Delta \varphi\|_{L^2}^{\frac{1}{2}} \|\varphi\|_{H^4}^{\frac{1}{2}} \\ & \quad + \|\Delta \varphi\|_{L^2} \|\varphi\|_{H^3} \|\varphi_t\|_{L^2}^{\frac{1}{2}} \|\Delta \varphi_t\|_{L^2}^{\frac{1}{2}}) \\ & \leq \epsilon_2 \|\varphi_{tt}\|_{L^2}^2 + C_{\epsilon_2} \|\Delta \varphi_t\|_{L^2}^2 + C_{\epsilon_2} \|\nabla \varphi_t\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} + C_{\epsilon_2} \|\varphi_t\|_{L^2}^2 \|\Delta \varphi\|_{L^2} \|\varphi\|_{H^4} \\ & \quad + C_{\epsilon_2} \|\Delta \varphi\|_{L^2}^2 \|\varphi\|_{H^3}^2 \|\varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2}. \end{aligned}$$

Assume $\epsilon_1 = \epsilon_2 = \frac{1}{4}$ in above estimates. We combine (3.29), (3.32), Lemma 2.2, and Theorem 3.1 to yield

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+1}{2}} \varphi_t\|_{L^2}^2 + \frac{1}{2} \|\varphi_{tt}\|_{L^2}^2 \\ & \leq C (\|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla \varphi\|_{L^2} \|\Delta \varphi\|_{L^2} + \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2) \\ & \quad + C \|\Delta \varphi_t\|_{L^2}^2 + C \|\nabla \varphi_t\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} \\ & \quad + C \|\varphi_t\|_{L^2}^2 \|\Delta \varphi\|_{L^2} \|\varphi\|_{H^4} + C \|\Delta \varphi\|_{L^2}^2 \|\varphi\|_{H^3}^2 \|\varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2} \\ & \leq C \|\nabla \varphi_t\|_{L^2}^2 + C \|\Delta \varphi_t\|_{L^2}^2 + C. \end{aligned} \quad (3.33)$$

Employing uniform Gronwall's lemma and utilizing Lemma 2.2 and Theorem 3.1, we obtain

$$\sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+1}{2}} \varphi_t\|_{L^2}^2 \leq C(r), \quad \int_t^{t+r} \|\varphi_{tt}\|_{L^2}^2 ds \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \quad (3.34)$$

Assuming $\psi = -\Delta w_t$ in (3.1)₂, we obtain

$$\|\Delta w_t\|_{L^2}^2 = ((\varphi_{tt}, \Delta w_t)) + ((u_t \cdot \nabla \varphi + u \cdot \nabla \varphi_t, \Delta w_t)). \quad (3.35)$$

By utilizing Lemma 2.4 and Young's inequality with ϵ , we obtain the following estimates for (3.35) with the method of energy estimates

$$\begin{aligned} & |((\varphi_{tt}, \Delta w_t))| + |((u_t \cdot \nabla \varphi + u \cdot \nabla \varphi_t, \Delta w_t))| \\ & \leq \|\Delta w_t\|_{L^2} \|\varphi_{tt}\|_{L^2} + C \|\Delta w_t\|_{L^2} (\|u_t\|_{L^2} \|\nabla \varphi\|_{L^\infty} + \|u\|_{L^4} \|\nabla \varphi_t\|_{L^4}) \\ & \leq \epsilon_1 \|\Delta w_t\|_{L^2}^2 + C_{\epsilon_1} \|\varphi_{tt}\|_{L^2}^2 + \epsilon_2 \|\Delta w_t\|_{L^2}^2 + C_{\epsilon_2} (\|u_t\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} \\ & \quad + \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2}) \\ & \leq \epsilon_1 \|\Delta w_t\|_{L^2}^2 + C_{\epsilon_1} \|\varphi_{tt}\|_{L^2}^2 + \epsilon_2 \|\Delta w_t\|_{L^2}^2 + C_{\epsilon_2} \|u_t\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} \\ & \quad + C_{\epsilon_2} \|u\|_{L^2} \|\nabla u\|_{L^2} (\|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2). \end{aligned}$$

Assume $\epsilon_1 = \epsilon_2 = \frac{1}{4}$ in above inequality. Then, combining Lemma 2.2 and Theorem 3.1, we obtain

$$\begin{aligned} \frac{1}{2} \|\Delta w_t\|_{L^2}^2 & \leq C \|\varphi_{tt}\|_{L^2}^2 + C \|u_t\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} + C \|u\|_{L^2} \|\nabla u\|_{L^2} (\|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2) \\ & \leq C \|\varphi_{tt}\|_{L^2}^2 + C (\|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2) + C. \end{aligned} \quad (3.36)$$

Integrating (3.36) from t to $t+r$ and combining (3.34) and Theorem 3.1, we obtain

$$\int_t^{t+r} \|\Delta w_t\|_{L^2}^2 ds \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \quad (3.37)$$

Assume $\psi = (-\Delta)^k \varphi_t$ in (3.1)₂. It follows that

$$\frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\frac{k}{2}} \varphi_t\|_{L^2}^2 + ((u_t \cdot \nabla \varphi + u \cdot \nabla \varphi_t, (-\Delta)^k \varphi_t)) = ((\Delta w_t, (-\Delta)^k \varphi_t)). \quad (3.38)$$

Furthermore, we have

$$\frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\frac{k}{2}} \varphi_t\|_{L^2}^2 \leq |((u_t \cdot \nabla \varphi + u \cdot \nabla \varphi_t, (-\Delta)^k \varphi_t))| + |((\Delta w_t, (-\Delta)^k \varphi_t))|. \quad (3.39)$$

Utilizing Lemma 2.4 and Young's inequality with ϵ , we have the following estimate for (3.39)

$$\begin{aligned} & |((\Delta w_t, (-\Delta)^k \varphi_t))| + |((u_t \cdot \nabla \varphi + u \cdot \nabla \varphi_t, (-\Delta)^k \varphi_t))| \\ & \leq \|\Delta w_t\|_{L^2} \|(-\Delta)^k \varphi_t\|_{L^2} + C \|(-\Delta)^k \varphi_t\|_{L^2} (\|u_t\|_{L^2} \|\nabla \varphi\|_{L^\infty} + \|u\|_{L^4} \|\nabla \varphi_t\|_{L^4}) \\ & \leq \epsilon_1 \|(-\Delta)^k \varphi_t\|_{L^2}^2 + C_{\epsilon_1} \|\Delta w_t\|_{L^2}^2 + \epsilon_2 \|(-\Delta)^k \varphi_t\|_{L^2}^2 \\ & \quad + C_{\epsilon_2} (\|u_t\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} + \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2}) \end{aligned}$$

$$\begin{aligned} &\leq \epsilon_1 \|(-\Delta)^k \varphi_t\|_{L^2}^2 + C_{\epsilon_1} \|\Delta w_t\|_{L^2}^2 + \epsilon_2 \|(-\Delta)^k \varphi_t\|_{L^2}^2 + C_{\epsilon_2} \|u_t\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} \\ &\quad + C_{\epsilon_2} \|u\|_{L^2} \|\nabla u\|_{L^2} (\|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2). \end{aligned}$$

Assume $\epsilon_1 = \epsilon_2 = \frac{1}{4}$ in above inequality. We combine (3.39) and Lemma 2.2 to obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\frac{k}{2}} \varphi_t\|_{L^2}^2 \\ &\leq \frac{1}{2} \|(-\Delta)^k \varphi_t\|_{L^2}^2 + C \|\Delta w_t\|_{L^2}^2 + C \|u_t\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} \\ &\quad + C \|u\|_{L^2} \|\nabla u\|_{L^2} (\|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2) \\ &\leq \frac{1}{2} \|(-\Delta)^k \varphi_t\|_{L^2}^2 + C \|\Delta w_t\|_{L^2}^2 + C (\|\nabla \varphi_t\|_{L^2}^2 + \|\Delta \varphi_t\|_{L^2}^2) + C. \end{aligned} \quad (3.40)$$

Applying uniform Gronwall's lemma in (3.40), and utilizing (3.37) and Theorem 3.1, we obtain the following result

$$\|(-\Delta)^{\frac{k}{2}} \varphi_t\|_{L^2}^2 \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \quad (3.41)$$

Therefore, from (3.29), (3.34), (3.37), and (3.41), the proof of Theorem 3.2 is completed. \square

4. (H^1, H^k) -Regularity of (u_{tt}, φ_{tt})

For further consideration of the regularity of the second derivative of the weak solution to the higher-order Navier-Stokes-Cahn-Hilliard system, we differentiate the Eq. (3.1) with respect to t . Then, we have

$$\begin{cases} ((u_{ttt}, v)) + ((u_{tt} \cdot \nabla u + 2u_t \cdot \nabla u_t + u \cdot \nabla u_{tt}, v)) + ((\nabla u_{tt}, \nabla v)) \\ - ((w_{tt} \nabla \varphi + 2w_t \nabla \varphi_t + w \nabla \varphi_{tt}, v)) = ((h_{tt}, v)), \\ ((\varphi_{ttt}, \psi)) + ((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, \psi)) + ((\nabla w_{tt}, \nabla \psi)) = 0, \\ \sum_{i=1}^k a_i (((-\Delta)^{\frac{i}{2}} \varphi_{tt}, (-\Delta)^{\frac{i}{2}} \xi)) + ((f''(\varphi) \varphi_t^2 + f'(\varphi) \varphi_{tt}, \xi)) - ((w_{tt}, \xi)) = 0. \end{cases} \quad (4.1)$$

Theorem 4.1. Assuming $h \in L^\infty(0, T; L^2(\Omega))$, $h_t \in L^\infty(0, T; L^2(\Omega))$ and $h_{tt} \in L^\infty(0, T; L^2(\Omega))$, we have $u_{tt} \in L^\infty(0, T; V_0(\Omega))$ and $\varphi_{tt} \in L^\infty(0, T; \mathcal{H}^{k-1}(\Omega))$. Furthermore, the weak solution (u, φ) satisfies

$$\begin{aligned} &\|(-\Delta)^{\frac{k-1}{2}} \varphi_{tt}\|_{L^2}^2 + \|\nabla u_{tt}\|_{L^2}^2 + \|\varphi_{tt}\|_{L^2}^2 + \|u_{tt}\|_{L^2}^2 \leq C(r), \\ &\int_t^{t+r} \|w_{tt}\|_{L^2}^2 + \|(-\Delta)^k \varphi_{tt}\|_{L^2}^2 + \|\Delta u_{tt}\|_{L^2}^2 ds \leq C(r), \end{aligned}$$

for $\forall t \geq 2r, \quad r > 0$.

Proof. Let $\psi = \varphi_{tt}$ in (4.1)₂, $\xi = -\Delta \varphi_{tt}$ in (4.1)₃. Then we have

$$\begin{cases} \frac{1}{2} \frac{d}{dt} \|\varphi_{tt}\|_{L^2}^2 + ((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, \varphi_{tt})) + ((\nabla w_{tt}, \nabla \varphi_{tt})) = 0, \\ \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+1}{2}} \varphi_{tt}\|_{L^2}^2 - ((\nabla w_{tt}, \nabla \varphi_{tt})) = ((f''(\varphi) \varphi_t^2 + f'(\varphi) \varphi_{tt}, \Delta \varphi_{tt})). \end{cases} \quad (4.2)$$

Summing the equations in (4.2) yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\varphi_{tt}\|_{L^2}^2 + \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+1}{2}} \varphi_{tt}\|_{L^2}^2 + ((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, \varphi_{tt})) \\ & = ((f''(\varphi)\varphi_t^2 + f'(\varphi)\varphi_{tt}, \Delta \varphi_{tt})). \end{aligned} \quad (4.3)$$

Thus,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\varphi_{tt}\|_{L^2}^2 + a_k \|(-\Delta)^{\frac{k+1}{2}} \varphi_{tt}\|_{L^2}^2 \\ & \leq \sum_{i=1}^{k-1} |a_i| \|(-\Delta)^{\frac{i+1}{2}} \varphi_{tt}\|_{L^2}^2 + |((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, \varphi_{tt}))| \\ & \quad + |((f''(\varphi)\varphi_t^2 + f'(\varphi)\varphi_{tt}, \Delta \varphi_{tt}))|. \end{aligned} \quad (4.4)$$

Using Lemma 2.4 and Young's inequality with ϵ , we present the following estimates for (4.4) by energy estimates.

The estimate of the first term of the right side in (4.4) is as follows

$$\begin{aligned} \sum_{i=1}^{k-1} |a_i| \|(-\Delta)^{\frac{i+1}{2}} \varphi_{tt}\|_{L^2}^2 & \leq C \sum_{i=1}^{k-1} |a_i| \|(-\Delta)^{\frac{k+1}{2}} \varphi_{tt}\|_{L^2}^{\frac{2(i+1)}{k+1}} \|\varphi_{tt}\|_{L^2}^{2-\frac{2(i+1)}{k+1}} \\ & \leq \epsilon_1 \|(-\Delta)^{\frac{k+1}{2}} \varphi_{tt}\|_{L^2}^{\frac{2(i+1)}{k+1} \cdot \frac{k+1}{i+1}} + C_{\epsilon_1} \|\varphi_{tt}\|_{L^2}^{(2-\frac{2(i+1)}{k+1}) \cdot \frac{k+1}{k-i}} \\ & \leq \epsilon_1 \|(-\Delta)^{\frac{k+1}{2}} \varphi_{tt}\|_{L^2}^2 + C_{\epsilon_1} \|\varphi_{tt}\|_{L^2}^2. \end{aligned}$$

The estimate of the second term of the right side in (4.4) is as follows

$$\begin{aligned} & |((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, \varphi_{tt}))| \\ & \leq |a(u_{tt}, \varphi, \varphi_{tt})| + 2|a(u_t, \varphi_t, \varphi_{tt})| \\ & \leq C \|u_{tt}\|_{L^2} \|\varphi_{tt}\|_{L^2} \|\nabla \varphi\|_{L^\infty} + C \|u_t\|_{L^4} \|\nabla \varphi_t\|_{L^4} \|\varphi_{tt}\|_{L^2} \\ & \leq C \|u_{tt}\|_{L^2} \|\varphi_{tt}\|_{L^2} \|\nabla \varphi\|_{L^2}^{\frac{1}{2}} \|\varphi\|_{H^3}^{\frac{1}{2}} + C \|u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{1}{2}} \|\nabla \varphi_t\|_{L^2}^{\frac{1}{2}} \|\Delta \varphi_t\|_{L^2}^{\frac{1}{2}} \|\varphi_{tt}\|_{L^2} \\ & \leq C \|\nabla \varphi\|_{L^2}^{\frac{1}{2}} \|\varphi\|_{H^3}^{\frac{1}{2}} (\|u_{tt}\|_{L^2}^2 + \|\varphi_{tt}\|_{L^2}^2) + C \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\varphi_{tt}\|_{L^2}^2 + C \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2}. \end{aligned}$$

The estimate of the third term of the right side in (4.4) is as follows

$$\begin{aligned} & |((f''(\varphi)\varphi_t^2 + f'(\varphi)\varphi_{tt}, \Delta \varphi_{tt}))| \\ & \leq C \|\Delta \varphi_{tt}\|_{L^2} (c_0 \|\varphi_t^2\|_{L^2} + c_0 \|\varphi_{tt}\|_{L^2}) \\ & \leq C \|\varphi_{tt}\|_{H^{k+1}} (\|\varphi_t\|_{L^2} \|\nabla \varphi_t\|_{L^2} + \|\varphi_{tt}\|_{L^2}) \\ & \leq \epsilon_2 \|\varphi_{tt}\|_{H^{k+1}}^2 + C_{\epsilon_2} (\|\varphi_t\|_{L^2}^2 \|\nabla \varphi_t\|_{L^2}^2 + \|\varphi_{tt}\|_{L^2}^2). \end{aligned}$$

Let $\epsilon_1 = \epsilon_2 = \frac{ak}{4}$ in above estimates. Then, we combine (4.4), Theorem 3.1 and Theorem 3.2 to yield

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\varphi_{tt}\|_{L^2}^2 + \frac{ak}{2} \|(-\Delta)^{\frac{k+1}{2}} \varphi_{tt}\|_{L^2}^2 \leq C \|\varphi_{tt}\|_{L^2}^2 + C \|\nabla \varphi\|_{L^2}^{\frac{1}{2}} \|\varphi\|_{H^3}^{\frac{1}{2}} (\|u_{tt}\|_{L^2}^2 + \|\varphi_{tt}\|_{L^2}^2) \\ & \quad + C \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\varphi_{tt}\|_{L^2}^2 + C \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2} + C \|\varphi_t\|_{L^2}^2 \|\nabla \varphi_t\|_{L^2}^2 \\ & \leq C \|\varphi_{tt}\|_{L^2}^2 + C \|u_{tt}\|_{L^2}^2 + C. \end{aligned} \quad (4.5)$$

Applying uniform Gronwall's lemma and utilizing Theorem 3.2, we obtain

$$\|\varphi_{tt}\|_{L^2}^2 \leq C(r), \quad \int_t^{t+r} \|(-\Delta)^{\frac{k+1}{2}} \varphi_{tt}\|_{L^2}^2 ds \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \quad (4.6)$$

Let $\psi = (-\Delta)^{k-1} \varphi_{tt}$ in (4.1)₂ and $\xi = (-\Delta)^k \varphi_{tt}$ in (4.1)₃. Then, we obtain

$$\begin{cases} ((\varphi_{ttt}, (-\Delta)^{k-1} \varphi_{tt})) + ((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, (-\Delta)^{k-1} \varphi_{tt})) + ((w_{tt}, (-\Delta)^k \varphi_{tt})) = 0, \\ \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+k}{2}} \varphi_{tt}\|_{L^2}^2 + ((f''(\varphi) \varphi_t^2 + f'(\varphi) \varphi_{tt}, (-\Delta)^k \varphi_{tt})) = ((w_{tt}, (-\Delta)^k \varphi_{tt})). \end{cases} \quad (4.7)$$

Sum the equations in (4.7) to obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\frac{k-1}{2}} \varphi_{tt}\|_{L^2}^2 + ((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, (-\Delta)^{k-1} \varphi_{tt})) \\ & + \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+k}{2}} \varphi_{tt}\|_{L^2}^2 + ((f''(\varphi) \varphi_t^2 + f'(\varphi) \varphi_{tt}, (-\Delta)^k \varphi_{tt})) = 0. \end{aligned}$$

Then, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\frac{k-1}{2}} \varphi_{tt}\|_{L^2}^2 + a_k \|(-\Delta)^k \varphi_{tt}\|_{L^2}^2 \\ & \leq |((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, (-\Delta)^{k-1} \varphi_{tt}))| \\ & + \sum_{i=1}^{k-1} |a_i| \|(-\Delta)^{\frac{i+k}{2}} \varphi_{tt}\|_{L^2}^2 + |((f''(\varphi) \varphi_t^2 + f'(\varphi) \varphi_{tt}, (-\Delta)^k \varphi_{tt}))|. \end{aligned} \quad (4.8)$$

Utilizing Lemma 2.4 and Young's inequality with ϵ , we obtain the following estimates for (4.8).

The estimate of the first term of the right side in (4.8) is as follows

$$\begin{aligned} & |((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, (-\Delta)^{k-1} \varphi_{tt}))| \\ & \leq C \|(-\Delta)^{k-1} \varphi_{tt}\|_{L^2} (\|u_{tt}\|_{L^2} \|\nabla \varphi\|_{L^\infty} + \|u_t\|_{L^4} \|\nabla \varphi_t\|_{L^4} + \|u\|_{L^4} \|\nabla \varphi_{tt}\|_{L^4}) \\ & \leq C \|(-\Delta)^k \varphi_{tt}\|_{L^2}^{\frac{k-1}{k}} \|\varphi_{tt}\|_{L^2}^{\frac{1}{k}} (\|u_{tt}\|_{L^2} \|\nabla \varphi\|_{L^2}^{\frac{1}{2}} \|\varphi\|_{H^3}^{\frac{1}{2}} + \|u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{1}{2}} \|\nabla \varphi_t\|_{L^2}^{\frac{1}{2}} \|\Delta \varphi_t\|_{L^2}^{\frac{1}{2}} \\ & \quad + \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla \varphi_{tt}\|_{L^2}^{\frac{1}{2}} \|\Delta \varphi_{tt}\|_{L^2}^{\frac{1}{2}}) \\ & \leq \frac{1}{2} \|(-\Delta)^k \varphi_{tt}\|_{L^2}^{\frac{2(k-1)}{k}} \|\varphi_{tt}\|_{L^2}^{\frac{2}{k}} + C (\|u_{tt}\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} + \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2} \\ & \quad + \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \varphi_{tt}\|_{L^2} \|\Delta \varphi_{tt}\|_{L^2}) \\ & \leq \epsilon_1 \|(-\Delta)^k \varphi_{tt}\|_{L^2}^{\frac{2(k-1)}{k} \cdot \frac{k}{k-1}} + C_{\epsilon_1} \|\varphi_{tt}\|_{L^2}^{\frac{2}{k} \cdot k} + C (\|u_{tt}\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} \\ & \quad + \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2} + \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \varphi_{tt}\|_{L^2} \|\Delta \varphi_{tt}\|_{L^2}) \\ & \leq \epsilon_1 \|(-\Delta)^k \varphi_{tt}\|_{L^2}^2 + C_{\epsilon_1} \|\varphi_{tt}\|_{L^2}^2 + C \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} \|u_{tt}\|_{L^2}^2 \\ & \quad + C \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2} + C \|u\|_{L^2} \|\nabla u\|_{L^2} (\|\nabla \varphi_{tt}\|_{L^2}^2 + \|\Delta \varphi_{tt}\|_{L^2}^2). \end{aligned}$$

The estimate of the second term of the right side in (4.8) is as follows

$$\begin{aligned} \sum_{i=1}^{k-1} |a_i| \|(-\Delta)^{\frac{i+k}{2}} \varphi_{tt}\|_{L^2}^2 &\leq C \sum_{i=1}^{k-1} |a_i| \|(-\Delta)^k \varphi_{tt}\|_{L^2}^{\frac{i+k}{k}} \|\varphi_{tt}\|_{L^2}^{2-\frac{i+k}{k}} \\ &\leq \epsilon_2 \|(-\Delta)^k \varphi_{tt}\|_{L^2}^{\frac{i+k}{k} \cdot \frac{2k}{i+k}} + C_{\epsilon_2} \|\varphi_{tt}\|_{L^2}^{(2-\frac{i+k}{k}) \cdot \frac{2k}{k-i}} \\ &\leq \epsilon_2 \|(-\Delta)^k \varphi_{tt}\|_{L^2}^2 + C_{\epsilon_2} \|\varphi_{tt}\|_{L^2}^2. \end{aligned}$$

The estimate of the third term of the right side in (4.8) is as follows

$$\begin{aligned} |((f''(\varphi)\varphi_t^2 + f'(\varphi)\varphi_{tt}), (-\Delta)^k \varphi_{tt})| &\leq C \|(-\Delta)^k \varphi_{tt}\|_{L^2} (c_0 \|\varphi_t^2\|_{L^2} + c_0 \|\varphi_{tt}\|_{L^2}) \\ &\leq C \|(-\Delta)^k \varphi_{tt}\|_{L^2} (c_0 \|\varphi_t\|_{L^2} \|\nabla \varphi_t\|_{L^2} + c_0 \|\varphi_{tt}\|_{L^2}) \\ &\leq \epsilon_3 \|(-\Delta)^k \varphi_{tt}\|_{L^2}^2 + (C_{\epsilon_3} \|\varphi_t\|_{L^2}^2 \|\nabla \varphi_t\|_{L^2}^2 + C_{\epsilon_3} \|\varphi_{tt}\|_{L^2}^2). \end{aligned}$$

Assume $\epsilon_1 = \epsilon_2 = \epsilon_3 = \frac{ak}{4}$ in above inequalities. Then, combining Lemma 2.2, Theorem 3.1, Theorem 3.2, (4.6), and (4.8), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\frac{k-1}{2}} \varphi_{tt}\|_{L^2}^2 + \frac{ak}{4} \|(-\Delta)^k \varphi_{tt}\|_{L^2}^2 \\ \leq C \|\varphi_{tt}\|_{L^2}^2 + C \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} \|u_{tt}\|_{L^2}^2 + C \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2} \\ + C \|u\|_{L^2} \|\nabla u\|_{L^2} (\|\nabla \varphi_{tt}\|_{L^2}^2 + \|\Delta \varphi_{tt}\|_{L^2}^2) + C \|\varphi_{tt}\|_{L^2}^2 + C \|\varphi_t\|_{L^2}^2 \|\nabla \varphi_t\|_{L^2}^2 + C \|\varphi_{tt}\|_{L^2}^2 \\ \leq C \|u_{tt}\|_{L^2}^2 + C (\|\nabla \varphi_{tt}\|_{L^2}^2 + \|\Delta \varphi_{tt}\|_{L^2}^2) + C. \end{aligned}$$

Applying uniform Gronwall's lemma and (4.6), we obtain

$$\begin{aligned} \|(-\Delta)^{\frac{k-1}{2}} \varphi_{tt}\|_{L^2}^2 &\leq C(r), \\ \int_t^{t+r} \|(-\Delta)^k \varphi_{tt}\|_{L^2}^2 ds &\leq C(r), \quad \forall t \geq 2r, \quad r > 0. \end{aligned} \tag{4.9}$$

By Lemma 2.4 and Young's inequality with ϵ , we obtain the following estimate

$$\begin{aligned} C \sum_{i=1}^{k-1} |a_i| \|(-\Delta)^i \varphi_{tt}\|_{L^2}^2 &\leq C \sum_{i=1}^{k-1} |a_i| \|(-\Delta)^k \varphi_{tt}\|_{L^2}^{\frac{2i}{k}} \|\varphi_{tt}\|_{L^2}^{2-\frac{2i}{k}} \\ &\leq \epsilon_1 \|(-\Delta)^k \varphi_{tt}\|_{L^2}^{\frac{2i}{k} \cdot \frac{k}{i}} + C_{\epsilon_1} \|\varphi_{tt}\|_{L^2}^{(2-\frac{2i}{k}) \cdot \frac{k}{k-i}} \\ &\leq \epsilon_1 \|(-\Delta)^k \varphi_{tt}\|_{L^2}^2 + C_{\epsilon_1} \|\varphi_{tt}\|_{L^2}^2, \quad \forall \epsilon_1 > 0. \end{aligned} \tag{4.10}$$

Assuming $\xi = -w_{tt}$ in (4.1)₃, we have

$$\|w_{tt}\|_{L^2}^2 = \sum_{i=1}^k a_i (((-\Delta)^{\frac{i}{2}} \varphi_{tt}, (-\Delta)^{\frac{i}{2}} w_{tt})) + ((f''(\varphi)\varphi_t^2 + f'(\varphi)\varphi_{tt}), w_{tt}).$$

Combining (4.10), we obtain the estimate

$$\begin{aligned}
\|w_{tt}\|_{L^2}^2 &= ((f''(\varphi)\varphi_t^2 + f'(\varphi)\varphi_{tt}, w_{tt})) + \sum_{i=1}^k a_i((-\Delta)^{\frac{i}{2}}\varphi_{tt}, (-\Delta)^{\frac{i}{2}}w_{tt}) \\
&\leq c_0\|\varphi_{tt}\|_{L^4}^2\|w_{tt}\|_{L^2} + c_0\|\varphi_{tt}\|_{L^2}\|w_{tt}\|_{L^2} + \sum_{i=1}^k a_i((-\Delta)^i\varphi_{tt}, w_{tt}) \\
&\leq \epsilon_2\|w_{tt}\|_{L^2}^2 + C_{\epsilon_2}\|\varphi_{tt}\|_{L^2}^2\|\nabla\varphi_{tt}\|_{L^2}^2 + \epsilon_3\|w_{tt}\|_{L^2}^2 \\
&\quad + C_{\epsilon_3}\|\varphi_{tt}\|_{L^2}^2 + \sum_{i=1}^k |a_i|\|(-\Delta)^i\varphi_{tt}\|_{L^2}\|w_{tt}\|_{L^2} \\
&\leq \epsilon_2\|w_{tt}\|_{L^2}^2 + C_{\epsilon_2}\|\varphi_{tt}\|_{L^2}^2\|\nabla\varphi_{tt}\|_{L^2}^2 + \epsilon_3\|w_{tt}\|_{L^2}^2 + C_{\epsilon_3}\|\varphi_{tt}\|_{L^2}^2 \\
&\quad + \epsilon_4\|w_{tt}\|_{L^2}^2 + C_{\epsilon_4}\sum_{i=1}^k |a_i|\|(-\Delta)^i\varphi_{tt}\|_{L^2}^2 \\
&\leq \epsilon_2\|w_{tt}\|_{L^2}^2 + C_{\epsilon_2}\|\varphi_{tt}\|_{L^2}^2\|\nabla\varphi_{tt}\|_{L^2}^2 + \epsilon_3\|w_{tt}\|_{L^2}^2 + C_{\epsilon_3}\|\varphi_{tt}\|_{L^2}^2 + \epsilon_4\|w_{tt}\|_{L^2}^2 \\
&\quad + C_{\epsilon_4}\sum_{i=1}^{k-1} |a_i|\|(-\Delta)^i\varphi_{tt}\|_{L^2}^2 + C_{\epsilon_4}a_k\|(-\Delta)^k\varphi_{tt}\|_{L^2}^2 \\
&\leq \epsilon_2\|w_{tt}\|_{L^2}^2 + C_{\epsilon_2}\|\varphi_{tt}\|_{L^2}^2\|\nabla\varphi_{tt}\|_{L^2}^2 + \epsilon_3\|w_{tt}\|_{L^2}^2 + C_{\epsilon_3}\|\varphi_{tt}\|_{L^2}^2 + \epsilon_4\|w_{tt}\|_{L^2}^2 \\
&\quad + (C_{\epsilon_4}a_k + \epsilon_1)\|(-\Delta)^k\varphi_{tt}\|_{L^2}^2 + C_{\epsilon_1, \epsilon_4}\|\varphi_{tt}\|_{L^2}^2.
\end{aligned}$$

Assume $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \frac{1}{4}$ in above estimate. From (4.9), we obtain

$$\|w_{tt}\|_{L^2}^2 \leq (C+1)\|(-\Delta)^k\varphi_{tt}\|_{L^2}^2 + C\|\varphi_{tt}\|_{L^2}^2\|\nabla\varphi_{tt}\|_{L^2}^2 + C_\epsilon\|\varphi_{tt}\|_{L^2}^2. \quad (4.11)$$

Integrating (4.11) from t to $t+r$, we obtain

$$\int_t^{t+r} \|w_{tt}\|_{L^2}^2 ds \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \quad (4.12)$$

Assuming $v = u_{tt}$ in (4.1)₁, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_{tt}\|_{L^2}^2 + ((u_{tt} \cdot \nabla u + 2u_t \cdot \nabla u_t + u \cdot \nabla u_{tt}, u_{tt})) + \|\nabla u_{tt}\|_{L^2}^2 \\
&= ((w_{tt}\nabla\varphi + 2w_t\nabla\varphi_t + w\nabla\varphi_{tt}, u_{tt})) + ((h_{tt}, u_{tt})).
\end{aligned} \quad (4.13)$$

Consequently, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_{tt}\|_{L^2}^2 + \|\nabla u_{tt}\|_{L^2}^2 &\leq |((u_{tt} \cdot \nabla u + 2u_t \cdot \nabla u_t + u \cdot \nabla u_{tt}, u_{tt}))| \\
&\quad + |((w_{tt}\nabla\varphi + 2w_t\nabla\varphi_t + w\nabla\varphi_{tt}, u_{tt}))| + |((h_{tt}, u_{tt}))|.
\end{aligned} \quad (4.14)$$

Owing to Lemma 2.4 and Young's inequality with ϵ , we obtain the following estimates for (4.14).

The estimate of the first term of the right side in (4.14) is as follows

$$\begin{aligned}
&|((u_{tt} \cdot \nabla u + 2u_t \cdot \nabla u_t + u \cdot \nabla u_{tt}, u_{tt}))| \\
&\leq C\|\nabla u\|_{L^2}\|u_{tt}\|_{L^4}^2 + C\|u_t\|_{L^\infty}\|\nabla u_t\|_{L^2}\|u_{tt}\|_{L^2}
\end{aligned}$$

$$\begin{aligned} &\leq C\|\nabla u\|_{L^2}\|u_{tt}\|_{L^2}\|\nabla u_{tt}\|_{L^2} + C\|u_t\|_{L^2}^{\frac{1}{2}}\|\Delta u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}\|u_{tt}\|_{L^2} \\ &\leq \epsilon\|\nabla u_{tt}\|_{L^2}^2 + C_\epsilon\|\nabla u\|_{L^2}^2\|u_{tt}\|_{L^2}^2 + C\|u_t\|_{L^2}^2\|\nabla u_t\|_{L^2}^4 + C\|\Delta u_t\|_{L^2}^2 + C\|u_{tt}\|_{L^2}^2. \end{aligned}$$

The estimate of the second term of the right side in (4.14) is as follows

$$\begin{aligned} &|((w_{tt}\nabla\varphi + 2w_t\nabla\varphi_t + w\nabla\varphi_{tt}, u_{tt}))| \\ &\leq C\|w_{tt}\|_{L^2}\|\nabla\varphi\|_{L^\infty}\|u_{tt}\|_{L^2} + C\|w_t\|_{L^\infty}\|\nabla\varphi_t\|_{L^2}\|u_{tt}\|_{L^2} + C\|w\|_{L^4}\|\nabla\varphi_{tt}\|_{L^4}\|u_{tt}\|_{L^2} \\ &\leq C\|w_{tt}\|_{L^2}\|\nabla\varphi\|_{L^2}^{\frac{1}{2}}\|\varphi\|_{H^3}^{\frac{1}{2}}\|u_{tt}\|_{L^2} + C\|w_t\|_{L^2}^{\frac{1}{2}}\|\Delta w_t\|_{L^2}^{\frac{1}{2}}\|\nabla\varphi_t\|_{L^2}\|u_{tt}\|_{L^2} \\ &\quad + C\|w\|_{L^2}^{\frac{1}{2}}\|\nabla w\|_{L^2}^{\frac{1}{2}}\|\nabla\varphi_{tt}\|_{L^2}^{\frac{1}{2}}\|\Delta\varphi_{tt}\|_{L^2}^{\frac{1}{2}}\|u_{tt}\|_{L^2} \\ &\leq C\|\nabla\varphi\|_{L^2}^{\frac{1}{2}}\|\varphi\|_{H^3}^{\frac{1}{2}}(\|w_{tt}\|_{L^2}^2 + \|u_{tt}\|_{L^2}^2) + C(\|w_t\|_{L^2}^2 + \|\Delta w_t\|_{L^2}^2) + C\|\nabla\varphi_t\|_{L^2}^2\|u_{tt}\|_{L^2}^2 \\ &\quad + C\|w\|_{L^2}\|\nabla w\|_{L^2}\|\nabla\varphi_{tt}\|_{L^2}\|\Delta\varphi_{tt}\|_{L^2} + C\|u_{tt}\|_{L^2}^2. \end{aligned}$$

The estimate of the third term of the right side in (4.14) is as follows

$$|((h_{tt}, u_{tt}))| \leq \frac{1}{2}\|u_{tt}\|_{L^2}^2 + \frac{1}{2}\|h_{tt}\|_{L^2}^2.$$

Let $\epsilon = \frac{1}{2}$ in above estimates. Then, combining (4.12), Lemma 2.2, Theorem 3.1, and Theorem 3.2, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_{tt}\|_{L^2}^2 + \frac{1}{2}\|\nabla u_{tt}\|_{L^2}^2 \\ &\leq C\|\nabla u\|_{L^2}^2\|u_{tt}\|_{L^2}^2 + C\|u_t\|_{L^2}^2\|\nabla u_t\|_{L^2}^4 + C\|\Delta u_t\|_{L^2}^2 + C\|u_{tt}\|_{L^2}^2 \\ &\quad + C\|\nabla\varphi\|_{L^2}^{\frac{1}{2}}\|\varphi\|_{H^3}^{\frac{1}{2}}(\|w_{tt}\|_{L^2}^2 + \|u_{tt}\|_{L^2}^2) + C(\|w_t\|_{L^2}^2 + \|\Delta w_t\|_{L^2}^2) + C\|\nabla\varphi_t\|_{L^2}^2\|u_{tt}\|_{L^2}^2 \\ &\quad + \|w\|_{L^2}\|\nabla w\|_{L^2}\|\nabla\varphi_{tt}\|_{L^2}\|\Delta\varphi_{tt}\|_{L^2} + C\|u_{tt}\|_{L^2}^2 + \frac{1}{2}\|u_{tt}\|_{L^2}^2 + \frac{1}{2}\|h_{tt}\|_{L^2}^2 \\ &\leq C\|u_{tt}\|_{L^2}^2 + C\|w_{tt}\|_{L^2}^2 + C\|\Delta u_t\|_{L^2}^2 + C(\|w_t\|_{L^2}^2 + \|\Delta w_t\|_{L^2}^2) + \frac{1}{2}\|h_{tt}\|_{L^2}^2 + C. \end{aligned} \quad (4.15)$$

Applying uniform Gronwall's lemma and utilizing (4.12), Theorem 3.1, and Theorem 3.2, we obtain

$$\|u_{tt}\|_{L^2}^2 \leq C(r), \quad \int_t^{t+r} \|\nabla u_{tt}\|_{L^2}^2 ds \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \quad (4.16)$$

Let $v = -\Delta u_{tt}$ in (4.1)₁. Thus, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla u_{tt}\|_{L^2}^2 + \|\Delta u_{tt}\|_{L^2}^2 + ((w_{tt}\nabla\varphi + 2w_t\nabla\varphi_t + w\nabla\varphi_{tt}, \Delta u_{tt})) \\ &= ((u_{tt} \cdot \nabla u + 2u_t \cdot \nabla u_t + u \cdot \nabla u_{tt}, \Delta u_{tt})) + ((h_{tt}, -\Delta u_{tt})). \end{aligned} \quad (4.17)$$

Furthermore, we obtain the following inequality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla u_{tt}\|_{L^2}^2 + \|\Delta u_{tt}\|_{L^2}^2 \leq |((u_{tt} \cdot \nabla u + 2u_t \cdot \nabla u_t + u \cdot \nabla u_{tt}, \Delta u_{tt}))| \\ &\quad + |((w_{tt}\nabla\varphi + 2w_t\nabla\varphi_t + w\nabla\varphi_{tt}, \Delta u_{tt}))| + |((h_{tt}, -\Delta u_{tt}))|. \end{aligned} \quad (4.18)$$

Applying interpolation inequalities in Lemma 2.4 and Young's inequality with ϵ , we obtain the following estimates for (4.18).

The estimate of the first term of the right side in (4.18) is as follows

$$\begin{aligned}
& |((u_{tt} \cdot \nabla u + 2u_t \cdot \nabla u_t + u \cdot \nabla u_{tt}, \Delta u_{tt}))| \\
& \leq C \|\Delta u_{tt}\|_{L^2} (\|u_{tt}\|_{L^4} \|\nabla u\|_{L^4} + \|u_t\|_{L^\infty} \|\nabla u_t\|_{L^2} + \|u\|_{L^4} \|\nabla u_{tt}\|_{L^4}) \\
& \leq C \|\Delta u_{tt}\|_{L^2} (\|u_{tt}\|_{L^2}^{\frac{1}{2}} \|\nabla u_{tt}\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}} + \|u_t\|_{L^2}^{\frac{1}{2}} \|\Delta u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}) \\
& \quad + C \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla u_{tt}\|_{L^2}^{\frac{1}{2}} \|\Delta u_{tt}\|_{L^2}^{\frac{3}{2}} \\
& \leq \epsilon_1 \|\Delta u_{tt}\|_{L^2}^2 + C_{\epsilon_1} \|u_{tt}\|_{L^2} \|\nabla u\|_{L^2} (\|\nabla u_{tt}\|_{L^2}^2 + \|\Delta u\|_{L^2}^2) + C_{\epsilon_1} (\|u_t\|_{L^2}^2 \|\nabla u_t\|_{L^2}^4 + \|\Delta u_t\|_{L^2}^2) \\
& \quad + \epsilon_2 \|\Delta u_{tt}\|_{L^2}^2 + C_{\epsilon_2} \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \|\nabla u_{tt}\|_{L^2}^2.
\end{aligned}$$

The estimate of the second term of the right side in (4.18) is as follows

$$\begin{aligned}
& |((w_{tt} \nabla \varphi + 2w_t \nabla \varphi_t + w \nabla \varphi_{tt}, \Delta u_{tt}))| \\
& \leq C \|\Delta u_{tt}\|_{L^2} (\|w_{tt}\|_{L^2} \|\nabla \varphi\|_{L^\infty} + \|w_t\|_{L^2} \|\nabla \varphi_t\|_{L^\infty} + \|w\|_{L^4} \|\nabla \varphi_{tt}\|_{L^4}) \\
& \leq \epsilon_3 \|\Delta u_{tt}\|_{L^2}^2 + C_{\epsilon_3} \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} \|w_{tt}\|_{L^2}^2 + C_{\epsilon_3} \|\nabla \varphi_t\|_{L^2} \|\varphi_t\|_{H^3} \|w_t\|_{L^2}^2 \\
& \quad + C_{\epsilon_3} \|w\|_{L^2} \|\nabla w\|_{L^2} \|\nabla \varphi_{tt}\|_{L^2} \|\Delta \varphi_{tt}\|_{L^2}.
\end{aligned}$$

The estimate of the third term of the right side in (4.18) is as follows

$$|((h_{tt}, -\Delta u_{tt}))| \leq \epsilon_4 \|\Delta u_{tt}\|_{L^2}^2 + C_{\epsilon_4} \|h_{tt}\|_{L^2}^2.$$

Let $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \frac{1}{8}$ in above estimates. Then, combining (4.18), Lemma 2.2, Theorem 3.1, and Theorem 3.2, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\nabla u_{tt}\|_{L^2}^2 + \frac{1}{2} \|\Delta u_{tt}\|_{L^2}^2 \\
& \leq C \|u_{tt}\|_{L^2} \|\nabla u\|_{L^2} (\|\nabla u_{tt}\|_{L^2}^2 + \|\Delta u\|_{L^2}^2) \\
& \quad + C (\|u_t\|_{L^2}^2 \|\nabla u_t\|_{L^2}^4 + \|\Delta u_t\|_{L^2}^2) + C \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \|\nabla u_{tt}\|_{L^2}^2 + C \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} \|w_{tt}\|_{L^2}^2 \\
& \quad + C \|\nabla \varphi_t\|_{L^2} \|\varphi_t\|_{H^3} \|w_t\|_{L^2}^2 + C \|w\|_{L^2} \|\nabla w\|_{L^2} \|\nabla \varphi_{tt}\|_{L^2} \|\Delta \varphi_{tt}\|_{L^2} + C \|h_{tt}\|_{L^2}^2 \\
& \leq C \|\nabla u_{tt}\|_{L^2}^2 + C \|\Delta u\|_{L^2}^2 + C \|\Delta u_t\|_{L^2}^2 + C \|w_{tt}\|_{L^2}^2 + C \|w_t\|_{L^2}^2 + C \|h_{tt}\|_{L^2}^2 + C.
\end{aligned} \tag{4.19}$$

Utilizing uniform Gronwall's Lemma, (4.12), (4.16), Theorem 3.1, and Theorem 3.2, we obtain

$$\|\nabla u_{tt}\|_{L^2}^2 \leq C(r), \quad \int_t^{t+r} \|\Delta u_{tt}\|_{L^2}^2 ds \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \tag{4.20}$$

Therefore, from (4.6), (4.12), (4.16) and (4.20), the proof of Theorem 4.1 is completed. \square

Theorem 4.2. Assume $h \in L^\infty(0, T; L^2(\Omega))$, $h_t \in L^\infty(0, T; L^2(\Omega))$ and $h_{tt} \in L^\infty(0, T; L^2(\Omega))$. Then, we have $\varphi_{tt} \in L^\infty(0, T; \mathcal{H}^k(\Omega))$. Furthermore, the weak solution (u, φ) satisfies

$$\begin{aligned} \|(-\Delta)^{\frac{k}{2}} \varphi_{tt}\|_{L^2}^2 &\leq C(r), \\ \int_t^{t+r} \|\varphi_{ttt}\|_{L^2}^2 + \|\Delta w_{tt}\|_{L^2}^2 ds &\leq C(r), \end{aligned}$$

for $\forall t \geq 2r$, $r > 0$.

Proof. Let $\psi = \varphi_{ttt}$ in (4.1)₂, $\xi = -\Delta \varphi_{ttt}$ in (4.1)₃. Thus, we obtain

$$\begin{cases} \|\varphi_{ttt}\|_{L^2}^2 + ((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, \varphi_{ttt})) + ((\nabla w_{tt}, \nabla \varphi_{ttt})) = 0, \\ \frac{1}{2} \frac{d}{dt} \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+1}{2}} \varphi_{tt}\|_{L^2}^2 + ((w_{tt}, \Delta \varphi_{ttt})) = ((f''(\varphi) \varphi_t^2 + f'(\varphi) \varphi_{tt}, \Delta \varphi_{ttt})). \end{cases} \quad (4.21)$$

Summing the two equalities in (4.21) yields

$$\begin{aligned} \|\varphi_{ttt}\|_{L^2}^2 + ((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, \varphi_{ttt})) + \frac{1}{2} \frac{d}{dt} \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+1}{2}} \varphi_{tt}\|_{L^2}^2 \\ = ((f''(\varphi) \varphi_t^2 + f'(\varphi) \varphi_{tt}, \Delta \varphi_{ttt})). \end{aligned} \quad (4.22)$$

Consequently, we have

$$\begin{aligned} \|\varphi_{ttt}\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+1}{2}} \varphi_{tt}\|_{L^2}^2 \\ \leq |((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, \varphi_{ttt}))| + |((f''(\varphi) \varphi_t^2 + f'(\varphi) \varphi_{tt}, \Delta \varphi_{ttt}))|. \end{aligned} \quad (4.23)$$

Applying Lemma 2.3, Sobolev imbedding theorem, and Young's inequality with ϵ to (4.23), we obtain the following estimate.

The estimate of the first term of the right side in (4.23) is as follows

$$\begin{aligned} &|((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, \varphi_{ttt}))| \\ &\leq C \|\varphi_{ttt}\|_{L^2} (\|u_{tt}\|_{L^2} \|\nabla \varphi\|_{L^\infty} + \|u_t\|_{L^2} \|\nabla \varphi_t\|_{L^\infty} + \|u\|_{L^4} \|\nabla \varphi_{tt}\|_{L^4}) \\ &\leq \epsilon_1 \|\varphi_{ttt}\|_{L^2}^2 + C_{\epsilon_1} (\|u_{tt}\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} + \|u_t\|_{L^2}^2 \|\nabla \varphi_t\|_{L^2} \|\varphi_t\|_{H^3} \\ &\quad + \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \varphi_{tt}\|_{L^2} \|\Delta \varphi_{tt}\|_{L^2}). \end{aligned}$$

The estimate of the second term of the right side in (4.23) is as follows

$$\begin{aligned} &|((f''(\varphi) \varphi_t^2 + f'(\varphi) \varphi_{tt}, \Delta \varphi_{ttt}))| \\ &= |((f^{(4)}(\varphi) |\nabla \varphi|^2 \varphi_t^2 + f^{(3)}(\varphi) \Delta \varphi \varphi_t^2 + 4f^{(3)}(\varphi) \nabla \varphi \varphi_t \nabla \varphi_t + 2f''(\varphi) (\varphi_t \Delta \varphi_t + |\nabla \varphi_t|^2) \\ &\quad + f^{(3)}(\varphi) |\nabla \varphi|^2 \varphi_{tt} + f''(\varphi) \Delta \varphi \varphi_{tt} + 2f''(\varphi) \nabla \varphi \nabla \varphi_{tt} + f'(\varphi) \Delta \varphi_{tt}, \varphi_{ttt}))| \\ &\leq C \|\varphi_{ttt}\|_{L^2} (\|\nabla \varphi\|_{L^8}^2 \|\varphi_t\|_{L^8}^2 + \|\Delta \varphi\|_{L^4} \|\varphi_t\|_{L^8}^2 + \|\nabla \varphi\|_{L^8} \|\varphi_t\|_{L^8} \|\nabla \varphi_t\|_{L^4} + \|\varphi_t\|_{L^4} \|\Delta \varphi_t\|_{L^4} \\ &\quad + \|\nabla \varphi_t\|_{L^4}^2 + \|\varphi_{tt}\|_{L^4} \|\nabla \varphi\|_{L^8}^2 + \|\Delta \varphi\|_{L^4} \|\varphi_{tt}\|_{L^4} + \|\nabla \varphi\|_{L^4} \|\nabla \varphi_{tt}\|_{L^4} + \|\Delta \varphi_{tt}\|_{L^2}) \\ &\leq C \|\varphi_{ttt}\|_{L^2} (\|\Delta \varphi\|_{L^2}^2 \|\nabla \varphi_t\|_{L^2}^2 + \|\varphi\|_{H^3} \|\nabla \varphi_t\|_{L^2}^2 + \|\varphi\|_{H^2} \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2} + \|\nabla \varphi_t\|_{L^2} \|\varphi_t\|_{H^3} \\ &\quad + \|\Delta \varphi_t\|_{L^2}^2 + \|\nabla \varphi_{tt}\|_{L^2} \|\Delta \varphi\|_{L^2}^2 + \|\varphi\|_{H^3} \|\nabla \varphi_{tt}\|_{L^2} + \|\Delta \varphi\|_{L^2} \|\Delta \varphi_{tt}\|_{L^2} + \|\Delta \varphi_{tt}\|_{L^2}) \\ &\leq \epsilon_2 \|\varphi_{ttt}\|_{L^2}^2 + C_{\epsilon_2} (\|\Delta \varphi\|_{L^2}^4 \|\nabla \varphi_t\|_{L^2}^4 + \|\varphi\|_{H^3}^2 \|\nabla \varphi_t\|_{L^2}^4 + \|\varphi\|_{H^2}^2 \|\nabla \varphi_t\|_{L^2}^2 \|\Delta \varphi_t\|_{L^2}^2) \end{aligned}$$

$$\begin{aligned}
& + \|\nabla \varphi_t\|_{L^2}^2 \|\varphi_t\|_{H^3}^2 + \|\Delta \varphi_t\|_{L^2}^4 + \|\nabla \varphi_{tt}\|_{L^2}^2 \|\Delta \varphi\|_{L^2}^4 + \|\varphi\|_{H^3}^2 \|\nabla \varphi_{tt}\|_{L^2}^2 \\
& + \|\Delta \varphi\|_{L^2}^2 \|\Delta \varphi_{tt}\|_{L^2}^2 + \|\Delta \varphi_{tt}\|_{L^2}^2.
\end{aligned}$$

Let $\epsilon_1 = \epsilon_2 = \frac{1}{4}$ in above estimates. Combining (4.23), Lemma 2.2, Theorem 3.1, Theorem 3.2, and Theorem 4.1, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+1}{2}} \varphi_{tt}\|_{L^2}^2 + \frac{1}{2} \|\varphi_{ttt}\|_{L^2}^2 \\
& \leq C(\|u_{tt}\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} + \|u_t\|_{L^2}^2 \|\nabla \varphi_t\|_{L^2} \|\varphi_t\|_{H^3} + \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \varphi_{tt}\|_{L^2} \|\Delta \varphi_{tt}\|_{L^2}) \\
& \quad + C(\|\Delta \varphi\|_{L^2}^4 \|\nabla \varphi_t\|_{L^2}^4 + \|\varphi\|_{H^3}^2 \|\nabla \varphi_t\|_{L^2}^4 + \|\varphi\|_{H^2}^2 \|\nabla \varphi_t\|_{L^2}^2 \|\Delta \varphi_t\|_{L^2}^2 + \|\nabla \varphi_t\|_{L^2}^2 \|\varphi_t\|_{H^3}^2 \\
& \quad + \|\Delta \varphi_t\|_{L^2}^4 + \|\nabla \varphi_{tt}\|_{L^2}^2 \|\Delta \varphi\|_{L^2}^4 + \|\varphi\|_{H^3}^2 \|\nabla \varphi_{tt}\|_{L^2}^2 + \|\Delta \varphi\|_{L^2}^2 \|\Delta \varphi_{tt}\|_{L^2}^2 + \|\Delta \varphi_{tt}\|_{L^2}^2) \\
& \leq C \|\Delta \varphi_{tt}\|_{L^2}^2 + C.
\end{aligned} \tag{4.24}$$

Utilizing uniform Gronwall's Lemma and Theorem 4.1, we obtain

$$\sum_{i=1}^k a_i \|(-\Delta)^{\frac{i+1}{2}} \varphi_{tt}\|_{L^2}^2 \leq C(r), \int_t^{t+r} \|\varphi_{ttt}\|_{L^2}^2 ds \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \tag{4.25}$$

Let $\psi = -\Delta w_{tt}$ in (4.1)₂. Thus, we obtain

$$\begin{aligned}
& \|\Delta w_{tt}\|_{L^2}^2 = ((\varphi_{ttt}, \Delta w_{tt})) + ((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, \Delta w_{tt})) \\
& \leq \epsilon_1 \|\Delta w_{tt}\|_{L^2}^2 + C_{\epsilon_1} \|\varphi_{ttt}\|_{L^2}^2 + C \|\Delta w_{tt}\|_{L^2} (\|u_{tt}\|_{L^2} \|\nabla \varphi\|_{L^\infty} \\
& \quad + \|u_t\|_{L^4} \|\nabla \varphi_t\|_{L^4} + \|u\|_{L^4} \|\nabla \varphi_{tt}\|_{L^4}) \\
& \leq \epsilon_1 \|\Delta w_{tt}\|_{L^2}^2 + C_{\epsilon_1} \|\varphi_{ttt}\|_{L^2}^2 + \epsilon_2 \|\Delta w_{tt}\|_{L^2}^2 + C_{\epsilon_2} (\|u_{tt}\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} \\
& \quad + \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2} + \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \varphi_{tt}\|_{L^2} \|\Delta \varphi_{tt}\|_{L^2}).
\end{aligned}$$

Let $\epsilon_1 = \epsilon_2 = \frac{1}{4}$ in above inequality. Then, combining Lemma 2.2, Theorem 3.1, Theorem 3.2, and Theorem 4.1, we obtain

$$\begin{aligned}
& \frac{1}{2} \|\Delta w_{tt}\|_{L^2}^2 \leq C \|\varphi_{ttt}\|_{L^2}^2 + C(\|u_{tt}\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} \\
& \quad + \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2} + \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \varphi_{tt}\|_{L^2} \|\Delta \varphi_{tt}\|_{L^2}) \\
& \leq C \|\varphi_{ttt}\|_{L^2}^2 + C.
\end{aligned} \tag{4.26}$$

Integrate (4.26) from t to $t+r$ and by (4.25) we get

$$\int_t^{t+r} \|\Delta w_{tt}\|_{L^2}^2 ds \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \tag{4.27}$$

Assume $\psi = (-\Delta)^k \varphi_{tt}$ in (4.1)₂, then we have

$$\frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\frac{k}{2}} \varphi_{tt}\|_{L^2}^2 + ((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, (-\Delta)^k \varphi_{tt})) = ((\Delta w_{tt}, (-\Delta)^k \varphi_{tt})). \tag{4.28}$$

Consequently, we obtain the following inequality

$$\frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\frac{k}{2}} \varphi_{tt}\|_{L^2}^2 \leq |((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, (-\Delta)^k \varphi_{tt}))| + |((\Delta w_{tt}, (-\Delta)^k \varphi_{tt}))|. \quad (4.29)$$

By interpolation inequalities in Lemma 2.4 and Young's inequality with ϵ , we obtain the following estimates for (4.29).

The estimate of the first term of the right side in (4.29) is as follows

$$\begin{aligned} & |((u_{tt} \cdot \nabla \varphi + 2u_t \cdot \nabla \varphi_t + u \cdot \nabla \varphi_{tt}, (-\Delta)^k \varphi_{tt}))| \\ & \leq C \|(-\Delta)^k \varphi_{tt}\|_{L^2} (\|u_{tt}\|_{L^2} \|\nabla \varphi\|_{L^\infty} + \|u_t\|_{L^4} \|\nabla \varphi_t\|_{L^4} + \|u\|_{L^4} \|\nabla \varphi_{tt}\|_{L^4}) \\ & \leq \frac{1}{2} \|(-\Delta)^k \varphi_{tt}\|_{L^2}^2 + C \|u_{tt}\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} + C \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2} \\ & \quad + C \|u\|_{L^2} \|\nabla u\|_{L^2} (\|\nabla \varphi_{tt}\|_{L^2}^2 + \|\Delta \varphi_{tt}\|_{L^2}^2). \end{aligned}$$

The estimate of the second term of the right side in (4.29) is as follows

$$|((\Delta w_{tt}, (-\Delta)^k \varphi_{tt}))| \leq \frac{1}{2} \|\Delta w_{tt}\|_{L^2}^2 + \frac{1}{2} \|(-\Delta)^k \varphi_{tt}\|_{L^2}^2.$$

Combining (4.29), Lemma 2.2, Theorem 3.1, Theorem 3.2, Theorem 4.1 and the two inequalities above, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(-\Delta)^{\frac{k}{2}} \varphi_{tt}\|_{L^2}^2 & \leq \|(-\Delta)^k \varphi_{tt}\|_{L^2}^2 + C \|u_{tt}\|_{L^2}^2 \|\nabla \varphi\|_{L^2} \|\varphi\|_{H^3} + C \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\nabla \varphi_t\|_{L^2} \|\Delta \varphi_t\|_{L^2} \\ & \quad + C \|u\|_{L^2} \|\nabla u\|_{L^2} (\|\nabla \varphi_{tt}\|_{L^2}^2 + \|\Delta \varphi_{tt}\|_{L^2}^2) + \frac{1}{2} \|\Delta w_{tt}\|_{L^2}^2 \\ & \leq \|(-\Delta)^k \varphi_{tt}\|_{L^2}^2 + \frac{1}{2} \|\Delta w_{tt}\|_{L^2}^2 + C. \end{aligned} \quad (4.30)$$

Integrating (4.30) from t to $t+r$ and utilizing (4.27) and Theorem 4.1, we obtain

$$\|(-\Delta)^{\frac{k}{2}} \varphi_{tt}\|_{L^2}^2 \leq C(r), \quad \forall t \geq 2r, \quad r > 0. \quad (4.31)$$

Therefore, from (4.25), (4.27) and (4.31), the proof of Theorem 4.2 is completed. \square

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