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# Properties of Isometric Mappings

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Some relations between isometry and linearity are examined. In particular, generalizations of the Mazur–Ulam theorem and open problems are discussed.
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## 1. INTRODUCTION

The main theme of this article is the relation between isometry and linearity, that is, the Mazur–Ulam theorem and its generalizations. For a discussion of several other aspects of isometries on Banach spaces see the article by Fleming and Jamison [19]. All vector spaces mentioned in this article are assumed to be real unless otherwise stated. Mazur and Ulam [31] proved the following result (see also Banach [5]).

THEOREM 1.1. *Let $U$ be an isometric transformation from a normed vector space $X$ onto a normed vector space $Y$ with $U(0) = 0$. Then $U$ is linear.*

Since continuity is implied by isometry it is only necessary to show that the isometric transformation $U$ is additive, and additivity follows easily providing one can prove that $U$ satisfies the functional equation

$$U\left(\frac{x_1 + x_2}{2}\right) = \frac{1}{2}U(x_1) + \frac{1}{2}U(x_2); \tag{1.1}$$

that is, $U$ maps the midpoint of the line segment joining $x_1$ and $x_2$ onto the midpoint of the line segment joining $U(x_1)$ and $U(x_2)$ for all $x_1$ and $x_2$



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in $X$. For the special case where both $X$ and $Y$ are strictly convex the proof of (1.1) is immediate. A normed vector space $E$ is called *strictly convex* (Clarkson [15]) if, for each pair $u, v$ of nonzero elements in $E$ such that $\|u + v\| = \|u\| + \|v\|$, it follows that $u = cv$ for some $c > 0$. When $X$ is strictly convex the only solution $m$ of the pair of equations

$$\|m - x_1\| = \|x_2 - m\| \quad \text{and} \quad \|m - x_1\| + \|x_2 - m\| = \|x_2 - x_1\| \tag{1.2}$$

is $m = \frac{1}{2}x_1 + \frac{1}{2}x_2$, with an analogous statement for $y_1 = U(x_1)$ and $y_2 = U(x_2)$ in $Y$.

Thus, for strictly convex spaces the equations (1.2) provide a metric characterization of the midpoint of the segment joining $x_1$ and $x_2$ in $X$ as well as that of the segment joining $U(x_1)$ and $U(x_2)$ in $Y$, so the linearity of $U$ is demonstrated. A well-known example of a space which is not strictly convex is $R^2$ with the norm $\|(a, b)\| = \max[|a|, |b|]$. If we take $x_1 = (0, 0)$ and $x_2 = (2, 0)$ in (1.2), we find that every point on the line segment joining the points $(1, -1)$ and $(1, 1)$ is a solution of (1.2).

In order to prove that $U$ is linear Mazur and Ulam needed to find a metric characterization of the midpoint of a pair of points that is valid for all normed spaces. They did this by constructing the following sequence of sets $H_m$, $m = 1, 2, 3, \ldots$, where $H_1$ is the set of solutions of (1.2); thus

$$H_1 = \left\{x \in X: \|x - x_1\| = \|x - x_2\| = \tfrac{1}{2}\|x_1 - x_2\|\right\}$$

and

$$H_m = \left\{x \in H_{m-1}: \|x - z\| \leq \tfrac{1}{2}\delta(H_{m-1}) \quad \text{for all } z \in H_{m-1}\right\},$$

$$m = 2, 3, \ldots,$$

where $\delta(H_{m-1})$ designates the diameter of the set $H_{m-1}$. The intersection of these sets $H_m$ is called the *metric center* of $x_1$ and $x_2$. Mazur and Ulam proved that it consists of a single point $\frac{1}{2}x_1 + \frac{1}{2}x_2$.

In Section 2 we often deal with metric vector spaces. For these we use the following terminology. A real vector space with a metric $d(\cdot, \cdot)$ satisfying $d(u + v, w + v) = d(u, w)$ for all $u, v, w$, and for which the operations of addition and scalar multiplication are jointly continuous will be called a *metric vector space*. A complete metric vector space will be called an *F-space*, following Banach ([5], p. 35). It is often convenient to introduce the functional $\|u\| = d(u, 0)$ and call it the *quasi-norm* for the space. It clearly has the properties

$$\begin{aligned}
&\text{(a)} \quad \|u\| = 0 \quad \text{if and only if } u = 0,\\
&\text{(b)} \quad \|u + v\| \leq \|u\| + \|v\|,\\
&\text{(c)} \quad \|-u\| = \|u\|,\\
&\text{(d)} \quad \beta_m \to \beta \quad \text{and} \quad \|u_m - u\| \to 0 \text{ imply } \|\beta_m u_m - \beta u\| \to 0
\end{aligned} \tag{1.3}$$

for $u, v, u_m$ in the space and $\beta, \beta_m$ real numbers.

We note that the quasi-norm fails to be a norm since it is not required to be positive homogeneous. With the above definition of an $F$-space, local convexity is not required. We will use some standard definitions concerning topological vector spaces, including those of a semi-norm and a bounded set. A topological vector space is called *locally bounded* if every neighborhood of the origin contains a bounded open set (see Hyers [22, 23]).

## 2. GENERALIZATIONS OF THE MAZUR–ULAM THEOREM

The question of whether the Mazur–Ulam theorem holds for all metric vector spaces still seems to be open. The finite dimensional case was dealt with by Charzynski [13]. He proved the theorem for a mapping $U$: $E \to H$ with $U(0) = 0$ which is an isometry between metric vector spaces $E$ and $H$ of the same finite dimension $n$. First he showed that it is sufficient to prove the theorem for the case $H = E$. Next, by a lengthy proof, he succeeded in defining a semi-norm $s(\cdot)$ on $E$ which satisfies the condition $s(U(w) - U(z)) = s(w - z)$ for all $w, z$ in $E$. In the special case $n = 1$ the semi-norm is actually a norm, and the linearity of $U$ follows from the Mazur–Ulam theorem. The rest of the proof is carried out by means of an induction on $n$.

Rolewicz [44, 45] considered the class of metric vector spaces which are locally bounded and whose quasi-norms are *concave*; that is, for each element $u$ of the space, the function $\varphi$: $R \to R$ defined by $\varphi(t) = \|tu\|$ is concave for positive $t$. When both $E$ and $H$ are $F$-spaces subject to these two conditons, Rolewicz proved that each surjective isometry $U$: $E \to H$ satisfying $U(0) = 0$ is linear. Using both local boundedness and the concavity condition he first proved the following:

LEMMA 2.1. *Let $E$ be a locally bounded $F$-space whose quasi-norm is concave, and put $m(r) = \sup\{\|u\|: \|2u\| \le r\}$. Then there exists $r_0 > 0$ such that $m(r) < r$ for $0 < r < r_0$.*

Rolewicz's proof now proceeds by a method analogous to that of Mazur–Ulam, but by first restricting points $u$ and $v$ of $E$ by the condition $\|u - v\| < r_0/2$ in characterizing $\frac{u}{2} + \frac{v}{2}$ as the metric center of $u$ and $v$ and later removing this restriction.

A different kind of generalization of the Mazur–Ulam theorem was given by Day [16, pp. 110, 111]. He adapted their proof to the following situation. Let $X$ and $Y$ be locally convex topological vector spaces and let $T$: $X \to Y$ carry a separating family $[p_s]$ of semi-norms to another such family $[p'_s]$ on $Y$ by the rule $p'_s(Tx - Ty) = p_s(x - y)$, where $T$ is surjective and $T(0) = 0$. Then $T$ is linear.

In each of the theorems quoted above on the linearity of an isometry (except Charzynski's), the hypothesis of surjectivity of the isometry was required. The necessity of this condition in general is shown by an example.

Figiel [18] cites the following. Let $R$ have the usual absolute value as norm and let $R^2$ have the norm $\|(y_1, y_2)\| = \max[|y_1|, |y_2|]$, with $T: R \to R^2$ given by $T(x) = (x, \sin x)$. Then $T$ is an isometry of $R$ into $R^2$ with $T(0) = 0$ which is not linear. Other examples are given by Baker [4], including that of a mapping $F: R^2 \to R^3$, where $R^2$ has the Euclidean norm while the norm for $R^3$ is $\|(x, y, z)\| = \max\{\sqrt{x^2 + y^2}, |z|\}$, such that $F$ is a nonlinear isometry and is homogeneous of degree one. In the same paper Baker proves that if $X$ and $Y$ are normed vector spaces and $Y$ is strictly convex, then every isometry $T: X \to Y$ with $T(0) = 0$ is linear. This provides another exception to the requirement of surjectivity in addition to Charzynski's theorem (see also Theorem 12, p. 145, of Aczél and Dhombres [1]).

Wobst [50] generalized Charzynski's theorem as follows.

THEOREM 2.2. *Let $E$ and $F$ be metric vector spaces with $E$ finite dimensional. Let $T: E \to F$ be a surjective isometry with $T(0) = 0$. Then $T$ is a linear mapping.*

He also proved that every isometry of a locally compact connected metric group with translation invariant metric into itself is surjective. In the same paper Wobst generalized Rolewicz's theorem cited above in the following way. Given a metric vector space $E$ with quasi-norm $\|\cdot\|$ and real numbers $r, r_0$ with $0 < r < r_0$, define

$$s_E(r, r_0) = \inf\left\{ \frac{\|2x\|}{\|x\|} : x \in E, r \le \|x\| < r_0 \right\},$$

and define $s_F(r, r_0)$ in the same way for another metric vector space $F$.

THEOREM 2.3 (Wobst [50, Theorem 4]). *Let $E$ and $F$ be metric vector spaces, and suppose that there exists an $r_0 > 0$ such that for all $r$ satisfying $0 < r < r_0$ we have*

$$s_E(r, r_0) > 1 \quad \text{and} \quad s_F(r, r_0) > 1. \tag{2.1}$$

*Then every isometry $T$ of $E$ onto $F$ with $T(0) = 0$ is linear.*

Wobst showed that if $E$ and $F$ are locally bounded and their quasi-norms are concave the conditions of Theorem 2.3 are satisfied and also that the conditions (2.1) imply local boundedness of $E$ and $F$. He gave an example of a metric space $E$ satisfying condition (2.1) pertaining to $E$, and yet

having a quasi-norm which is not concave, to show that Theorem 2.3 is a true generalization of the theorem of Rolewicz.

Can anything be said regarding linearity in the case of isometric embeddings of normed vector spaces in general (i.e., the nonsurjective case)? An answer has been provided by Figiel [18], who demonstrated the following:

THEOREM 2.4. *Given an isometric embedding* $\varphi$: $X \to Y$ *of a normed vector space* $X$ *into a normed vector space* $Y$ *with* $\varphi(0) = 0$, *there exists a linear* (*not necessarily continuous*) *mapping* $F$: $Y \to X$ *such that* $F \circ \varphi$ *is the identity on* $X$ *and the restriction of* $F$ *to the linear span of* $\varphi(X)$ *is of norm one*.

Vogt [48] gave a generalization of the Mazur–Ulam theorem in which he considered, instead of isometries, mappings which preserve equality of distance.

DEFINITION 2.5. A mapping $f$: $X \to Y$ between two normed vector spaces is said to *preserve equality of distance* if there exists a function $p$ whose domain and range is the real interval $[0, \infty)$ such that $\|f(x) - f(y)\| = p(\|x - y\|)$. Such mappings were studied by Schoenberg [46] and by von Neumann and Schoenberg [35] for Hilbert spaces. Vogt's result is:

THEOREM 2.6. *Let* $X$ *and* $Y$ *be normed vector spaces, where the dimension of* $X$ *is greater than one*. *If* $f$: $X \to Y$ *is a surjective mapping which preserves equality of distance with* $f(0) = 0$, *then* $f$ *is linear and* $f = \beta T$, *where* $\beta \neq 0$ *is a real number and* $T$ *is an isometry from* $X$ *onto* $Y$. *The same result holds for* $\dim X = 1$ *providing that* $f$ *is continuous*.

In proving this theorem Vogt adapted the method of Mazur and Ulam to his purposes by first demonstrating the following theorem in metric theory, a theorem similar to one stated by Aronszajn [3]. We give an outline of Vogt's proof to illustrate his approach, which will be important later.

THEOREM 2.7. *Let* $(M, d)$ *be a bounded metric space*. *Suppose that there exists an element* $m$ *in* $M$, *a surjective isometry* $g$: $M \to M$, *and a constant* $k > 1$ *such that for all* $x$ *in* $M$, $d(gx, x) \geq kd(m, x)$. *Then* $m$ *is a fixed point for every surjective isometry* $h$: $M \to M$.

*Proof.* Since metric isometries are injective, $h^{-1}$ and $g^{-1}$ exist and are bijective isometries along with $h, g$ and combinations of these four mappings. Define a sequence of bijective isometries $g_n$: $M \to M$ and elements $m_n$ as follows:

$$g_1 = g, \qquad g_2 = hgh^{-1}, \qquad g_{n+1} = g_{n-1} g_n g_{n-1}^{-1}, \qquad n \geq 2,$$
$$m_1 = m, \qquad m_2 = hm, \qquad m_{n+1} = g_{n-1} m_n, \qquad n \geq 2.$$

A straightforward induction shows that

$$d(g_n x, x) \geq kd(m_n, x) \text{ for all } x \text{ in } M \quad \text{and} \quad n \geq 1. \tag{2.2}$$

Put $x = m_{n+1}$ in (2.2) to obtain

$$d(m_{n+2}, m_{n+1}) = d(g_n m_{n+1}, m_{n+1}) \geq kd(m_n, m_{n+1}) = kd(m_{n+1}, m_n).$$

Another induction gives $d(m_{n+2}, m_{n+1}) \geq k^n d(m_2, m_1)$ for $n \geq 1$. Since $M$ is a bounded metric space there exists a positive number $s$ such that $s \geq d(m_{n+2}, m_{n+1})$ for all $n \geq 1$. Hence, $s/k^n \geq d(m_2, m_1)$ for all $n$ and since $k > 1$ by hypothesis, it follows that $d(m_2, m_1) = 0$, so that $m_2 = hm_1 = hm = m$ and $m$ is a fixed point of $h$.

To illustrate how Theorem 2.7 may be applied, we shall use it in proving the following statement (i.e., Theorem 4′ of Wobst [50]).

THEOREM 2.8. *Let $E$ and $F$ be metric vector spaces and let $T\colon E \to F$ be a surjective isometry with $T(0) = 0$. Suppose that there exist real numbers $r > 0, a > 1, b > 1$ such that $d_E(2u, 0) \geq ad_E(u, 0)$ and $d_F(2v, 0) \geq bd_F(v, 0)$ for all $u$ in $E$ with $d_E(u, 0) < r$ and all $v$ in $F$ with $d_F(v, 0) < r$. Then $T$ is linear.*

*Proof.* For convenience we will use quasi-norms for the spaces $E$ and $F$, both designated by $\|\cdot\|$, and we will use Vogt's method in the proof of the linearity of $T$. As indicated in the Introduction, it is sufficient to show that $T$ satisfies

$$T\left(\frac{x+y}{2}\right) = \frac{T(x)}{2} + \frac{T(y)}{2} \tag{2.3}$$

for all $x$ and $y$ in $E$. We shall prove this first, subject to the restriction that $x$ and $y$ satisfy the inequality $\|x - y\| < \frac{ar}{4}$. Making a change of variable (a translation) in (2.3) we have

$$T\left(\frac{y-x}{2}\right) = \frac{T(0)}{2} + \frac{T(y-x)}{2}.$$

Put $2u = y - x$ and use the fact that $T(0) = 0$ to obtain

$$2T(u) = T(2u) \tag{2.4}$$

instead of (2.3). Note that the inequalities $a\|u\| \leq \|2u\| = \|y - x\| < \frac{ar}{4}$ are true if and only if $\|u\| < r/4$. In order to prove (2.4) let $u$ be fixed in $E$ and satisfy $\|u\| < r/4$, and define a set $M$ as

$$M = \left\{v \in F\colon \|v\| = \|2Tu - v\| < 2\|Tu\| < \tfrac{r}{2}\right\}.$$

Set $m = Tu$ and define $g\colon M \to M$ by $gv = 2Tu - v$. It follows easily that $M$ is a bounded metric subspace of $F$, that $m$ belongs to $M$, that $g$ is a surjective isometry of $M$ onto itself, and that for $v$ in $M$ we have $d(gv, v) = \|2Tu - 2v\| = \|2(m - v)\| \geq bd(m, v)$, where $b > 1$ by hypothesis. Thus, the conditions of theorem are satisfied. Next, let $u_0$ be fixed in $T^{-1}(2Tu)$ with $\|u_0\| < \frac{r}{4}$, and let

$$h\colon M \to M \text{ be defined by } hv = T\big(u_0 - T^{-1}(v)\big).$$

Clearly $h$ is an isometry taking $M$ into $M$ and $h$ is surjective since it is its own inverse. Hence, by Theorem 2.7, $m$ is a fixed point of $h$, so that

$$Tu = m = hm = T\big(u_0 - T^{-1}(m)\big) = T(u_0 - u),$$

and we have

$$\begin{aligned} 0 &= \|Tu - T(u_0 - u)\| = \|u - (u_0 - u)\| = \|2u - u_0\| \\ &= \|T(2u) - T(u_0)\| = \|T(2u) - 2T(u)\|. \end{aligned}$$

This proves (2.4) for all $u$ satisfying $\|u\| < \frac{r}{4}$, and so (2.3) is demonstrated for all $x$ and $y$ in $E$ subject to the inequality $\|x - y\| < \frac{ar}{4}$. The proof for all $x$ and $y$ in $E$ is completed by following the extension method of Rolewicz [44] (see also pages 243 and 244 of Rolewicz [45]).

## 3. APPROXIMATE ISOMETRIES

An *$\varepsilon$-isometry* of one metric space $E$ into another $F$ is a mapping $T\colon E \to F$ which changes distances by at most $\varepsilon > 0$; that is

$$|d(Tx, Ty) - d(x, y)| < \varepsilon \quad \text{for all } x, y \quad \text{in } E.$$

For such mappings we may ask the following "stability" question. Given $\eta > 0$ does there exist an $\varepsilon > 0$ and a true isometry $U\colon E \to F$ such that $d(T(x), U(x)) < \eta$ for all $x$ in $E$? In the case where $E$ is a Hilbert space and $F = E$ the answer was given by Hyers and Ulam [25], who showed that for a given $\varepsilon > 0$ and for a surjective $\varepsilon$-isometry $T$ with $T(0) = 0$

$$U(x) = \lim_{n \to \infty} \frac{T(2^n x)}{2^n} \tag{3.1}$$

exists for all $x$ in $E$ and that

$$\|T(x) - U(x)\| < k\varepsilon \tag{3.2}$$

for all $x$ in $E$, with $k = 10$. A similar stability theorem for spaces of continuous functions on metric spaces which are compact was proved by Hyers and Ulam [26]. For a discussion of these and other special cases see Section 4 of Hyers [24]. We now turn to the problem for general Banach spaces $E$ and $F$.

Gruber [21] studied this stability problem for arbitrary normed vector spaces and obtained the following result on the size of $k$ in (3.2).

THEOREM 3.1. *Given normed vector spaces $E$ and $F$, suppose that $T$: $E \to F$ is a surjective $\varepsilon$-isometry with $T(0) = 0$ and that $U$: $E \to F$ is an isometry such that $U(0) = 0$ and that $\|T(x) - U(x)\|/\|x\|$ tends to 0 uniformly as $\|x\| \to \infty$. Then $U$ is a surjective linear isometry and both* (3.1) *and* (3.2) *hold with $k = 5$. Also, when $T$ is continuous, $k = 3$.*

In his proof of this theorem Gruber made use of Figiel's theorem (Theorem 2.4 above) as well as a lemma of Renz (see Bourgin [12, p. 317]). In the same paper Gruber proved the stability of isometry when the normed vector spaces are finite dimensional and $T$: $E \to F$ is a surjective $\varepsilon$-isometry.

The breakthrough to the general case was obtained by Gevirtz [20], who proved the following:

THEOREM 3.2. *Given Banach spaces $X$ and $Y$, let $T$: $X \to Y$ be a surjective $\varepsilon$-isometry with $T(0) = 0$. Then there exists a surjective isometry $U$: $X \to Y$ given by* (3.1) *which satisfies* (3.2) *with $k = 5$.*

In order to prove this theorem, Gevirtz needed to demonstrate the inequality for the $\varepsilon$-isometry $T$,

$$\left\| T\left(\frac{x+y}{2}\right) - \frac{T(x) + T(y)}{2} \right\| \leq 10(\varepsilon\|x - y\|)^{1/2} + 20\varepsilon, \quad (3.3)$$

for all $x$ and $y$ in $X$. With the help of this inequality and Theorem 3.1 above, Theorem 3.2 is easily proved. The difficult part of the proof is establishing (3.3). In doing this Gevirtz succeeded in adapting Vogt's approach, as described in Section 2, to the present situation where $T$ is an $\varepsilon$-isometry. Instead of inverses he used the concept of $\delta$-inverses. Given a surjective mapping $T$: $X \to Y$ and a $\delta > 0$, a mapping $S$: $Y \to X$ for which $\|TS(z) - z\| < \delta$ for all $z$ in $Y$ is called a *$\delta$-inverse* of $T$ (see Bourgin [12]).

Omladič and Šemrl [36] have shown the following result.

THEOREM 3.3. *Let $X$ and $Y$ be real Banach spaces. Given $\varepsilon > 0$, let $f$: $X \to Y$ be a surjective $\varepsilon$-isometry with $f(0) = 0$. Then there exists a surjective*

*isometry* $U$: $X \to Y$ *such that*

$$\|f(x) - U(x)\| \leq 2\varepsilon \quad \text{for all } x \text{ in } X. \tag{3.4}$$

These authors gave examples to prove that $2\varepsilon$ is the best possible result. Thus, the inequality is sharp.

A wider concept of *approximate isometry* than that of $\varepsilon$-isometry was defined and studied by Lindenstrauss and Szankowski [29]. Given a surjective mapping between Banach spaces $X$ and $Y$, define the function $\varphi_T(s)$ for $s \geq 0$ by

$$\varphi_T(s) = \sup\{|\,\|Tu - Tv\| - \|u - v\|\,|: \|u - v\| \leq s \quad \text{or} \quad \|Tu - Tv\| \leq s\}. \tag{3.5}$$

Lindenstrauss and Szankowski investigated the question: What order of growth for the function $\varphi_T(s)$ as $s$ tends to infinity is permissible in order that the stability result of Gevirtz (Theorem 3.2) may be suitably generalized? Their principal theorem is as follows.

THEOREM 3.4. *Let* $T$: $X \to Y$ *be a surjective mapping and suppose that* $\varphi_T$ *as defined by* (3.5) *satisfies the condition*

$$\int_1^\infty \frac{\varphi_T(s)}{s^2}\, ds < \infty. \tag{3.6}$$

*Then there exists a surjective isometry* $U$: $X \to Y$ *such that*

$$\|Tx - Ux\| \leq K\left(\|x\| \int_{\|x\|}^\infty \frac{\varphi(s)}{s^2}\, ds + \int_1^{\|x\|} \frac{\varphi(s)}{s}\, ds\right), \tag{3.7}$$

*where* $K$ *is a universal constant and* $\varphi(s) = \max\{1, \varphi_T(s)\}$.

These authors also show that this result is a best possible one by giving a counterexample showing that the condition (3.6) is necessary. They note that the first summand in (3.7) has to appear. However, if we apply (3.7) to the special case where $T$ is an $\varepsilon$-isometry with $\varepsilon \geq 1$, we get

$$\|Tx - Ux\| \leq K\varepsilon + K\varepsilon \log(\|x\|).$$

Thus, by Theorem 3.2, the second summand in (3.7) is not always necessary. Incidentally, the necessity of the first summand shows that (3.7) cannot be replaced by an inequality of the form

$$\|Tx - Ux\| < C\varphi_T(\|x\|)$$

as in the case where $T$ is an $\varepsilon$-isometry.

Fickett [17] gave another generalization of the theorem of Hyers and Ulam [25] by considering the case where the $\varepsilon$-isometry $T$ is defined only on a bounded subset of $R^n$. For this case one cannot apply the direct method (3.1) to the construction of a true isometry $U$ from $T$ because the domain of $T$ is bounded. He used an interesting method, whose basic idea comes from geometry, to construct a true isometry from the "$\varepsilon$-isometry."

THEOREM 3.5. *Given $\varepsilon > 0$ and $D \subset R^n$ of diameter 1, let $T: D \to R^n$ be a mapping such that*

$$\big|\|T(x) - T(y)\| - \|x - y\|\big| \le \varepsilon \tag{3.8}$$

*for all $x, y \in D$. Then there exists an isometry $U: D \to R^n$ with*

$$\|T(x) - U(x)\| \le 27\varepsilon^{2^{-n}}$$

*for every $x \in D$.*

Using the stability result of Theorem 3.5, Fickett could partially prove Ulam's conjecture in measure theory.

In contrast to Theorem 3.5, Jung [27] dealt with the case where the "$\varepsilon$-isometry" $T$ is defined on a restricted but unbounded domain. He modified the method (3.1) slightly to prove the existence of a unique isometry which differs from $T$ by no more than $9\varepsilon$.

THEOREM 3.6. *Let $E$ be a finite dimensional, strictly convex, real Hilbert space and define $D_u = \{x \in E : \|x\| > d\} \cup \{0\}$ for a given $d \ge 1$. Given $\varepsilon > 0$, let $T: D_u \to E$ be a mapping which satisfies $T(0) = 0$ and the inequality* (3.8) *for all $x, y \in D_u$. Then there exists a unique isometry $U$: $E \to E$ such that*

$$\|T(x) - U(x)\| \le 9\varepsilon$$

*for all $x \in D_u$.*

As Lindenstrauss and Szankowski [29] investigated the inequality (3.5), in which the upper bound is not constant, in connection with the stability of isometries, Jung [27] considered the inequality

$$\big|\|T(x) - T(y)\| - \|x - y\|\big| \le \theta\|x - y\|^p. \tag{3.9}$$

THEOREM 3.7. *Let $E$ be a finite dimensional, strictly convex, real Hilbert space and define $D_b = \{x \in E : \|x\| \le d\}$ for a $d \in (0, 1]$. Given $p \in (1, \infty)$ and $\theta \in [0, 1]$, let $T: D_b \to E$ be a mapping satisfying $T(0) = 0$ as well as the inequality* (3.9) *for all $x, y \in D_b$. Then there exists a unique isometry $U$: $E \to E$ such that*

$$\|T(x) - U(x)\| \le \frac{2^{(1+p)/2}\sqrt{\theta}}{2^{(p-1)/2} - 1}\|x\|^{(1+p)/2}$$

*for each $x \in D_b$.*

## 4. THE PROBLEM OF CONSERVATIVE DISTANCES

Given metric spaces $E$ and $F$ and a mapping $U$: $E \to F$, one may ask the general question, what do we really need to know to ensure that $U$ is an isometry? In Section 3 we found that it is sufficient if $U$ is the limit of an approximate isometry when $E$ and $F$ are Banach spaces. In the present section we consider the following situation. For some fixed positive number $a$ suppose that $U$: $E \to F$ preserves the distance $a$; i.e., for all pairs $x, y$ in $E$ with $d_E(x, y) = a$ we have $d_E(Ux, Uy) = a$. Then $a$ is called a *conservative distance* for $U$. In case $E$ and $F$ are normed vector spaces we may assume without loss of generality that $a = 1$. The basic *problem of conservative distances* is whether the existence of a single conservative distance for $U$ implies that $U$ is an isometry of $E$ into $F$.

Beckman and Quarles [6] proved that in the case where $E = F$ is Euclidean $m$-space $E^m$ with $2 \leq m < \infty$, and if $a > 0$ is a conservative distance for $U$: $E^m \to E^m$, then $U$ is a surjective isometry. They also observed that this theorem is false for $m = 1$ and $m = \infty$ (i.e., real Hilbert space). A counterexample for $m = 1$ is the mapping which translates each integral point one unit to the right and leaves all other points fixed. Their counterexample for $m = \infty$ is also discontinuous (see Rassias [38]).

Some interesting variations and extensions of the Beckman–Quarles theorem were given by Kuz'minyh [28].

Other remarks and problems concerning this problem of conservative distances are mentioned in Rassias [40]. A discussion of the problem for finite dimensional spaces with various metrics can be found in the paper by Ciesielski and Rassias [14]. As indicated above it is known that not every mapping $U$: $E^\infty \to E^\infty$ which preserves the distance one is an isometry, but the counterexample was discontinuous. What happens if we impose continuity conditions on $U$? An answer was given by Mielnik and Rassias [33] as follows. *Every homeomorphism* $U$: $E^\infty \to E^\infty$ *with a conservative distance* $a > 0$ *is an isometry.*

The problem of conservative distances for mappings between general normed vector spaces $X$ and $Y$ was studied by Rassias and Šemrl [41]. They deal with mappings $f$: $X \to Y$ which have the *strong distance one preserving property* (SDOPP); i.e., for all $x, y$ in $X$ with $\|x - y\| = 1$ it follows that $\|f(x) - f(y)\| = 1$, and that the converse is also true. They show that when either $X$ or $Y$ has dimension greater than one and $f$: $X \to Y$ is surjective with SDOPP, then $f$ satisfies the inequality

$$\big|\|f(x) - f(y)\| - \|x - y\|\big| < 1 \quad \text{for all } x, y \text{ in } X,$$

and also that $f$ preserves the distance $m$ in both directions for all positive integers $m$. Another result in the same paper demonstrates that if a

surjective mapping $f: X \to Y$ with SDOPP satisfies the Lipschitz condition $\|f(x) - f(y)\| \leq \|x - y\|$ for all $x, y$ in $X$ then $f$ is an isometry.

So far we have limited ourselves to mappings between real vector spaces. The last two sections of the paper of Rassias and Sharma [42] deal with isometries between complex Hilbert spaces, including a proof of Wigner's theorem, which is important in the theory of quantum mechanics. A discussion of isometries between Hilbert spaces over the skew field of quaternions is also given.

## 5. REMARKS

The question of whether the Mazur–Ulam theorem can be generalized to isometries between any two metric vector spaces remains unsolved. In the theorems of Section 2 above, the behavior of the ratio $d(2x, 0)/d(x, 0)$ plays an important role, and the condition of local boundedness is either explicit or implied by the assumptions. The stability question of isometries between $F$-spaces remains open. Various unanswered questions concerning the problem of conservative distances are discussed by Rassias [37–40] and by Ciesielski and Rassias [14].

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## REFERENCES

1. J. Aczél and J. Dhombres, ''Functional Equations in Several Variables,'' *in* Encyclopedia of Mathematics and Its Applications, Vol. 31, Cambridge Univ. Press, Cambridge, 1989.
2. A. D. Alexandrov, Mappings of families of sets, *Soviet Math. Dokl.* **11** (1970), 116–120.
3. N. Aronszajn, Characterization métrique de $L$-espace de Hiblert, des espaces vectoriels et de certains groupes métriques, *C. R. Acad. Sci. Paris*, **201** (1935), 811–813.
4. J. A. Baker, Isometries in normed spaces, *Amer. Math. Monthly* **78** (1971), 655–658.
5. S. Banach, ''Théorie des Opérations Linéaires,'' Monografie Matematyczne, Vol. 1, Warszawa, 1932; reprint, Chelsea, New York, 1955.
6. F. S. Beckman and D. A. Quarles, On isometries of Euclidean spaces, *Proc. Amer. Math. Soc.* **4** (1953), 810–815.
7. C. Bessaga, A. Pelczynski, and S. Rolewicz, Some properties of the norm in $F$-spaces, *Studia Math.* **16** (1957), 183–192.
8. W. A. Beyer, Approximately Lorentz transformations and $p$-adic fields, Report LA-DC-9486, Los Alamos National Laboratory, March 1968.
9. D. G. Bourgin, Approximate isometries, *Bull. Amer. Math. Soc.* **52** (1946), 704–714.
10. D. G. Bourgin, Approximately isometric and multiplicative transformations on continuous function rings, *Duke Math. J.* **16** (1949), 385–397.

11. D. G. Bourgin, Two dimensional $\varepsilon$-isometries, *Trans. Amer. Math. Soc.* **244** (1978), 85–102.
12. R. D. Bourgin, Approximate isometries on finite dimensional Banach spaces, *Trans. Amer. Math. Soc.* **207** (1975), 309–328.
13. Z. Charzynski, Sur les transformations isométriques des espaces du type $(F)$, *Studia Math.* **13** (1953), 94–121.
14. K. Ciesielski and Th. M. Rassias, On some properties of isometric mappings, *Facta Univ. Ser. Math. Inform.* **7** (1992), 107–115.
15. J. A. Clarkson, Uniformly convex spaces, *Trans. Amer. Math. Soc.* **40** (1936), 396–414.
16. M. M. Day, "Normed Linear Spaces," Springer-Verlag, Berlin, 1958.
17. J. W. Fickett, Approximate isometries on bounded sets with an application to measure theory, *Studia Math.* **72** (1981), 37–46.
18. Figiel, T., On non-linear isometric embedding of normed linear spaces, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys.* **16** (1968), 185–188.
19. R. J. Fleming and J. E. Jamison, Isometries on Banach Spaces—A survey, *in* "Analysis, Geometry and Groups: A Riemann Legacy Volume" (H. M. Srivastava and Th. M. Rassias, eds.), pp. 52–123, Hadronic Press, Palm Harbor, FL, 1993.
20. J. Gevirtz, Stability of isometries on Banach spaces, *Proc. Amer. Math. Soc.* **89** (1983), 633–636.
21. P. M. Gruber, Stability of isometries, *Trans. Amer. Math. Soc.* **245** (1978), 263–277.
22. D. H. Hyers, A note on linear topological spaces, *Bull. Amer. Math. Soc.* **44** (1938), 76–80.
23. D. H. Hyers, Locally bounded linear topological spaces, Rev. Cienc. (Lima) **41** (1939), 555–574.
24. D. H. Hyers, The stability of homomorphisms and related topics, *in* "Global Analysis—Analysis on Manifolds" (Th. M. Rassias, Ed.), pp. 140–153, Teubner-Texte zur Math., Vol. 57, Teubner, Leipzig, 1983.
25. D. H. Hyers and S. M. Ulam, On approximate isometries, *Bull. Amer. Math. Soc.* **51** (1945), 288–292.
26. D. H. Hyers and S. M. Ulam, Approximate isometries of the space of continuous functions, *Ann. of Math.* **48** (1947), 285–289.
27. S.-M. Jung, Hyers–Ulam–Rassias stability of isometries on restricted domains, submitted.
28. A. V. Kuz'minyh, On a characteristic property of isometric mappings, *Soviet Math. Dokl.* **17** (1976), 43–45.
29. J. Lindenstrauss and A. Szankowski, Nonlinear perturbations of isometries, *in* "Colloq. in Honor of Laurent Schwartz (Palaiseau, 1983)," Vol. 1, Asterique, No. 131, pp. 357–371, Soc. Math. France, Marseille, 1985.
30. S. Mazur and W. Orlicz, Sur les espaces métriques linéaires (I), *Studia Math.* **10** (1948), 184–208.
31. S. Mazur and S. M. Ulam, Sur les transformations isométriques des espaces vectoriels normés, *C. R. Acad. Sci. Paris* **194** (1932), 946–948.
32. B. Mielnik, Phenomenon of mobility in non-linear theories, *Comm. Math. Phys.* **101** (1985), 323–339.
33. B. Mielnik and Th. M. Rassias, On the Aleksandrov problem of conservative distances, *Proc. Amer. Math. Soc.* **116** (1992), 1115–1118.
34. P. S. Modenov and A. S. Parkhomenko, "Geometric Transformations," Vol. 1, Academic Press, New York, 1965.
35. J. Von Neumann and I. J. Schoenberg, Fourier integrals and metric geometry, *Trans. Amer. Math. Soc.* **50** (1941), 226–251.
36. M. Omladić and P. Šemrl, On non-linear perturbations of isometries, preprint.

37. Th. M. Rassias, Is a distance one preserving mapping between metric spaces always an isometry? *Amer. Math. Monthly* **90** (1983), 200.
38. Th. M. Rassias, Some remarks on isometric mappings, *Facta Univ. Ser. Math. Inform.* **2** (1987), 49–52.
39. Th. M. Rassias, On the stability of mappings, *Rend. Sem. Mat. Fis. Milano* **58** (1988), 91–99.
40. Th. M. Rassias, Problems and remarks, *in* "Report of Meeting—The 27th International Symposium on Functional Equations," *Aequationes Math.* **39** (1990), 304 & 320–321.
41. Th. M. Rassias and P. Šemrl, On the Mazur–Ulam theorem and the Aleksandrov problem for unit distance preserving mappings, *Proc. Amer. Math. Soc.* **118** (1993), 919–925.
42. Th. M. Rassias and C. S. Sharma, Properties of isometries, *J. Natur. Geom.* **3** (1993), 1–38.
43. S. Rolewicz, On a certain class of linear metric spaces, *Bull. Acad. Polon. Sci.* **5** (1957), 479–484.
44. S. Rolewicz, A generalization of the Mazur–Ulam theorem, *Studia. Math.* **31** (1968), 501–505.
45. S. Rolewicz, "Metric Linear Spaces," *Monogr. Mat.*, Vol. 56, Polish Scientific Publishers, Warszawa, Poland, 1972.
46. I. J. Schoenberg, Metric spaces and completely monotone functions, *Ann. of Math.* **39** (1938), 811–841.
47. R. Swain, Approximate isometries in bounded spaces, *Proc. Amer. Math.* **2** (1951), 727–729.
48. A. Vogt, Maps which preserve equality of distance, *Studia Math.* **45** (1973), 43–48.
49. E. P. Wigner, On unitary representations of the inhomogeneous Lorentz group, *Ann. of Math.* **40** (1939), 149–204.
50. R. Wobst, Isometrien in metrischen Vektorräumen, *Studia Math.* **54** (1975), 41–54.
51. K. Yosida, "Functional Analysis," 2nd ed., Springer-Verlag, New York, 1968.