



Weak type (p, p) bounds for Schrödinger groups via generalized Gaussian estimates

Zhijie Fan

Department of Mathematics, Sun Yat-sen (Zhongshan) University, Guangzhou, 510275, PR China



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ABSTRACT

Let L be a non-negative self-adjoint operator acting on $L^2(X)$, where X is a space of homogeneous type with a dimension n . Suppose that the heat operator e^{-tL} satisfies the generalized Gaussian (p_0, p'_0) -estimates of order m for some $1 \leq p_0 < 2$. It is known that the operator $(I+L)^{-s}e^{itL}$ is bounded on $L^p(X)$ for $s \geq n|1/2 - 1/p|$ and $p \in (p_0, p'_0)$ (see for example, [5,7,9,10,13,26]). In this paper we study the endpoint case $p = p_0$ and show that for $s_0 = n|\frac{1}{2} - \frac{1}{p_0}|$, the operator $(I+L)^{-s_0}e^{itL}$ is of weak type (p_0, p_0) , that is, there is a constant $C > 0$, independent of t and f so that

$$\mu(\{x : |(I+L)^{-s_0}e^{itL}f(x)| > \alpha\}) \leq C(1+|t|)^{n(1-\frac{p_0}{2})} \left(\frac{\|f\|_{p_0}}{\alpha}\right)^{p_0}, \quad t \in \mathbb{R}$$

for $\alpha > 0$ when $\mu(X) = \infty$, and $\alpha > (\|f\|_{p_0}/\mu(X))^{p_0}$ when $\mu(X) < \infty$. Our results can be applied to Schrödinger operators with rough potentials and higher order elliptic operators with bounded measurable coefficients although in general, their semigroups fail to satisfy Gaussian upper bounds.

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1. Introduction

Throughout the paper we suppose that (X, d, μ) is a metric measure space with a distance function d and a measure μ . We say that (X, d, μ) satisfies the doubling property (see Chapter 3, [8]) if there exists a constant $C > 0$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \quad \forall r > 0, x \in X. \quad (1.1)$$

Note that the doubling property implies the following strong homogeneity property,

$$\mu(B(x, \lambda r)) \leq C\lambda^n \mu(B(x, r)), \quad (1.2)$$

E-mail address: fanzhj3@mail2.sysu.edu.cn.

for some $C, n > 0$ uniformly for all $\lambda \geq 1$ and $x \in X$. In Euclidean space with Lebesgue measure, the parameter n corresponds to the dimension of the space, but in our more abstract setting, the optimal n is not necessarily an integer.

Let L be a non-negative self-adjoint operator on the Hilbert space $L^2(X)$. Consider the Schrödinger equation in $X \times \mathbb{R}$,

$$\begin{cases} i\partial_t u + Lu = 0, \\ u|_{t=0} = f \end{cases}$$

with initial data f . Then the solution can be formally written as

$$u(x, t) = e^{itL} f(x) = \int_0^\infty e^{it\lambda} dE_L(\lambda) f(x), \quad t \in \mathbb{R} \quad (1.3)$$

for $f \in L^2(X)$, where E_L denotes the resolution of the identity associated with L . By the spectral theorem ([25]), the operator e^{itL} is continuous on $L^2(X)$, and forms the Schrödinger group. A natural problem is to study the mapping properties of families of operators derived from the Schrödinger group on various functional spaces defined on X . This has attracted a lot of attention in the last decades, and has been a very active research topic in harmonic analysis and partial differential equations— see for example, [1,5–7,9,13,16–20,22,23,26,27,29].

In this paper, we consider a non-negative self-adjoint operator L and numbers $m \geq 2$ and $1 \leq p_0 \leq 2$. Following [3,5,10], we say that the semigroup e^{-tL} generated by L , satisfies the generalized Gaussian (p_0, p'_0) -estimate of order m , if there exist constants $C, c > 0$ such that

$$\|P_{B(x, t^{1/m})} e^{-tL} P_{B(y, t^{1/m})}\|_{p_0 \rightarrow p'_0} \leq C \mu(B(x, t^{1/m}))^{-(\frac{1}{p_0} - \frac{1}{p'_0})} \exp\left(-c \left(\frac{d(x, y)^m}{t}\right)^{\frac{1}{m-1}}\right) \quad (\text{GGE}_{p_0, p'_0, m})$$

for every $t > 0$ and $x, y \in X$. Note that condition $(\text{GGE}_{p_0, p'_0, m})$ for the special case $p_0 = 1$ is equivalent to m -th order Gaussian estimates (see for example, [5]). This means that the semigroup e^{-tL} has integral kernels $p_t(x, y)$ satisfying the following Gaussian upper estimate:

$$|p_t(x, y)| \leq \frac{C}{\mu(B(x, t^{1/m}))} \exp\left(-c \left(\frac{d(x, y)^m}{t}\right)^{\frac{1}{m-1}}\right) \quad (\text{GE}_m)$$

for every $t > 0, x, y \in X$, where c, C are two positive constants and $m \geq 2$. There are numbers of operators which satisfy generalized Gaussian estimates and, among them, there exist many for which classical Gaussian estimates (GE_m) fail. This happens, e.g., for Schrödinger operators with rough potentials [28], second order elliptic operators with rough lower order terms [24], or higher order elliptic operators with bounded measurable coefficients [14].

Recently, under the assumption $(\text{GGE}_{p_0, p'_0, m})$, Chen, Duong, Li and Yan [10] showed that if L satisfies the estimate $(\text{GGE}_{p_0, p'_0, m})$ for some $m \geq 2$ and $1 \leq p_0 < 2$, then for every $p \in (p_0, p'_0)$, there exists a constant $C = C(n, p) > 0$ independent of t and f such that

$$\|(I + L)^{-s} e^{itL} f\|_{L^p(X)} \leq C(1 + |t|)^s \|f\|_{L^p(X)}, \quad t \in \mathbb{R}, \quad s \geq n \left| \frac{1}{2} - \frac{1}{p} \right|. \quad (1.4)$$

See also [5,7,9,13,26].

In this paper, we extend the previous result to the endpoint case $p = p_0$ and obtain the following result.

Theorem 1.1. Suppose that (X, d, μ) is a space of homogeneous type with a dimension n and that L satisfies the property $(\text{GGE}_{p_0, p'_0, m})$ for some $1 \leq p_0 < 2$ and $m \geq 2$. Then for $s_0 = n|\frac{1}{2} - \frac{1}{p_0}|$, the operator $(I + L)^{-s_0} e^{itL}$ is of weak type (p_0, p_0) , that is, there is a constant $C > 0$, independent of t and f so that

$$\mu(\{x : |(I + L)^{-s_0} e^{itL} f(x)| > \alpha\}) \leq C(1 + |t|)^{n(1 - \frac{p_0}{2})} \left(\frac{\|f\|_{p_0}}{\alpha} \right)^{p_0}, \quad t \in \mathbb{R}$$

for $\alpha > 0$ when $\mu(X) = \infty$ and $\alpha > (\|f\|_{p_0}/\mu(X))^{p_0}$ when $\mu(X) < \infty$.

We would like to mention that when L satisfies the Gaussian estimates (GE_m) , Chen, Duong, Li, Song and Yan [12, Theorem 1.1] proved that the operator $(I + L)^{-n/2} e^{itL}$ is of weak type $(1, 1)$. In their proof, it heavily relies on the following Plancherel-type estimate

$$\int_X |K_{e^{-(1-i\tau)R^{-m}L}}(x, y)|^2 d(x, y)^s d\mu(y) \leq C\mu(B(x, 1/R))^{-1} R^{-s} (1 + |\tau|)^s, \quad (1.5)$$

for some constant $C > 0$ independent of $s \geq 0$, $R > 0$ and $\tau \in \mathbb{R}$, where $K_{e^{-(1-i\tau)R^{-m}L}}(x, y)$ denotes the integral kernel of the operators $e^{-(1-i\tau)R^{-m}L}$. In our setting, we do not have such an estimate at our disposal. To overcome this difficulty, we apply the Phragmén-Lindelöf theorem to show that the generalized Gaussian estimates $(\text{GGE}_{p_0, p'_0, m})$ implies the following $L^{p_0} - L^2$ off-diagonal estimates of the operator $e^{-(1-i\tau)R^{-1}L}$:

$$\|\chi_{C_\nu(B)} e^{-(1-i\tau)R^{-1}L} \chi_B\|_{p_0 \rightarrow 2} \leq C\mu(B(x_B, R^{-1/m}))^{-(\frac{1}{p_0} - \frac{1}{2})} \exp\left(-c\left(\frac{\sqrt[m]{R} 2^\nu r}{1 + |\tau|}\right)^{\frac{m}{m-1}}\right),$$

for some constant $C > 0$ independent of $\tau \in \mathbb{R}$, $R > 0$ and balls $B \subset X$ with center x_B and radius r . This estimate is a suitable substitute of (1.5) and it helps us deduce that for any $\frac{n}{2} < s < mM$, there exists a positive constant C independent of $k > k_0$, $t > 0$, $\nu \geq \nu_0$, and any ball B with radius $r \sim 2^k$, such that

$$\|\chi_{C_\nu(B)} e^{itL} F_k(L) \chi_B f\|_2 \leq C 2^{-\nu s} \mu(B)^{-(\frac{1}{p_0} - \frac{1}{2})} (1 + |t|)^{p_0(\frac{1}{p_0} - \frac{1}{2})(s + \frac{n}{2})} \|f\|_{p_0}, \quad (1.6)$$

where $F_k(L) = (I + L)^{-(\frac{1}{p_0} - \frac{1}{2})n} (I - e^{-2^{mk}L})^M \varphi_0(2^{-m(k-k_0)/(m-1)}L)$, and φ_0 is a smooth function with $\text{supp } \varphi_0 \subset [0, 1]$, $\varphi_0(\lambda) = 1$ on $[0, 1/2]$, and $k_0 \sim 2^{-1}p_0 \log_2(1 + |t|)$. This inequality plays a crucial role in obtaining the sharp growth $(1 + |t|)^{(1 - \frac{p_0}{2})n}$ in the setting of homogeneous space (see Remark 2.6).

The paper is organized as follows. In section 2 we present some $L^{p_0} - L^2$ off-diagonal estimates for heat semigroups, resolvent and compactly supported spectral multipliers. In section 3, we apply the off-diagonal estimates obtained in section 2, combined with the Hardy-Littlewood maximal function and duality argument, to show Theorem 1.1.

2. Preliminary results

We now set some notations and common concepts to be used throughout the course of the paper. For $1 \leq p \leq +\infty$, we denote the norm of a function $f \in L^p(X, d\mu)$ by $\|f\|_p$. We let $\langle \cdot, \cdot \rangle$ be the scalar product in $L^2(X, d\mu)$. If T is a bounded linear operator from $L^p(X, d\mu)$ to $L^q(X, d\mu)$, $1 \leq p, q \leq +\infty$, we write $\|T\|_{p \rightarrow q}$ for the operator norm of T . The indicator function of a subset $E \subseteq X$ is denoted by χ_E . Let f be a tempered distribution, then the Fourier transform \hat{f} is defined by

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx, \quad \xi \in \mathbb{R}^n.$$

Next, let $B(x, r) = \{y \in X, d(x, y) < r\}$ be the open ball with center $x \in X$ and radius $r > 0$. To simplify notation we often just use B instead of $B(x, r)$. Let V_s be a multiplier operator defined by $V_s f(x) := \mu(B(x, s))f(x)$, and \mathcal{M} will be denoted by the Hardy-Littlewood maximal function. Also, for $1 \leq p_0 \leq 2$, let

$$\sigma_{p_0} := \frac{1}{p_0} - \frac{1}{2}.$$

For simplicity, we write

$$C_1(B) := 2B \quad \text{and} \quad C_\nu(B) := 2^\nu B - 2^{\nu-1}B, \quad \nu = 2, 3, \dots$$

In this section we will prove $L^{p_0} - L^2$ off-diagonal estimates for resolvent and compactly spectral multipliers, which play crucial roles in obtaining the sharp growth $(1 + |t|)^{p_0 \sigma_{p_0} n}$ for the operator norm $\|e^{itL}(I + L)^{-\sigma_{p_0} n}\|_{L^{p_0} \rightarrow L^{p_0, \infty}}$ in the setting of homogeneous space. To begin with, we show the following $L^{p_0} - L^2$ off-diagonal estimates for heat semigroups.

Lemma 2.1. *There exists a constant $C > 0$ such that for any ball $B \subset X$ with center x_B and radius r and any $\lambda > 0$, $\nu \in \mathbb{N}$, the following estimate holds:*

$$\|\chi_{C_\nu(B)} e^{-\lambda L} \chi_B\|_{p_0 \rightarrow 2} \leq C \mu(B(x_B, \lambda^{1/m}))^{-\sigma_{p_0}} \exp\left(-c \left(\frac{d(C_\nu(B), B)}{\lambda^{1/m}}\right)^{m/m-1}\right). \quad (2.1)$$

Proof. The proof was essentially proved in [4, Theorem 1.2]. We give a brief argument of this proof for completeness and the convenience of readers.

By [4, Theorem 1.2], property $(\text{GGE}_{p_0, p'_0, m})$ implies the following two ball estimate:

$$\|\chi_{B_1} V_{\lambda^{1/m}}^a e^{-\lambda L} V_{\lambda^{1/m}}^b \chi_{B_2}\|_{p_0 \rightarrow 2} \leq C \exp\left(-c \left(\frac{d(B_1, B_2)}{\lambda^{1/m}}\right)^{\frac{m}{m-1}}\right), \quad \text{for any balls } B_1, B_2,$$

where $a, b \geq 0$ such that $a + b = \sigma_{p_0}$. Therefore, estimate (2.1) holds for $\nu = 0$. For any $\nu \geq 1$, we note that there exist c_n balls $\{B_\nu^{(j)}\}_{j=1}^{c_n}$ such that $C_\nu(B) \subset \bigcup_{j=1}^{c_n} B_\nu^{(j)}$ and $d(C_\nu(B), B) \sim d(B_\nu^{(j)}, B)$, for all $j = 1, 2, \dots, c_n$. Hence

$$\begin{aligned} \|\chi_{C_\nu(B)} e^{-\lambda L} \chi_B\|_{p_0 \rightarrow 2} &\leq \mu(B(x_B, \lambda^{1/m}))^{-\sigma_{p_0}} \sum_{j=1}^{c_n} \|\chi_{B_\nu^{(j)}} e^{-\lambda L} V_{\lambda^{1/m}}^{\sigma_{p_0}} \chi_B\|_{p_0 \rightarrow 2} \\ &\leq C \mu(B(x_B, \lambda^{1/m}))^{-\sigma_{p_0}} \exp\left(-c \left(\frac{d(C_\nu(B), B)}{\lambda^{1/m}}\right)^{\frac{m}{m-1}}\right). \quad \square \end{aligned}$$

2.1. $L^{p_0} - L^2$ off-diagonal estimates for resolvent

Proposition 2.2. *For any $N \in \mathbb{N}$, there exists a positive constant C such that for any $\nu \geq 2$,*

$$\|\chi_{C_\nu(B)} (I + L)^{-\sigma_{p_0} n} (I - e^{-r^m L})^M \chi_B\|_{p_0 \rightarrow 2} \leq C 2^{-\nu N} \mu(B)^{-\sigma_{p_0}}, \quad (2.2)$$

and there exists a positive constant C such that

$$\|\chi_{2B} (I + L)^{-\sigma_{p_0} n} (I - e^{-r^m L})^M \chi_{2B}\|_{p_0 \rightarrow 2} \leq C \max\{1, r^{\sigma_{p_0} n}\} \mu(B)^{-\sigma_{p_0}}, \quad (2.3)$$

where M is a fixed parameter chosen to be bigger than $\frac{N}{m}$ and the constant C is independent of $B = B(x_B, r)$.

Proof. Note that

$$(I + L)^{-\sigma_{p_0} n} = \frac{1}{\Gamma(\sigma_{p_0} n)} \int_0^\infty e^{-\lambda L} e^{-\lambda} \lambda^{\sigma_{p_0} n - 1} d\lambda.$$

From it we use the change of variables to obtain

$$\begin{aligned} (I + L)^{-\sigma_{p_0} n} (I - e^{-r^m L})^M &= \frac{1}{\Gamma(\sigma_{p_0} n)} \int_0^\infty e^{-\lambda L} (I - e^{-r^m L})^M e^{-\lambda} \lambda^{\sigma_{p_0} n - 1} d\lambda \\ &= \frac{1}{\Gamma(\sigma_{p_0} n)} \int_0^\infty g_{r^m}(\lambda) e^{-\lambda L} d\lambda, \end{aligned} \quad (2.4)$$

where

$$g_s(\lambda) = \sum_{\ell=0}^M C_M^\ell (-1)^\ell \chi_{\{\lambda > \ell s\}}(\lambda) (\lambda - \ell s)^{\sigma_{p_0} n - 1} e^{-(\lambda - \ell s)}. \quad (2.5)$$

Then, it can be verified that

$$|g_{r^m}(\lambda)| \leq \begin{cases} C \lambda^{\sigma_{p_0} n - 1} e^{-\lambda}, & 0 < \lambda < r^m, \\ C(r^m)^{\sigma_{p_0} n - 1} e^{-r^m} + (\lambda - \beta r^m)^{\sigma_{p_0} n - 1} e^{-(\lambda - \beta r^m)}, & \beta r^m \leq \lambda < (\beta + 1)r^m, 1 \leq \beta \leq M, \\ C r^m M \lambda^{\sigma_{p_0} n - 1 - M} e^{-\frac{\lambda}{2(M+1)}}, & \lambda \geq (M + 1)r^m. \end{cases}$$

Now let us prove (2.2). By the formula (2.4), Lemma 2.1 and doubling condition (1.2), we get

$$\begin{aligned} &\left\| \chi_{C_\nu(B)} (I + L)^{-\sigma_{p_0} n} (I - e^{-r^m L})^M \chi_B \right\|_{p_0 \rightarrow 2} \\ &\leq C \mu(B)^{-\sigma_{p_0}} \int_0^\infty |g_{r^m}(\lambda)| \left(1 + \frac{r}{\lambda^{1/m}}\right)^{\sigma_{p_0} n} \exp\left(-c\left(\frac{2^\nu r}{\lambda^{1/m}}\right)^{\frac{m}{m-1}}\right) d\lambda \\ &= C \mu(B)^{-\sigma_{p_0}} \left(\int_0^{r^m} + \int_{r^m}^{(M+1)r^m} + \int_{(M+1)r^m}^\infty \right) |g_{r^m}(\lambda)| \left(1 + \frac{r}{\lambda^{1/m}}\right)^{\sigma_{p_0} n} \exp\left(-c\left(\frac{2^\nu r}{\lambda^{1/m}}\right)^{\frac{m}{m-1}}\right) d\lambda \\ &=: I + II + III. \end{aligned}$$

For the term I , we use the property of g_m to obtain

$$\begin{aligned} I &\leq C \mu(B)^{-\sigma_{p_0}} \int_0^{r^m} \lambda^{\sigma_{p_0} n - 1} e^{-\lambda} \left(1 + \frac{r}{\lambda^{1/m}}\right)^{\sigma_{p_0} n} \exp\left(-c\left(\frac{2^\nu r}{\lambda^{1/m}}\right)^{\frac{m}{m-1}}\right) d\lambda \\ &\leq C 2^{-\nu N} \mu(B)^{-\sigma_{p_0}} \int_0^1 \lambda^{-1 + \frac{N - \sigma_{p_0} n}{m}} d\lambda \\ &\leq C 2^{-\nu N} \mu(B)^{-\sigma_{p_0}}, \end{aligned} \quad (2.6)$$

for some large $N > \sigma_{p_0} n$. For the term II , we have

$$\begin{aligned}
II &\leq C\mu(B)^{-\sigma_{p_0}} \sum_{\ell=1}^M \int_{\ell r^m}^{(\ell+1)r^m} (r^m)^{\sigma_{p_0}n-1} e^{-r^m} \left(1 + \frac{r}{\lambda^{1/m}}\right)^{\sigma_{p_0}n} \exp\left(-c\left(\frac{2^\nu r}{\lambda^{1/m}}\right)^{\frac{m}{m-1}}\right) d\lambda \\
&\quad + C\mu(B)^{-\sigma_{p_0}} \sum_{\ell=1}^M \int_{\ell r^m}^{(\ell+1)r^m} (\lambda - \ell r^m)^{\sigma_{p_0}n-1} e^{-(\lambda - \ell r^m)} \left(1 + \frac{r}{\lambda^{1/m}}\right)^{\sigma_{p_0}n} \exp\left(-c\left(\frac{2^\nu r}{\lambda^{1/m}}\right)^{\frac{m}{m-1}}\right) d\lambda \\
&\leq C2^{-\nu N} \mu(B)^{-\sigma_{p_0}}.
\end{aligned} \tag{2.7}$$

Consider the term III . Since $M > \frac{N}{m}$, we conclude that

$$\begin{aligned}
III &\leq C\mu(B)^{-\sigma_{p_0}} \int_{(M+1)r^m}^{\infty} r^{mM} \lambda^{\sigma_{p_0}n-1-M} e^{-\frac{\lambda}{2(M+1)}} \left(1 + \frac{r}{\lambda^{1/m}}\right)^{\sigma_{p_0}n} \exp\left(-c\left(\frac{2^\nu r}{\lambda^{1/m}}\right)^{\frac{m}{m-1}}\right) d\lambda \\
&\leq C2^{-\nu N} \mu(B)^{-\sigma_{p_0}} \int_1^{\infty} \lambda^{-1-M+\frac{N}{m}} d\lambda \\
&\leq C2^{-\nu N} \mu(B)^{-\sigma_{p_0}}.
\end{aligned}$$

This, in combination with estimates (2.6) and (2.7), shows the desired estimate (2.2).

To show estimate (2.3), we see that

$$\begin{aligned}
&\left\| \chi_{2B}(I+L)^{-\sigma_{p_0}n} (I - e^{-r^m L})^M \chi_{2B} \right\|_{p_0 \rightarrow 2} \\
&\leq C\mu(B)^{-\sigma_{p_0}} \int_0^{\infty} |g_{r^m}(\lambda)| \left(1 + \frac{r}{\lambda^{1/m}}\right)^{\sigma_{p_0}n} d\lambda \\
&= C\mu(B)^{-\sigma_{p_0}} \left(\int_0^{r^m} + \int_{r^m}^{(M+1)r^m} + \int_{(M+1)r^m}^{\infty} \right) |g_{r^m}(\lambda)| \left(1 + \frac{r}{\lambda^{1/m}}\right)^{\sigma_{p_0}n} d\lambda.
\end{aligned}$$

Then, we can deduce that the last two parts are no larger than $C\mu(B)^{-\sigma_{p_0}}$ by the same way. Also, it can be shown by a simple modification of the estimate of III that the first part can be bounded by $C \max\{1, r^{\sigma_{p_0}n}\} \mu(B)^{-\sigma_{p_0}}$. We omit the details and leave it to the readers. \square

2.2. $L^{p_0} - L^2$ off-diagonal estimates for compactly supported spectral multipliers

To begin with, we state the following version of Phragmen-Lindelöf Theorem.

Lemma 2.3. Suppose that function F is analytic in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and that

$$\begin{aligned}
|F(z)| &\leq a_1 (\operatorname{Re} z)^{-\beta_1}, \\
|F(t)| &\leq a_1 t^{-\beta_1} \exp(-a_2 t^{-\beta_2}),
\end{aligned}$$

for some $a_1, a_2 > 0$, $\beta_1 \geq 0$, $\beta_2 \in (0, 1]$, all $t > 0$ and $z \in \mathbb{C}_+$. Then

$$|F(z)| \leq a_1 2^{\beta_1} (\operatorname{Re} z)^{-\beta_1} \exp\left(-\frac{a_2 \beta_2}{2} |z|^{-\beta_2-1} \operatorname{Re} z\right)$$

for all $z \in \mathbb{C}_+$.

Proof. For the proof, we refer it to [15, Lemma 9]. \square

By Lemma 2.3, we can modify the argument from [11, Lemma 3.3] to extend the off-diagonal estimate (2.1) from real times $t > 0$ to complex times $z = -(1 - i\tau)R^{-1}$ for any $R > 0$.

Lemma 2.4. *There exists a constant $C > 0$ such that for any ball $B \subset X$ with center x_B and radius r , integer $\nu \geq 2$, real number $R > 0$, the following estimate holds:*

$$\|\chi_{C_\nu(B)} e^{-(1-i\tau)R^{-1}L} \chi_B\|_{p_0 \rightarrow 2} \leq C \mu(B(x_B, R^{-1/m}))^{-\sigma_{p_0}} \exp\left(-c\left(\frac{\sqrt[m]{R} 2^\nu r}{1+|\tau|}\right)^{\frac{m}{m-1}}\right). \quad (2.8)$$

Proof. For simplicity, we write $z = -(1 - i\tau)R^{-1}$. For any ball $B \subset X$, consider the analytic function $F : \mathbb{C}_+ \rightarrow \mathbb{R}$ defined by

$$F(z) := e^{-Rz} \mu(B(x_B, R^{-1/m}))^{\sigma_{p_0}} \langle e^{-zL} f_1, f_2 \rangle,$$

where $\text{supp} f_1 \subset B$ and $\text{supp} f_2 \subset C_\nu(B)$.

It was shown in [5, Proposition 3.1] that

$$\|e^{-\frac{\text{Re} z}{4}L} V_{(\text{Re} z)^{1/m}}^{\sigma_{p_0}}\|_{p_0 \rightarrow 2} \leq C.$$

This, together with the spectral theorem, shows that

$$\|e^{-zL} V_{(\text{Re} z)^{1/m}}^{\sigma_{p_0}}\|_{p_0 \rightarrow 2} \leq \|e^{-\frac{\text{Re} z}{4}L}\|_{2 \rightarrow 2} \|e^{-(z - \frac{\text{Re} z}{2})L}\|_{2 \rightarrow 2} \|e^{-\frac{\text{Re} z}{4}L} V_{(\text{Re} z)^{1/m}}^{\sigma_{p_0}}\|_{p_0 \rightarrow 2} \leq C.$$

Hence,

$$|\langle e^{-zL} f_1, f_2 \rangle| \leq \|e^{-zL} V_{(\text{Re} z)^{1/m}}^{\sigma_{p_0}}\|_{p_0 \rightarrow 2} \|V_{(\text{Re} z)^{1/m}}^{-\sigma_{p_0}} f_1\|_{p_0} \|f_2\|_2 \leq C \mu(B(x_B, (\text{Re} z)^{1/m}))^{-\sigma_{p_0}} \|f_1\|_{p_0} \|f_2\|_2. \quad (2.9)$$

This, in combination with doubling condition (1.2), yields

$$\begin{aligned} |F(z)| &\leq C e^{-R\text{Re} z} \left(\frac{\mu(B(x_B, R^{-1/m}))}{\mu(B(x_B, (\text{Re} z)^{1/m}))} \right)^{\sigma_{p_0}} \|f_1\|_{p_0} \|f_2\|_2 \\ &\leq C e^{-R\text{Re} z} \left(1 + \frac{1}{R\text{Re} z} \right)^{\frac{\sigma_{p_0} n}{m}} \|f_1\|_{p_0} \|f_2\|_2 \\ &\leq C (R\text{Re} z)^{-\frac{\sigma_{p_0} n}{m}} \|f_1\|_{p_0} \|f_2\|_2. \end{aligned} \quad (2.10)$$

Similarly, by Lemma 2.1,

$$\begin{aligned} |F(\lambda)| &\leq e^{-R\lambda} \mu(B(x_B, R^{-1/m}))^{\sigma_{p_0}} \|\chi_{C_\nu(B)} e^{-\lambda L} \chi_B\|_{p_0 \rightarrow 2} \|f_1\|_{p_0} \|f_2\|_2 \\ &\leq C e^{-R\lambda} \left(\frac{\mu(B(x_B, R^{-1/m}))}{\mu(B(x_B, \lambda^{1/m}))} \right)^{\sigma_{p_0}} \exp\left(-c\left(\frac{2^\nu r}{\lambda^{1/m}}\right)^{\frac{m}{m-1}}\right) \|f_1\|_{p_0} \|f_2\|_2 \\ &\leq C e^{-R\lambda} \left(1 + \frac{1}{R\lambda} \right)^{\frac{\sigma_{p_0} n}{m}} \exp\left(-c\left(\frac{2^\nu r}{\lambda^{1/m}}\right)^{\frac{m}{m-1}}\right) \|f_1\|_{p_0} \|f_2\|_2 \\ &\leq C (R\lambda)^{-\frac{\sigma_{p_0} n}{m}} \exp\left(-c\left(\frac{2^\nu r}{\lambda^{1/m}}\right)^{\frac{m}{m-1}}\right) \|f_1\|_{p_0} \|f_2\|_2. \end{aligned} \quad (2.11)$$

Next, combining (2.10) with (2.11), we choose $z = (1 - i\tau)R^{-1}$, $a_1 = CR^{-\frac{\sigma_{p_0}n}{m}}\|f_1\|_{p_0}\|f_2\|_2$, $a_2 = c(2^\nu r)^{\frac{m}{m-1}}$, $\beta_1 = \frac{\sigma_{p_0}n}{m}$ and $\beta_2 = 1/(m-1)$ in Lemma 2.3 to obtain

$$|F((1 - i\tau)R^{-1})| \leq C \exp\left(-c\left(\frac{\sqrt[m]{R}2^\nu r}{1 + |\tau|}\right)^{\frac{m}{m-1}}\right) \|f_1\|_{p_0} \|f_2\|_2,$$

which yields the estimate (2.8). \square

We define a Besov type norm of F by

$$\|F\|_{B^s} := \int_{-\infty}^{\infty} |\hat{F}(\tau)|(1 + |\tau|)^s d\tau,$$

where \hat{F} denotes the Fourier transform of F . Applying Fubini theorem, we can easily check that for every functions F and G ,

$$\begin{aligned} \|FG\|_{B^s} &= \int_{-\infty}^{\infty} |\widehat{FG}(\tau)|(1 + |\tau|)^s d\tau \\ &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{F}(\tau - \eta)\hat{G}(\eta)|(1 + |\tau - \eta|)^s (1 + |\eta|)^s d\eta d\tau \\ &\leq \|F\|_{B^s} \|G\|_{B^s}. \end{aligned} \quad (2.12)$$

In the sequel, for any $R > 0$, we denote the dilation of a function F by $\delta_R F(\cdot) := F(R\cdot)$.

Proposition 2.5. *For every $s \geq 0$, there exists a constant $C > 0$ such that for every $\nu \geq 2$,*

$$\|\chi_{C_\nu(B)} F(L) \chi_B\|_{p_0 \rightarrow 2} \leq C \mu(B(x_B, R^{-1/m}))^{-\sigma_{p_0}} (\sqrt[m]{R} 2^\nu r)^{-s} \|\delta_R F\|_{B^s} \quad (2.13)$$

for all balls $B := B(x_B, r) \subseteq X$, and all Borel functions F such that $\text{supp} F \subseteq [-R, R]$.

Proof. Let $G(\lambda) = (\delta_R F)(\lambda) e^\lambda$. In virtue of the Fourier inversion formula

$$F(L) = G(L/R) e^{-L/R} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{(i\tau-1)R^{-1}L} \hat{G}(\tau) d\tau.$$

This, in combination with estimate (2.8) yields that

$$\begin{aligned} \|\chi_{C_\nu(B)} F(L) \chi_B\|_{p_0 \rightarrow 2} &\leq \frac{1}{2\pi} \int_{\mathbb{R}} \|\chi_{C_\nu(B)} e^{(i\tau-1)R^{-1}L} \chi_B\|_{p_0 \rightarrow 2} |\hat{G}(\tau)| d\tau \\ &\leq C \mu(B(x_B, R^{-1/m}))^{-\sigma_{p_0}} \int_{\mathbb{R}} \exp\left(-c\left(\frac{\sqrt[m]{R} 2^\nu r}{1 + |\tau|}\right)^{\frac{m}{m-1}}\right) |\hat{G}(\tau)| d\tau \\ &\leq C \mu(B(x_B, R^{-1/m}))^{-\sigma_{p_0}} (\sqrt[m]{R} 2^\nu r)^{-s} \|G\|_{B^s}. \end{aligned}$$

Note that $\text{supp} \delta_R F \subseteq [-1, 1]$. Thus by taking a smooth cutoff function ψ such that $\text{supp} \psi \subset [-2, 2]$ and $\psi(\lambda) = 1$ for $\lambda \in [-1, 1]$, we have

$$G(\lambda) = (\delta_R F)(\lambda)e^\lambda = (\delta_R F)(\lambda)\psi(\lambda)e^\lambda.$$

Hence, by (2.12),

$$\|G\|_{B^s} \leq C\|\delta_R F\|_{B^s}\|\psi(\lambda)e^\lambda\|_{B^s} \leq C\|\delta_R F\|_{B^s}.$$

This ends the proof of Proposition 2.5. \square

Remark 2.6. In [21, Lemma 2.5], the author used some techniques introduced by Blunck [5] to show that for any $\nu \geq 2$, e^{-zL} satisfies the following off-diagonal estimate:

$$\|\chi_{C_\nu(B)}e^{-zL}\chi_B\|_{p_0 \rightarrow 2} \leq C \frac{1}{\mu(B(x_B, (\operatorname{Re} z))^{\frac{1}{m}-1}|z|)^{\sigma_{p_0}}} \left(\frac{|z|}{\operatorname{Re} z}\right)^{\sigma_{p_0} n} 2^{\nu n} \exp\left(-c\left(\frac{2^\nu r}{(\operatorname{Re} z)^{\frac{1}{m}-1}|z|}\right)^{\frac{m}{m-1}}\right).$$

It follows that for any $\nu \geq 2$, $R > 0$, $\tau \in \mathbb{R}$,

$$\begin{aligned} & \|\chi_{C_\nu(B)}e^{-(1-i\tau)R^{-1}L}\chi_B\|_{p_0 \rightarrow 2} \\ & \leq C \frac{1}{\mu(B(x_B, R^{-1/m}\sqrt{1+\tau^2}))^{\sigma_{p_0}}} (1+\tau^2)^{\frac{\sigma_{p_0} n}{2}} 2^{\nu n} \exp\left(-c\left(\frac{2^\nu r}{\sqrt{1+\tau^2}}\right)^{\frac{m}{m-1}}\right), \end{aligned} \quad (2.14)$$

for all $B = B(x_B, r) \subseteq X$. In our Lemma 2.4, we made an important improvement in obtaining the upper bound on the right hand side of (2.14) without the factor “ $2^{\nu n}$ ”, which plays a key role in obtaining the sharp growth $(1+|t|)^{p_0\sigma_{p_0}n}$ in the proof of Theorem 1.1. Also, in the setting of homogeneous space, it seems hard to remove $\sqrt{1+\tau^2}$ directly from the volume term $\mu(B(x_B, R^{-1/m}\sqrt{1+\tau^2}))^{\sigma_{p_0}}$ appeared on the right hand side of (2.14) by the doubling condition (1.2). Instead, our Lemma 2.4 provide a slight different but much more subtle upper bound such that the factor $\sqrt{1+\tau^2}$ doesn't appear on the volume term, which is helpful to obtain off-diagonal estimates (2.13) of compactly supported spectral multipliers.

3. Proof of Theorem 1.1

Fix $f \in L^{p_0}(X)$. For $\alpha > \mu(X)^{-p_0}\|f\|_{p_0}^{p_0}$, we apply the Calderón-Zygmund decomposition at height α to $|f|$. Then there exist constants C and K so that

- (i) $f = g + b = g + \sum_j b_j$;
- (ii) $\|g\|_{p_0} \leq C\|f\|_{p_0}$, $\|g\|_\infty \leq C\alpha$;
- (iii) b_j is supported in B_j and $\#\{j : x \in 2B_j\} \leq K$ for all $x \in X$;
- (iv) $\int_X |b_j|^{p_0} d\mu \leq C\alpha^{p_0}\mu(B_j)$ and $\sum_j \mu(B_j) \leq C\alpha^{-p_0}\|f\|_{p_0}^{p_0}$.

Let r_{B_j} be the radius of B_j and let

$$J_k = \{j : 2^k \leq r_{B_j} < 2^{k+1}\}, \text{ for } k \in \mathbb{Z}.$$

We decompose the bad function $b(x)$ as follows

$$b(x) = \sum_{k \leq k_0} \sum_{j \in J_k} b_j(x) + \sum_{k > k_0} \sum_{j \in J_k} b_j(x) =: h_1(x) + h_2(x),$$

where k_0 is an integer such that $2^{k_0} \leq (1+|t|)^{\frac{p_0}{2}} < 2^{k_0+1}$. Then it is enough to show that there exists a constant $C > 0$ independent of α and t such that

$$\mu\left(\left\{x : |e^{itL}(I+L)^{-\sigma_{p_0}n}g(x)| > \alpha\right\}\right) \leq C\alpha^{-p_0}(1+|t|)^{p_0\sigma_{p_0}n}\|f\|_{p_0}^{p_0} \quad (3.1)$$

and such that for $i = 1, 2$,

$$\mu\left(\left\{x : |e^{itL}(I+L)^{-\sigma_{p_0}n}h_i(x)| > \alpha\right\}\right) \leq C\alpha^{-p_0}(1+|t|)^{p_0\sigma_{p_0}n}\|f\|_{p_0}^{p_0}. \quad (3.2)$$

By the property (ii) and spectral theory,

$$\mu\left(\left\{x : |e^{itL}(I+L)^{-\sigma_{p_0}n}g(x)| > \alpha\right\}\right) \leq \alpha^{-2}\|e^{itL}(I+L)^{-\sigma_{p_0}n}g\|_2^2 \leq \alpha^{-2}\|g\|_2^2 \leq C\alpha^{-p_0}\|f\|_{p_0}^{p_0},$$

which proves (3.1).

Proof of (3.2) for $i = 1$.

Since the Schrödinger group e^{-itL} is bounded on $L^2(X)$, we have

$$\begin{aligned} \mu\left(\left\{x : |e^{itL}(I+L)^{-\sigma_{p_0}n}h_1(x)| > \alpha\right\}\right) &\leq \alpha^{-2}\|e^{itL}(I+L)^{-\sigma_{p_0}n}h_1\|_2^2 \\ &\leq \alpha^{-2}\|(I+L)^{-\sigma_{p_0}n}h_1\|_2^2 \\ &\leq \alpha^{-2}\left\|\sum_{k \leq k_0} \sum_{j \in J_k} (I+L)^{-\sigma_{p_0}n} (I - e^{-r_{B_j}^m L})^M b_j(x)\right\|_2^2 \\ &\quad + \alpha^{-2}\left\|\sum_{k \leq k_0} \sum_{j \in J_k} (I+L)^{-\sigma_{p_0}n} [I - (I - e^{-r_{B_j}^m L})^M] b_j(x)\right\|_2^2 \\ &=: I + II. \end{aligned}$$

Now we borrow the argument from [2] to estimate these two parts. We first estimate the term I by duality:

$$\begin{aligned} I &= \alpha^{-2} \sup_{\|u\|_2=1} \left(\int_X u(x) \sum_{k \leq k_0} \sum_{j \in J_k} (I+L)^{-\sigma_{p_0}n} (I - e^{-r_{B_j}^m L})^M b_j(x) d\mu(x) \right)^2 \\ &\leq \alpha^{-2} \sup_{\|u\|_2=1} \left(\sum_{k \leq k_0} \sum_{j \in J_k} \sum_{\nu=1}^{\infty} A_{\nu j} \right)^2, \end{aligned} \quad (3.3)$$

where $A_{\nu j} := \int_{C_\nu(B_j)} |(I+L)^{-\sigma_{p_0}n} (I - e^{-r_{B_j}^m L})^M b_j(x)| |u(x)| d\mu(x)$. By Proposition 2.2 and the fact that $r_{B_j} \leq 2^{k_0+1} \leq 2(1+|t|)^{\frac{p_0}{2}}$, for any $\nu \geq 1$,

$$\begin{aligned} \|(I+L)^{-\sigma_{p_0}n} (I - e^{-r_{B_j}^m L})^M b_j\|_{L^2(C_\nu(B_j))} &\leq \|\chi_{C_\nu(B_j)} (I+L)^{-\sigma_{p_0}n} (I - e^{-r_{B_j}^m L})^M \chi_{B_j}\|_{p_0 \rightarrow 2} \|b_j\|_{p_0} \\ &\leq C(1+|t|)^{\frac{p_0\sigma_{p_0}n}{2}} 2^{-\nu N} \alpha \mu(B_j)^{\frac{1}{2}}. \end{aligned} \quad (3.4)$$

Also for any $y \in B_j$ and any $\nu \geq 1$,

$$\|u\|_{L^2(C_\nu(B_j))} \leq \|u\|_{L^2((\nu+1)B_j)} \leq \mu((\nu+1)B_j)^{\frac{1}{2}} \mathcal{M}(|u|^2)(y)^{\frac{1}{2}} \leq C2^{\frac{\nu n}{2}} \mu(B_j)^{\frac{1}{2}} \mathcal{M}(|u|^2)(y)^{\frac{1}{2}}, \quad (3.5)$$

where in the last inequality we used the doubling condition (1.2). We apply Hölder's inequality, one obtains

$$\begin{aligned} A_{\nu j} &\leq \|(I+L)^{-\sigma_{p_0}n} (I - e^{-r_{B_j}^m L})^M b_j\|_{L^2(C_\nu(B_j))} \|u\|_{L^2(C_\nu(B_j))} \\ &\leq C(1+|t|)^{\frac{p_0\sigma_{p_0}n}{2}} 2^{-(N-\frac{n}{2})\nu} \alpha \mu(B_j) \mathcal{M}(|u|^2)(y)^{\frac{1}{2}}. \end{aligned}$$

Averaging over B_j yields

$$A_{\nu j} \leq C(1 + |t|)^{\frac{p_0 \sigma_{p_0} n}{2}} 2^{-(N - \frac{n}{2})\nu} \alpha \int_{B_j} \mathcal{M}(|u|^2)(y)^{\frac{1}{2}} d\mu(y).$$

Choosing N sufficiently large and then Summing over $\nu \geq 1$ and j , we have

$$\begin{aligned} I &\leq C(1 + |t|)^{p_0 \sigma_{p_0} n} \sup_{\|u\|_2=1} \left(\int_{\bigcup_j B_j} \mathcal{M}(|u|^2)(y)^{\frac{1}{2}} d\mu(y) \right)^2 \\ &\leq C(1 + |t|)^{p_0 \sigma_{p_0} n} \sup_{\|u\|_2=1} \mu\left(\bigcup_j B_j\right) \|u^2\|_1 \\ &\leq C(1 + |t|)^{p_0 \sigma_{p_0} n} \alpha^{-p_0} \|f\|_{p_0}^{p_0}, \end{aligned}$$

where in the next to last inequality, we use Kolmogorov's lemma and the weak type (1,1) of the Hardy-Littlewood maximal function, and in the last one we apply the property (iv) in the Calderón-Zygmund decomposition.

Next, we estimate the second part II , by spectral theorem,

$$\begin{aligned} II &= \alpha^{-2} \left\| \sum_{k \leq k_0} \sum_{j \in J_k} (I + L)^{-\sigma_{p_0} n} [I - (I - e^{-r_{B_j}^m L})^M] b_j(x) \right\|_2^2 \\ &\leq \alpha^{-2} \left\| \sum_{k \leq k_0} \sum_{j \in J_k} [I - (I - e^{-r_{B_j}^m L})^M] b_j(x) \right\|_2^2 \end{aligned}$$

Observe that $[I - (I - e^{-r_{B_j}^m L})^M]$ is a finite combination of the terms $e^{-jr_{B_j}^m L}$, $j = 1, \dots, M$ and that, by the doubling condition (1.2) and Lemma 2.1, semigroup $e^{-jr_{B_j}^m L}$ satisfies the following estimate

$$\|\chi_{C_\nu(B_j)} e^{-jr_{B_j}^m L} \chi_{B_j}\|_{p_0 \rightarrow 2} \leq C 2^{-\nu N} \mu(B_j)^{-\sigma_{p_0}}.$$

We can easily apply the same duality argument as estimating the term I to show that

$$II \leq C \alpha^{-p_0} \|f\|_{p_0}^{p_0}.$$

Combining the estimates for I and II , we obtain (3.2) for $i = 1$, i.e.,

$$\mu\left(\left\{x : |e^{itL}(I + L)^{-\sigma_{p_0} n} h_1(x)| > \alpha\right\}\right) \leq C \alpha^{-p_0} (1 + |t|)^{p_0 \sigma_{p_0} n} \|f\|_{p_0}^{p_0}.$$

Proof of (3.2) for $i = 2$.

Set $\Omega_t := \bigcup_j 2(1 + |t|)^{p_0 \sigma_{p_0}} B_j$. By (iv) in the Calderón-Zygmund decomposition,

$$\begin{aligned} \mu\left(\left\{x \in \Omega_t : |e^{itL}(I + L)^{-\sigma_{p_0} n} h_2(x)| > \alpha\right\}\right) &\leq C \mu\left(\bigcup_j 2(1 + |t|)^{p_0 \sigma_{p_0}} B_j\right) \\ &\leq C(1 + |t|)^{p_0 \sigma_{p_0} n} \sum_j \mu(B_j) \\ &\leq C \alpha^{-p_0} (1 + |t|)^{p_0 \sigma_{p_0} n} \|f\|_{p_0}^{p_0}. \end{aligned}$$

Next we show that

$$\mu\left(\left\{x \in \Omega_t^c : |e^{itL}(I + L)^{-\sigma_{p_0} n} h_2(x)| > \alpha\right\}\right) \leq C \alpha^{-p_0} (1 + |t|)^{p_0 \sigma_{p_0} n} \|f\|_{p_0}^{p_0}.$$

Note that for every $j \in J_k$, $k > k_0$, the function b_j is supported in B_j , and the radius of the ball B_j is equivalent to 2^k . We decompose $(I + L)^{-\sigma_{p_0}n} e^{itL} b_j$ into two parts:

$$\begin{aligned} e^{itL}(I + L)^{-\sigma_{p_0}n} b_j &= e^{itL}(I + L)^{-\sigma_{p_0}n} [I - (I - e^{-2^{mk}L})^M] b_j \\ &\quad + e^{itL}(I + L)^{-\sigma_{p_0}n} (I - e^{-2^{mk}L})^M b_j, \end{aligned} \quad (3.6)$$

where M is a fixed parameter chosen to be bigger than $\frac{n}{2m}$.

Consider the term $e^{itL}(I + L)^{-\sigma_{p_0}n} [I - (I - e^{-2^{mk}L})^M] b_j$. By the spectral theorem,

$$\begin{aligned} &\mu\left(\left\{x \in \Omega_t^c : \left| \sum_{k>k_0} \sum_{j \in J_k} e^{itL}(I + L)^{-\sigma_{p_0}n} [I - (I - e^{-2^{mk}L})^M] b_j(x) \right| > \alpha \right\}\right) \\ &\leq \alpha^{-2} \left\| \sum_{k>k_0} \sum_{j \in J_k} e^{itL}(I + L)^{-\sigma_{p_0}n} [I - (I - e^{-2^{mk}L})^M] b_j(x) \right\|_2^2 \\ &\leq \alpha^{-2} \left\| \sum_{k>k_0} \sum_{j \in J_k} [I - (I - e^{-2^{mk}L})^M] b_j(x) \right\|_2^2. \end{aligned}$$

Then we follow the similar procedure as estimating the term II to show that

$$\mu\left(\left\{x \in \Omega_t^c : \left| \sum_{k>k_0} \sum_{j \in J_k} e^{itL}(I + L)^{-\sigma_{p_0}n} [I - (I - e^{-2^{mk}L})^M] b_j(x) \right| > \alpha \right\}\right) \leq C \alpha^{-p_0} \|f\|_{p_0}^{p_0}.$$

For the term $e^{itL}(I + L)^{-\sigma_{p_0}n} (I - e^{-2^{mk}L})^M b_j$ in (3.6), we let φ_1 be a smooth function such that $\text{supp } \varphi_1 \subset [1/2, \infty]$ and $\varphi_1(\lambda) = 1$ on $[1, \infty]$. Also, let $\varphi_0(\lambda) = 1 - \varphi_1(\lambda)$. We further decompose this part as follows

$$e^{itL}(I + L)^{-\sigma_{p_0}n} (I - e^{-2^{mk}L})^M b_j = e^{itL} F_k(L) b_j + e^{itL} G_k(L) b_j,$$

where

$$F_k(L) = (I + L)^{-\sigma_{p_0}n} (I - e^{-2^{mk}L})^M \varphi_0(2^{-m(k-k_0)/(m-1)} L)$$

and

$$G_k(L) = (I + L)^{-\sigma_{p_0}n} (I - e^{-2^{mk}L})^M \varphi_1(2^{-m(k-k_0)/(m-1)} L).$$

Hence

$$\begin{aligned} &\mu\left(\left\{x \in \Omega_t^c : \left| \sum_{k>k_0} \sum_{j \in J_k} e^{itL}(I + L)^{-\sigma_{p_0}n} (I - e^{-2^{mk}L})^M b_j(x) \right| > \alpha \right\}\right) \\ &\leq \mu\left(\left\{x \in \Omega_t^c : \left| \sum_{k>k_0} \sum_{j \in J_k} e^{itL} F_k(L) b_j(x) \right| > \frac{\alpha}{2} \right\}\right) \\ &\quad + \mu\left(\left\{x \in \Omega_t^c : \left| \sum_{k>k_0} \sum_{j \in J_k} e^{itL} G_k(L) b_j(x) \right| > \frac{\alpha}{2} \right\}\right) =: III + IV. \end{aligned}$$

The estimate of the term III is delicate. We define $\nu_0 \in \mathbb{N}$ such that $2^{\nu_0} \leq 2(1 + |t|)^{p_0 \sigma_{p_0}} < 2^{\nu_0+1}$. It is easy to see that ν_0 is well-defined. Next we show the following lemma, which plays a crucial role in estimating the term III .

Lemma 3.1. *With the notation above, then for any $\frac{n}{2} < s < mM$, there exists a positive constant C independent of $k > k_0$, $j \in J_k$ and $\nu \geq \nu_0$, such that*

$$\|\chi_{C_\nu(B_j)} e^{itL} F_k(L) \chi_{B_j} f\|_2 \leq C 2^{-\nu s} \mu(B_j)^{-\sigma_{p_0}} (1 + |t|)^{p_0 \sigma_{p_0} (s + \frac{n}{2})} \|f\|_{p_0}. \quad (3.7)$$

Proof. Let ϕ be a non-negative C_c^∞ function on \mathbb{R} such that $\text{supp} \phi \subseteq (1/4, 1)$ and

$$\sum_{\ell \in \mathbb{Z}} \phi(2^{-\ell} \lambda) = 1, \quad \forall \lambda > 0,$$

and let $\phi_\ell(\lambda)$ denote the function $\phi(2^{-\ell} \lambda)$.

By the spectral theory, one writes

$$e^{itL} F_k(L) = \sum_{\ell=-\infty}^{m(k-k_0)/(m-1)} e^{itL} (I + L)^{-\sigma_{p_0} n} (I - e^{-2^{mk} L})^M \varphi_0(2^{-m(k-k_0)/(m-1)} L) \phi(2^{-\ell} L).$$

Set $F_{k,\ell}(\lambda) := e^{it\lambda} (1 + \lambda)^{-\sigma_{p_0} n} (I - e^{-2^{mk} \lambda})^M \varphi_0(2^{-m(k-k_0)/(m-1)} \lambda) \phi(2^{-\ell} \lambda)$, then we apply Minkowski's inequality, estimate (2.13) and the doubling condition (1.2) to get that

$$\begin{aligned} \|\chi_{C_\nu(B_j)} e^{itL} F_k(L) \chi_{B_j}\|_{p_0 \rightarrow 2} &\leq \sum_{\ell=-\infty}^{m(k-k_0)/(m-1)} \|\chi_{C_\nu(B_j)} F_{k,\ell}(L) \chi_{B_j}\|_{p_0 \rightarrow 2} \\ &\leq C \sum_{\ell=-\infty}^{m(k-k_0)/(m-1)} \mu(B(x_{B_j}, 2^{-\ell/m}))^{-\sigma_{p_0}} (2^{\ell/m} 2^\nu r_{B_j})^{-s} \|\delta_{2^\ell} F_{k,\ell}\|_{B^s} \\ &\leq C 2^{-\nu s} \mu(B_j)^{-\sigma_{p_0}} \sum_{\ell=-\infty}^{m(k-k_0)/(m-1)} (1 + 2^{\ell/m} r_{B_j})^{\sigma_{p_0} n} (2^{\ell/m} r_{B_j})^{-s} \|\delta_{2^\ell} F_{k,\ell}\|_{B^s}. \end{aligned} \quad (3.8)$$

To go on, we claim that

$$\|\delta_{2^\ell} F_{k,\ell}\|_{B^s} \leq C \min\{1, 2^{(\ell+mk)M}\} \min\{1, 2^{-\sigma_{p_0} n \ell}\} \max\{1, (2^\ell(1 + |t|))^s\}. \quad (3.9)$$

Let us show the claim (3.9). Now we let $\eta \in C_c^\infty(\mathbb{R})$ with $\text{supp} \eta \subset [1/8, 2]$ and $\eta(\lambda) = 1$ for $\lambda \in [1/4, 1]$. One has

$$\begin{aligned} &\|\delta_{2^\ell} F_{k,\ell}(\lambda)\|_{B^s} \\ &= \|e^{it2^\ell \lambda} (1 + 2^\ell \lambda)^{-\sigma_{p_0} n} (1 - e^{-2^{mk+\ell} \lambda})^M \varphi_0(2^{-m(k-k_0)/(m-1)} 2^\ell \lambda) \phi(\lambda)\|_{B^s} \\ &\leq \|\eta(\lambda) \varphi_0(2^{-m(k-k_0)/(m-1)} 2^\ell \lambda)\|_{B^s} \|\eta(\lambda) (1 - e^{-2^{mk+\ell} \lambda})^M\|_{B^s} \|\phi(\lambda) e^{it2^\ell \lambda} (1 + 2^\ell \lambda)^{-\sigma_{p_0} n}\|_{B^s} \\ &\leq C \|\eta(\lambda) \varphi_0(2^{-m(k-k_0)/(m-1)} 2^\ell \lambda)\|_{C^{s+2}} \|\eta(\lambda) (1 - e^{-2^{mk+\ell} \lambda})^M\|_{C^{s+2}} \|\phi(\lambda) e^{it2^\ell \lambda} (1 + 2^\ell \lambda)^{-\sigma_{p_0} n}\|_{B^s}. \end{aligned}$$

Note that $\ell \leq m(k - k_0)/(m - 1)$ and $\text{supp} \eta \subset [1/8, 2]$,

$$\|\eta(\lambda) \varphi_0(2^{-m(k-k_0)/(m-1)} 2^\ell \lambda)\|_{C^{s+2}} \leq C$$

and

$$\|\eta(\lambda)(1 - e^{-2^{mk+\ell}\lambda})^M\|_{C^{s+2}} \leq C \min\{1, 2^{(\ell+mk)M}\}$$

with C independent of k and ℓ .

As for the third term $\|\phi(\lambda)e^{it2^\ell\lambda}(1 + 2^\ell\lambda)^{-\sigma_{p_0}n}\|_{B^s}$, we note that the Fourier transform $\mathcal{F}(\phi e^{it2^\ell\cdot}(1 + 2^\ell\cdot)^{-\sigma_{p_0}n})(\tau)$ of $\phi(\lambda)e^{it2^\ell\lambda}(1 + 2^\ell\lambda)^{-\sigma_{p_0}n}$ is given by

$$\mathcal{F}(\phi e^{it2^\ell\cdot}(1 + 2^\ell\cdot)^{-\sigma_{p_0}n})(\tau) = \int_{\mathbb{R}} \phi(\lambda) \frac{e^{i(2^\ell t - \tau)\lambda}}{(1 + 2^\ell\lambda)^{\sigma_{p_0}n}} d\lambda.$$

Integration by parts gives for every $N \in \mathbb{N}$,

$$|\mathcal{F}(\phi e^{it2^\ell\cdot}(1 + 2^\ell\cdot)^{-\sigma_{p_0}n})(\tau)| \leq C(1 + 2^\ell)^{-\sigma_{p_0}n}(1 + |2^\ell t - \tau|)^{-N},$$

which yields that,

$$\begin{aligned} \|\phi(\lambda)e^{it2^\ell\lambda}(1 + 2^\ell\lambda)^{-\sigma_{p_0}n}\|_{B^s} &\leq C \min\{1, 2^{-\sigma_{p_0}n\ell}\} \int_{\mathbb{R}} (1 + |2^\ell t - \tau|)^{-N} (1 + |\tau|)^s d\tau \\ &\leq C \min\{1, 2^{-\sigma_{p_0}n\ell}\} (1 + 2^\ell t)^s \\ &\leq C \min\{1, 2^{-\sigma_{p_0}n\ell}\} \max\{1, (2^\ell(1 + |t|))^s\}. \end{aligned}$$

Hence, (3.9) holds.

It follows from (3.8) and (3.9) that

$$\begin{aligned} &\|\chi_{C_\nu(B_j)} e^{itL} F_k(L) \chi_{B_j}\|_{p_0 \rightarrow 2} \\ &\leq C 2^{-\nu s} \mu(B_j)^{-\sigma_{p_0}} \\ &\quad \times \sum_{\ell=-\infty}^{m(k-k_0)/(m-1)} (1 + 2^{\ell/m} r_{B_j})^{\sigma_{p_0}n} (2^{\ell/m} r_{B_j})^{-s} \min\{1, 2^{(\ell+mk)M}\} \min\{1, 2^{-\sigma_{p_0}n\ell}\} \max\{1, (2^\ell(1 + |t|))^s\} \\ &\leq C 2^{-\nu s} \mu(B_j)^{-\sigma_{p_0}} r_{B_j}^{\sigma_{p_0}n-s} \sum_{\ell=1}^{m(k-k_0)/(m-1)} 2^{(s-\sigma_{p_0}n)(1-\frac{1}{m})\ell} (1 + |t|)^s + C_s 2^{-\nu s} \mu(B_j)^{-\sigma_{p_0}} r_{B_j}^{\sigma_{p_0}n-s} \\ &\quad \times \sum_{\ell=-2k_0/p_0}^0 2^{(s+\frac{\sigma_{p_0}n-s}{m})\ell} (1 + |t|)^s \\ &\quad + C 2^{-\nu s} \mu(B_j)^{-\sigma_{p_0}} r_{B_j}^{\sigma_{p_0}n-s} \sum_{\ell=-mk}^{-2k_0/p_0} 2^{\frac{\ell}{m}(\sigma_{p_0}n-s)} + C_s 2^{-\nu s} \mu(B_j)^{-\sigma_{p_0}} \sum_{\ell=-\infty}^{-mk} 2^{(\ell+mk)(M-\frac{s}{m})} \\ &\leq C 2^{-\nu s} \mu(B_j)^{-\sigma_{p_0}} (1 + |t|)^{p_0\sigma_{p_0}(s+\frac{n}{2})}, \end{aligned}$$

where in the last inequality we used the fact that $\frac{n}{2} < s < mM$.

This finishes the proof of Lemma 3.1. \square

Back to the estimate of the term *III*, we now apply Cauchy-Schwartz inequality to obtain

$$III = \mu\left(\left\{x \in \Omega_t^c : \left| \sum_{k>k_0} \sum_{j \in J_k} e^{itL} F_k(L) b_j(x) \right| > \alpha \right\}\right)$$

$$\begin{aligned}
&\leq \alpha^{-1} \sum_{k>k_0} \sum_{j \in J_k} \sum_{\nu=\nu_0}^{\infty} \int_{C_\nu(B_j)} |e^{itL} F_k(L) b_j(x)| d\mu(x) \\
&\leq \alpha^{-1} \sum_{k>k_0} \sum_{j \in J_k} \sum_{\nu=\nu_0}^{\infty} \mu(C_\nu(B_j))^{\frac{1}{2}} \left(\int_{C_\nu(B_j)} |e^{itL} F_k(L) b_j(x)|^2 d\mu(x) \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.10}$$

This, in combination with the doubling condition (1.2) and Lemma 3.1, we conclude that

$$\begin{aligned}
\text{RHS of (3.10)} &\leq C\alpha^{-1} \sum_{k>k_0} \sum_{j \in J_k} \sum_{\nu=\nu_0}^{\infty} 2^{\frac{\nu n}{2}} \mu(B_j)^{\frac{1}{2}} \|\chi_{C_\nu(B_j)} e^{itL} F_k(L) \chi_{B_j}\|_{p_0 \rightarrow 2} \|b_j\|_{p_0} \\
&\leq C \sum_{k>k_0} \sum_{j \in J_k} \sum_{\nu=\nu_0}^{\infty} \mu(B_j) 2^{\nu(\frac{n}{2}-s)} (1+|t|)^{p_0 \sigma_{p_0}(s+\frac{n}{2})} \\
&\leq C(1+|t|)^{p_0 \sigma_{p_0} n} \sum_j \mu(B_j) \\
&\leq C(1+|t|)^{p_0 \sigma_{p_0} n} \alpha^{-p_0} \|f\|_{p_0}^{p_0}.
\end{aligned}$$

Concerning the term IV , since the Schrödinger group e^{-itL} is bounded on $L^2(X)$, we have

$$\begin{aligned}
IV &\leq C\alpha^{-2} \left\| \sum_{k>k_0} \sum_{j \in J_k} e^{itL} G_k(L) b_j \right\|_2^2 \\
&\leq C\alpha^{-2} \left\| \sum_{k>k_0} \sum_{j \in J_k} G_k(L) b_j \right\|_2^2 \\
&\leq C\alpha^{-2} \left\| \sum_{k>k_0} \sum_{j \in J_k} \chi_{2B_j} G_k(L) b_j \right\|_2^2 + C\alpha^{-2} \left\| \sum_{k>k_0} \sum_{j \in J_k} \chi_{(2B_j)^c} G_k(L) b_j \right\|_2^2 =: IV_1 + IV_2.
\end{aligned}$$

To handle the term IV_1 , we first note that

$$\begin{aligned}
\|G_k(L) b_j\|_2 &\leq \|G_k(L) (I + 2^{-m(k-k_0)/(m-1)} L)^{\sigma_{p_0} n}\|_{2 \rightarrow 2} \|(I + 2^{-m(k-k_0)/(m-1)} L)^{-\sigma_{p_0} n} b_j\|_2 \\
&\leq \|(1+\lambda)^{-\sigma_{p_0} n} (1 - e^{-2^{mk}\lambda})^M \varphi_1(2^{-m(k-k_0)/(m-1)} \lambda) (1 + 2^{-m(k-k_0)/(m-1)} \lambda)^{\sigma_{p_0} n}\|_\infty \\
&\quad \times \|(I + 2^{-m(k-k_0)/(m-1)} L)^{-\sigma_{p_0} n} b_j\|_2 \\
&\leq C 2^{-\frac{m(k-k_0)}{m-1} \sigma_{p_0} n} \|(I + 2^{-m(k-k_0)/(m-1)} L)^{-\sigma_{p_0} n} b_j\|_2.
\end{aligned}$$

Since the $2B_j$'s have bounded overlaps, we apply Minkowski's inequality to obtain

$$\begin{aligned}
&\left\| \sum_{k>k_0} \sum_{j \in J_k} \chi_{2B_j} G_k(L) b_j \right\|_2^2 \\
&\leq C \sum_{k>k_0} \sum_{j \in J_k} \int_X |G_k(L) b_j(x)|^2 d\mu(x) \\
&\leq C \sum_{k>k_0} \sum_{j \in J_k} 2^{-\frac{2m(k-k_0)}{m-1} \sigma_{p_0} n} \|(I + 2^{-m(k-k_0)/(m-1)} L)^{-\sigma_{p_0} n} b_j\|_2^2 \\
&\leq C \sum_{k>k_0} \sum_{j \in J_k} 2^{-\frac{2m(k-k_0)}{m-1} \sigma_{p_0} n} \|\chi_{2B_j} (I + 2^{-m(k-k_0)/(m-1)} L)^{-\sigma_{p_0} n} \chi_{2B_j}\|_{p_0 \rightarrow 2}^2 \|b_j\|_{p_0}^2
\end{aligned}$$

$$+ C \sum_{k > k_0} \sum_{j \in J_k} \sum_{\nu=2}^{\infty} 2^{-\frac{2m(k-k_0)}{m-1} \sigma_{p_0} n} \|\chi_{C_\nu(B_j)} (I + 2^{-m(k-k_0)/(m-1)} L)^{-\sigma_{p_0} n} \chi_{B_j}\|_{p_0 \rightarrow 2}^2 \|b_j\|_{p_0}^2. \quad (3.11)$$

By applying the representation formula

$$(I + 2^{-m(k-k_0)/(m-1)} L)^{-\sigma_{p_0} n} = \frac{1}{\Gamma(\sigma_{p_0} n)} \int_0^\infty e^{-\lambda 2^{-m(k-k_0)/(m-1)} L} e^{-\lambda} \lambda^{\sigma_{p_0} n - 1} d\lambda$$

and Lemma 2.1, we conclude that

$$\begin{aligned} \|\chi_{2B_j} (I + 2^{-m(k-k_0)/(m-1)} L)^{-\sigma_{p_0} n} \chi_{2B_j}\|_{p_0 \rightarrow 2} &\leq C \int_0^\infty \|\chi_{2B_j} e^{-\lambda 2^{-m(k-k_0)/(m-1)} L} \chi_{2B_j}\|_{p_0 \rightarrow 2} e^{-\lambda} \lambda^{\sigma_{p_0} n - 1} d\lambda \\ &\leq C \int_0^\infty \mu(B(x_{B_j}, \lambda^{\frac{1}{m}} 2^{-\frac{k-k_0}{m-1}}))^{-\sigma_{p_0}} e^{-\lambda} \lambda^{\sigma_{p_0} n - 1} d\lambda \end{aligned}$$

and that for any $\nu \geq 2$,

$$\begin{aligned} \|\chi_{C_\nu(B_j)} (I + 2^{-m(k-k_0)/(m-1)} L)^{-\sigma_{p_0} n} \chi_{B_j}\|_{p_0 \rightarrow 2} &\leq C \int_0^\infty \|\chi_{C_\nu(B_j)} e^{-\lambda 2^{-m(k-k_0)/(m-1)} L} \chi_{B_j}\|_{p_0 \rightarrow 2} e^{-\lambda} \lambda^{\sigma_{p_0} n - 1} d\lambda \\ &\leq C \int_0^\infty \mu(B(x_{B_j}, \lambda^{\frac{1}{m}} 2^{-\frac{k-k_0}{m-1}}))^{-\sigma_{p_0}} \exp\left(-c \left(\frac{2^{\frac{k-k_0}{m-1}} 2^\nu r_{B_j}}{\lambda^{1/m}}\right)^{\frac{m}{m-1}}\right) e^{-\lambda} \lambda^{\sigma_{p_0} n - 1} d\lambda. \end{aligned}$$

The doubling condition (1.2) implies that

$$\begin{aligned} \frac{1}{\mu(B(x_{B_j}, \lambda^{\frac{1}{m}} 2^{-\frac{k-k_0}{m-1}}))} &= \frac{\mu(B(x_{B_j}, 2^{-\frac{k-k_0}{m-1}}))}{\mu(B(x_{B_j}, \lambda^{\frac{1}{m}} 2^{-\frac{k-k_0}{m-1}}))} \cdot \frac{\mu(B(x_{B_j}, 2^k))}{\mu(B(x_{B_j}, 2^{-\frac{k-k_0}{m-1}}))} \cdot \frac{1}{\mu(B(x_{B_j}, 2^k))} \\ &\leq C 2^{\frac{m(k-k_0)}{m-1} n} (1 + |t|)^{\frac{p_0 n}{2}} \left(1 + \frac{1}{\lambda^{1/m}}\right)^n \frac{1}{\mu(B_j)}. \end{aligned}$$

This indicates that

$$\|\chi_{2B_j} (I + 2^{-\frac{m(k-k_0)}{m-1}} L)^{-\sigma_{p_0} n} \chi_{2B_j}\|_{p_0 \rightarrow 2} \leq C 2^{\frac{m(k-k_0)}{m-1} \sigma_{p_0} n} (1 + |t|)^{\frac{p_0 \sigma_{p_0} n}{2}} \mu(B_j)^{-\sigma_{p_0}}, \quad (3.12)$$

and that

$$\begin{aligned} \|\chi_{C_\nu(B_j)} (I + 2^{-\frac{m(k-k_0)}{m-1}} L)^{-\sigma_{p_0} n} \chi_{B_j}\|_{p_0 \rightarrow 2} &\leq C 2^{\frac{m(k-k_0)}{m-1} \sigma_{p_0} n} (1 + |t|)^{\frac{p_0 \sigma_{p_0} n}{2}} \mu(B_j)^{-\sigma_{p_0}} \\ &\quad \times \int_0^\infty \exp\left(-c \left(\frac{2^{\frac{k-k_0}{m-1}} 2^\nu r_{B_j}}{\lambda^{1/m}}\right)^{\frac{m}{m-1}}\right) e^{-\lambda} \left(1 + \frac{1}{\lambda^{1/m}}\right)^{\sigma_{p_0} n} \lambda^{\sigma_{p_0} n - 1} d\lambda \\ &\leq C 2^{\frac{m(k-k_0)}{m-1} \sigma_{p_0} n} (1 + |t|)^{\frac{p_0 \sigma_{p_0} n}{2}} \mu(B_j)^{-\sigma_{p_0}} \left(2^{\frac{k-k_0}{m-1}} 2^\nu r_{B_j}\right)^{-N}. \end{aligned} \quad (3.13)$$

Observing that $2^{\frac{k-k_0}{m-1}} r_{B_j} \geq 1$ for $k > k_0$, we combine (3.11), (3.12) with (3.13) to conclude that

$$\begin{aligned}
 \left\| \sum_{k>k_0} \sum_{j \in J_k} \chi_{2B_j} G_k(L) b_j \right\|_2^2 &\leq C(1+|t|)^{p_0 \sigma_{p_0} n} \sum_{k>k_0} \sum_{j \in J_k} \mu(B_j)^{-2\sigma_{p_0}} \|b_j\|_{p_0}^2 \\
 &\quad + C(1+|t|)^{p_0 \sigma_{p_0} n} \sum_{k>k_0} \sum_{j \in J_k} \sum_{\nu=2}^{\infty} \mu(B_j)^{-2\sigma_{p_0}} \|b_j\|_{p_0}^2 \left(2^{\frac{k-k_0}{m-1}} 2^\nu r_{B_j} \right)^{-N} \\
 &\leq C(1+|t|)^{p_0 \sigma_{p_0} n} \sum_{k>k_0} \sum_{j \in J_k} \mu(B_j)^{-2\sigma_{p_0}} \|b_j\|_{p_0}^2 \\
 &\leq C(1+|t|)^{p_0 \sigma_{p_0} n} \sum_{k>k_0} \sum_{j \in J_k} \mu(B_j)^{-2\sigma_{p_0}} \|b_j\|_{p_0}^{2-p_0} \|b_j\|_{p_0}^{p_0} \\
 &\leq C(1+|t|)^{p_0 \sigma_{p_0} n} \alpha^{2-p_0} \sum_j \int_X |b_j(y)|^{p_0} d\mu(y) \\
 &\leq C(1+|t|)^{p_0 \sigma_{p_0} n} \alpha^{2-p_0} \|f\|_{p_0}^{p_0}.
 \end{aligned}$$

Finally, it remains to show that

$$\left\| \sum_{k>k_0} \sum_{j \in J_k} \chi_{(2B_j)^c} G_k(L) b_j \right\|_2^2 \leq C \alpha^{2-p_0} \|f\|_{p_0}^{p_0}. \quad (3.14)$$

To continue, we claim that for any $N > \sigma_{p_0} n$, there exists a positive constant C such that for any $\nu \geq 2$ and $k > k_0$, we have

$$\|\chi_{C_\nu(B_j)} G_k(L) \chi_{B_j}\|_{p_0 \rightarrow 2} \leq C 2^{-\nu N} \mu(B_j)^{-\sigma_{p_0}}. \quad (3.15)$$

To show (3.15), we apply (2.13) to obtain that

$$\begin{aligned}
 \|\chi_{C_\nu(B_j)} G_k(L) \chi_{B_j}\|_{p_0 \rightarrow 2} &\leq \sum_{\ell=-1}^{\infty} \|\chi_{C_\nu(B_j)} G_{k,\ell}(L) \chi_{B_j}\|_{p_0 \rightarrow 2} \\
 &\leq C \sum_{\ell=-1}^{\infty} \mu(B(x_{B_j}, 2^{-\ell/m}))^{-\sigma_{p_0}} (2^{\ell/m} 2^\nu r_{B_j})^{-N} \|\delta_{2^\ell} G_{k,\ell}\|_{B^N}, \quad (3.16)
 \end{aligned}$$

where $G_{k,\ell}(\lambda) := G_k(\lambda) \phi(2^{-\ell} \lambda)$ satisfies

$$\begin{aligned}
 \|\delta_{2^\ell} G_{k,\ell}\|_{B^N} &= \|(1+2^\ell \lambda)^{-\sigma_{p_0} n} (1-e^{-2^{m k + \ell} \lambda})^M \varphi_1(2^{-\frac{m(k-k_0)}{m-1}} 2^\ell \lambda) \phi(\lambda)\|_{B^N} \\
 &\leq \|\phi(\lambda) (1+2^\ell \lambda)^{-\sigma_{p_0} n}\|_{B^N} \|\eta(\lambda) (1-e^{-2^{m k + \ell} \lambda})^M\|_{B^N} \|\eta(\lambda) \varphi_1(2^{-\frac{m(k-k_0)}{m-1}} 2^\ell \lambda)\|_{B^N} \\
 &\leq C \|\phi(\lambda) (1+2^\ell \lambda)^{-\sigma_{p_0} n}\|_{C^{N+2}} \|\eta(\lambda) (1-e^{-2^{m k + \ell} \lambda})^M\|_{C^{N+2}} \|\eta(\lambda) \varphi_1(2^{-\frac{m(k-k_0)}{m-1}} 2^\ell \lambda)\|_{C^{N+2}} \\
 &\leq C 2^{-\ell \sigma_{p_0} n}.
 \end{aligned}$$

This, in combination with (3.16) and the doubling condition (1.2), yields

$$\begin{aligned}
 \|\chi_{C_\nu(B_j)} G_k(L) \chi_{B_j}\|_{p_0 \rightarrow 2} &\leq C \sum_{\ell=-1}^{\infty} \mu(B(x_{B_j}, 2^{-\ell/m}))^{-\sigma_{p_0}} (2^{\ell/m} 2^\nu r_{B_j})^{-N} 2^{-\ell \sigma_{p_0} n} \\
 &\leq C 2^{-\nu N} \mu(B_j)^{-\sigma_{p_0}} \sum_{\ell=-1}^{\infty} (2^{\ell/m} r_{B_j})^{\sigma_{p_0} n - N} 2^{-\ell \sigma_{p_0} n}
 \end{aligned}$$

$$\leq C2^{-\nu N}\mu(B_j)^{-\sigma_{p_0}},$$

where in the last inequality we used the fact that $r_{B_j} \geq 1$ when $k \geq k_0$. This ends the proof of estimate (3.15).

Now we follow the similar procedure as estimating the term I to show (3.14).

$$\begin{aligned} \left\| \sum_{k>k_0} \sum_{j \in J_k} \chi_{(2B_j)^c} G_k(L) b_j \right\|_2^2 &= \sup_{\|u\|_2=1} \left(\int_X u(x) \sum_{k \leq k_0} \sum_{j \in J_k} \chi_{(2B_j)^c} G_k(L) b_j(x) d\mu(x) \right)^2 \\ &\leq \sup_{\|u\|_2=1} \left(\sum_{k \leq k_0} \sum_{j \in J_k} \sum_{\nu=2}^{\infty} B_{\nu j} \right)^2, \end{aligned} \quad (3.17)$$

where $B_{\nu j} := \int_{C_\nu(B_j)} |G_k(L) b_j(x)| |u(x)| d\mu(x)$.

By (3.5) and (3.15), one obtains

$$B_{\nu j} \leq \|G_k(L) b_j\|_{L^2(C_\nu(B_j))} \|u\|_{L^2(C_\nu(B_j))} \leq C2^{-(N-\frac{n}{2})\nu} \alpha \mu(B_j) \mathcal{M}(|u|^2)(y)^{\frac{1}{2}}.$$

Averaging over B_j yields

$$B_{\nu j} \leq C2^{-(N-\frac{n}{2})\nu} \alpha \int_{B_j} \mathcal{M}(|u|^2)(y)^{\frac{1}{2}} d\mu(y).$$

Choosing N sufficiently large and then summing over $\nu \geq 1$ and j , we have

$$\begin{aligned} \left\| \sum_{k>k_0} \sum_{j \in J_k} \chi_{(2B_j)^c} G_k(L) b_j \right\|_2^2 &\leq C\alpha^2 \sup_{\|u\|_2=1} \left(\int_{\bigcup_j B_j} \mathcal{M}(|u|^2)(y)^{\frac{1}{2}} d\mu(y) \right)^2 \\ &\leq C\alpha^2 \sup_{\|u\|_2=1} \mu\left(\bigcup_j B_j\right) \|u^2\|_1 \\ &\leq C\alpha^{2-p_0} \|f\|_{p_0}^{p_0}. \end{aligned}$$

The proof of Theorem 1.1 is complete.

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