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# Variance of real zeros of random orthogonal polynomials

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ABSTRACT

We determine the asymptotics for the variance of the number of zeros of random linear combinations of orthogonal polynomials of degree $\leq n$ in subintervals $[a, b]$ of the support of the underlying orthogonality measure $\mu$. We show that, as $n \to \infty$, this variance is asymptotic to $cn$, for some explicit constant $c > 0$.



## 1. Introduction and main results

Let $\mu$ be a positive Borel measure compactly supported in the real line, whose support contains infinitely many points. For $n \geq 0,\ n \in \mathbb{Z}$, we consider the $n$th orthonormal polynomial

$$p_n(x) = \gamma_n x^n + \dots \tag{1.1}$$

for $\mu$, with $\gamma_n > 0$, so that

$$\int p_n(x) p_m(x)\, d\mu(x) = \delta_{mn}, \quad m, n \geq 0.$$

Define the ensemble of random orthogonal polynomials of the form

$$G_n(x) = \sum_{j=0}^{n} a_j p_j(x), \quad n \geq 0, \tag{1.2}$$

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where $\{a_j\}_{j=0}^{\infty}$ are standard Gaussian $\mathcal{N}(0,1)$ i.i.d. random variables. For any interval $[a,b] \subset \mathbb{R}$, let $N_n([a,b])$ (resp. $N_n(\mathbb{R})$) denote the number of zeros of $G_n$ lying in $[a,b]$ (resp. total number of real zeros).

Real zeros of high degree random polynomials have been studied since the 1930s. The early work concentrated on the expected number of real zeros $\mathbb{E}[N_n(\mathbb{R})]$ for $P_n(x) = \sum_{k=0}^{n} a_k x^k$, where $\{a_k\}_{k=0}^{n}$ are i.i.d. random variables. Bloch and Pólya [9] gave the upper bound $\mathbb{E}[N_n(\mathbb{R})] = O(\sqrt{n})$ for polynomials with coefficients in $\{-1,0,1\}$. Improvements and generalizations were obtained by Littlewood and Offord [26,27], Erdős and Offord [14] and others. Kac [22] introduced the "Kac-Rice formula" to establish the important asymptotic result

$$\mathbb{E}[N_n(\mathbb{R})] = (2/\pi + o(1)) \log n \quad \text{as } n \to \infty,$$

for polynomials with independent real Gaussian coefficients.

More precise forms of this asymptotic were obtained by Kac [23], Edelman and Kostlan [13], Wilkins [40] and others. For related further directions, see [7] and [16]. Maslova [32] proved that the variance of real zeros for Kac polynomials $\sum_{k=0}^{n} a_k z^k$ satisfies

$$\mathrm{Var}[N_n(\mathbb{R})] = \frac{4}{\pi}\left(1 - \frac{2}{\pi}\right) \log n + o(\log n)$$

for i.i.d. coefficients with mean 0, variance 1 and $\mathbb{P}(a_k = 0) = 0$. This result was recently generalized by Nguyen and Vu [33].

Das [10] considered random Legendre polynomials corresponding to Lebesgue measure $d\mu(x) = dx$ on $[-1,1]$, and found that $\mathbb{E}[N_n([-1,1])]$ is asymptotically equal to $n/\sqrt{3}$. Wilkins [39] estimated the error term in this asymptotic relation. For random Jacobi polynomials, Das and Bhatt [11] established that $\mathbb{E}[N_n([-1,1])]$ is asymptotically equal to $n/\sqrt{3}$ too. Farahmand [15], [16], [17] considered the expected number of the level crossings of random sums of Legendre polynomials with coefficients having different distributions. These results were generalized to wide classes of random orthogonal polynomials by Lubinsky, Pritsker and Xie [30] and [31]. In particular, they showed that the first term in the asymptotics for $\mathbb{E}[N_n(\mathbb{R})]$ remains the same as for the Legendre case.

The asymptotic variance and the Gaussianity for real zeros of random trigonometric polynomials were established by Granville and Wigman [19], and subsequently by Azaïs and León [2] via different methods. Su and Shao [35] found the asymptotic variance for the real zeros of random cosine polynomials, while Azaïs, Dalmao and León [1] gave a different proof. Xie [41] showed that the variance of real zeros for a general class of random orthogonal polynomials is $o(n^2)$. A recent paper of Do, H. Nguyen and O. Nguyen [12] studied dependence of the variance on the distribution of the i.i.d. random coefficients in the trigonometric case.

In this paper our main goal is determining the asymptotic for the variance of the number of real zeros for the ensemble of random orthogonal polynomials of the form (1.2). To state our results, we require the following definition:

**Definition 1.1.** We say that a measure is regular in the sense of Stahl, Totik, and Ullman, if the leading coefficients $\{\gamma_j\}$ of the orthonormal polynomials in (1.1) satisfy

$$\lim_{j\to\infty} \gamma_j^{1/j} = \frac{1}{cap\,(\mathrm{supp}\,[\mu])},$$

where $cap\,(\mathrm{supp}\,[\mu])$ denotes the logarithmic capacity of $\mathrm{supp}\,[\mu]$.

While not a transparent condition, it is a weak one. For example, if the support of $\mu$ consists of finitely many intervals, and $\mu'$ is positive a.e. in each of those intervals, then $\mu$ is regular. However, much less is needed [34]. We let $\nu$ denote the equilibrium measure $\nu$ for $\operatorname{supp}[\mu]$ in the sense of potential theory, and let $\omega(x) = \frac{d\nu}{dx}$. In any open subinterval of $\operatorname{supp}[\mu]$, $\omega$ exists, and is positive and continuous [34]. For example, when $\operatorname{supp}[\mu] = [-1, 1]$,

$$\omega(x) = \frac{1}{\pi\sqrt{1-x^2}}.$$

Let

$$S(u) = \frac{\sin \pi u}{\pi u}; \tag{1.3}$$

$$F(u) = \det \begin{bmatrix} 1 & S(u) & 0 & S'(u) \\ S(u) & 1 & -S'(u) & 0 \\ 0 & -S'(u) & -S''(0) & -S''(u) \\ S'(u) & 0 & -S''(u) & -S''(0) \end{bmatrix}; \tag{1.4}$$

$$G(u) = \det \begin{bmatrix} 1 & S(u) & -S'(u) \\ S(u) & 1 & 0 \\ -S'(u) & 0 & -S''(0) \end{bmatrix}; \tag{1.5}$$

$$H(u) = \det \begin{bmatrix} 1 & S(u) & 0 \\ S(u) & 1 & -S'(u) \\ S'(u) & 0 & -S''(u) \end{bmatrix}. \tag{1.6}$$

Sylvester's determinant identity and the fact that $G(-u) = G(u)$ show that

$$\left(1 - S(u)^2\right) F(u) = G(u)^2 - H(u)^2.$$

Also let

$$\Xi(u) = \frac{1}{\pi^2} \left\{ \frac{\sqrt{F(u)}}{1 - S(u)^2} + \frac{1}{\left(1 - S(u)^2\right)^{3/2}} H(u) \arcsin\left(\frac{H(u)}{G(u)}\right) \right\} - \frac{1}{3} \tag{1.7}$$

and

$$c = \int_{-\infty}^{\infty} \Xi(u)\, du + \frac{1}{\sqrt{3}}. \tag{1.8}$$

**Theorem 1.2.** *Let $\mu$ be a measure with compact support on the real line, that is regular in the sense of Stahl, Totik, and Ullmann. Let $\omega$ denote the Radon-Nikodym derivative of the equilibrium measure for the support of $\mu$. Let $[a', b']$ be a subinterval in the support of $\mu$, such that $\mu$ is absolutely continuous there, and its Radon-Nikodym derivative $\mu'$ is positive and continuous there. Assume moreover, that*

$$\sup_{n \geq 1} \|p_n\|_{L_\infty[a', b']} < \infty. \tag{1.9}$$

*If $[a, b] \subset (a', b')$ then*

$$\lim_{n\to\infty} \frac{1}{n} Var[N_n([a, b])] = c \left( \int_a^b \omega(y)\, dy \right). \tag{1.10}$$

Note that the limit does not depend on the particular measure $\mu$, but involves the equilibrium density of the support of $\mu$. The bounds for the orthonormal polynomials are known for example when $\mu'$ satisfies a Dini-Lipschitz condition. Therefore an application of Theorem 1.2 gives:

**Corollary 1.3.** *Let $\mu$ be a measure supported on $[-1,1]$ satisfying the Szegő condition*

$$\int_{-1}^{1} \log \mu'(x) \frac{dx}{\pi\sqrt{1-x^2}} > -\infty.$$

*Let $[a',b']$ be a subinterval of $(-1,1)$, in which $\mu$ is absolutely continuous, while $\mu'$ is positive and continuous in $[a',b']$. Assume moreover that its local modulus of continuity,*

$$\Omega(t) = \sup\{|\mu'(x) - \mu'(y)| : x, y \in [a',b'] \text{ and } |x-y| \le t\},\ t > 0,$$

*satisfies the Dini-Lipshitz condition*

$$\int_0^1 \frac{\Omega(t)}{t} dt < \infty.$$

*If $[a,b] \subset (a',b')$, then*

$$\lim_{n\to\infty} \frac{1}{n} Var[N_n([a,b])] = c\left(\int_a^b \frac{1}{\pi\sqrt{1-y^2}} dy\right). \tag{1.11}$$

**Remarks.**

(a) We believe that this result is new even for the Legendre weight $\mu' = 1$.

(b) The hypotheses of Theorem 1.2 are also satisfied for exponential weights investigated in [25] that do not satisfy the Szegő condition. For example, the conclusion of Theorem 1.2 holds for any $[a,b] \subset (-1,1)$, when

$$\mu'(x) = \exp\left(-\exp_k \left(1-x^2\right)^{-\alpha}\right),\ x \in (-1,1),$$

where $\alpha > 0$ and $\exp_k = \exp(\exp(\ldots\exp()))$ denotes the $k$th iterated exponential.

(c) For a class of weights supported on several disjoint intervals, in a classic paper, Widom [38] established asymptotics of the orthonormal polynomials under some smoothness conditions on the weight. These imply the uniform boundedness of the orthonormal polynomials in subintervals of the interior of the support, so that Theorem 1.2 applies to Widom's weights.

(d) As noted above, the analogous limit for trigonometric polynomials was established by Granville and Wigman in [19]. We have indications that our results are related to those of [19] via the same limiting Paley-Wiener process.

(e) Azaïs, Dalmao and León [1, Theorem 1] found the asymptotics for the variance of zeros of random cosine polynomials $\sum_{k=0}^{n} a_k \cos ky$ on $[0,\pi]$. These random cosine polynomials are equivalent to the random Chebyshev polynomials $\sum_{k=0}^{n} a_k T_k(x)$ on $[-1,1]$ by the change of variable $y = \arccos x$. Our asymptotic variance result of Theorem 1.2 for the random Chebyshev polynomials agrees with that of [1, Theorem 1] for random cosine polynomials.

This paper is organized as follows: in Section 2, we state the Kac-Rice formula for the variance, and prove Theorem 1.2 and Corollary 1.3, deferring technical details to later sections. In Section 3, we record some technical estimates and gather results from elsewhere. In Section 4, we estimate the "tail term" with $|x-y| \geq \frac{\Lambda}{n}$ in the integral defining the main term in the variance. In Section 5, we handle the "central term" where $x$ and $y$ are close, which gives the dominant contribution to the integral. In Section 6, the appendix, we prove the formula for the variance.

In the sequel, $C, C_1, C_2, ...$ denote constants independent of $n, x, y$. The same symbol may be different in different occurrences.

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## 2. The proofs of Theorem 1.2 and Corollary 1.3

We begin with the Kac-Rice formulas for the expectation and variance. These involve the reproducing kernel

$$K_n(x,y) = \sum_{j=0}^{n-1} p_j(x)\, p_j(y) \tag{2.1}$$

and for nonnegative integers $r, s$, its derivatives

$$K_n^{(r,s)}(x,y) = \sum_{j=0}^{n-1} p_j^{(r)}(x)\, p_j^{(s)}(y). \tag{2.2}$$

**Lemma 2.1.** *Let $[a,b] \subset \mathbb{R}$, and let $G_n$ be defined by (1.2). Then the expected number of real zeros for $G_n$ is expressed by*

$$\mathbb{E}\left[N_n([a,b])\right] = \frac{1}{\pi}\int_a^b \rho_1(x)\; dx, \tag{2.3}$$

*where*

$$\rho_1(x) = \frac{1}{\pi}\sqrt{\frac{K_{n+1}^{(1,1)}(x,x)}{K_{n+1}(x,x)} - \left(\frac{K_{n+1}^{(0,1)}(x,x)}{K_{n+1}(x,x)}\right)^2}. \tag{2.4}$$

**Proof.** See [30]. □

We note that $\rho_1$ depends on $n$, but we omit this dependence to simplify the notation. The same applies to $\rho_2$ below. The variance of real zeros of $G_n$ is found from the following formula, which was derived in [41] by using the method of [19].

**Lemma 2.2.** *Let $[a,b] \subset \mathbb{R}$, and let $G_n$ be defined by (1.2).*

$$\mathrm{Var}\left[N_n([a,b])\right] = \int_a^b\int_a^b \left\{\rho_2(x,y) - \rho_1(x)\,\rho_1(y)\right\} dxdy + \int_a^b \rho_1(x)\, dx, \tag{2.5}$$

*where*

$$\rho_2(x,y) = \frac{1}{\pi^2\sqrt{\Delta}}\left(\sqrt{\Omega_{11}\Omega_{22}-\Omega_{12}^2}+\Omega_{12}\arcsin\left(\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}\right)\right). \tag{2.6}$$

*Here*

$$\Delta(x,y) := K_{n+1}(x,x)K_{n+1}(y,y) - K_{n+1}^2(x,y) \tag{2.7}$$

*and* $\Omega$ *is the covariance matrix of the random vector* $(P_n'(x), P_n'(y))$ *conditional upon* $P_n(x) = P_n(y) = 0$*:*

$$\Omega = \begin{bmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{12} & \Omega_{22}\end{bmatrix},$$

*with*

$$\begin{aligned}&\Omega_{11}(x,y) := K_{n+1}^{(1,1)}(x,x) - \\ &\frac{1}{\Delta}\left(K_{n+1}(y,y)(K_{n+1}^{(0,1)}(x,x))^2 - 2K_{n+1}(x,y)K_{n+1}^{(0,1)}(x,x)K_{n+1}^{(0,1)}(y,x) + K_{n+1}(x,x)(K_{n+1}^{(0,1)}(y,x))^2\right),\end{aligned} \tag{2.8}$$

$$\begin{aligned}&\Omega_{22}(x,y) := K_{n+1}^{(1,1)}(y,y) - \\ &\frac{1}{\Delta}\left(K_{n+1}(y,y)(K_{n+1}^{(0,1)}(x,y))^2 - 2K_{n+1}(x,y)K_{n+1}^{(0,1)}(x,y)K_{n+1}^{(0,1)}(y,y) + K_{n+1}(x,x)(K_{n+1}^{(0,1)}(y,y))^2\right),\end{aligned} \tag{2.9}$$

$$\begin{aligned}&\Omega_{12}(x,y) := K_{n+1}^{(1,1)}(x,y) - \\ &\frac{1}{\Delta}[K_{n+1}(y,y)K_{n+1}^{(0,1)}(x,x)K_{n+1}^{(0,1)}(x,y) - K_{n+1}(x,y)K_{n+1}^{(0,1)}(x,y)K_{n+1}^{(0,1)}(y,x) \\ &-K_{n+1}(x,y)K_{n+1}^{(0,1)}(x,x)K_{n+1}^{(0,1)}(y,y) + K_{n+1}(x,x)K_{n+1}^{(0,1)}(y,x)K_{n+1}^{(0,1)}(y,y)].\end{aligned} \tag{2.10}$$

**Proof.** See the Appendix. It is also shown there that the matrix $\Omega$ is nonnegative definite, so that the square root defining $\rho_2$ is well defined. □

To prove Theorem 1.2, we split the first integral in (2.5) into a central term that provides the main contribution, and a tail term: for some large enough $\Lambda$, write

$$\begin{aligned}&\int_a^b\int_a^b \{\rho_2(x,y) - \rho_1(x)\rho_1(y)\}\,dx\,dy \\ &= \left[\iint\limits_{\{(x,y):x,y\in[a,b],|x-y|\geq\Lambda/n\}} + \iint\limits_{\{(x,y):x,y\in[a,b],|x-y|<\Lambda/n\}}\right]\{\rho_2(x,y)-\rho_1(x)\rho_1(y)\}\,dx\,dy \\ &= \text{Tail } + \text{Central}.\end{aligned}$$

We handle the tail term by proving the following estimate and a simple consequence:

**Lemma 2.3.**

*(a) There exist $C_1, n_0$, and $\Lambda_0$ such that for $n \geq n_0$ and $|x-y| \geq \frac{\Lambda_0}{n}$,*

$$|\rho_2(x,y) - \rho_1(x)\rho_1(y)| \leq \frac{C_1}{|x-y|^2}. \tag{2.11}$$

*(b) There exist $C_2, n_0$, and $\Lambda_0$ such that for $n \geq n_0$ and $\Lambda \geq \Lambda_0$,*

$$\iint_{\{(x,y):x,y\in[a,b],|x-y|\geq\Lambda/n\}} |\rho_2(x,y) - \rho_1(x)\rho_1(y)|\, dx\, dy \leq C_2\frac{n}{\Lambda}. \tag{2.12}$$

**Proof.** See Section 4. □

Recall that $\Xi$ is defined by (1.7). For the central term we will prove:

**Lemma 2.4.**

*(a) Uniformly for $u$ in compact subsets of $\mathbb{C}\setminus\{0\}$, and $x \in [a,b]$ and $y = x + \frac{u}{n\omega(x)}$,*

$$\left(\frac{1}{n\omega(x)}\right)^2 \{\rho_2(x,y) - \rho_1(x)\rho_1(y)\} = \Xi(u) + o(1). \tag{2.13}$$

*(b) Let $\eta > 0$. There exists $C$ such that for $x \in [a,b]$, $y = x + \frac{u}{n\omega(x)}$, $u \in [-\eta,\eta]$ and $n \geq 1$,*

$$|\rho_2(x,y) - \rho_1(x)\rho_1(y)| \leq Cn^2.$$

**Proof.** See Section 5. □

The second integral in (2.5) is simpler:

**Lemma 2.5.**

$$\frac{1}{n}\int_a^b \rho_1(x)\,dx = \frac{1}{\sqrt{3}}\int_a^b \omega(x)\,dx + o(1). \tag{2.14}$$

**Proof.** See Section 5. □

**Proof of Theorem 1.2.** We fix $\Lambda > \eta > 0$ and split

$$\begin{aligned}&\int_a^b\int_a^b \{\rho_2(x,y) - \rho_1(x)\rho_1(y)\}\,dy\,dx\\ &= \int_a^b\left[\int_I + \int_J + \int_K\right]\{\rho_2(x,y) - \rho_1(x)\rho_1(y)\}\,dy\,dx,\end{aligned} \tag{2.15}$$

where for a given $x$,

$$\begin{aligned} I &= \{y \in [a,b] : |y-x| \geq \Lambda / (n\omega(x))\}; \\ J &= \{y \in [a,b] : \eta / (n\omega(x)) \leq |y-x| < \Lambda / (n\omega(x))\}; \\ K &= \{y \in [a,b] : |y-x| < \eta / (n\omega(x))\}. \end{aligned}$$

If $\omega_0$ is the maximum of $\omega(x)$ in $[a,b]$, (recall that $\omega$ is positive and continuous in $[a,b]$) then

$$\begin{aligned} &\left| \int_a^b \int_I \{\rho_2(x,y) - \rho_1(x)\rho_1(y)\} \, dy \, dx \right| \\ &\leq \iint_{\{(x,y): x,y \in [a,b], |x-y| \geq \Lambda/(n\omega_0)\}} |\rho_2(x,y) - \rho_1(x)\rho_1(y)| \, dy \, dx \\ &\leq C_1 \frac{n\omega_0}{\Lambda}, \end{aligned} \tag{2.16}$$

by Lemma 2.3(b), provided $\Lambda/\omega_0 \geq \Lambda_0$. Next,

$$\begin{aligned} &\frac{1}{n} \int_a^b \int_J \{\rho_2(x,y) - \rho_1(x)\rho_1(y)\} \, dy \, dx \\ &= \int_a^b \omega(x) \int_{\substack{\eta \leq |u| \leq \Lambda, \\ x + \frac{u}{n\omega(x)} \in [a,b]}} \left\{ \rho_2\left(x, x + \frac{u}{n\omega(x)}\right) - \rho_1(x)\rho_1\left(x + \frac{u}{n\omega(x)}\right) \right\} \frac{1}{(n\omega(x))^2} du \, dx. \end{aligned}$$

Note that if $\eta \leq |u| \leq \Lambda$ and $x \in [a,b]$ but $x + \frac{u}{n\omega(x)} \notin [a,b]$, then $x$ is at a distance of $O\left(\frac{\Lambda}{n}\right)$ to $a$ or $b$, and in view of Lemma 2.4(b), the integral over such $(x,u)$ is $O\left(\frac{1}{n}\right)$. Using Lemma 2.4(a), we deduce that

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \int_a^b \int_J \{\rho_2(x,y) - \rho_1(x)\rho_1(y)\} \, dy dx \\ &= \left( \int_a^b \omega(x) \, dx \right) \left( \int_{\eta \leq |u| \leq \Lambda} \Xi(u) \, du \right). \end{aligned} \tag{2.17}$$

Finally, from Lemma 2.4(b), (but with a different fixed $\eta$ there),

$$\frac{1}{n} \left| \int_a^b \int_K \{\rho_2(x,y) - \rho_1(x)\rho_1(y)\} \, dy dx \right| \leq C\eta, \tag{2.18}$$

where $C$ is independent of $n, \eta$. Combining the three estimates (2.16)–(2.18) over $I, J, K$, with (2.15) and Lemma 2.5, we obtain

$$\begin{aligned} &\limsup_{n\to\infty} \left| \frac{1}{n} Var[N_n([a,b])] - \left( \int_a^b \omega(x) \, dx \right) \left( \int_{\eta \leq |u| \leq \Lambda} \Xi(u) \, du + \frac{1}{\sqrt{3}} \right) \right| \\ &\leq C\left( \frac{1}{\Lambda} + \eta \right), \end{aligned}$$

where $C$ is independent of $n, \Lambda, \eta$. Now if $B > A \geq \Lambda_0$, then Lemma 2.3(b) and Lemma 2.4(a) show that

$$\left(\int_a^b \omega(x)\,dx\right) \left|\int_{A\leq u\leq B} \Xi(u)\,du\right|$$
$$= \lim_{n\to\infty} \frac{1}{n}\left|\int_a^b \int_{\{y\in[a,b]:A\omega(x)/n\leq y-x<B\omega(x)/n\}} \{\rho_2(x,y)-\rho_1(x)\rho_1(y)\}\,dy\,dx\right| \leq C_1/A.$$

It follows that $\int_{\Lambda_0}^{\infty} \Xi(u)\,du$ converges. Similarly, $\int_{-\infty}^{-\Lambda_0} \Xi(u)\,du$ converges. So we may let $\Lambda\to\infty$ above and deduce that

$$\limsup_{n\to\infty}\left|\frac{1}{n}Var\left[N_n([a,b])\right] - \left(\int_a^b \omega(x)\,dx\right)\left(\int_{|u|\geq\eta} \Xi(u)\,du + \frac{1}{\sqrt{3}}\right)\right|$$
$$\leq C\eta.$$

On the other hand, Lemma 2.4(a) and Lemma 2.4(b) show that if $0<\delta<\eta$,

$$\left(\int_a^b \omega(x)\,dx\right) \left|\int_{\delta\leq u\leq \eta} \Xi(u)\,du\right|$$
$$= \lim_{n\to\infty} \frac{1}{n}\left|\int_a^b \int_{\{y\in[a,b]:\delta\omega(x)/n\leq y-x<\eta\omega(x)/n\}} \{\rho_2(x,y)-\rho_1(x)\rho_1(y)\}\,dy\,dx\right| \leq C_2\eta.$$

It follows that $\int_0^{\eta} \Xi(u)\,du$ converges. Similarly, $\int_{-\eta}^{0} \Xi(u)\,du$ converges. So we may let $\eta\to 0+$ above to deduce the result. □

**Proof of Corollary 1.3.** Under the hypotheses of this theorem, Badkov even established asymptotics for the orthonormal polynomials [4, p. 42, Corollary 2] that trivially imply (1.9). Also, as noted above, since $\mu'$ satisfies Szegő's condition and so is positive a.e. in $[-1,1]$, it is regular [34, Corollary 4.1.3]. Then the result follows from Theorem 1.2. □

## 3. Auxiliary results

Throughout this section, we assume that $\mu$ is as in Theorem 1.2. We begin by recording some determinantal and other formulae: let $\Delta, \Omega_{11}, \Omega_{12}, \Omega_{22}$ be as in (2.7)–(2.10). Also let

$$\Sigma = \begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) & K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(0,1)}(x,y) \\ K_{n+1}(x,y) & K_{n+1}(y,y) & K_{n+1}^{(0,1)}(y,x) & K_{n+1}^{(0,1)}(y,y) \\ K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(0,1)}(y,x) & K_{n+1}^{(1,1)}(x,x) & K_{n+1}^{(1,1)}(x,y) \\ K_{n+1}^{(0,1)}(x,y) & K_{n+1}^{(0,1)}(y,y) & K_{n+1}^{(1,1)}(x,y) & K_{n+1}^{(1,1)}(y,y) \end{bmatrix}. \tag{3.1}$$

**Lemma 3.1.**
*(a)*

$$\Delta(x,y) = \det \begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) \\ K_{n+1}(y,x) & K_{n+1}(y,y) \end{bmatrix}; \tag{3.2}$$

*(b)*

$$\Delta\Omega_{11} = \det \begin{bmatrix} K_{n+1}(y,y) & K_{n+1}(y,x) & K_{n+1}^{(0,1)}(y,x) \\ K_{n+1}(x,y) & K_{n+1}(x,x) & K_{n+1}^{(0,1)}(x,x) \\ K_{n+1}^{(1,0)}(x,y) & K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(1,1)}(x,x) \end{bmatrix}; \tag{3.3}$$

*(c)*

$$\Delta\Omega_{22} = \det \begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) & K_{n+1}^{(0,1)}(x,y) \\ K_{n+1}(y,x) & K_{n+1}(y,y) & K_{n+1}^{(0,1)}(y,y) \\ K_{n+1}^{(1,0)}(y,x) & K_{n+1}^{(1,0)}(y,y) & K_{n+1}^{(1,1)}(y,y) \end{bmatrix}; \tag{3.4}$$

*(d)*

$$\Delta\Omega_{12} = \det \begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) & K_{n+1}^{(0,1)}(x,x) \\ K_{n+1}(y,x) & K_{n+1}(y,y) & K_{n+1}^{(0,1)}(y,x) \\ K_{n+1}^{(1,0)}(y,x) & K_{n+1}^{(0,1)}(y,y) & K_{n+1}^{(1,1)}(y,x) \end{bmatrix}. \tag{3.5}$$

*(e) Let $\Sigma$ be given by (3.1). Then*

$$\left(\Omega_{11}\Omega_{22} - \Omega_{12}^2\right)\Delta = \det(\Sigma). \tag{3.6}$$

**Proof.** (a)–(d): These follow by expanding the determinants for example along the bottom row.
(e) This can be established using Sylvester's determinant identity [5, p. 24, Thm. 1.4.1] on the matrix $\Sigma$ defined by (3.1):

$$\det(\Sigma)\det(\Sigma_{3,4;3,4}) = \det(\Sigma_{3;3})\det(\Sigma_{4;4}) - \det(\Sigma_{3;4})\det(\Sigma_{4;3}),$$

where $\Sigma_{3,4;3,4}$ denotes the $2\times 2$ matrix formed from $\Sigma$ by removing the 3rd and 4th rows and columns of $\Sigma$, while $\Sigma_{r;s}$ denotes the $3\times 3$ matrix formed from $\Sigma$ by removing the $r$th row and $s$th column. This identity and (a–d) yield

$$\det(\Sigma)\Delta = (\Delta\Omega_{22})(\Delta\Omega_{11}) - (\Delta\Omega_{12})^2.$$

Note that in identifying $\det(\Sigma_{4;4})$ with $\Delta\Omega_{11}$, we have to swap the 1st and 2nd rows and columns. Moreover, we use that $\Sigma_{4;3}^T = \Sigma_{3;4}$. □

Next, we record some estimates on the reproducing kernels and their derivatives:

**Lemma 3.2.** *Let $[a,b]$ be a subinterval of $(a',b')$. Then for $r,s = 0,1$ and $r=2, s=0$; and for all $n \geq 1$ and $x,y \in [a,b]$,*

$$\left|K_n^{(r,s)}(x,y)\right| \leq \frac{Cn^{r+s}}{|x-y| + \frac{1}{n}}. \tag{3.7}$$

**Proof.** First we note that since $\mu$ has compact support [18, p. 41],

$$C_2 = \sup_{n\geq 1} \frac{\gamma_{n-1}}{\gamma_n} < \infty.$$

The Christoffel-Darboux formula asserts that

$$K_n(x,y)=\frac{\gamma_{n-1}}{\gamma_n}\frac{p_n(x)\,p_{n-1}(y)-p_{n-1}(x)\,p_n(y)}{x-y},$$

so that using our bound $|p_n(x)|\leq C_1$ for $x,y\in[a',b']$,

$$|K_n(x,y)|\leq\frac{2C_2C_1^2}{|x-y|}.$$

Moreover, by Cauchy-Schwartz,

$$|K_n(x,y)|\leq\left(\sum_{j=0}^{n-1}p_j^2(x)\right)^{1/2}\left(\sum_{j=0}^{n-1}p_j^2(y)\right)^{1/2}\leq C_1^2n.$$

Combining the last two inequalities gives

$$|K_n(x,y)|\leq C_1^2\min\left\{\frac{2C_2}{|x-y|},n\right\},$$

so that (for example, using the inequality between arithmetic and harmonic means) we have the result (3.7) for $r=s=0$. Next,

$$\begin{aligned}&K_n^{(1,0)}(x,y)\\&=\frac{\gamma_{n-1}}{\gamma_n}\left(\frac{p_n'(x)\,p_{n-1}(y)-p_{n-1}'(x)\,p_n(y)}{x-y}+\frac{p_{n-1}(x)\,p_n(y)-p_{n-1}(y)\,p_n(x)}{(x-y)^2}\right).\end{aligned}\tag{3.8}$$

To estimate the derivatives, we use Bernstein's inequality for derivatives, namely for polynomials of degree $\leq n$,

$$|P'(x)|\leq\frac{n}{\sqrt{1-x^2}}\|P\|_{L_\infty[-1,1]},x\in(-1,1).$$

This has the following consequence: for $j,n\geq1$ and polynomials $P$ of degree $\leq n$,

$$\left\|P^{(j)}\right\|_{L_\infty[a,b]}\leq C_3n^j\|P\|_{L_\infty[a',b']}.$$

Here $C_3$ depends on $j,a,b,a',b'$ but not on $P$ nor on the degree $n$ of $P$. It then follows that for $j=0,1,2$,

$$C_4=\sup_{n\geq1}\left\|p_n^{(j)}\right\|_{L_\infty[a,b]}/n^j<\infty.$$

Also then, from (3.8), for $x,y\in[a,b]$,

$$\left|K_n^{(1,0)}(x,y)\right|\leq2C_2\left\{\frac{C_1C_4n}{|x-y|}+\frac{C_1^2}{|x-y|^2}\right\}.$$

Next, by Cauchy-Schwartz,

$$\left|K_n^{(1,0)}(x,y)\right|\leq\left(\sum_{j=0}^{n-1}p_j'(x)^2\right)^{1/2}\left(\sum_{j=0}^{n-1}p_j^2(y)\right)^{1/2}\leq C_4C_1n^2.$$

Thus

$$\left|K_n^{(1,0)}(x,y)\right| \leq C_5 \min\left\{\frac{n}{|x-y|} + \frac{1}{|x-y|^2}, n^2\right\}.$$

This yields (3.7) for $r=1, s=0$. Of course $r=0, s=1$ follows by symmetry. Finally,

$$K_n^{(1,1)}(x,y) = \frac{\gamma_{n-1}}{\gamma_n}\left(\frac{p_n'(x)\,p_{n-1}'(y) - p_{n-1}'(x)\,p_n'(y)}{x-y} + \frac{p_n'(x)\,p_{n-1}(y) - p_{n-1}'(x)\,p_n(y)}{(x-y)^2}\right.$$
$$\left. + \frac{p_{n-1}(x)\,p_n'(y) - p_{n-1}'(y)\,p_n(x)}{(x-y)^2} + 2\frac{p_{n-1}(x)\,p_n(y) - p_{n-1}(y)\,p_n(x)}{(x-y)^3}\right).$$

Thus using our bounds on $\left\{p_n^{(j)}\right\}$, $j=0,1,2$, gives for $x,y \in [a,b]$,

$$\left|K_n^{(1,1)}(x,y)\right| \leq C_6\left\{\frac{n^2}{|x-y|} + \frac{n}{|x-y|^2} + \frac{1}{|x-y|^3}\right\},$$

and again Cauchy-Schwartz gives

$$\left|K_n^{(1,1)}(x,y)\right| \leq \left(\sum_{j=0}^{n-1} p_j'(x)^2\right)^{1/2}\left(\sum_{j=0}^{n-1} p_j'(x)^2\right)^{1/2} \leq C_7 n^3.$$

This and the previous inequality give (3.7) for $r=s=1$. The case $r=2, s=0$ is similar. □

Next, we record some universality limits. Recall that $S$ is defined by (1.3):

**Lemma 3.3.** *Let $[a',b']$ be a subinterval in the support of $\mu$ such that $\mu$ is absolutely continuous there, and $\mu'$ is positive and continuous there. Let $[a,b] \subset (a',b')$. Let $r,s$ be non-negative integers. Then*

*(a) Uniformly for $x \in [a,b]$ and $u,v$ in compact subsets of $\mathbb{C}$,*

$$\lim_{n\to\infty} \frac{K_n^{(r,s)}\left(x+\frac{u}{n\omega(x)}, x+\frac{v}{n\omega(x)}\right)}{K_n(x,x)}\left(\frac{1}{n\omega(x)}\right)^{r+s} = (-1)^s S^{(r+s)}(u-v). \tag{3.9}$$

*(b) Let*

$$\tau_{r,s} = \begin{cases} 0, & r+s \text{ odd} \\ \frac{(-1)^{(r-s)/2}}{r+s+1}, & r+s \text{ even} \end{cases}. \tag{3.10}$$

*Then uniformly for $x \in [a,b]$,*

$$\lim_{n\to\infty} \frac{1}{n^{r+s+1}} K_n^{(r,s)}(x,x)\,\mu'(x) = \pi^{r+s}\omega(x)^{r+s+1}\tau_{r,s} \tag{3.11}$$

*and*

$$\lim_{n\to\infty} \frac{1}{n^{r+s}} \frac{K_n^{(r,s)}(x,x)}{K_n(x,x)} = (\pi\omega(x))^{r+s}\tau_{r,s}. \tag{3.12}$$

*(c) In particular, uniformly for $x \in [a, b]$,*

$$\lim_{n\to\infty} \frac{1}{n^2} K_n^{(1,0)}(x, x) = 0 \tag{3.13}$$

*and for $r = 0, 1$,*

$$K_n^{(r,r)}(x, x) \geq C n^{2r+1}. \tag{3.14}$$

*(d)*

$$S''(0) = -\frac{\pi^2}{3}. \tag{3.15}$$

**Proof.** (a) We start with a result of Totik [37, Theorem 2.2]: uniformly for $x \in [a, b]$, and $u, v$ in compact subsets of $\mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} K_n\left(x + \frac{u}{n}, x + \frac{v}{n}\right) \mu'(x) / \omega(x) = S((u - v)\,\omega(x)). \tag{3.16}$$

In particular, it then follows that uniformly for $x \in [a, b]$, and $u$ in compact subsets of $\mathbb{R}$,

$$\lim_{n\to\infty} \frac{K_n\left(x + \frac{u}{n}, x + \frac{u}{n}\right)}{K_n(x, x)} = 1.$$

Theorem 1.1 in [28, p. 375] then asserts that uniformly for $x \in [a, b]$, and $u, v$ in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{K_n\left(x + \frac{u}{K_n(x,x)\mu'(x)}, x + \frac{v}{K_n(x,x)\mu'(x)}\right)}{K_n(x, x)} = S(u - v).$$

Here the uniformity and Totik's (3.16) allows us to replace $K_n(x, x)\,\mu'(x)$ by $n\omega(x)$: uniformly for $x \in [a, b]$, and $u, v$ in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{K_n\left(x + \frac{u}{n\omega(x)}, x + \frac{v}{n\omega(x)}\right)}{K_n(x, x)} = S(u - v). \tag{3.17}$$

This is the case $r = s = 0$ of (3.9). Because the limit holds uniformly for $u, v$ in compact subsets of $\mathbb{C}$, we may differentiate this asymptotic with respect to $u, v$ to get the general case of (3.9).
(b) For the special case where the support of $\mu$ is $[-1, 1]$, this is Corollary 1.3 in [29, p. 917] (see also [36]). There it was shown that [29, p. 937]

$$S(u - v) = \sum_{j,k=0}^{\infty} \frac{u^j}{j!} \frac{v^k}{k!} \pi^{j+k} \tau_{j,k}, \tag{3.18}$$

so we can reformulate (3.9) for $r = s = 0$ as

$$\lim_{n\to\infty} \sum_{j,k=0}^{\infty} \frac{\left(\frac{u}{n\omega(x)}\right)^j}{j!} \frac{\left(\frac{v}{n\omega(x)}\right)^k}{k!} \frac{K_n^{(j,k)}(x, x)}{K_n(x, x)} = \sum_{j,k=0}^{\infty} \frac{u^j}{j!} \frac{v^k}{k!} \pi^{j+k} \tau_{j,k}.$$

Comparing coefficients of like powers of $u, v$ gives (3.12). That this holds uniformly in $x$ for a given $r, s$ follows easily from the uniformity of the original limit in $x$ (cf. [29, p. 938]). Finally Totik's limit (3.16) gives

$$\lim_{n\to\infty} \frac{1}{n} K_n(x,x) \mu'(x) / \omega(x) = 1,$$

uniformly for $x \in [a,b]$, so we also obtain the first asymptotic (3.11).
(c) This follows directly from (b).
(d) From (3.18),

$$S(u) = \sum_{j=0}^{\infty} \frac{u^j}{j!} \pi^j \tau_{j,0}. \tag{3.19}$$

So $S''(0) = \pi^2 \tau_{2,0} = -\frac{\pi^2}{3}$. $\square$

## 4. The tail term - Lemma 2.3

Recall that $\rho_1, \rho_2$ are defined by (2.4) and (2.6). First write

$$\rho_1(x) = \frac{1}{\pi K_{n+1}(x,x)} \sqrt{\Psi(x)} \tag{4.1}$$

where

$$\Psi(x) = K_{n+1}^{(1,1)}(x,x) K_{n+1}(x,x) - K_{n+1}^{(0,1)}(x,x)^2. \tag{4.2}$$

Next, write

$$\rho_2(x,y) - \rho_1(x)\rho_1(y) = T_1 + T_2 + T_3, \tag{4.3}$$

where

$$\begin{aligned} T_1 &= \frac{1}{\pi^2 \Delta} \left( \sqrt{\left(\Omega_{11}\Omega_{22} - \Omega_{12}^2\right)\Delta} - \sqrt{\Psi(x)\Psi(y)} \right); \\ T_2 &= \frac{1}{\pi^2 \sqrt{\Delta}} \left|\Omega_{12}\right| \arcsin\left( \frac{\left|\Omega_{12}\right|}{\sqrt{\Omega_{11}\Omega_{22}}} \right); \\ T_3 &= \frac{1}{\pi^2} \left( \frac{1}{\Delta} - \frac{1}{K_{n+1}(x,x) K_{n+1}(y,y)} \right) \sqrt{\Psi(x)\Psi(y)}. \end{aligned} \tag{4.4}$$

We estimate each $T$ term separately. It is the following lemma that contains the main idea, namely cancellation using Laplace's determinant formula:

**Lemma 4.1.** *There exist $n_0$ and $\Lambda_0 > 0$ such that for $n \geq n_0$ and all $x, y \in [a,b]$, with $|x-y| \geq \Lambda_0/n$,*

$$|T_1| \leq \frac{C}{\left(|x-y| + \frac{1}{n}\right)^2}. \tag{4.5}$$

**Proof.** Write

$$T_1 = \frac{\left(\Omega_{11}\Omega_{22} - \Omega_{12}^2\right)\Delta - \Psi(x)\Psi(y)}{\pi^2 \Delta \left[ \sqrt{\left(\Omega_{11}\Omega_{22} - \Omega_{12}^2\right)\Delta} + \sqrt{\Psi(x)\Psi(y)} \right]} = \frac{\text{Num}}{\text{Denom}}.$$

The numerator is (recall (3.6))

$$\begin{aligned}
\mathrm{Num} &= \left(\Omega_{11}\Omega_{22} - \Omega_{12}^2\right)\Delta - \Psi(x)\Psi(y) \\
&= \det(\Sigma) - \Psi(x)\Psi(y) \\
&= \det\begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) & K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(0,1)}(x,y) \\ K_{n+1}(x,y) & K_{n+1}(y,y) & K_{n+1}^{(0,1)}(y,x) & K_{n+1}^{(0,1)}(y,y) \\ K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(0,1)}(y,x) & K_{n+1}^{(1,1)}(x,x) & K_{n+1}^{(1,1)}(x,y) \\ K_{n+1}^{(0,1)}(x,y) & K_{n+1}^{(0,1)}(y,y) & K_{n+1}^{(1,1)}(x,y) & K_{n+1}^{(1,1)}(y,y) \end{bmatrix} \\
&\quad - \det\begin{bmatrix} K_{n+1}(x,x) & K_{n+1}^{(0,1)}(x,x) \\ K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(1,1)}(x,x) \end{bmatrix} \det\begin{bmatrix} K_{n+1}(y,y) & K_{n+1}^{(0,1)}(y,y) \\ K_{n+1}^{(0,1)}(y,y) & K_{n+1}^{(1,1)}(y,y) \end{bmatrix}.
\end{aligned}$$

Let $\Sigma$ be the $4 \times 4$ matrix above. Then we can write this as

$$\mathrm{Num} = \det[\Sigma] - \det\left[\Sigma\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}\right] \det\left[\Sigma\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}\right]$$

where $\Sigma\begin{pmatrix} r & s \\ j & k \end{pmatrix}$ denotes the matrix formed from $\Sigma$ by taking the elements that lie in rows $r, s$ and columns $j, k$. Now let us use Laplace's determinant expansion [24, p. 37]: we have chosen rows $1, 3$. Laplace's expansion gives

$$\det(\Sigma) = \sum_{1 \le j < k \le 4} (-1)^{1+3+j+k} \det\left[\Sigma\begin{pmatrix} 1 & 3 \\ j & k \end{pmatrix}\right] \det\left[\Sigma^c\begin{pmatrix} 1 & 3 \\ j & k \end{pmatrix}\right],$$

where $\Sigma^c$ is formed from the complimentary rows and columns. The choices for $(j,k)$ are $\{(1,2),(1,3),(1,4),(2,3)\,(2,4),(3,4)\}$, This gives $\det(\Sigma)$ as a sum of 6 terms, one of which is $\det\left[\Sigma\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}\right] \det\left[\Sigma\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}\right]$. So

$$\begin{aligned}
\mathrm{Num} = &- \det\left[\Sigma\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}\right] \det\left[\Sigma\begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix}\right] \\
&- \det\left[\Sigma\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}\right] \det\left[\Sigma\begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}\right] - \det\left[\Sigma\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}\right] \det\left[\Sigma\begin{pmatrix} 2 & 4 \\ 1 & 4 \end{pmatrix}\right] \\
&+ \det\left[\Sigma\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\right] \det\left[\Sigma\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}\right] - \det\left[\Sigma\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}\right] \det\left[\Sigma\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\right] \\
= &- \det\begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) \\ K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(0,1)}(y,x) \end{bmatrix} \det\begin{bmatrix} K_{n+1}^{(0,1)}(y,x) & K_{n+1}^{(0,1)}(y,y) \\ K_{n+1}^{(1,1)}(x,y) & K_{n+1}^{(1,1)}(y,y) \end{bmatrix} \\
&- \det\begin{bmatrix} K_{n+1}(x,x) & K_{n+1}^{(0,1)}(x,y) \\ K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(1,1)}(x,y) \end{bmatrix} \det\begin{bmatrix} K_{n+1}(y,y) & K_{n+1}^{(0,1)}(y,x) \\ K_{n+1}^{(0,1)}(y,y) & K_{n+1}^{(1,1)}(x,y) \end{bmatrix} \\
&- \det\begin{bmatrix} K_{n+1}(x,y) & K_{n+1}^{(0,1)}(x,x) \\ K_{n+1}^{(0,1)}(y,x) & K_{n+1}^{(1,1)}(x,x) \end{bmatrix} \det\begin{bmatrix} K_{n+1}(x,y) & K_{n+1}^{(0,1)}(y,y) \\ K_{n+1}^{(0,1)}(x,y) & K_{n+1}^{(1,1)}(y,y) \end{bmatrix} \\
&+ \det\begin{bmatrix} K_{n+1}(x,y) & K_{n+1}^{(0,1)}(x,y) \\ K_{n+1}^{(0,1)}(y,x) & K_{n+1}^{(1,1)}(x,y) \end{bmatrix} \det\begin{bmatrix} K_{n+1}(x,y) & K_{n+1}^{(0,1)}(y,x) \\ K_{n+1}^{(0,1)}(x,y) & K_{n+1}^{(1,1)}(x,y) \end{bmatrix}
\end{aligned}$$

$$- \det \begin{bmatrix} K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(0,1)}(x,y) \\ K_{n+1}^{(1,1)}(x,x) & K_{n+1}^{(1,1)}(x,y) \end{bmatrix} \det \begin{bmatrix} K_{n+1}(x,y) & K_{n+1}(y,y) \\ K_{n+1}^{(0,1)}(x,y) & K_{n+1}^{(0,1)}(y,y) \end{bmatrix}.$$

Using the estimate (3.7) and that $\left(|x-y| + \frac{1}{n}\right)^{-1} \le n$, we continue this as

$$\begin{aligned}
&= - \det \begin{bmatrix} O(n) & O\left(\frac{1}{|x-y|+\frac{1}{n}}\right) \\ O(n^2) & O\left(\frac{n}{|x-y|+\frac{1}{n}}\right) \end{bmatrix} \det \begin{bmatrix} O\left(\frac{n}{|x-y|+\frac{1}{n}}\right) & O(n^2) \\ O\left(\frac{n^2}{|x-y|+\frac{1}{n}}\right) & O(n^3) \end{bmatrix} \\
&\quad - \det \begin{bmatrix} O(n) & O\left(\frac{n}{|x-y|+\frac{1}{n}}\right) \\ O(n^2) & O\left(\frac{n^2}{|x-y|+\frac{1}{n}}\right) \end{bmatrix} \det \begin{bmatrix} O(n) & O\left(\frac{n}{|x-y|+\frac{1}{n}}\right) \\ O(n^2) & O\left(\frac{n^2}{|x-y|+\frac{1}{n}}\right) \end{bmatrix} \\
&\quad - \det \begin{bmatrix} O\left(\frac{1}{|x-y|+\frac{1}{n}}\right) & O(n^2) \\ O\left(\frac{n}{|x-y|+\frac{1}{n}}\right) & O(n^3) \end{bmatrix} \det \begin{bmatrix} O\left(\frac{1}{|x-y|+\frac{1}{n}}\right) & O(n^2) \\ O\left(\frac{n}{|x-y|+\frac{1}{n}}\right) & O(n^3) \end{bmatrix} \\
&\quad + \det \begin{bmatrix} O\left(\frac{1}{|x-y|+\frac{1}{n}}\right) & O\left(\frac{n}{|x-y|+\frac{1}{n}}\right) \\ O\left(\frac{n}{|x-y|+\frac{1}{n}}\right) & O\left(\frac{n^2}{|x-y|+\frac{1}{n}}\right) \end{bmatrix} \det \begin{bmatrix} O\left(\frac{1}{|x-y|+\frac{1}{n}}\right) & O\left(\frac{n}{|x-y|+\frac{1}{n}}\right) \\ O\left(\frac{n}{|x-y|+\frac{1}{n}}\right) & O\left(\frac{n^2}{|x-y|+\frac{1}{n}}\right) \end{bmatrix} \\
&\quad - \det \begin{bmatrix} O(n^2) & O\left(\frac{n}{|x-y|+\frac{1}{n}}\right) \\ O(n^3) & O\left(\frac{n^2}{|x-y|+\frac{1}{n}}\right) \end{bmatrix} \det \begin{bmatrix} O\left(\frac{1}{|x-y|+\frac{1}{n}}\right) & O(n) \\ O\left(\frac{n}{|x-y|+\frac{1}{n}}\right) & O(n^2) \end{bmatrix} \\
&= O\left(\frac{n^6}{\left(|x-y|+\frac{1}{n}\right)^2}\right).
\end{aligned}$$

Thus

$$\text{Num} = O\left(\frac{n^6}{\left(|x-y|+\frac{1}{n}\right)^2}\right). \tag{4.6}$$

Also

$$\begin{aligned}
\text{Denom} &= \pi^2 \Delta \left[\sqrt{\left(\Omega_{11}\Omega_{22} - \Omega_{12}^2\right)\Delta} + \sqrt{\Psi(x)\Psi(y)}\right] \\
&\ge \pi^2 \Delta \sqrt{\Psi(x)\Psi(y)}.
\end{aligned}$$

Here from (3.14) and (3.13), for $n$ large enough,

$$\Psi(x) = K_{n+1}^{(1,1)}(x,x) K_n(x,x) - K_n^{(0,1)}(x,x)^2 \ge Cn^4 - o\left(n^4\right) \ge Cn^4.$$

Also from (3.14) and (3.7),

$$\begin{aligned}
1 - \frac{\Delta}{K_n(x,x) K_n(y,y)} &= \frac{K_n^2(x,y)}{K_n(x,x) K_n(y,y)} \\
&\le \frac{C}{\left(|x-y|+\frac{1}{n}\right)^2 n^2} \\
&= \frac{C}{\left(n|x-y|+1\right)^2} \le \frac{1}{2},
\end{aligned}$$

if $|x-y| \geq \Lambda_0/n$ with $\Lambda_0$ large enough. Then

$$\Delta \geq \frac{1}{2} K_n(x,x) K_n(y,y) \geq Cn^2 \tag{4.7}$$

and

$$\text{Denom} \geq Cn^6. \tag{4.8}$$

Then combined with (4.6), this yields

$$|T_1| = \left|\frac{\text{Num}}{\text{Denom}}\right| \leq \frac{C}{\left(|x-y| + \frac{1}{n}\right)^2}. \quad \square$$

Next, let us deal with $T_2$:

**Lemma 4.2.** *There exist $n_0$ and $\Lambda_0$ such that for $n \geq n_0$ and all $x, y \in [a,b]$, with $|x-y| \geq \Lambda_0/n$,*

$$|T_2| \leq \frac{C}{\left(|x-y| + \frac{1}{n}\right)^2}. \tag{4.9}$$

**Proof.** Recall that

$$|T_2| = T_2 = \frac{1}{\pi^2 \sqrt{\Delta}} |\Omega_{12}| \arcsin\left(\frac{|\Omega_{12}|}{\sqrt{\Omega_{11}\Omega_{22}}}\right).$$

From $|\sin u| \geq \frac{2}{\pi}|u|$, $|u| \leq \frac{\pi}{2}$, we obtain for $|v| \leq 1$,

$$\frac{2}{\pi} |\arcsin v| \leq |v|$$

so

$$|T_2| \leq \frac{1}{2\pi \Delta^{3/2}} \frac{|\Omega_{12}\Delta|^2}{\sqrt{\Omega_{11}\Omega_{22}\Delta^2}}. \tag{4.10}$$

Here from Lemma 3.1(d) and Lemma 3.2,

$$\begin{aligned}
\Omega_{12}\Delta &= \det \begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) & K_{n+1}^{(0,1)}(x,x) \\ K_{n+1}(y,x) & K_{n+1}(y,y) & K_{n+1}^{(0,1)}(y,x) \\ K_{n+1}^{(1,0)}(y,x) & K_{n+1}^{(0,1)}(y,y) & K_{n+1}^{(1,1)}(y,x) \end{bmatrix} \\
&= \det \begin{bmatrix} O(n) & O\left(\frac{1}{|x-y|+\frac{1}{n}}\right) & O(n^2) \\ O\left(\frac{1}{|x-y|+\frac{1}{n}}\right) & O(n) & O\left(\frac{n}{|x-y|+\frac{1}{n}}\right) \\ O\left(\frac{n}{|x-y|+\frac{1}{n}}\right) & O(n^2) & O\left(\frac{n^2}{|x-y|+\frac{1}{n}}\right) \end{bmatrix}
\end{aligned}$$

We expand by the first row and continue this as

$$\Omega_{12}\Delta = O\left(\frac{n^4}{|x-y| + \frac{1}{n}}\right). \tag{4.11}$$

Next, we examine $\Omega_{11}$ and $\Omega_{22}$. From Lemma 3.1(b), followed by (3.7), (3.13),

$$\Omega_{11}\Delta = \det\begin{bmatrix} K_{n+1}(y,y) & K_{n+1}(y,x) & K_{n+1}^{(0,1)}(y,x) \\ K_{n+1}(x,y) & K_{n+1}(x,x) & K_{n+1}^{(0,1)}(x,x) \\ K_{n+1}^{(1,0)}(x,y) & K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(1,1)}(x,x) \end{bmatrix}$$
$$= \det\begin{bmatrix} K_{n+1}(y,y) & O\left(\frac{1}{|x-y|+\frac{1}{n}}\right) & O\left(\frac{n}{|x-y|+\frac{1}{n}}\right) \\ O\left(\frac{1}{|x-y|+\frac{1}{n}}\right) & K_{n+1}(x,x) & o\left(n^2\right) \\ O\left(\frac{n}{|x-y|+\frac{1}{n}}\right) & o\left(n^2\right) & K_{n+1}^{(1,1)}(x,x) \end{bmatrix}.$$

Expanding by the first row, and using $K_{n+1}^{(r,r)}(x,x) = O\left(n^{2r+1}\right)$, we see that

$$\Omega_{11}\Delta = K_{n+1}(y,y)\left\{K_{n+1}(x,x)\,K_{n+1}^{(1,1)}(x,x) - o\left(n^4\right)\right\}$$
$$-O\left(\frac{1}{|x-y|+\frac{1}{n}}\right)\left\{O\left(\frac{n^3}{|x-y|+\frac{1}{n}}\right) + o\left(\frac{n^3}{|x-y|+\frac{1}{n}}\right)\right\}$$
$$+O\left(\frac{n}{|x-y|+\frac{1}{n}}\right)\left\{O\left(\frac{n^2}{|x-y|+\frac{1}{n}}\right) + O\left(\frac{n^2}{|x-y|+\frac{1}{n}}\right)\right\},$$

so if $|x-y| \geq \Lambda_0/n$, and $\Lambda_0 \geq 1$,

$$\Omega_{11}\Delta = K_{n+1}(y,y)\,K_{n+1}(x,x)\,K_{n+1}^{(1,1)}(x,x) - o\left(n^5\right) + O\left(\frac{n^5}{\Lambda_0^2}\right)$$
$$\geq Cn^5 - o\left(n^5\right) + O\left(\frac{n^5}{\Lambda_0^2}\right) \geq C_1 n^5 \tag{4.12}$$

if $\Lambda_0$ and $n$ are large enough, say $n \geq n_0$, by (3.13) and (3.14). Of course the constant $C_1$ depends on the size of $C$, and the decay of the $o\left(n^5\right)$ term, as does $n_0$. In much the same way,

$$\Omega_{22}\Delta = \det\begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) & K_{n+1}^{(0,1)}(x,y) \\ K_{n+1}(y,x) & K_{n+1}(y,y) & K_{n+1}^{(0,1)}(y,y) \\ K_{n+1}^{(1,0)}(y,x) & K_{n+1}^{(1,0)}(y,y) & K_{n+1}^{(1,1)}(y,y) \end{bmatrix}$$
$$= K_{n+1}(x,x)\,K_{n+1}(y,y)\,K_{n+1}^{(1,1)}(y,y) - o\left(n^5\right) + O\left(\frac{n^5}{\Lambda_0^2}\right)$$
$$\geq C_1 n^5. \tag{4.13}$$

Again the threshholds $n_0$ and $\Lambda_0$ influence the choice of $C_1$. Then combining (4.10)–(4.13), followed by (4.7),

$$T_2 \leq C\left(\frac{n^4}{|x-y|+\frac{1}{n}}\right)^2 \frac{1}{\Delta^{3/2}}\frac{1}{n^5} \leq C\left(\frac{1}{|x-y|+\frac{1}{n}}\right)^2. \quad \square$$

Next, we handle $T_3$:

**Lemma 4.3.** *There exist $n_0$ and $\Lambda_0$ such that for $n \geq n_0$ and all $x, y \in [a,b]$, with $|x-y| \geq \Lambda_0/n$,*

$$|T_3| \leq \frac{C}{\left(|x-y|+\frac{1}{n}\right)^2}. \tag{4.14}$$

**Proof.** Note first from (4.2), (3.13), and (3.14),

$$\Psi(x) = O\left(n^4\right) - o\left(n^4\right) = O\left(n^4\right).$$

Next, recall from (4.4),

$$\begin{aligned} T_3 &= \frac{1}{\pi^2}\left(\frac{1}{\Delta} - \frac{1}{K_{n+1}(x,x)\,K_{n+1}(y,y)}\right)\sqrt{\Psi(x)\,\Psi(y)} \\ &= \frac{1}{\pi^2}\frac{K_{n+1}^2(x,y)}{\Delta K_{n+1}(x,x)\,K_{n+1}(y,y)}\sqrt{\Psi(x)\,\Psi(y)} \\ &\leq \frac{C}{\left(|x-y|+\frac{1}{n}\right)^2 \Delta n^2} n^4 \\ &\leq \frac{C}{\left(|x-y|+\frac{1}{n}\right)^2}, \end{aligned}$$

by (4.7). Note too that $T_3 \geq 0$. □

**Proof of Lemma 2.3(a).** Just combine the estimates for $T_1, T_2, T_3$ from Lemmas 4.1, 4.2, 4.3 and recall (4.3). □

**Proof of Lemma 2.3(b).** From Lemma 2.3(a), for $y \in [a,b]$,

$$\begin{aligned} &\int\limits_{\{x\in[a,b],|x-y|\geq\Lambda/n\}} |\rho_2(x,y) - \rho_1(x)\,\rho_1(y)|\,dx \\ &\leq \int\limits_{\{x\in[a,b],|x-y|\geq\Lambda/n\}} \frac{C}{|x-y|^2}dx \\ &\leq \int\limits_{\{x\in[a,b],|x-y|\geq\Lambda/n\}} \frac{2C}{|x-y|^2+\left(\frac{\Lambda}{n}\right)^2}dx \\ &\leq \int\limits_{-\infty}^{\infty} \frac{2C}{|x-y|^2+\left(\frac{\Lambda}{n}\right)^2}dx. \end{aligned}$$

We make the substitution $x - y = \frac{\Lambda}{n}t$ in the integral:

$$= \frac{n}{\Lambda}\int\limits_{-\infty}^{\infty}\frac{2C}{t^2+1}dt.$$

Then (2.12) follows. □

## 5. The central term - Lemma 2.4

Recall that $\Delta, \Omega_{11}, \Omega_{22}, \Omega_{12}$ were defined in (2.7)–(2.10), while $S, F, G, H$ were defined in (1.3)–(1.6):

**Lemma 5.1.** *Uniformly for $u$ in compact subsets of the plane, and uniformly for $x \in [a,b]$ and $y = x + \frac{u}{n\omega(x)}$,*

*(a)*

$$\frac{\left(\Omega_{11}\Omega_{22}-\Omega_{12}^{2}\right)\Delta}{K_{n+1}\left(x,x\right)^{4}}\left(\frac{1}{n\omega\left(x\right)}\right)^{4}=F\left(u\right)+o\left(1\right); \tag{5.1}$$

*(b)*

$$\frac{\Delta}{K_{n+1}\left(x,x\right)^{2}}=1-S\left(u\right)^{2}+o\left(1\right); \tag{5.2}$$

*(c)*

$$\frac{\Delta\Omega_{11}}{K_{n+1}\left(x,x\right)^{3}}\left(\frac{1}{n\omega\left(x\right)}\right)^{2}=G\left(u\right)+o\left(1\right); \tag{5.3}$$

*(d)*

$$\frac{\Delta\Omega_{22}}{K_{n+1}\left(x,x\right)^{3}}\left(\frac{1}{n\omega\left(x\right)}\right)^{2}=G\left(u\right)+o\left(1\right); \tag{5.4}$$

*(e)*

$$\frac{\Omega_{12}\Delta}{K_{n+1}\left(x,x\right)^{3}}\left(\frac{1}{n\omega\left(x\right)}\right)^{2}=H\left(u\right)+o\left(1\right). \tag{5.5}$$

**Proof.** We repeatedly use that $\frac{K_{n+1}(y,y)}{K_{n+1}(x,x)}=1+o\left(1\right)$, as follows from (3.11).
(a) Recall that $\Sigma$ was defined by (3.1). Then (3.6) gives

$$\begin{aligned}
&\frac{\left[\left(\Omega_{11}\Omega_{22}-\Omega_{12}^{2}\right)\Delta\right]}{K_{n+1}\left(x,x\right)^{4}}\left(\frac{1}{n\omega\left(x\right)}\right)^{4}\\
&=\frac{\det\Sigma}{K_{n+1}\left(x,x\right)^{4}}\left(\frac{1}{n\omega\left(x\right)}\right)^{4}\\
&=\det\begin{bmatrix}
1 & \frac{K_{n+1}(x,y)}{K_{n+1}(x,x)} & \frac{K_{n+1}^{(0,1)}(x,x)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} & \frac{K_{n+1}^{(0,1)}(x,y)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)}\\
\frac{K_{n+1}(x,y)}{K_{n+1}(x,x)} & \frac{K_{n+1}(y,y)}{K_{n+1}(x,x)} & \frac{K_{n+1}^{(0,1)}(y,x)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} & \frac{K_{n+1}^{(0,1)}(y,y)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)}\\
\frac{K_{n+1}^{(0,1)}(x,x)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} & \frac{K_{n+1}^{(0,1)}(y,x)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} & \frac{K_{n+1}^{(1,1)}(x,x)}{K_{n+1}(x,x)}\left(\frac{1}{n\omega(x)}\right)^{2} & \frac{K_{n+1}^{(1,1)}(x,y)}{K_{n+1}(x,x)}\left(\frac{1}{n\omega(x)}\right)^{2}\\
\frac{K_{n+1}^{(0,1)}(x,y)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} & \frac{K_{n+1}^{(0,1)}(y,y)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} & \frac{K_{n+1}^{(1,1)}(x,y)}{K_{n+1}(x,x)}\left(\frac{1}{n\omega(x)}\right)^{2} & \frac{K_{n+1}^{(1,1)}(y,y)}{K_{n+1}(x,x)}\left(\frac{1}{n\omega(x)}\right)^{2}
\end{bmatrix}.
\end{aligned}$$

Here we have factored in $\frac{1}{n\omega(x)}$ into the 3rd and 4th rows and columns. Using (3.9) and recalling that $y=x+\frac{u}{n\omega(x)}$, we continue this as

$$\begin{aligned}
&=\det\begin{bmatrix}
1 & S\left(-u\right) & -S'\left(0\right) & -S'\left(-u\right)\\
S\left(-u\right) & 1 & -S'\left(u\right) & -S'\left(0\right)\\
-S'\left(0\right) & -S'\left(u\right) & -S''\left(0\right) & -S''\left(-u\right)\\
-S'\left(-u\right) & -S'\left(0\right) & -S''\left(-u\right) & -S''\left(0\right)
\end{bmatrix}+o\left(1\right)\\
&=\det\begin{bmatrix}
1 & S\left(u\right) & 0 & S'\left(u\right)\\
S\left(u\right) & 1 & -S'\left(u\right) & 0\\
0 & -S'\left(u\right) & -S''\left(0\right) & -S''\left(u\right)\\
S'\left(u\right) & 0 & -S''\left(u\right) & -S''\left(0\right)
\end{bmatrix}+o\left(1\right)=F\left(u\right)+o\left(1\right)
\end{aligned}$$

as $S$ is even, so $S'$ is odd and $S''$ is even.

(b) From (3.9),

$$\frac{\Delta}{K_{n+1}(x,x)^2} = \det\begin{bmatrix} 1 & \frac{K_{n+1}(x,y)}{K_{n+1}(x,x)} \\ \frac{K_{n+1}(x,y)}{K_{n+1}(x,x)} & \frac{K_{n+1}(y,y)}{K_{n+1}(x,x)} \end{bmatrix} = \det\begin{bmatrix} 1 & S(-u) \\ S(-u) & 1 \end{bmatrix} + o(1).$$

(c) From (3.3) and then (3.9),

$$\begin{aligned} &\frac{\Delta\Omega_{11}}{K_{n+1}(x,x)^3}\left(\frac{1}{n\omega(x)}\right)^2 \\ &= \det\begin{bmatrix} \frac{K_{n+1}(y,y)}{K_{n+1}(x,x)} & \frac{K_{n+1}(y,x)}{K_{n+1}(x,x)} & \frac{K_{n+1}^{(0,1)}(y,x)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} \\ \frac{K_{n+1}(x,y)}{K_{n+1}(x,x)} & 1 & \frac{K_{n+1}^{(0,1)}(x,x)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} \\ \frac{K_{n+1}^{(1,0)}(x,y)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} & \frac{K_{n+1}^{(0,1)}(x,x)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} & \frac{K_{n+1}^{(1,1)}(x,x)}{K_{n+1}(x,x)}\left(\frac{1}{n\omega(x)}\right)^2 \end{bmatrix} \\ &= \det\begin{bmatrix} 1 & S(u) & -S'(u) \\ S(-u) & 1 & -S'(0) \\ S'(-u) & -S'(0) & -S''(0) \end{bmatrix} + o(1) \\ &= \det\begin{bmatrix} 1 & S(u) & -S'(u) \\ S(u) & 1 & 0 \\ -S'(u) & 0 & -S''(0) \end{bmatrix} + o(1) = G(u) + o(1), \end{aligned}$$

recall (1.5).

(d) From (3.4) and then (3.9),

$$\begin{aligned} &\frac{\Delta\Omega_{22}}{K_{n+1}(x,x)^3}\left(\frac{1}{n\omega(x)}\right)^2 \\ &= \det\begin{bmatrix} 1 & \frac{K_{n+1}(x,y)}{K_{n+1}(x,x)} & \frac{K_{n+1}^{(0,1)}(x,y)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} \\ \frac{K_{n+1}(y,x)}{K_{n+1}(x,x)} & \frac{K_{n+1}(y,y)}{K_{n+1}(x,x)} & \frac{K_{n+1}^{(0,1)}(y,y)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} \\ \frac{K_{n+1}^{(1,0)}(y,x)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} & \frac{K_{n+1}^{(1,0)}(y,y)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} & \frac{K_{n+1}^{(1,1)}(y,y)}{K_{n+1}(x,x)}\left(\frac{1}{n\omega(x)}\right)^2 \end{bmatrix} \\ &= \det\begin{bmatrix} 1 & S(-u) & -S'(-u) \\ S(u) & 1 & -S'(0) \\ S'(u) & S'(0) & -S''(0) \end{bmatrix} + o(1) = G(u) + o(1) \end{aligned}$$

as $S'$ is odd, and we can multiply both the 3rd row and 3rd column by $-1$.

(e) From (3.5) and then (3.9),

$$\begin{aligned} &\frac{\Omega_{12}\Delta}{K_{n+1}(x,x)^3}\left(\frac{1}{n\omega(x)}\right)^2 \\ &= \det\begin{bmatrix} 1 & \frac{K_{n+1}(x,y)}{K_{n+1}(x,x)} & \frac{K_{n+1}^{(0,1)}(x,x)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} \\ \frac{K_{n+1}(y,x)}{K_{n+1}(x,x)} & \frac{K_{n+1}(y,y)}{K_{n+1}(x,x)} & \frac{K_{n+1}^{(0,1)}(y,x)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} \\ \frac{K_{n+1}^{(1,0)}(y,x)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} & \frac{K_{n+1}^{(0,1)}(y,y)}{K_{n+1}(x,x)}\frac{1}{n\omega(x)} & \frac{K_{n+1}^{(1,1)}(y,x)}{K_{n+1}(x,x)}\left(\frac{1}{n\omega(x)}\right)^2 \end{bmatrix} \\ &= \det\begin{bmatrix} 1 & S(-u) & 0 \\ S(u) & 1 & -S'(u) \\ S'(u) & 0 & -S''(u) \end{bmatrix} + o(1) = H(u) + o(1), \end{aligned}$$

recall (1.6). $\square$

Now we can obtain the asymptotics for $\rho_2(x,y) - \rho_1(x)\rho_1(y)$ stated in (2.13):

**Proof of Lemma 2.4(a).** Recall as in (4.3)–(4.4), that

$$\begin{aligned}
&\left(\frac{1}{n\omega(x)}\right)^2 \{\rho_2(x,y) - \rho_1(x)\rho_1(y)\} \\
&= \left(\frac{1}{n\omega(x)}\right)^2 \{T_1 + T_2 + T_3\}. \qquad (5.6)
\end{aligned}$$

We handle the terms $T_j, j = 1,2,3$ one by one:

**Step 1:** $T_1$

Firstly from (3.9), (3.10), (3.15), and (4.2),

$$\begin{aligned}
&\frac{\Psi(x)}{K_{n+1}(x,x)^2}\left(\frac{1}{n\omega(x)}\right)^2 \\
&= \left(\frac{K_{n+1}^{(1,1)}(x,x)}{K_{n+1}(x,x)} - \left(\frac{K_{n+1}^{(0,1)}(x,x)}{K_{n+1}(x,x)}\right)^2\right)\left(\frac{1}{n\omega(x)}\right)^2 \\
&= -S''(0) + o(1) = \frac{\pi^2}{3} + o(1).
\end{aligned}$$

Also, from (3.9), uniformly for $u$ in compact subsets of $\mathbb{C}$,

$$\begin{aligned}
&\frac{\Psi(y)}{K_{n+1}(x,x)^2}\left(\frac{1}{n\omega(x)}\right)^2 \\
&= \left(\frac{K_{n+1}^{(1,1)}(y,y)}{K_{n+1}(x,x)}\frac{K_{n+1}(y,y)}{K_{n+1}(x,x)} - \left(\frac{K_{n+1}^{(0,1)}(y,y)}{K_{n+1}(x,x)}\right)^2\right)\left(\frac{1}{n\omega(x)}\right)^2 \\
&= -S''(0) + o(1) = \frac{\pi^2}{3} + o(1). \qquad (5.7)
\end{aligned}$$

Then

$$\begin{aligned}
&\left(\frac{1}{n\omega(x)}\right)^4 \frac{\Psi(x)\Psi(y)}{\Delta^2} \\
&= \frac{K_{n+1}(x,x)^4}{\Delta^2}\left[\frac{\Psi(x)}{K_{n+1}(x,x)^2}\left(\frac{1}{n\omega(x)}\right)^2\right]\left[\frac{\Psi(y)}{K_{n+1}(x,x)^2}\left(\frac{1}{n\omega(x)}\right)^2\right] \\
&= \frac{1}{\left(1 - S(u)^2\right)^2}\left(\frac{\pi^2}{3}\right)^2 + o(1),
\end{aligned}$$

by the above and Lemma 5.1(b). Hence also with an obvious choice of branches, uniformly for $u$ in compact subsets of $\mathbb{C}\setminus\{0\}$,

$$\left(\frac{1}{n\omega(x)}\right)^2 \frac{1}{\Delta}\sqrt{\Psi(x)\Psi(y)} = \frac{1}{1 - S(u)^2}\left(\frac{\pi^2}{3}\right) + o(1). \qquad (5.8)$$

(Note that $\Delta$ occurs outside the square root, and only it leads to the pole at 0). Then from (5.1) and (5.8), and recalling the definition of $T_1$ at (4.4),

$$\begin{aligned}
&\left(\frac{1}{n\omega(x)}\right)^2 T_1 \\
&= \frac{K_{n+1}(x,x)^2}{\pi^2\Delta}\sqrt{\frac{\left(\Omega_{11}\Omega_{22}-\Omega_{12}^2\right)\Delta}{K_{n+1}(x,x)^4\,(n\omega(x))^4}} - \left(\frac{1}{n\omega(x)}\right)^2\frac{1}{\pi^2\Delta}\sqrt{\Psi(x)\,\Psi(y)} \\
&= \frac{1}{\pi^2\left(1-S(u)^2\right)}\left(\sqrt{F(u)}-\frac{\pi^2}{3}\right)+o(1).
\end{aligned}$$

**Step 2:** $T_2$
From (4.4),

$$\begin{aligned}
&\left(\frac{1}{n\omega(x)}\right)^2 T_2 \\
&= \frac{K_{n+1}(x,x)^3}{\pi^2\Delta^{3/2}}\left[\frac{\Omega_{12}\Delta}{K_{n+1}(x,x)^3}\left(\frac{1}{n\omega(x)}\right)^2\right]\arcsin\left(\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}\right) \\
&= \frac{1}{\pi^2\left(1-S(u)^2\right)^{3/2}}H(u)\arcsin\left(\frac{H(u)}{G(u)}\right)+o(1),
\end{aligned}$$

by (5.2)–(5.5).
**Step 3:** $T_3$
From (4.4) and (5.5),

$$\begin{aligned}
&\left(\frac{1}{n\omega(x)}\right)^2 T_3 \\
&= \left(\frac{1}{n\omega(x)}\right)^2\frac{1}{\pi^2}\left(\frac{K_{n+1}(x,y)^2}{\Delta K_{n+1}(x,x)\,K_{n+1}(y,y)}\right)\sqrt{\Psi(x)\,\Psi(y)} \\
&= \frac{1}{\pi^2}\left(\frac{K_{n+1}(x,y)}{K_{n+1}(x,x)}\right)^2\frac{K_{n+1}(x,x)}{K_{n+1}(y,y)}\left[\frac{1}{(n\omega(x))^2\,\Delta}\sqrt{\Psi(x)\,\Psi(y)}\right] \\
&= \frac{1}{\pi^2}\left(\frac{S(u)^2}{1-S(u)^2}\right)\frac{\pi^2}{3}+o(1),
\end{aligned}$$

by (5.8) and (3.9). Substituting the asymptotics for $T_j, j=1,2,3$ into (5.6) gives

$$\begin{aligned}
&\left(\frac{1}{n\omega(x)}\right)^2\left\{\rho_2(x,y)-\rho_1(x)\,\rho_1(y)\right\} \\
&= \frac{1}{\pi^2\left(1-S(u)^2\right)}\left\{\sqrt{F(u)}-\frac{\pi^2}{3}\left(1-S(u)^2\right)+\frac{H(u)}{\sqrt{1-S(u)^2}}\arcsin\left(\frac{H(u)}{G(u)}\right)\right\}+o(1) \\
&= \Xi(u)+o(1),
\end{aligned}$$

recall (1.7). $\square$

We next deal with $u$ near 0, which turns out to be challenging. First, we prove

**Lemma 5.2.**

*(a)* $\Delta\left(x, x+\frac{u}{n\omega(x)}\right)$ *has a double zero at* $u=0$*, and there is* $\rho>0$ *such that for all* $x\in[a,b]$ *and* $n$ *large enough,* $\Delta\left(x, x+\frac{u}{n\omega(x)}\right)$ *has no other zeros in* $|u|\leq\rho$*. Moreover, uniformly for* $u$ *in compact subsets of* $\mathbb{C}$*, and* $x\in[a,b]$*,*

$$\lim_{n\to\infty}\frac{\Delta\left(x, x+\frac{u}{n\omega(x)}\right)}{K_{n+1}(x,x)^2u^2}=\frac{1-S(u)^2}{u^2}. \tag{5.9}$$

*The right-hand side is interpreted as its limiting value at* $u=0$*.*

*(b)* $\left[\left(\Omega_{11}\Omega_{22}-\Omega_{12}^2\right)\Delta\right]\left(x, x+\frac{u}{n\omega(x)}\right)$ *has a zero of even order at least 4 at* $u=0$*. Moreover, uniformly for* $u$ *in compact subsets of* $\mathbb{C}$*, and* $x\in[a,b]$*,*

$$\lim_{n\to\infty}\frac{\left(\Omega_{11}\Omega_{22}-\Omega_{12}^2\right)}{\Delta}\left(\frac{1}{n\omega(x)}\right)^4=\frac{F(u)}{\left(1-S(u)^2\right)^2}.$$

*The right-hand side is interpreted as its limiting value at* $u=0$*.*

**Proof.** (a) First,

$$\Delta\left(x, x+\frac{u}{n\omega(x)}\right)$$
$$=K_{n+1}(x,x)\,K_{n+1}\left(x+\frac{u}{n\omega(x)}, x+\frac{u}{n\omega(x)}\right)-K_{n+1}\left(x, x+\frac{u}{n\omega(x)}\right)^2$$

is a polynomial in $u$, and by Cauchy-Schwarz is non-negative for real $u$, with a zero at $u=0$. This then must be a zero of even multiplicity. But since

$$\lim_{n\to\infty}\frac{\Delta\left(x, x+\frac{u}{n\omega(x)}\right)}{K_{n+1}(x,x)^2}=1-S(u)^2,$$

uniformly in compact sets by Lemma 5.1(b) and (3.9), and the right-hand side has an isolated double zero at 0, it follows from Hurwitz' Theorem and the considerations above, that necessarily for large enough $n$, $\Delta\left(x, x+\frac{u}{n\omega(x)}\right)$ has a double zero at 0, and no other zeros in some neighborhood of 0 that is independent of $n$. Since the convergence is uniform in $x$, the neighborhood may also be taken independent of $x$. But then $\left\{\frac{\Delta\left(x,x+\frac{u}{n\omega(x)}\right)}{K_{n+1}(x,x)^2u^2}\right\}_{n\geq1}$ is a sequence of polynomials in $u$ that converges uniformly in compact subsets of $\mathbb{C}\backslash\{0\}$ and hence also in compact subsets of $\mathbb{C}$.

(b) Recall (3.6):

$$\left(\Omega_{11}\Omega_{22}-\Omega_{12}^2\right)\Delta=\det(\Sigma).$$

Here $\det(\Sigma)$ is also a polynomial in $u$ when $y=x+\frac{u}{n\omega(x)}$. As in the proof of Lemma 2.2 in the Appendix, $\Sigma$ is a positive definite matrix when $x\neq y$, so is nonegative definite for all $x,y$. Then $\det(\Sigma)\geq0$ for real $x,y$ while $\det(\Sigma)=0$ when $u=0$. Thus as a polynomial in $u$, $\det(\Sigma)$ can only have an even multiplicity zero at

$u = 0$. We need to show that it has a zero of multiplicity at least 4 when $u = 0$. By a classical inequality for determinants of positive definite matrices and their leading submatrices [6, p. 63, Thm. 7], when $y$ is real,

$$0 \leq \det(\Sigma) \leq \Delta(x, y) \det \begin{bmatrix} K_{n+1}^{(1,1)}(x,x) & K_{n+1}^{(1,1)}(x,y) \\ K_{n+1}^{(1,1)}(x,y) & K_{n+1}^{(1,1)}(y,y) \end{bmatrix}.$$

We already know that $\Delta$ has a double zero at $u = 0$ for $y = x + \frac{u}{n\omega(x)}$. But the second determinant also vanishes when $y = x$, that is $u = 0$. It follows that necessarily as a polynomial in $u$, $\det(\Sigma)$ has a zero of multiplicity at least 4 at $u = 0$. Then

$$\frac{\Omega_{11}\Omega_{22} - \Omega_{12}^2}{\Delta} = \frac{\det(\Sigma)}{\Delta^2}$$

has a removable singularity at 0, since the zero of multiplicity 4 in the denominator is cancelled by the zero of multiplicity $\geq 4$ in the numerator. Then from (5.1), (5.2), uniformly for $x \in [a, b]$ and $u$ in some neighborhood of 0,

$$\begin{aligned} &\frac{\Omega_{11}\Omega_{22} - \Omega_{12}^2}{\Delta} \left(\frac{1}{n\omega(x)}\right)^4 \\ &= \frac{\left(\Omega_{11}\Omega_{22} - \Omega_{12}^2\right)\Delta}{K_{n+1}(x,x)^4} \left(\frac{1}{n\omega(x)}\right)^4 \left[\frac{K_{n+1}(x,x)^2}{\Delta}\right]^2 \\ &= \frac{F(u)}{\left(1 - S(u)^2\right)^2} + o(1). \end{aligned}$$

Moreover, since $S(u) = 1$ only at $u = 0$, this limit actually holds uniformly for $u$ in compact subsets of $\mathbb{C}$. $\square$

Next, we deal with the most difficult term $\Omega_{12}$:

**Lemma 5.3.** $(\Omega_{12}\Delta)\left(x, x + \frac{u}{n\omega(x)}\right)$ *has a zero of multiplicity at least* 3 *at* $u = 0$. *Moreover, uniformly for* $u$ *in compact subsets of* $\mathbb{R}$, *and* $x \in [a, b]$,

$$\lim_{n\to\infty} \frac{\Omega_{12}}{\sqrt{\Delta}} \left(\frac{1}{n\omega(x)}\right)^2 = \frac{H(u)}{\left(1 - S^2(u)\right)^{3/2}}.$$

*The right-hand side is interpreted as its limiting value at* $u = 0$. *In addition, uniformly for* $u$ *in compact subsets of* $\mathbb{R}$, *and* $x \in [a, b]$,

$$\frac{|\Omega_{12}|}{\sqrt{\Delta}} \arcsin\left(\frac{|\Omega_{12}|}{\sqrt{\Omega_{11}\Omega_{22}}}\right) \left(\frac{1}{n\omega(x)}\right)^2 \leq C.$$

**Proof.** We first perform row and column operations in the determinant defining $\Delta_{12}$ and then expand using Taylor series. More precisely, we subtract the first row from the second; then the first column from the second; and then we subtract $\frac{1}{y-x}$ $\times$ the second row from the third:

$$\begin{aligned} &\Omega_{12}\Delta \\ &= \det \begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) & K_{n+1}^{(0,1)}(x,x) \\ K_{n+1}(y,x) & K_{n+1}(y,y) & K_{n+1}^{(0,1)}(y,x) \\ K_{n+1}^{(1,0)}(y,x) & K_{n+1}^{(0,1)}(y,y) & K_{n+1}^{(1,1)}(y,x) \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \det \begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) & K_{n+1}^{(0,1)}(x,x) \\ K_{n+1}(y,x) - K_{n+1}(x,x) & K_{n+1}(y,y) - K_{n+1}(x,y) & K_{n+1}^{(0,1)}(y,x) - K_{n+1}^{(0,1)}(x,x) \\ K_{n+1}^{(1,0)}(y,x) & K_{n+1}^{(0,1)}(y,y) & K_{n+1}^{(1,1)}(y,x) \end{bmatrix} \\
&= \det \begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) - K_{n+1}(x,x) & K_{n+1}^{(0,1)}(x,x) \\ K_{n+1}(y,x) - K_{n+1}(x,x) & \begin{array}{c} K_{n+1}(y,y) + K_{n+1}(x,x) \\ -2K_{n+1}(x,y) \end{array} & K_{n+1}^{(0,1)}(y,x) - K_{n+1}^{(0,1)}(x,x) \\ K_{n+1}^{(1,0)}(y,x) & K_{n+1}^{(1,0)}(y,y) - K_{n+1}^{(1,0)}(y,x) & K_{n+1}^{(1,1)}(y,x) \end{bmatrix} \\
&= \det \begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) - K_{n+1}(x,x) & K_{n+1}^{(0,1)}(x,x) \\ K_{n+1}(y,x) - K_{n+1}(x,x) & \begin{array}{c} K_{n+1}(y,y) + K_{n+1}(x,x) \\ -2K_{n+1}(x,y) \end{array} & K_{n+1}^{(0,1)}(y,x) - K_{n+1}^{(0,1)}(x,x) \\ \begin{array}{c} K_{n+1}^{(1,0)}(y,x) \\ -\frac{K_{n+1}(y,x) - K_{n+1}(x,x)}{y-x} \end{array} & \begin{array}{c} K_{n+1}^{(1,0)}(y,y) - K_{n+1}^{(1,0)}(y,x) \\ -\frac{K_{n+1}(y,y) + K_{n+1}(x,x) - 2K_{n+1}(x,y)}{y-x} \end{array} & \begin{array}{c} K_{n+1}^{(1,1)}(y,x) \\ -\frac{K_{n+1}^{(0,1)}(y,x) - K_{n+1}^{(0,1)}(x,x)}{y-x} \end{array} \end{bmatrix}.
\end{aligned}
\tag{5.10}
$$

Let us examine the entries in the second and third rows. First, for some $t$ between $x, y$,

$$K_{n+1}(y,x) - K_{n+1}(x,x) = K_{n+1}^{(1,0)}(t,x)(y-x) = O\left(n^2(y-x)\right)$$

by Lemma 3.2. Second, using the estimates from that lemma, for some $r, s, v$ between $x, y$,

$$
\begin{aligned}
& K_{n+1}(y,y) + K_{n+1}(x,x) - 2K_{n+1}(x,y) \\
&= K_{n+1}(x,x) + (y-x)\, 2K_{n+1}^{(1,0)}(x,x) + \frac{1}{2}(y-x)^2\, 2\left\{K_{n+1}^{(1,1)}(r,r) + K_{n+1}^{(2,0)}(r,r)\right\} \\
&\quad + K_{n+1}(x,x) - 2\left\{K_{n+1}(x,x) + (y-x) K_{n+1}^{(0,1)}(x,x) + \frac{1}{2}(y-x)^2 K_{n+1}^{(0,2)}(x,s)\right\} \\
&= (y-x)^2 \left\{K_{n+1}^{(1,1)}(r,r) + K_{n+1}^{(2,0)}(r,r) - K_{n+1}^{(0,2)}(x,s)\right\} \\
&= O\left(n^3 (y-x)^2\right).
\end{aligned}
$$

Third,

$$K_{n+1}^{(0,1)}(y,x) - K_{n+1}^{(0,1)}(x,x) = O\left(n^3(y-x)\right).$$

Fourth, for some $t$ between $y, x$,

$$
\begin{aligned}
& K_{n+1}^{(1,0)}(y,x) - \frac{K_{n+1}(y,x) - K_{n+1}(x,x)}{y-x} \\
&= K_{n+1}^{(1,0)}(y,x) - K_{n+1}^{(1,0)}(t,x) = O\left(n^3(y-x)\right).
\end{aligned}
$$

Fifth, for some $r, \zeta, s$ between $y, x$, with $r, s$ as above,

$$
\begin{aligned}
& K_{n+1}^{(1,0)}(y,y) - K_{n+1}^{(1,0)}(y,x) - \frac{K_{n+1}(y,y) + K_{n+1}(x,x) - 2K_{n+1}(x,y)}{y-x} \\
&= (y-x) K_{n+1}^{(1,1)}(y,\zeta) - (y-x)\left\{K_{n+1}^{(1,1)}(r,r) + K_{n+1}^{(2,0)}(r,r) - K_{n+1}^{(0,2)}(x,s)\right\} \\
&= (y-x)\left\{K_{n+1}^{(1,1)}(y,\zeta) - K_{n+1}^{(1,1)}(r,r) - K_{n+1}^{(2,0)}(r,r) + K_{n+1}^{(0,2)}(x,s)\right\} \\
&= O\left(n^4 (y-x)^2\right),
\end{aligned}
$$

by the estimates in Lemma 3.2. Sixth, for some $\xi$ between $x, y$,

$$\begin{aligned} & K_{n+1}^{(1,1)}(y,x) - \frac{K_{n+1}^{(0,1)}(y,x) - K_{n+1}^{(0,1)}(x,x)}{y-x} \\ & = K_{n+1}^{(1,1)}(y,x) - K_{n+1}^{(1,1)}(\xi,x) = O\left(n^4 (y-x)\right). \end{aligned}$$

Then substituting all these into (5.10),

$$\begin{aligned} & \Omega_{12}\Delta \\ & = \det \begin{bmatrix} O(n) & O\left(n^2 (y-x)\right) & O\left(n^2\right) \\ O\left(n^2 (y-x)\right) & O\left(n^3 (y-x)^2\right) & O\left(n^3 (y-x)\right) \\ O\left(n^3 (y-x)\right) & O\left(n^4 (y-x)^2\right) & O\left(n^4 (y-x)\right) \end{bmatrix} \\ & = (y-x)^3 \det \begin{bmatrix} O(n) & O\left(n^2\right) & O\left(n^2\right) \\ O\left(n^2\right) & O\left(n^3\right) & O\left(n^3\right) \\ O\left(n^3\right) & O\left(n^4\right) & O\left(n^4\right) \end{bmatrix} = O\left(n^8 (y-x)^3\right). \end{aligned}$$

Here we extracted factors of $y-x$ from the second and third rows, and then the second column. It follows that as a polynomial in $u$, $(\Omega_{12}\Delta)\left(x, x + \frac{u}{n\omega(x)}\right)$ has a zero of multiplicity at least 3 at 0. Then $\frac{\Omega_{12}\Delta}{u^3}$ is a polynomial in $u$, and $\frac{\Omega_{12}}{\sqrt{\Delta}} = \frac{\Omega_{12}\Delta}{u^3}\left(\frac{u^2}{\Delta}\right)^{3/2}$, which is analytic in some neighborhood of 0 that is independent of $n, x, u$. The uniform convergence in (5.5) gives uniformly for $u$ in compact subsets of $\mathbb{R}$,

$$\begin{aligned} & \frac{\Omega_{12}}{\sqrt{\Delta}}\left(\frac{1}{n\omega(x)}\right)^2 \\ & = \left[\frac{\Omega_{12}\Delta}{K_{n+1}(x,x)^3}\left(\frac{1}{n\omega(x)}\right)^2\right]\frac{K_{n+1}(x,x)^3}{\Delta^{3/2}} \\ & = \frac{H(u)}{\left(1 - S^2(u)\right)^{3/2}} + o(1). \end{aligned}$$

Also then, $H(u)$ necessarily has a zero of multiplicity $\geq 3$ at 0. Finally, uniformly for $u$ in compact subsets of $\mathbb{R}$,

$$\begin{aligned} & \frac{|\Omega_{12}|}{\sqrt{\Delta}} \arcsin\left(\frac{|\Omega_{12}|}{\sqrt{\Omega_{11}\Omega_{22}}}\right)\left(\frac{1}{n\omega(x)}\right)^2 \\ & \leq \frac{|\Omega_{12}|}{\sqrt{\Delta}} \frac{\pi}{2}\left(\frac{1}{n\omega(x)}\right)^2 \leq C. \quad \square \end{aligned}$$

Now we can deduce the desired bound near the diagonal:

**Proof of Lemma 2.4(b).** Recall from (2.6) that

$$\begin{aligned} & |\rho_2(x,y)|\left(\frac{1}{n\omega(x)}\right)^2 \\ & \leq \frac{1}{\pi^2}\left(\sqrt{\frac{\Omega_{11}\Omega_{22} - \Omega_{12}^2}{\Delta}} + \frac{|\Omega_{12}|}{\sqrt{\Delta}} \arcsin\left(\frac{|\Omega_{12}|}{\sqrt{\Omega_{11}\Omega_{22}}}\right)\right)\left(\frac{1}{n\omega(x)}\right)^2 \leq C, \end{aligned}$$

by Lemma 5.1(a), (b) and Lemmas 5.2–5.3. Next, from (4.1), followed by (5.7), (with $u = 0$ there)

$$\frac{\rho_1(x)}{n\omega(x)} = \frac{1}{\pi}\sqrt{\frac{\Psi(x)}{K_{n+1}(x,x)^2\,(n\omega(x))^2}} = \frac{1}{\sqrt{3}} + o(1)\,, \tag{5.11}$$

and a similar asymptotic holds for $\rho_1(y)$. It follows that

$$|\rho_2(x,y) - \rho_1(x)\,\rho_1(y)| \left(\frac{1}{n\omega(x)}\right)^2 \le C,$$

which gives the result, since $\omega$ is positive and continuous in $[a,b]$. $\square$

**Proof of Lemma 2.5.** This follows directly from (5.11). $\square$

## 6. Appendix - proof of Lemma 2.2

In this section, we prove Lemma 2.2. The functions $\rho_2(x,y)$ and $\rho_1(x)$ arising in (2.5), are called the second and the first intensities, or the two-point and one-point correlation functions of zeros, see, e.g., [20, pp. 7–8]. By their defining properties, we have

$$\mathbb{E}\left[N_n([a,b])\right] = \int_a^b \rho_1(x)\,dx$$

and

$$\mathbb{E}\left[N_n([a,b])\,(N_n([a,b]) - 1)\right] = \int_a^b\int_a^b \rho_2(x,y)\;dx\;dy.$$

Thus the variance of real zeros of random orthogonal polynomials in an interval $[a,b] \subset \mathbb{R}$ can be written as in (2.5) by completing the following steps:

$$\begin{aligned}
&\mathrm{Var}\left[N_n([a,b])\right] \\
&= \mathbb{E}\left[N_n([a,b])^2 - \mathbb{E}\left[N_n([a,b])\right]^2\right] \\
&= \mathbb{E}\left[N_n([a,b])\,(N_n([a,b]) - 1)\right] - \mathbb{E}\left[N_n([a,b])\right]^2 + \mathbb{E}\left[N_n([a,b])\right] \\
&= \int_a^b\int_a^b \rho_2(x,y)\;dx\;dy - \int_a^b\int_a^b \rho_1(x)\,\rho_1(y)\;dx\;dy + \int_a^b \rho_1(x)\,dx \\
&= \int_a^b\int_a^b \{\rho_2(x,y) - \rho_1(x)\,\rho_1(y)\}\;dx\;dy + \int_a^b \rho_1(x)\,dx.
\end{aligned}$$

We follow the argument of [19] in several parts of this proof. For $x, y \in \mathbb{R}$, define the random vector

$$V = V(x,y) := \left(G_n(x), G_n(y), G_n'(x), G_n'(y)\right)^T,$$

and observe that the components of this vector are Gaussian random variables satisfying

$$\mathbb{E}[G_n(x)] = \mathbb{E}[G_n'(x)] = 0,\ \operatorname{Var}[G_n(x)] = K_{n+1}(x,x) \text{ and } \operatorname{Var}[G_n'(x)] = K_{n+1}^{(1,1)}(x,x).$$

The covariance matrix $\Sigma$ of $V$ is defined by

$$\begin{aligned}
\Sigma &= \Sigma(x,y) \\
&:= \begin{bmatrix} \operatorname{Var}[G_n(x)] & \operatorname{Cov}[G_n(x),G_n(y)] & \operatorname{Cov}[G_n(x),G_n'(x)] & \operatorname{Cov}[G_n(x),G_n'(y)] \\ \operatorname{Cov}[G_n(y),G_n(x)] & \operatorname{Var}[G_n(y)] & \operatorname{Cov}[G_n(y),G_n'(x)] & \operatorname{Cov}[G_n(y),G_n'(y)] \\ \operatorname{Cov}[G_n'(x),G_n(x)] & \operatorname{Cov}[G_n'(x),G_n(y)] & \operatorname{Var}[G_n'(x)] & \operatorname{Cov}[G_n'(x),G_n'(y)] \\ \operatorname{Cov}[G_n'(y),G_n(x)] & \operatorname{Cov}[G_n'(y),G_n(y)] & \operatorname{Cov}[G_n'(y),G_n'(x)] & \operatorname{Var}[G_n'(y)] \end{bmatrix} \\
&= \begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) & K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(0,1)}(x,y) \\ K_{n+1}(x,y) & K_{n+1}(y,y) & K_{n+1}^{(0,1)}(y,x) & K_{n+1}^{(0,1)}(y,y) \\ K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(0,1)}(y,x) & K_{n+1}^{(1,1)}(x,x) & K_{n+1}^{(1,1)}(x,y) \\ K_{n+1}^{(0,1)}(x,y) & K_{n+1}^{(0,1)}(y,y) & K_{n+1}^{(1,1)}(x,y) & K_{n+1}^{(1,1)}(y,y) \end{bmatrix},
\end{aligned} \tag{6.1}$$

exactly as in (3.1). When $x = y$, the first row of $\Sigma$ is the same as the second row, and hence $\det \Sigma = 0$. Our first goal is to show that $V$ has the multivariate normal distribution with mean zero and the covariance matrix $\Sigma$ when $x \neq y$ and $n \geq 3$. This follows in a standard way, e.g., from [21, Corollary 16.2], by proving that $\Sigma$ is positive definite, which amounts to showing that $\vec{v}^T \Sigma \vec{v} > 0$ for all nonzero $\vec{v} \in \mathbb{R}^4$. Recall that any covariance matrix is positive semi-definite [21, Theorem 12.4], i.e., $\vec{v}^T \Sigma \vec{v} \geq 0$ for all $\vec{v} \in \mathbb{R}^4$. This means we only need to demonstrate that $\vec{v}^T \Sigma \vec{v} = 0$ implies $\vec{v} = \vec{0}$. For a vector $\vec{v} = [v_1 \quad v_2 \quad v_3 \quad v_4]^T$, observe that

$$\vec{v}^T \Sigma \vec{v} = \operatorname{Var}[\vec{v}^T V] = \sum_{k=0}^{n} (v_1 p_k(x) + v_2 p_k(y) + v_3 p_k'(x) + v_4 p_k'(y))^2.$$

It is clear now that $\vec{v}^T \Sigma \vec{v} = 0$ if and only if

$$v_1 p_k(x) + v_2 p_k(y) + v_3 p_k'(x) + v_4 p_k'(y) = 0, \quad k = 0, \ldots, n. \tag{6.2}$$

But this system of equations has only trivial solution $\vec{v} = \vec{0}$. Indeed, if we write

$$Q_n(t) = \sum_{j=0}^{n} b_j p_j(t),$$

where $\{b_j\}_{j=0}^n \subset \mathbb{R}$ is arbitrary, then (6.2) implies that

$$v_1 Q_n(x) + v_2 Q_n(y) + v_3 Q_n'(x) + v_4 Q_n'(y) = 0. \tag{6.3}$$

Since $\{p_j(x)\}_{j=0}^n$ is a basis for the vector space of all polynomials of degree at most $n$ with real coefficients, the set of all polynomials $Q_n(t)$ coincides with this space. In particular, since $n \geq 3$ and $x \neq y$, we use the following choices for $Q_n$ in (6.3) to conclude that

$$\begin{aligned}
Q_n(t) = (t-x)(t-y)^2 &\Rightarrow v_3 = 0; \\
Q_n(t) = (t-x)^2(t-y) &\Rightarrow v_4 = 0; \\
Q_n(t) = t-y &\Rightarrow v_1 = 0; \\
Q_n(t) = t-x &\Rightarrow v_2 = 0.
\end{aligned}$$

We now write $\Sigma$ in the following block form

$$\Sigma = \begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) & K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(0,1)}(x,y) \\ K_{n+1}(x,y) & K_{n+1}(y,y) & K_{n+1}^{(0,1)}(y,x) & K_{n+1}^{(0,1)}(y,y) \\ K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(0,1)}(y,x) & K_{n+1}^{(1,1)}(x,x) & K_{n+1}^{(1,1)}(x,y) \\ K_{n+1}^{(0,1)}(x,y) & K_{n+1}^{(0,1)}(y,y) & K_{n+1}^{(1,1)}(x,y) & K_{n+1}^{(1,1)}(y,y) \end{bmatrix} =: \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}, \tag{6.4}$$

where $A$, $B$ and $C$ are the corresponding $2 \times 2$ matrices. Note that $\det A = \Delta = 0$ if and only if $x = y$ by the equality case in the Cauchy-Schwarz inequality. Thus we define $\Omega = C - B^T A^{-1} B$ for $x \neq y$, and write

$$\Sigma = \begin{bmatrix} A & \mathbf{0} \\ B^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & A^{-1}B \\ \mathbf{0} & \Omega \end{bmatrix}.$$

The latter implies that

$$\det \Sigma = \det A \, \det \Omega = \Delta \, \det \Omega.$$

Since $\Sigma$ is invertible for $x \neq y$, so is $\Omega$ and thus $\det \Omega > 0$ if $x \neq y$. It also follows from (6.4) by direct algebraic manipulations that the elements of the matrix

$$\Omega = C - B^T A^{-1} B = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{bmatrix}$$

are as defined in (2.8)–(2.10).

Since the random vector $V = V(x,y)$ has the multivariate normal distribution $\mathcal{N}(\mathbf{0}, \Sigma)$ with a non-singular covariance matrix $\Sigma$, we compute the density of its distribution by [21, p. 130] in the form

$$\begin{aligned} p_{x,y}(0,0,t_1,t_2) &= \frac{\exp\left(-\frac{1}{2}(0,0,t_1,t_2)\,\Sigma^{-1}(0,0,t_1,t_2)^T\right)}{(2\pi)^2(\det\Sigma)^{1/2}} \\ &= \frac{\exp\left(-\frac{1}{2}(t_1,t_2)\,\Omega^{-1}(t_1,t_2)^T\right)}{(2\pi)^2(\det\Sigma)^{1/2}}. \end{aligned}$$

Using matrix algebra, we further obtain that

$$\Sigma^{-1} = \begin{bmatrix} [A - BC^{-1}B^T]^{-1} & -A^{-1}B[C - B^TA^{-1}B]^{-1} \\ -C^{-1}B^T[A - BC^{-1}B^T]^{-1} & [C - B^TA^{-1}B]^{-1} \end{bmatrix}.$$

Theorem 3.2 of [3, p. 71] states that if $(a,b) \subset \mathbb{R}$, then

$$\mathbb{E}[N_n([a,b])\,(N_n([a,b]) - 1)] = \iint\limits_D \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} |t_1 t_2| p_{x,y}(0,0,t_1,t_2)\, dt_1 dt_2 dx dy,$$

where $D = \{(x,y) \in \mathbb{R}^2 |\ a \le x, y \le b\}$. Hence

$$\begin{aligned} &\mathbb{E}[N_n([a,b])(N_n([a,b]) - 1)] \\ &= \iint\limits_D \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} |t_1 t_2| \frac{\exp\left(-\frac{1}{2}(t_1,t_2)\Omega^{-1}(t_1,t_2)^T\right)}{(2\pi)^2(\det\Sigma)^{1/2}}\, dt_1 dt_2 dx dy, \\ &= \iint\limits_D \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} |t_1 t_2| \frac{\exp\left(-\frac{1}{2}(t_1,t_2)\Omega^{-1}(t_1,t_2)^T\right)}{(2\pi)^2(\Delta\, \det\Omega)^{1/2}}\, dt_1 dt_2 dx dy, \\ &= \frac{1}{4\pi^2} \iint\limits_D \frac{I(x,y)}{\sqrt{\Delta\, \det\Omega}}\, dx dy, \end{aligned}$$

where the inner integral is

$$I(x,y)=\int_{\mathbb{R}}\int_{\mathbb{R}}|t_1t_2|\exp\left(-\frac{1}{2}(t_1,t_2)\Omega^{-1}(t_1,t_2)^T\right)dt_1dt_2.$$

Note if $x\neq y$, we have $\det\Omega=\Omega_{11}\Omega_{22}-\Omega_{12}^2>0$ and

$$\Omega^{-1}=\frac{1}{\det\Omega}\begin{bmatrix}\Omega_{22} & -\Omega_{12}\\ -\Omega_{12} & \Omega_{11}\end{bmatrix}.$$

It follows that

$$(t_1,t_2)\,\Omega^{-1}\,(t_1,t_2)^T=\frac{\Omega_{22}}{\det\Omega}\,t_1^2-2\,\frac{\Omega_{12}}{\det\Omega}\,t_1t_2+\frac{\Omega_{11}}{\det\Omega}\,t_2^2.$$

Applying the result of [8, (3.9)], we evaluate the inner integral as

$$I(x,y)=\frac{4(\det\Omega)^2}{\Omega_{11}\Omega_{22}(1-\delta^2)}\left(1+\frac{\delta}{\sqrt{1-\delta^2}}\arcsin\delta\right),$$

with

$$\delta=-\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}.$$

Finally, putting everything together, we obtain

$$\begin{aligned}&\mathbb{E}[N_n([a,b])(N_n([a,b])-1)]\\ &=\frac{1}{4\pi^2}\iint_D\frac{4(\det\Omega)^2}{\Omega_{11}\Omega_{22}(1-\delta^2)}\left(1+\frac{\delta}{\sqrt{1-\delta^2}}\arcsin\delta\right)\frac{dx\,dy}{\sqrt{\Delta}\,\det\Omega}\\ &=\frac{1}{\pi^2}\iint_D\sqrt{\Omega_{11}\Omega_{22}-\Omega_{12}^2}\left(1-\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}-\Omega_{12}^2}}\arcsin\left(-\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}\right)\right)\frac{dx_1\,dx_2}{\sqrt{\Delta}}\\ &=\frac{1}{\pi^2}\iint_D\left(\sqrt{\Omega_{11}\Omega_{22}-\Omega_{12}^2}+\Omega_{12}\arcsin\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}\right)\frac{dx\,dy}{\sqrt{\Delta}}.\end{aligned}$$

This and Lemma 2.1 give the result. □

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