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Non-surjective coarse isometries between L_p spaces ☆

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ABSTRACT

In this paper, we study stability and weak stability of coarse isometries of Banach spaces. As a result, we show that if a coarse isometry $f : L_p(\Omega_1, \Sigma_1, \mu_1) \to L_p(\Omega_2, \Sigma_2, \mu_2)$ $(1 < p < \infty)$ is weakly stable at each point of a Schauder basis, then there is a linear isometry $U : L_p(\Omega_1, \Sigma_1, \mu_1) \to L_p(\Omega_2, \Sigma_2, \mu_2)$, where $(\Omega_j, \Sigma_j, \mu_j)$ $(j = 1, 2)$ are σ-finite measure spaces. Furthermore, if f is uniformly weakly stable, then $\|f(x) - Ux\| = o(\|x\|)$ when $\|x\| \to \infty$. As an application, we obtain that $\|Pf(x) - Ux\| = o(\|x\|)$ is equivalent to $\|f(x) - Ux\| = o(\|x\|)$ as $\|x\| \to \infty$, where $P : Y \to U(X)$ is a projection with $\|P\| = 1$.

1. Introduction

Throughout this paper, X and Y denote real Banach spaces. We say a mapping $f : X \to Y$ is standard if $f(0) = 0$. The properties of isometries have been studied since the celebrated Mazur-Ulam theorem ([15] 1932): Every surjective standard isometry between Banach spaces is necessarily linear. For non-surjective isometry, T. Figiel [11] showed the remarkable result in 1968: every standard isometry $f : X \to Y$ admits a linear left-inverse $T : \overline{span} f(X) \to X$ with $\|T\| = 1$ such that $T \circ f = I_X$.

A mapping $f : X \to Y$ is called an ε-isometry for $\varepsilon \geq 0$ if $\big|\|f(x) - f(y)\| - \|x - y\|\big| \leq \varepsilon$ whenever $x, y \in X$. In 1983, J. Gevirtz [12] proved that if $f : X \to Y$ is a surjective standard ε-isometry, then there exists a surjective linear isometry $U : X \to Y$ such that

$$\|f(x) - Ux\| \leq 5\varepsilon \quad \text{for all } x \in X. \tag{1.1}$$

In 1995, M. Omladič and P. Šemrl [16] showed the constant 5 in (1.1) can be replaced by 2 which is the best constant (see, also, Y. Benyamini and J. Lindenstauss [2, Theorem 15.2]). Upon T. Figiel [11] theorem

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and other remarkable results for non-surjective ε-isometry (see [17,18]), in 2013, L. Cheng, Y. Dong and W. Zhang [5] established the weak stability formula:

Theorem 1.1. *(Cheng, Dong and Zhang) Suppose that $f : X \to Y$ is a standard ε-isometry. Then for every $x^* \in X^*$ there exists $\phi \in Y^*$ with $\|x^*\| = \|\phi\| \equiv r$ so that*

$$|\langle x^*, x\rangle - \langle \phi, f(x)\rangle| \leq 4r\varepsilon \quad \text{for all } x \in X. \tag{1.2}$$

It has played an important role in the study of stability properties of ε-isometries (see [3–5,7–9,19,20]).

We say a mapping $f : X \to Y$ is a coarse isometry if $\varepsilon_f(t) = o(t)$ as $t \to \infty$, where

$$\varepsilon_f(t) = \sup_{x,y\in X, \|x-y\|\leq t} \{\big|\|f(x) - f(y)\| - \|x - y\|\big|\} \quad \text{for } t \geq 0.$$

In particular, every ε-isometry mapping is a coarse isometry. In 1985, J. Lindenstrauss and A. Szankowski [14] first introduced the following larger perturbation function for a surjective standard mapping $f : X \to Y$:

$$\varphi_f(t) = \sup_{x,y\in X} \{\big|\|f(x) - f(y)\| - \|x - y\|\big| : \|x - y\| \leq t \ \text{ or } \ \|f(x) - f(y)\| \leq t\} \quad \text{for } t \geq 0$$

and obtained an asymptotical stability result, which generalizes the result of J. Gevirtz [12] about ε-isometry:

Theorem 1.2. *(Lindenstrauss and Szankowski) Let f be a surjective standard map from a Banach space X onto a Banach space Y. If*

$$\int_1^\infty \frac{\varphi_f(t)}{t^2} dt < \infty, \tag{1.3}$$

then there is a linear isometry U from X onto Y so that

$$\|f(x) - Ux\| = o(\|x\|) \quad \text{as } \|x\| \to \infty.$$

At the same time, they showed that condition (1.3) can not be removed. Note that for each $t \geq 0$, $\varepsilon_f(t) \leq \varphi_f(t)$ and (1.3) implies $\varphi_f(t) = o(t)$ when $t \to \infty$, then f is a coarse isometry.

In 2000, G. Dolinar [10] noted that Theorem 1.2 also holds if $\varphi_f(t)$ is substituted by $\varepsilon_f(t)$ in the integral convergence condition (1.3).

As far as we know, since it is difficult to ensure the existence of $\lim\limits_{r\to\infty} \frac{f(rx)}{r}$ for each $x \in X$, the representation of non-surjective coarse isometries has not been studied until 2019. L. Cheng et al. [6] first investigated the non-surjective coarse isometry $f : X \to Y$ and obtained the following result, where Y is a uniformly convex Banach space of power type p.

Theorem 1.3. *(Cheng, Fang, Luo and Sun) Suppose that $f : X \to Y$ is a standard coarse isometry and that Y is uniformly convex with convexity of type p. If*

$$\int_1^\infty \frac{\varepsilon_f(s)^{\frac{1}{p}}}{s^{1+\frac{1}{p}}} ds < \infty, \tag{1.4}$$

then there is a linear isometry $U : X \to Y$ so that

$$\|f(x) - Ux\| = o(\|x\|) \quad \text{as } \|x\| \to \infty.$$

In this paper, we shall study stability properties of non-surjective coarse isometry $f : X \to Y$ by assuming $X = L_p(\Omega_1, \Sigma_1, \mu_1)$ and $Y = L_p(\Omega_2, \Sigma_2, \mu_2)$, where $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ are two σ-finite measure spaces and $1 < p < \infty$. Besides the integral convergence condition (1.4) we use the pointwise weak stability formula and uniform weak stability formula to guarantee the stability of a non-surjective coarse isometry. The details are as follows: Suppose that $\{e_k\}$ is a Schauder basis in X and that $f : X \to Y$ is a standard coarse isometry, if for every $k \in \mathbb{N}$ and $x^* \in S_{X^*}$, there exists $\phi \in S_{Y^*}$ such that

$$\lim_{r\to\infty} |\,\langle x^*, e_k\rangle - \langle \phi, \frac{f(re_k)}{r}\rangle| = 0,$$

then there is a linear isometry $U : X \to Y$ satisfying

$$Ux = \lim_{r\to\infty} \frac{f(rx)}{r} \quad \text{for all } x \in X.$$

Moreover, if for every $x^* \in S_{X^*}$, there exists $\phi \in S_{Y^*}$ such that

$$|\,\langle x^*, x\rangle - \langle \phi, f(x)\rangle\,| = o(\|x\|) \quad \text{as } \|x\| \to \infty,$$

then there is a linear isometry $U : X \to Y$ satisfying

$$\|f(x) - Ux\| = o(\|x\|) \quad \text{as } \|x\| \to \infty.$$

As an application, we obtain a stability result of basic sequences via coarse isometries and prove that $\|Pf(x) - Ux\| = o(\|x\|)$ is equivalent to $\|f(x) - Ux\| = o(\|x\|)$ as $\|x\| \to \infty$, where $P : Y \to U(X)$ is a projection with $\|P\| = 1$.

In this paper, the letters X, Y are used to denote real Banach spaces, and X^*, Y^* are their dual spaces. For a real Banach space X, we denote by S_X and B_X the unit sphere and the closed unit ball of X respectively. $\partial\|\cdot\| : X \to 2^{X^*}$ stands for the subdifferential mapping of the norm $\|\cdot\|$. Given a bounded linear operator $T : X \to Y$, $T^* : Y^* \to X^*$ is its dual operator.

2. Main results

To begin with, we give a definition.

Definition 2.1. Let X and Y be Banach spaces and let $f : X \to Y$ be a standard coarse isometry.

(1) f is said to be pointwisely weakly stable if for each $x \in X$ and $x^* \in S_{X^*}$, there exists $\phi \in S_{Y^*}$ such that

$$\lim_{r\to\infty} |\,\langle x^*, x\rangle - \langle \phi, \frac{f(rx)}{r}\rangle| = 0;$$

(2) f is called pointwisely stable if there exists a linear isometry $U : X \to Y$ so that

$$Ux = \lim_{r\to\infty} \frac{f(rx)}{r} \quad \text{for all } x \in X;$$

(3) f is said to be uniformly weakly stable if for each $x^* \in S_{X^*}$, there exists $\phi \in S_{Y^*}$ such that

$$\lim_{r\to\infty} |\,\langle x^*, x\rangle - \langle \phi, \frac{f(rx)}{r}\rangle| = 0 \quad \text{uniformly for } x \in S_X;$$

(4) f is called uniformly stable if there exists a linear isometry $U : X \to Y$ so that

$$Ux = \lim_{r\to\infty} \frac{f(rx)}{r} \quad \text{uniformly for } x \in S_X.$$

Proposition 2.2. *Let X and Y be Banach spaces and let $f : X \to Y$ be a standard coarse isometry. If f is uniformly stable (pointwisely stable), then it is uniformly weakly stable (pointwisely weakly stable).*

Proof. Let $U : X \to Y$ be a linear isometry so that

$$Ux = \lim_{r\to\infty} \frac{f(rx)}{r} \quad \text{uniformly for } x \in S_X.$$

Since U is a linear isometry, $U^* : Y^* \to X^*$ is a surjective bounded linear operator with $\|U^*\| = 1$. Note that for each $x^* \in S_{X^*}$, there exists $\phi \in S_{Y^*}$ such that $U^*\phi = x^*$. Indeed, due to the surjective of U^*, there exists $\varphi \in Y^*$ such that $U^*\varphi = x^*$. We put

$$\langle \phi, y\rangle = \begin{cases} \langle \varphi, y\rangle, & \text{if } y \in U(X); \\ 0, & \text{otherwise.} \end{cases}$$

Then $\phi \in Y^*$, $U^*\phi = x^*$ and

$$\|\phi\| = \sup_{y\in S_Y} |\langle \phi, y\rangle| = \sup_{y\in S_{U(X)}} |\langle \varphi, y\rangle| = \sup_{x\in S_X} |\langle \varphi, Ux\rangle| = \sup_{x\in S_X} |\langle x^*, x\rangle| = \|x^*\|.$$

On the other hand,

$$\begin{aligned} |\,\langle x^*, x\rangle - \langle \phi, \frac{f(rx)}{r}\rangle| &= |\,\langle U^*\phi, x\rangle - \langle \phi, \frac{f(rx)}{r}\rangle| \\ &= |\langle \phi, Ux - \frac{f(rx)}{r}\rangle| \\ &\le \|Ux - \frac{f(rx)}{r}\|, \end{aligned}$$

then

$$\lim_{r\to\infty} |\,\langle x^*, x\rangle - \langle \phi, \frac{f(rx)}{r}\rangle| = 0 \quad \text{uniformly for } x \in S_X.$$

The pointwise case is similar. □

The next lemma is essential for our main results, and one can refer to [13].

Lemma 2.3. *[13, Theorem 3.3] Let $1 \le p < \infty$. If $(\Omega_i, \Sigma_i, \mu_i)$ are measure spaces, then there is a measure space (Ω, Σ, μ) such that $(L_p(\mu_i))_{\mathcal{U}}$ is isometric and order isomorphic to $L_p(\mu)$.*

The following theorem says that a standard coarse isometry f is pointwisely stable when it is pointwisely weakly stable on a Schauder basis.

Recall that a Banach space X is said to have the Kadec-Klee property if the weak topology and the norm topology of X agree on the unit sphere S_X of X.

Theorem 2.4. *Let $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ be two σ-finite measure spaces and $1 < p < \infty$. Suppose that $X = L_p(\Omega_1, \Sigma_1, \mu_1)$ with a Schauder basis $\{e_k\}$, $Y = L_p(\Omega_2, \Sigma_2, \mu_2)$ and that $f : X \to Y$ is a standard coarse isometry. If for every $k \in \mathbb{N}$ and $x^* \in S_{X^*}$, there exists $\phi \in S_{Y^*}$ such that*

$$\lim_{r\to\infty} |\langle x^*, e_k\rangle - \langle \phi, \frac{f(re_k)}{r}\rangle| = 0, \tag{2.1}$$

then there is a linear isometry $U : X \to Y$ satisfying

$$Ux = \lim_{r\to\infty} \frac{f(rx)}{r} \quad \text{for all } x \in X.$$

Proof. Our proof is divided into three steps.

Step I. We first show that

$$\lim_{r\to\infty} \frac{f(re_k)}{r} \text{ exists and } \lim_{r\to\infty} \frac{f(are_k)}{r} = a \lim_{r\to\infty} \frac{f(re_k)}{r} \quad \text{for all } k \in \mathbb{N},\ a \in \mathbb{R}.$$

Let

$$I = \{\{\lambda_n\} \subseteq \mathbb{R} : \lambda_n \to \infty,\ \text{as } n \to \infty\},$$

and for each $x \in X$,

$$V_x = \{u \in Y : u = w\text{-}\lim_{n\to\infty} \frac{f(\lambda_n x)}{\lambda_n} \text{ for some } \{\lambda_n\} \in I\}.$$

Since f is a coarse isometry, for every $x \in X$, there exists $N \in \mathbb{N}$ so that $\{\frac{f(rx)}{r}\}_{r\geq N}$ is a bounded subset of Y. Indeed,

$$\left|\|\frac{f(rx)}{r}\| - \|x\|\right| = \frac{1}{|r|}\left|\|f(rx)\| - \|rx\|\right| \leq \frac{\varepsilon_f(|r|\ \|x\|)}{|r|} \to 0 \quad \text{uniformly for } x \in S_X,\ \text{as } r \to \infty. \tag{2.2}$$

As Y is reflexive, it follows that $\{\frac{f(rx)}{r}\}_{r\geq N}$ is relatively weakly compact and hence it is relatively weakly sequentially compact by the Eberlein-Šmulian theorem. Thus for each $k \in \mathbb{N}$, $V_{e_k} \neq \emptyset$, and for each $u \in V_{e_k}$, there exists $\{\lambda_n\} \in I$ such that $u = w\text{-}\lim_{n\to\infty} \frac{f(\lambda_n e_k)}{\lambda_n}$. Given $x^* \in \partial\|e_k\|$, according to (2.1), there exists $\phi \in S_{Y^*}$ such that

$$\lim_{n\to\infty} |\langle x^*, e_k\rangle - \langle \phi, \frac{f(\lambda_n e_k)}{\lambda_n}\rangle| = 0.$$

Consequently,

$$\|e_k\| = \lim_{n\to\infty} \langle \phi, \frac{f(\lambda_n e_k)}{\lambda_n}\rangle = \langle \phi, u\rangle \leq \|\phi\|\ \|u\| = \|u\|,$$

and

$$\|u\| = \|w\text{-}\lim_{n\to\infty} \frac{f(\lambda_n e_k)}{\lambda_n}\| \leq \lim_{n\to\infty} \|\frac{f(\lambda_n e_k)}{\lambda_n}\| = \|e_k\|.$$

Hence, $\|u\| = \|e_k\|$. It follows that for every $k \in \mathbb{N}$ and for every $x^* \in \partial\|e_k\|$, there exists $\phi \in S_{Y^*}$ such that

$$V_{e_k} \subseteq \{u \in Y : \langle \phi, u\rangle = \|u\| = \|e_k\|\}.$$

By the smoothness property of Y^*, V_{e_k} is a singleton. We denote it by e'_k. Then $\|e_k\| = \|e'_k\|$.

Note that for every $\{\lambda_n\} \in I$, $w\text{-}\lim\limits_{n\to\infty} \frac{f(\lambda_n e_k)}{\lambda_n} = e'_k$. Otherwise, we can find a w-neighborhood W of e'_k and a subsequence $\{\lambda_{n_l}\}$ of $\{\lambda_n\}$ such that $\frac{f(\lambda_{n_l} e_k)}{\lambda_{n_l}} \notin W$ for all $l \in \mathbb{N}$. However, $\{\frac{f(\lambda_{n_l} e_k)}{\lambda_{n_l}}\}$ is also bounded, then there exists a subsequence $\{r_m\}$ of $\{\lambda_{n_l}\}$ such that $w\text{-}\lim\limits_{m\to\infty} \frac{f(r_m e_k)}{r_m} = e'_k$. This is contradiction to $\frac{f(\lambda_{n_l} e_k)}{\lambda_{n_l}} \notin W$ for all $l \in \mathbb{N}$. Hence for every $\{\lambda_n\} \in I$, $w\text{-}\lim\limits_{n\to\infty} \frac{f(\lambda_n e_k)}{\lambda_n} = e'_k$. Due to (2.2) and Kadec-Klee property of Y, we have $\lim\limits_{n\to\infty} \frac{f(\lambda_n e_k)}{\lambda_n} = e'_k$. Arbitrariness of $\{\lambda_n\}$ entails

$$\lim_{r\to\infty} \frac{f(re_k)}{r} = e'_k. \tag{2.3}$$

By the proof above, similarly, we have

$$\lim_{r\to\infty} \frac{f(-re_k)}{r} = (-e_k)' \tag{2.4}$$

Using (2.1), (2.3) and (2.4), for the $x^* \in S_{X^*}$, $\phi \in S_{Y^*}$ above, we have

$$\langle x^*, -e_k\rangle - \langle \phi, (-e_k)'\rangle = 0 \quad \text{and} \quad \langle x^*, -e_k\rangle - \langle \phi, -e'_k\rangle = 0.$$

Therefore, $\langle \phi, (-e_k)'\rangle = \langle \phi, -e'_k\rangle$. This and smoothness of Y^* imply $(-e_k)' = -e'_k$ and then $\lim\limits_{r\to\infty} \frac{f(are_k)}{r} = a \lim\limits_{r\to\infty} \frac{f(re_k)}{r}$ for all $a \in \mathbb{R}$.

Step II. Next, we show that

$$Ux = \sum_{k=1}^{\infty} a_k e'_k$$

defines a linear isometry $U : X \to Y$, where $x = \sum\limits_{k=1}^{\infty} a_k e_k$, $a_k \in \mathbb{R}$.

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and $\{\lambda_n\} \in I$. We define an isometry mapping $\tilde{f} : (X)_{\mathcal{U}} \to (Y)_{\mathcal{U}}$ as follows:

$$\tilde{f}(u_1, u_2, ..., u_n, ...) = \left(\frac{f(\lambda_1 u_1)}{\lambda_1}, \frac{f(\lambda_2 u_2)}{\lambda_2}, ..., \frac{f(\lambda_n u_n)}{\lambda_n}, ...\right),$$

where $u = (u_1, u_2, ..., u_n, ...) \in (X)_{\mathcal{U}}$. Indeed, given $u \in (X)_{\mathcal{U}}$, $v \in (Y)_{\mathcal{U}}$,

$$\begin{aligned} \big|\|\tilde{f}(u) - \tilde{f}(v)\| - \|u - v\|\big| &= \left|\| \left(\frac{f(\lambda_n u_n) - f(\lambda_n v_n)}{\lambda_n}\right) \| - \|(u_n - v_n)\|\right| \\ &= \lim_{\mathcal{U}} \left|\|\frac{f(\lambda_n u_n) - f(\lambda_n v_n)}{\lambda_n}\| - \|u_n - v_n\|\right| \\ &\le \lim_{\mathcal{U}} \frac{\varepsilon_f(|\lambda_n|\, \|u_n - v_n\|)}{|\lambda_n|} = 0. \end{aligned}$$

According to Lemma 2.3, $(Y)_{\mathcal{U}} = L_p(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ) and then $(Y)_{\mathcal{U}}$ is strictly convex. This and $f(0) = 0$ entail that $\tilde{f}$ is a linear isometry. Hence, for each $u \in (X)_{\mathcal{U}}$, $v \in (Y)_{\mathcal{U}}$, $\tilde{f}(u + v) = \tilde{f}(u) + \tilde{f}(v)$. By the definition of $\tilde{f}$, for each $x, y \in X$, we have

$$\lim_{\mathcal{U}} \|\frac{f(\lambda_n(x + y)) - f(\lambda_n x) - f(\lambda_n y)}{\lambda_n}\| = 0, \tag{2.5}$$

and then

$$w\text{-}\lim_{\mathcal{U}} \frac{f(\lambda_n(x+y)) - f(\lambda_n x) - f(\lambda_n y)}{\lambda_n} = 0.$$

Therefore,

$$w\text{-}\lim_{\mathcal{U}} \frac{f(\lambda_n(x+y))}{\lambda_n} = w\text{-}\lim_{\mathcal{U}} \frac{f(\lambda_n x)}{\lambda_n} + w\text{-}\lim_{\mathcal{U}} \frac{f(\lambda_n y)}{\lambda_n} \tag{2.6}$$

Note that for each $x_0 = \sum\limits_{k=1}^{m} a_k e_k$, $\lim\limits_{r\to\infty} \frac{f(ra_k e_k)}{r} = a_k \lim\limits_{r\to\infty} \frac{f(re_k)}{r} = a_k e_k'$, where $a_k \in \mathbb{R}$, $m \in \mathbb{N}$, then

$$w\text{-}\lim_{\mathcal{U}} \frac{f(\lambda_n a_k e_k)}{\lambda_n} = a_k e_k'. \tag{2.7}$$

By (2.6) and (2.7), we obtain

$$w\text{-}\lim_{\mathcal{U}} \frac{f(\lambda_n x_0)}{\lambda_n} = w\text{-}\lim_{\mathcal{U}} \frac{f(\lambda_n \sum_{k=1}^{m} a_k e_k)}{\lambda_n} = \sum_{k=1}^{m} w\text{-}\lim_{\mathcal{U}} \frac{f(\lambda_n a_k e_k)}{\lambda_n} = \sum_{k=1}^{m} a_k e_k'.$$

On the other hand, (2.5) implies

$$\lim_{\mathcal{U}} \|\frac{f(\lambda_n x_0)}{\lambda_n}\| = \lim_{\mathcal{U}} \|\sum_{k=1}^{m} \frac{f(\lambda_n a_k e_k)}{\lambda_n}\| = \|\sum_{k=1}^{m} a_k e_k'\|.$$

By the Kadec-Klee property of Y,

$$\lim_{\mathcal{U}} \frac{f(\lambda_n x_0)}{\lambda_n} = \sum_{k=1}^{m} a_k e_k'.$$

Arbitrariness of $\{\lambda_n\}$ entails $\lim\limits_{r\to\infty} \frac{f(rx_0)}{r} = \sum\limits_{k=1}^{m} a_k e_k'$. Since f is a standard coarse isometry, it follows that $\lim\limits_{r\to\infty} \frac{f(rx_0)}{r}$ defines a linear isometry from a dense subspace of X into Y. Given $x = \sum\limits_{k=1}^{\infty} a_k e_k$, where $a_k \in \mathbb{R}$, let $x_m = \sum\limits_{k=1}^{m} a_k e_k$. We define $Ux = \lim\limits_{m\to\infty} \lim\limits_{r\to\infty} \frac{f(rx_m)}{r}$. Clearly, $U : X \to Y$ is a linear isometry and

$$Ux = \sum_{k=1}^{\infty} a_k e_k'.$$

Step III. Finally, we prove

$$Ux = \lim_{r\to\infty} \frac{f(rx)}{r} \quad \text{for all } x \in X.$$

Let $x = \sum\limits_{k=1}^{\infty} a_k e_k$ and $\varepsilon > 0$ be given. Then there exist $n_0, n_1 \in \mathbb{N}$ such that for all $r > n_1$, $\|\sum\limits_{k=n_0+1}^{\infty} a_k e_k\| < \frac{\varepsilon}{4}$, $\|\sum\limits_{k=n_0+1}^{\infty} a_k e_k'\| < \frac{\varepsilon}{4}$, $\frac{\varepsilon_f(r\frac{\varepsilon}{4})}{r} < \frac{\varepsilon}{4}$ and $\|\frac{f(r\sum\limits_{k=1}^{n_0} a_k e_k)}{r} - \sum\limits_{k=1}^{n_0} a_k e_k'\| < \frac{\varepsilon}{4}$. Hence, if $r > n_1$, then

$$
\begin{aligned}
\|\frac{f(rx)}{r} - Ux\| &= \|\frac{f(r\sum\limits_{k=1}^{\infty} a_k e_k)}{r} - \sum_{k=1}^{\infty} a_k e'_k\| \\
&\leq \|\frac{f(r\sum\limits_{k=1}^{\infty} a_k e_k)}{r} - \frac{f(r\sum\limits_{k=1}^{n_0} a_k e_k)}{r}\| + \|\frac{f(r\sum\limits_{k=1}^{n_0} a_k e_k)}{r} - \sum_{k=1}^{n_0} a_k e'_k\| + \|\sum_{k=1}^{n_0} a_k e'_k - \sum_{k=1}^{\infty} a_k e'_k\| \\
&\leq \frac{\varepsilon_f(r\|\sum\limits_{k=n_0+1}^{\infty} a_k e_k\|)}{r} + \|\sum_{k=n_0+1}^{\infty} a_k e_k\| + \|\sum_{k=n_0+1}^{\infty} a_k e'_k\| + \frac{\varepsilon}{4} \\
&< \varepsilon.
\end{aligned}
$$

It follows that

$$
Ux = \lim_{r\to\infty} \frac{f(rx)}{r} \quad \text{for all } x \in X. \quad \square
$$

By the theorem above, we have the following projection result.

Theorem 2.5. *Let $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ be two σ-finite measure spaces and $1 < p < \infty$. Suppose that $X = L_p(\Omega_1, \Sigma_1, \mu_1)$, $Y = L_p(\Omega_2, \Sigma_2, \mu_2)$ and that $f : X \to Y$ is a standard coarse isometry. If $U : X \to Y$ is a linear isometry and $P : Y \to U(X)$ is a projection with $\|P\| = 1$, then the following statements are equivalent:*

(i) $Ux = \lim\limits_{r\to\infty} \frac{f(rx)}{r}$ for all $x \in X$;

(ii) $Ux = \lim\limits_{r\to\infty} \frac{Pf(rx)}{r}$ for all $x \in X$.

Proof. (i) ⇒ (ii).

$$
\|\frac{Pf(rx)}{r} - Ux\| = \|P(\frac{f(rx)}{r} - Ux)\| \leq \|\frac{f(rx)}{r} - Ux\| \to 0 \quad \text{as } r \to \infty.
$$

(ii) ⇒ (i). Let $T = U^{-1}P : Y \to X$, then $\|T\| = 1$ and for all $x \in X$

$$
\|\frac{Tf(rx)}{r} - x\| = \|\frac{U^{-1}Pf(rx)}{r} - x\| = \|\frac{Pf(rx)}{r} - Ux\| \to 0 \quad \text{as } r \to \infty. \tag{2.8}
$$

By Theorem 2.4, we only need to prove f is pointwisely weakly stable. Given $x^* \in S_{X^*}$, for $x \in \partial\|x^*\|$, we have

$$
\|x^*\| = \langle x^*, x\rangle = \lim_{r\to\infty} \langle x^*, T\frac{f(rx)}{r}\rangle = \lim_{r\to\infty} \langle T^*x^*, \frac{f(rx)}{r}\rangle \leq \|T^*x^*\| \lim_{r\to\infty} \|\frac{f(rx)}{r}\| = \|T^*x^*\| \leq \|x^*\|.
$$

Hence $T^*x^* \in S_{Y^*}$. Let $\phi = T^*x^*$, then $\phi \in S_{Y^*}$ and for all $x \in X$

$$
|\langle x^*, x\rangle - \langle \phi, \frac{f(rx)}{r}\rangle| = |\langle x^*, x\rangle - \langle T^*x^*, \frac{f(rx)}{r}\rangle| = |\langle x^*, x\rangle - \langle x^*, \frac{Tf(rx)}{r}\rangle| \leq \|\frac{Tf(rx)}{r} - x\|
$$

Therefore, combining (2.8) and the inequality above, (i) holds. $\square$

Remark 2.6. According to the proof of Theorem 2.4, we can see that the theorem also holds when X is replaced by a general Banach space with a Schauder basis.

By Theorem 2.4 and Remark 2.6 above, obviously, we have the following results.

Corollary 2.7. *Let X be a Banach space with a sequence $\{z_n\}_{n=1}^{\infty}$ where $\overline{span}\{z_n\}_{n=1}^{\infty} = X$ and $Y = L_p(\Omega, \Sigma, \mu)$ $(1 < p < \infty)$ for some σ-finite measure space (Ω, Σ, μ). Suppose that $f : X \to Y$ is a standard coarse isometry. If for every $n \in \mathbb{N}$ and $x^* \in S_{X^*}$, there exists $\phi \in S_{Y^*}$ such that*

$$\lim_{r\to\infty} |\langle x^*, z_n\rangle - \langle \phi, \frac{f(rz_n)}{r}\rangle| = 0,$$

then there is a linear isometry $U : X \to Y$ satisfying

$$Ux = \lim_{r\to\infty} \frac{f(rx)}{r} \quad \text{for all } x \in X.$$

Corollary 2.8. *Let X be a Banach space with a δ-net M and $Y = L_p(\Omega, \Sigma, \mu)$ $(1 < p < \infty)$ for some σ-finite measure space (Ω, Σ, μ). Suppose that $f : X \to Y$ is a standard coarse isometry. If for every $z \in M$ and $x^* \in S_{X^*}$, there exists $\phi \in S_{Y^*}$ such that*

$$\lim_{r\to\infty} |\langle x^*, z\rangle - \langle \phi, \frac{f(rz)}{r}\rangle| = 0,$$

then there is a linear isometry $U : X \to Y$ satisfying

$$Ux = \lim_{r\to\infty} \frac{f(rx)}{r} \quad \text{for all } x \in X. \tag{2.9}$$

Proof. Since M is a δ-net, for every $x \in X$ and every positive integer n, there exists $z_n \in M$ such that $\|nx - z_n\| < \delta$. Hence $\lim_{n\to\infty} \frac{z_n}{n} = x$. It follows that the linear span of M is dense in X. Hence, we know (2.9) holds by the proof of Theorem 2.4. $\square$

In 2018, D. Dai [8] proved the following result about the stability of basic sequences via nonlinear ε-isometries between Banach spaces whenever Y is a strictly convex Banach space admitting the Kadec-Klee property.

Recall that a sequence $\{e_k\}_{k=1}^{\infty}$ in a Banach space X is called a basic sequence if it is a basis for the closed linear span of $\{e_k\}_{k=1}^{\infty}$. Two bases (or basic sequences) $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in the respective Banach spaces X and Y are equivalent if whenever we take a sequence of scalars $\{a_n\}_{n=1}^{\infty}$, then $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=1}^{\infty} a_n y_n$ converges.

Theorem 2.9. *[1, Theorem 1.3.9] (Principle of small perturbations) Let $\{x_n\}_{n=1}^{\infty}$ be a basic sequence in a Banach space X with basis constant K_b. If $\{y_n\}_{n=1}^{\infty}$ is a sequence in X such that*

$$2K_b \sum_{n=1}^{\infty} \frac{\|x_n - y_n\|}{\|x_n\|} = \theta < 1,$$

then $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are congruent. In particular:

(i) $\{y_n\}_{n=1}^{\infty}$ is a basic sequence with basis constant at most $K_b(1 + \theta)(1 - \theta)^{-1}$.

(ii) If $\{x_n\}_{n=1}^{\infty}$ is a basis, so is $\{y_n\}_{n=1}^{\infty}$.

(iii) If the closed linear span of $\{x_n\}_{n=1}^{\infty}$ is complemented in X, then so is the closed linear span of $\{y_n\}_{n=1}^{\infty}$.

Using the above result and Corollary 2.8, we obtain the stability of basic sequences by non-surjective coarse isometry.

Theorem 2.10. *Let X be a Banach space with a δ-net M and $Y = L_p(\Omega, \Sigma, \mu)$ $(1 < p < \infty)$ for some σ-finite measure space (Ω, Σ, μ). Suppose that $f : X \to Y$ is a standard coarse isometry and that $\{x_n\} \subseteq X$ is a basic sequence. If for every $z \in M$ and $x^* \in S_{X^*}$, there exists $\phi \in S_{Y^*}$ such that*

$$\lim_{r\to\infty} |\langle x^*, z\rangle - \langle \phi, \frac{f(rz)}{r}\rangle| = 0,$$

then there exists $\{\lambda_n\} \in I$ such that $\{f(\lambda_n x_n)\}$ is a basic sequence which is equivalent to $\{\lambda_n x_n\}$.

Proof. By Corollary 2.8, there is a linear isometry $U : X \to Y$ satisfying

$$Ux = \lim_{r\to\infty} \frac{f(rx)}{r} \quad \text{for all } x \in X.$$

Hence $\{Ux_n\}$ is a basic sequence equivalent to $\{x_n\}$ and we can choose $\{\lambda_n\} \in I$ such that

$$2K \sum_{n=1}^{\infty} \frac{\|Ux_n - \frac{f(\lambda_n x_n)}{\lambda_n}\|}{\|Ux_n\|} = \theta < 1,$$

where K is the basis constant of $\{x_n\}$. Consequently, by Theorem 2.9 and the discussion above, $\{\frac{f(\lambda_n x_n)}{\lambda_n}\}$ is a basic sequence equivalent to $\{x_n\}$. This entails $\{f(\lambda_n x_n)\}$ is a basic sequence equivalent to $\{\lambda_n x_n\}$. $\square$

Corollary 2.11. *Let X be a Banach space with a basic sequence $\{x_n\}$ and $Y = L_p(\Omega, \Sigma, \mu)$ $(1 < p < \infty)$ for some σ-finite measure space. If $f : X \to Y$ is a standard ε-isometry. Then there exists $\{\lambda_n\} \in I$ such that $\{f(\lambda_n x_n)\}$ is a basic sequence which is equivalent to $\{\lambda_n x_n\}$.*

The following result states that if $Y = L_p(\Omega, \Sigma, \mu)$ $(1 < p < \infty)$ for some σ-finite measure space (Ω, Σ, μ), then the converse version of Proposition 2.2 is also true for uniform case.

Theorem 2.12. *Let X be a Banach space and $Y = L_p(\Omega, \Sigma, \mu)$ $(1 < p < \infty)$ for some σ-finite measure space (Ω, Σ, μ). Suppose that $f : X \to Y$ is a standard coarse isometry. If for every $x^* \in S_{X^*}$, there exists $\phi \in S_{Y^*}$ such that*

$$|\,\langle x^*, x\rangle - \langle \phi, f(x)\rangle\,| = o(\|x\|) \quad \text{as } \|x\| \to \infty, \tag{2.10}$$

then there is a linear isometry $U : X \to Y$ satisfying

$$\|f(x) - Ux\| = o(\|x\|) \quad \text{as } \|x\| \to \infty.$$

Proof. According to (2.10) and Step I in the proof of Theorem 2.4, there is a linear isometry $U : X \to Y$ satisfying

$$Ux = \lim_{r\to\infty} \frac{f(rx)}{r} \quad \text{for all } x \in X.$$

It remains to show that

$$Ux = \lim_{r\to\infty} \frac{f(rx)}{r} \quad \text{uniformly for } x \in S_X. \tag{2.11}$$

Given $\{\lambda_n\} \in I$. Because of (2.2), we have

$$\|Ux\| = \lim_{n\to\infty} \|\frac{f(\lambda_n x)}{\lambda_n}\| = 1 \quad \text{uniformly for } x \in S_X. \tag{2.12}$$

Due to (2.10), for each $x^* \in S_{X^*}$, there exists $\phi_{x^*} \in S_{Y^*}$ such that

$$\lim_{n\to\infty} |\langle x^*, x\rangle - \langle \phi_{x^*}, \frac{f(\lambda_n x)}{\lambda_n}\rangle| = 0 \quad \text{uniformly for } x \in S_X.$$

Consequently, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$ and $x \in S_X$,

$$\left|\|\frac{f(\lambda_n x)}{\lambda_n}\| - 1\right| < \varepsilon \quad \text{and} \quad \left|\langle x^*, x\rangle - \langle \phi_{x^*}, \frac{f(\lambda_n x)}{\lambda_n}\rangle\right| < \varepsilon.$$

On the one hand,

$$\|\frac{f(\lambda_n x)}{\lambda_n} + Ux\| - 2 \le \|\frac{f(\lambda_n x)}{\lambda_n}\| - 1 \le \varepsilon.$$

On the other hand,

$$\begin{aligned}\|\frac{f(\lambda_n x)}{\lambda_n} + Ux\| - 2 &\ge \sup_{x^*\in X^*} \langle \phi_{x^*}, \frac{f(\lambda_n x)}{\lambda_n} + Ux\rangle - 2 \\ &\ge 2 \sup_{x^*\in X^*} \langle x^*, x\rangle - 2 - \varepsilon \\ &= -\varepsilon.\end{aligned}$$

Hence,

$$\lim_{n\to\infty} \|\frac{f(\lambda_n x)}{\lambda_n} + Ux\| = 2 \quad \text{uniformly for } x \in S_X. \tag{2.13}$$

Since Y is uniformly convex, combining (2.12) and (2.13), we get

$$Ux = \lim_{n\to\infty} \frac{f(\lambda_n x)}{\lambda_n} \quad \text{uniformly for } x \in S_X.$$

Therefore, (2.11) holds. □

Remark 2.13. Theorem 2.12 also holds when the image space Y is replaced by a uniformly convex Banach space.

By the theorem above, similarly, we obtain a uniform version of Theorem 2.5.

Theorem 2.14. *Let $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ be two σ-finite measure spaces and $1 < p < \infty$. Suppose that $X = L_p(\Omega_1, \Sigma_1, \mu_1)$, $Y = L_p(\Omega_2, \Sigma_2, \mu_2)$ and that $f : X \to Y$ is a standard coarse isometry. If $U : X \to Y$ is a linear isometry and $P : Y \to U(X)$ is a projection with $\|P\| = 1$, then the following statements are equivalent:*

(i) $\|f(x) - Ux\| = o(\|x\|)$ as $\|x\| \to \infty$;

(ii) $\|Pf(x) - Ux\| = o(\|x\|)$ as $\|x\| \to \infty$.

Proof. (i) ⇒ (ii).

$$\|Pf(x) - Ux\| = \|P(f(x) - Ux)\| \le \|f(x) - Ux\| = o(\|x\|) \quad \text{as } \|x\| \to \infty.$$

(ii) $\Rightarrow$ (i). Let $T = U^{-1}P : Y \to X$, then $\|T\| = 1$ and

$$\|Tf(x) - x\| = \|U^{-1}Pf(x) - x\| = \|Pf(x) - Ux\| = o(\|x\|) \quad \text{as } \|x\| \to \infty. \tag{2.14}$$

By Theorem 2.12, we only need to prove (2.10). Given $x^* \in S_{X^*}$, for $x \in \partial\|x^*\|$, we have

$$\|x^*\| = \langle x^*, x\rangle = \lim_{r\to\infty} \langle x^*, T\frac{f(rx)}{r}\rangle = \lim_{r\to\infty} \langle T^*x^*, \frac{f(rx)}{r}\rangle \le \|T^*x^*\| \lim_{r\to\infty} \|\frac{f(rx)}{r}\| = \|T^*x^*\| \le \|x^*\|.$$

Hence $T^*x^* \in S_{Y^*}$. Let $\phi = T^*x^*$, then $\phi \in S_{Y^*}$ and

$$|\langle x^*, x\rangle - \langle \phi, f(x)\rangle| = |\langle x^*, x\rangle - \langle T^*x^*, f(x)\rangle| = |\langle x^*, x\rangle - \langle x^*, Tf(x)\rangle| \le \|Tf(x) - x\|$$

Therefore, combining (2.14) and the inequality above, we obtain (2.10) and then (i) holds. $\square$

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