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NOTE

Minimal Positive Solutions to Some Singular Second-Order Differential Equations

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The existence of a minimal $C^1[0,1]$ positive solution is established for some second-order singular boundary value and initial value problems by new schemes, which are related to x'. Our nonlinearity may be singular at $t = 0, 1, x = 0$, or $x' = 0$.

Key Words: minimal positive solutions; Helly selection principle; fixed point theorems; singular problems.

1. INTRODUCTION

Recently, much attention has been focused on singular boundary value problems (see, e.g., [1–3, 6, 7, 9, 10, 12]) and initial value problems (see, e.g., [4, 8]) in ordinary differential equations.

Under the conditions of superlinearity, semilinearity, delay, etc., some well-known results have been obtained with $x'' + f(t, x) = 0$ (see, e.g., [1, 3, 6, 9]).

If $f(t, x, x')$ is related to x', the study of a positive solution is difficult. Some results have been obtained (see, e.g., [2, 4, 7, 10, 12]). But the assumptions are rather strict; thus verification is not convenient.

Analyzing the results mentioned above (see, e.g., [2, 4, 7, 8, 10] and the references therein), in order to obtain the solution of singular problems, the

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Arzela–Ascoli theorem is usually applied to the approximate solutions of singular problems. This implies that the approximate solutions are equicontinuous. Hence some assumptions must be strict and complicated.

In this paper, by using the fixed point theorem in cones and new schemes in place of the Arzela–Ascoli theorem, we study a minimal $C^1[0, 1]$ positive solution of the boundary value problem

$$\begin{aligned} x'' + f(t, x, x') = 0, \qquad 0 < t < 1, \\ x(0) = x'(1) = 0, \end{aligned} \tag{1.1}$$

with simple conditions. $f(t, x, x')$ is related to x' and has some singularities at $t = 0,\ 1,\ x = 0$, or $x' = 0$ (may be higher order singularity of x'):

$$\lim_{y \to 0^+} f(\cdot, \cdot, y) = +\infty. \tag{1.2}$$

In Section 2, we state some preliminaries for use later in the paper. In Section 3, the existence of a minimal $C^1[0, 1]$ positive solution of (1.1) is proved and the same result is applied to the initial value problem (1.3) with singularities in (1.2):

$$\begin{aligned} x'' = f(t, x, x'), \qquad 0 < t < 1, \\ x(0) = x'(0) = 0. \end{aligned} \tag{1.3}$$

Finally, an example is given.

The goal of this paper is to study the existence of a $C^1[0, 1]$ positive solution of (1.1) or (1.3) without using the Arzela–Ascoli theorem and any assumption of monotonicity, concavity, or convexity. Hence new results are obtained. As far as we know, the study of minimal positive solutions is rather rare, too. The results of this paper will fill this gap in the literature.

2. PRELIMINARIES

A function $x(t)$ is said to be the positive solution of (1.1) if and only if

(i) $x(t) > 0$ for $t \in (0, 1]$ and $x(t)$ belongs to $C[0, 1] \cap C^2(0, 1)$;

(ii) $x(t)$ satisfies (1.1) with boundary conditions.

If $x(t)$ is a positive solution of (1.1) and $x(t) \in C^1[0, 1]$, then $x(t)$ is said to be a $C^1[0, 1]$ positive solution of (1.1).

A function $x_0(t)$ is said to be a minimal positive solution of (1.1) if

(i) $x_0(t)$ is a positive solution of (1.1);

(ii) there is no positive solution $x(t)$ of (1.1) such that $x(t) \le x_0(t)$, $x(t) \ne x_0(t)$.

Throughout this paper, we assume that the following conditions with f are satisfied:

(H1) $f \in C((0,1) \times (0,+\infty)^2, (0,+\infty))$, $f(t,x,y) \leq k(t)p(x)q(y)$, $k(t) > 0$, $t \in (0,1)$, $k(t) \in L^1[0,1]$, $p(x), q(y) \in C((0,+\infty),(0,+\infty))$.

(H2) Define $F(r) = \int_0^r (1/q(y))\,dy$. There exists $R_0 > 0$ such that $F(R_0) > \int_0^1 k(t)\,dt \sup p(0, R_0+1]$.

(H3) $\lim_{r\to+\infty} F(r) = +\infty$.

Remark 2.1. (H1) and (1.2) imply that $\lim_{y\to 0^+} q(y) = +\infty$. (H3) is applied only to a minimal $C^1[0,1]$ positive solution.

By (H2), there exists a positive constant r_0, $r_0 < R_0$, such that $\int_{r_0}^{R_0}(1/q(y))\,dy \geq \int_0^1 k(t)\,dt \sup p(0, R_0+1]$. (H1) and (H2) imply that $\sup p(0, R_0+1] < +\infty$ and $\sup p(0,s] < +\infty$ hold for any fixed $s > 0$.

Consider the integral equation in $C[0,1]$

$$Ty(t) = \int_t^1 f\big(s, Ay(s), y(s)\big)\,ds = y(t), \qquad t \in [0,1], \tag{2.1}$$

where $Ay(t) = \int_0^t y(s)\,ds$. For convenience, $y(t)$, which satisfies (2.1), $y(t) \in C[0,1]$ and $y(t) > 0$, $t \in [0,1)$, is said to be a positive solution of (2.1).

LEMMA 2.1. *Let $x(t) = \int_0^t y(s)\,ds$. Then $x(t)$ is a $C^1[0,1]$ positive solution of* (1.1) *if and only if $y(t)$ is a positive solution of* (2.1).

Proof. Let $x(t)$ be a $C^1[0,1]$ positive solution of (1.2). Next, we will prove that $y(t) = x'(t) > 0$ for any $t \in [0,1)$. In fact, $x(t) = \int_0^t y(s)\,ds > 0$ for any $t \in (0,1]$. If there exists $t \in (0,1)$ such that $x'(t) \leq 0$, then there must be $t_0 \in (0,1)$ such that $x'(t_0) = 0$. Choose $\{t_n\} \in (0,1)$ and $t_0 \in (0,1)$, $t_n \to t_0$, such that $x'(t_n) > 0$ and $x'(t_n) \to x'(t_0) = 0$. This shows that $x''(t_n) = -f(t_n, Ay(t_n), y(t_n)) \to -\infty$ $(t_n \to t_0)$, which contradicts $x(t) \in C^2(0,1)$, $t \in (0,1)$. $y(t) = x'(t)$ satisfies (2.1) and thus is a positive solution of (2.1).

If $y(t)$ is a positive solution of (2.1), then $x(t) = \int_0^t y(s)\,ds$ is a $C^1[0,1]$ positive solution of (1.1).

Lemma 2.1 is proved. ∎

By Lemma 2.1, to determine a $C^1[0,1]$ positive solution of (1.1), we need only determine a positive solution of (2.1).

LEMMA 2.2. *Let $\{x_n(t)\}$ be an infinite sequence of bounded variation function on $[a,b]$ and $\{x_n(t_0)\}$ $(t_0 \in [a,b])$ and $\{V(x_n)\}$ be bounded ($V(x)$ denotes the total variation of x). Then there exists a subsequence $\{x_{n_k}(t)\}$ of $\{x_n(t)\}$, $i \neq j$, $n_i \neq n_j$, such that $\{x_{n_k}(t)\}$ converges everywhere to some bounded variation function $x(t)$ on $[a,b]$.*

Lemma 2.2 is the Helly selection principle (see, e.g., [11]).

Next, we consider an approximate problem of (1.1)

$$\begin{aligned} &x'' + f(t, x + m^{-1}, x') = 0, \qquad 0 < t < 1, \\ &x(0) = 0, \qquad x'(1) = m^{-1}, \end{aligned} \tag{2.2}$$

where m is a natural number.

It is easy to verify that if $y(t)$ is a positive solution of

$$\begin{aligned} y(t) &= \int_t^1 f\big(s, Ay(s) + m^{-1}, y(s)\big)\, ds + m^{-1}, \\ Ay(t) &= \int_0^t y(s)\, ds, \end{aligned} \tag{2.3}$$

then $x(t) = \int_0^t y(s)\, ds$ is a $C^1[0, 1]$ positive solution of (2.2).

Put $D[0, 1] = \{y(t) \in C([0, 1], [0, +\infty)),\ y(t)$ is decreasing$\}$, $\|x\| = \max\{x(t) : 0 \le t \le 1\}$. For $y(t) \in D[0, 1]$, define

$$T_m y(t) = \int_t^1 f\big(s, Ay(s) + m^{-1}, y(s)\big)\, ds + m^{-1}, \qquad Ay(t) = \int_0^t y(s)\, ds,$$

and

$$Iy(t) = \begin{cases} y(t) & \text{if } y(1) \ge m^{-1}, \\ y(t) + \big(m^{-1} - y(1)\big) & \text{if } y(1) < m^{-1}. \end{cases}$$

Obviously, a fixed point of T_m is a solution of (2.3). Notice that, for $y(t) \in D[0, 1]$, $T_m I y(1) \ge m^{-1}$ holds. It is easy to show that a fixed point of $T_m I$ must be one of T_m. Hence we will actually study the existence of a fixed point of $T_m I$ in $D[0, 1]$ in the following discussion.

LEMMA 2.3. *Suppose condition* (H1) *holds. Then $T_m I$ is a continuous and compact operator in* $D[0, 1]$.

Proof. Let $y(t)$ be in $D[0, 1]$. Then $\|y\| + m^{-1} \ge Iy(s) \ge m^{-1} > 0$ and $\|y\| + 2m^{-1} \ge AIy(s) + m^{-1} \ge m^{-1}$. (H1) guarantees the existence of $T_m I y(t)$.

Let $y_n(t)$, $y_0(t) \in D[0, 1]$, $\|y_n - y_0\| \to 0$. Then there exists a constant $h > 0$, such that $\|y_n\| \le h$ and $\|y_0\| \le h$ hold.

$$\begin{aligned} &|T_m I y_n(t) - T_m I y_0(t)| \\ &\quad = \left| \int_t^1 f(s, AIy_n(s) + m^{-1}, Iy_n(s))\, ds - \int_t^1 f(s, AIy_0(s) + m^{-1}, Iy_0(s))\, ds \right| \\ &\quad = \left| \int_t^1 [f(s, AIy_n(s) + m^{-1}, Iy_n(s)) - f(s, AIy_0(s) + m^{-1}, Iy_0(s))]\, ds \right| \\ &\quad \le \int_0^1 |f(s, AIy_n(s) + m^{-1}, Iy_n(s)) - f(s, AIy_0(s) + m^{-1}, Iy_0(s))|\, ds. \end{aligned}$$

(H1) implies that $\{f(s, AIy_n(s)+m^{-1}, Iy_n(s))\}$ converges to $f(s, AIy_0(s)+m^{-1}, Iy_0(s))$ for $s \in (0,1)$. By the Lebesgue dominated convergence theorem (the dominated function $F(s) = k(s)\max p[m^{-1}, h+2m^{-1}] \max q[m^{-1}, h+m^{-1}]$), $\|T_mIy_n(t) - T_mIy_0(t)\| \to 0$, i.e., T_mI is a continuous operator.

Let C be a bounded set in $D[0,1]$; i.e., there exists $h_1 > 0$ such that $\|y\| \le h_1$ for any $y(t) \in C$, $t_1, t_2 \in [0,1]$, $t_1 < t_2$, $y(t) \in D$.

$$\begin{aligned}
&|T_mIy(t_2) - T_mIy(t_1)| \\
&\quad = \int_{t_1}^{t_2} f(s, AIy(s)+m^{-1}, Iy(s))\, ds \\
&\quad \le \int_{t_1}^{t_2} k(s)\, ds \max p[m^{-1}, h_1+2m^{-1}] \max q[m^{-1}, h_1+m^{-1}].
\end{aligned}$$

According to the absolute continuity of the Lebesgue integral, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, when $|t_2 - t_1| < \delta$, $|\int_{t_1}^{t_2} k(s)\, ds| < \varepsilon$ holds. Therefore $\{T_mIy(t) : y(t) \in D\}$ is equicontinuous. This completes the proof. ∎

The following lemma is from [5].

LEMMA 2.4. *Let E be a Banach space and let K be a cone of E. $B_R(0) = \{x \in K : \|x\| \le R\}$. F: $K \cap B_R(0) = K_R \to K$ is a continuous and compact operator. Assume that the following conditions are satisfied*:

(1) *$Fx \ne \lambda x$ for any $\|x\| = R$ and $\lambda > 1$.*

(2) *There exists $r \in (0,R)$ such that $Fx \ne \lambda x$ for any $\|x\| = r$ and $\lambda < 1$.*

(3) $\inf\{\|Fx\| : \|x\| = r\} > 0$.

Then F has a fixed point in $\overline{K}_R \backslash K_r$.

Remark 2.2. Clearly, (2) and (3) in Lemma 2.4 may be replaced by (2′):

(2′) *There exists $r \in (0,R)$ such that $\|Fx\| \ge \|x\|$ for any $\|x\| = r$.*

3. MAIN RESULTS

THEOREM 3.1. *Suppose* (H1) *and* (H2) *hold. Then there exists $M_0 > 0$ such that T_mI has a fixed point in $D[0,1]$ for any $M > M_0$.*

Proof. First, $T_m Iy \neq \mu y$ for any $\mu > 1$ and $\|y\| = R_0$ with $m > \max\{2, 2/r_0\}$. Consider that

$$\lambda T_m Iy(t) = \lambda \int_t^1 f\big(s, AIy(s) + m^{-1}, Iy(s)\big)\, ds + \frac{\lambda}{m} = y(t), \qquad 0 < \lambda < 1. \tag{3.1}$$

If $y(t)$ a solution of (3.1), $\|y\| = R_0$, and $y(t)$ is decreasing in $[0,1]$, $\|y\| = y(0) = R_0$, and $y(1) = \lambda/m < 1/m$. From (3.1),

$$-y'(t) = \lambda f(t, AIy(t) + m^{-1}, Iy(t)) \leq \lambda k(t) p(AIy(t) + m^{-1}) q(Iy(t)),$$

i.e.,

$$-[q(Iy(t))]^{-1} y'(t) \leq \lambda k(t) p(AIy(t) + m^{-1}). \tag{3.2}$$

Put $\tau = \tau|_{y(\tau) = r_0 - 1/m + \lambda/m}$. Notice that $Iy(t) = y(t) + (1/m - \lambda/m)$, $t \in [0,1]$. Integrating for (3.2) from 0 to τ, we obtain

$$\int_0^\tau - \frac{1}{q\big[y(t) + (1/m - \lambda/m)\big]} y'(t)\, dt \leq \lambda \times \int_0^\tau k(t) p\left[A\left(y(t) + \left(\frac{1}{m} - \frac{\lambda}{m}\right)\right) + \frac{1}{m}\right] dt,$$

i.e.,

$$\int_{r_0}^{R_0 + (1/m - \lambda/m)} \frac{1}{q(y)}\, dy \leq \lambda \int_0^\tau k(t) p\left[A\left(y(t) + \left(\frac{1}{m} - \frac{\lambda}{m}\right)\right) + \frac{1}{m}\right] dt.$$

If $\tau < 1$, then

$$\int_{r_0}^{R_0 + (1/m - \lambda/m)} \frac{1}{q(y)}\, dy \leq \lambda \int_0^\tau k(t)\, dt \sup p(0, R_0 + 1] < \int_0^1 k(t)\, dt \sup p(0, R_0 + 1].$$

From this, we obtain

$$\int_{r_0}^{R_0} \frac{1}{q(y)}\, dy \leq \int_{r_0}^{R_0 + (1/m - \lambda/m)} \frac{1}{q(y)}\, dy \leq \int_0^1 k(t)\, dt \sup p(0, R_0 + 1],$$

which contradicts (H2). Hence, for $\tau \geq 1$, $y(t) \geq r_0 - 1/m + \lambda/m \geq r_0/2$ on $[0,1]$. On the other hand, $y(1) = \lambda/m < r_0/2$. This shows that $T_m Iy \neq \mu y$ for any $\mu > 1$ and $\|y\| = R_0$.

Second, by virtue of (1.2), choose $r > 0$, $r < 1$, $r < R_0$. When $0 < y \leq r$, $f(\cdot, \cdot, y) \geq 1$. Assume that $M_0 \geq \max\{2/r, 2/r_0, 2\}$ is fixed. For any $\|y\| = r/2$, when $m > M_0$, $0 < Iy(t) \leq y(t) + 1/m \leq r$,

$$\begin{aligned} T_m Iy(t) &= \int_t^1 f(s, AIy(s) + m^{-1}, Iy(s))\, ds + m^{-1} \\ &\geq \int_t^1 1\, ds + m^{-1} = 1 - t + m^{-1}. \end{aligned}$$

Thus $\|T_m Iy\| \geq 1 \geq r \geq r/2$. By Lemmas 2.3 and 2.4 and Remark 2.2, $T_m I$ has a fixed point $y_m(t)$ in $D[0,1]$, $y_m(1) = m^{-1}$. The proof is complete. ∎

THEOREM 3.2. *Suppose* (H1) *and* (H2) *hold. Then* (2.1) *has a positive solution in* $D[0,1]$ *at least.*

Proof. By Theorem 3.1, T_m has a fixed point $y_m(t)$ ($m \geq M_0$) in $D[0,1]$, $\|y_m\| \leq R_0$. According to Lemma 2.2, there exists a subsequence $\{y_{m_k}(t)\}$ of $\{y_m(t)\}$, $i \neq j$, $m_i \neq m_j$, which converges everywhere on $[0,1]$. Without loss of generality, let $\{y_{m_k}(t)\}$ be itself of $\{y_m(t)\}$. Put $y(t) = \lim_{m\to+\infty} y_m(t)$, $t \in [0,1]$, $y(t)$ is a decreasing function. Next, we will prove that $y(t) > 0$, $t \in [0,1)$. In fact, if there exists $t_0 \in [0,1)$ such that $y(t_0) = 0$. From (1.2), for any $N > 0$, there exists $\delta > 0$ such that, when $0 < y < \delta$, $f(\cdot, \cdot, y) \geq N$. Since $\{y_m(t_0)\}$ converges to $y(t_0)$, there exists $M > 0$, such that, when $m \geq M$, $|y_m(t_0)| = |y_m(t_0) - y(t_0)| < \delta$. From the decrease in $y_m(t)$, $0 < y_m(t) < \delta$ holds, $t \in [t_0, 1]$, $m \geq M$. Hence

$$\begin{aligned} y_m(t_0) &= \int_{t_0}^1 f(s, Ay_m(s) + m^{-1}, y_m(s))\, ds + m^{-1} \\ &\geq N(1 - t_0), \qquad m \geq M. \end{aligned}$$

This contradicts the boundedness of $\{\|y_m\|\}$. If $t \in (0,1)$, $y(t) > 0$, $Ay(t) = \int_0^t y(s)\, ds > 0$, then $\{f(s, Ay_m(s) + m^{-1}, y_m(s))\}$ converges to $f(s, Ay(s), y(s))$, $s \in (0,1)$. By Fatou's theorem in Lebesgue integral, we obtain

$$\begin{aligned} \int_0^1 f(s, Ay(s), y(s))\, ds &= \int_0^1 \lim_{m\to+\infty} f(s, Ay_m(s) + m^{-1}, y_m(s))\, ds \\ &\leq \varliminf_{m\to+\infty} \int_0^1 f(s, Ay_m(s) + m^{-1}, y_m(s))\, ds \leq R_0. \end{aligned}$$

This shows that $f(s, Ay(s), y(s)) \in L^1[0,1]$. Now we will prove that $y(t)$ is a positive solution of (2.1), i.e.,

$$y(t) = \int_t^1 f(s, Ay(s), y(s))\, ds, \qquad t \in [0,1],\ y(t) \in C[0,1]. \tag{3.3}$$

For any $t \in (0,1)$, choose $b \in (0,1)$ such that $t \in (0,b]$. Since $\{y_m(b)\}$ converges to $y(b)$, there exists, for $M > 0$, $y_m(b) \geq y(b)/2 > 0$ $(m \geq M)$. By the decrease in $y_m(t)$, $y_m(t) \geq y(b)/2$, $t \in (0,b]$, $m \geq M$. $Ay(t) = \int_0^t y(s)\,ds > 0$, $t \in (0,b]$, and for any $s \in (0,b]$, $\{f(s, Ay_m(s)+m^{-1}, y_m(s))\}$ converges to $f(s, Ay(s), y(s))$. By the dominated convergence theorem,

$$\begin{aligned} y(t) - y(b) &= \lim_{m\to+\infty} y_m(t) - \lim_{m\to+\infty} y_m(b) \\ &= \lim_{m\to+\infty} (y_m(t) - y_m(b)) \\ &= \lim_{m\to+\infty} \int_t^b f(s, Ay_m(s) + m^{-1}, y_m(s))\,ds \\ &= \int_t^b f(s, Ay(s), y(s))\,ds \end{aligned}$$

(the dominated function is $k(s)\sup p(0, R_0+1]\max q[y(b)/2, R_0]$), i.e.,

$$y(t) - y(b) = \int_t^b f(s, Ay(s), y(s))\,ds. \tag{3.4}$$

Next, we will prove that $y(1^-) = \lim_{t\to 1^-} y(t) = 0$.

In fact, if $y(1^-) > 0$, then $y(t) \geq y(1^-)$, $t \in [0,1)$, from the decrease in $y(t)$. From

$$y_m(t) = \int_t^1 f(s, Ay_m(s) + m^{-1}, y_m(s))\,ds + m^{-1},$$

we have

$$\int_t^1 \frac{-y_m'(s)}{q(y_m(s))}\,ds \leq \int_t^1 k(s)\,ds \sup p(0, R_0+1],$$

i.e.,

$$\int_{1/m}^{y_m(t)} \frac{1}{q(y)}\,dy \leq \int_t^1 k(s)\,ds \sup p(0, R_0+1].$$

Letting $m \to +\infty$, we obtain

$$\int_0^{y(t)} \frac{dy}{q(y)} \leq \int_t^1 k(s)\,ds \sup p(0, R_0+1]. \tag{3.5}$$

Let $t \to 1^-$ in (3.5). Then $\int_0^{y(1^-)} (1/q(y))\,dy = 0$. This is a contradiction.

Letting $b \to 1^-$ in (3.4), we obtain

$$y(t) = \int_t^1 f(s, Ay(s), y(s))\,ds, \qquad t \in (0,1]. \tag{3.6}$$

For $t = 0$, put $0 < z < b < 1$, where b is fixed. From (3.6),

$$\begin{aligned}
y(0) &= \lim_{m\to+\infty} y_m(0) \\
&= \lim_{m\to+\infty} \left\{ \int_0^1 f(s, Ay_m(s) + m^{-1}, y_m(s))\, ds + \frac{1}{m} \right\} \\
&= \lim_{m\to+\infty} \left\{ \int_z^1 f(s, Ay_m(s) + m^{-1}, y_m(s))\, ds + \frac{1}{m} \right\} \\
&\quad + \lim_{m\to+\infty} \left\{ \int_0^z f(s, Ay_m(s) + m^{-1}, y_m(s))\, ds + \frac{1}{m} \right\} \\
&= \lim_{m\to+\infty} y_m(z) + \lim_{m\to+\infty} \int_0^z f(s, Ay_m(s) + m^{-1}, y_m(s))\, ds \\
&= y(z) + \lim_{m\to+\infty} \int_0^z f(s, Ay_m(s) + m^{-1}, y_m(s))\, ds \\
&= \int_z^1 f(s, Ay(s), y(s))\, ds + \lim_{m\to+\infty} \int_0^z f(s, Ay_m(s) + m^{-1}, y_m(s))\, ds.
\end{aligned}$$

Choose $M > 0$ such that, when $m \geq M$, $y_m(b) \geq y(b)/2$. By the decrease in $y_m(t)$, $y_m(t) \geq y(b)/2$, $t \in [0, b]$. Notice that, when $m \geq M$,

$$\int_0^z f(s, Ay_m(s) + m^{-1}, y_m(s))\, ds \leq \int_0^z k(s)\, ds \sup p(0, R_0 + 1] \times \max q\left[\frac{y(b)}{2}, R_0\right].$$

Let $z \to 0$. Then (3.3) holds at $t = 0$. Obviously, $y(t)$ is continuous on $[0, 1]$ from (3.3), and $y(t) \in D[0, 1]$. The proof is complete. ∎

THEOREM 3.3. *Suppose* (H1)–(H3) *hold. Then* (1.1) *has a minimal* $C^1[0, 1]$ *positive solution.*

Proof. Let $\Omega = \{x(t) : x(t)$ is a $C^1[0, 1]$ positive solution of $(1.1)\}$. Theorem 3.2 and Lemma 2.1 implies that Ω is nonempty.

Define a partially ordered $\leq$ in $\Omega : x \leq y$ iff $x(t) \leq y(t)$ for any $t \in [0, 1]$. We prove only that any chain in $\langle \Omega, \leq \rangle$ has a lower bound in Ω. The rest is obtained from Zorn's lemma.

Let $\{x_\alpha(t)\}$ be a chain in $\langle \Omega, \leq \rangle$. Since $C[0, 1]$ is a separable Banach space, there exists denumerable set at most $\{x_n(t)\}$, which is dense in $\{x_\alpha(t)\}$. Without loss of generality, we may assume that $\{x_n(t)\} \subseteq \{x_\alpha(t)\}$. Put $z_n(t) = \min\{x_1(t), x_2(t), \dots, x_n(t)\}$. Since $\{x_\alpha(t)\}$ is a chain, $z_n(t) \in \Omega$ for any n (in fact, $z_n(t)$ equals one of $\{x_n(t)\}$) and $z_{n+1}(t) \leq z_n(t)$ for any n. Put $z(t) = \lim_{m\to+\infty} z_n(t)$. We prove that $z(t) \in \Omega$.

By Lemma 2.1, there exists $y_n(t)$ (e.g., $y_n(t)$ may be $z_n'(t)$), which is a positive solution of (2.1), such that $z_n(t) = \int_0^t y_n(s)\,ds$. Notice that $\|z_n\| \le \|z_1\|$ from (2.1), we obtain

$$\begin{aligned} -y_n'(t) &\le k(t)p(Ay_n(t))q(y_n(t)) \\ &= k(t)p(z_n(t))q(y_n(t)) \\ &\le k(t)\sup p(0, \|z_1\|]q(y_n(t)). \end{aligned}$$

Consequently,

$$-\frac{y_n'(t)}{q(y_n(t))} \le k(t)\sup p(0, \|z_1\|]. \tag{3.7}$$

Integrating (3.7) from 0 to 1, we obtain

$$\int_0^{\|y_n\|} \frac{1}{q(y)}\,dy \le \int_0^1 k(t)\sup p(0, \|z_1\|]. \tag{3.8}$$

Equation (3.8) and (H3) imply that $\{\|y_n\|\}$ is bounded. From Lemma 2.2, there exists a subsequence $\{y_{n_k}(t)\}$ of $\{y_n(t)\}$, $i \neq j$, $n_i \neq n_j$, which converges everywhere on $[0, 1]$. Without loss of generality, let $\{y_{n_k}(t)\}$ be itself of $\{y_n(t)\}$. Put $y_0(t) = \lim_{m\to+\infty} y_n(t)$, $t \in [0, 1]$. Use $\{y_n(t)\}$, $y_0(t)$, and 0 in place of $\{y_m(t)\}$, $y(t)$, and $1/m$ in Theorem 3.2, respectively. In the very same way, we obtain that $y_0(t)$ is a positive solution of (2.1).

The boundedness of $\{\|y_n\|\}$ leads to $z(t) = \lim_{n\to+\infty} z_n(t) = \lim_{n\to+\infty} \times \int_0^t y_n(s)\,ds = \int_0^t y_0(s)\,ds$. Hence $z(t) \in \Omega$ by Lemma 2.1. For any $x(t) \in \{x_\alpha\}$, there exists $\{x_{n_k}(t)\} \subseteq \{x_n(t)\}$ such that $\|x_{n_k} - x\| \to 0$. Notice that $x_{n_k}(t) \ge z_{n_k}(t) \ge z(t)$, $t \in [0, 1]$. Let $n_k \to +\infty$. Then $x(t) \ge z(t)$, $t \in [0, 1]$; i.e., $\{x_\alpha\}$ has lower boundedness in Ω. Zorn's lemma shows that (1.1) has a minimal $C^1[0, 1]$ positive solution.

The proof is complete. ∎

THEOREM 3.4. *Suppose* (H1)–(H3) *hold. Then* (1.3) *has a minimal* $C^1[0, 1]$ *positive solution.*

Proof. This is essentially the same as the proof of Theorem 3.3. It is the core of Theorem 3.4 to prove the existence of a positive solution of

$$Sz(t) = \int_0^t f(s, Az(s), z(s))\,ds = z(t), \qquad t \in [0, 1]. \tag{3.9}$$

(The definition of a $C^1[0, 1]$ positive solution of (1.3) is the same as that of (1.1); the positive solution of (3.9) $z(t)$ satisfies (3.9), $z(t) \in C[0, 1]$ and $z(t) > 0, t \in (0, 1]$). Assume that $D_1[0, 1] = \{z(t) \in C([0, 1], [0, +\infty)),$

$z(t)$ is increasing$\}$, $\|x\| = \max\{x(t) : 0 \le t \le 1\}$. In place of T_m and I in Section 2, for $z(t) \in D_1[0, 1]$, define

$$S_m z(t) = \int_0^t f(s, Az(s) + m^{-1}, z(s))\,ds + m^{-1}, \qquad Az(t) = \int_0^t z(s)\,ds$$

and

$$Iz(t) = \begin{cases} z(t) & \text{if } z(0) \ge m^{-1}, \\ z(t) + (m^{-1} - z(0)) & \text{if } z(0) < m^{-1}. \end{cases}$$

Similar to the proofs of Lemma 2.3 and Theorem 3.1, $S_m I$ is continuous and compact. $S_m I$ has a fixed point $z_m(t)$ in $D_1[0, 1]$ (integrating from τ to 1 in place of 0 to τ in Theorem 3.1, $\tau > 0$ leads to the contradiction), $\|z_m\| \le R_0$, $z(0) = m^{-1}$. Without loss of generality, let $\{z_m(t)\}$ converge everywhere on $[0, 1]$ by Lemma 2.2. Put $z(t) = \lim_{m\to+\infty} z_m(t)$, $t \in [0, 1]$, $z(t)$ is a increasing function, $z(t) > 0$, $t \in (0, 1]$, and $f(s, Az(s), z(s)) \in L^1[0, 1]$ (see the proof $y(t) > 0$, $t \in [0, 1)$, $f(s, Ay(s), y(s)) \in L^1[0, 1]$ in Theorem 3.2). For $t \in (0, 1]$, choose $h \in (0, 1]$ to fit $t \in [h, 1]$. Since $\{z_m(h)\}$ converges to $z(h)$, there exists $M > 0$ such that for $m \ge M$, $z_m(h) \ge z(h)/2 > 0$. The increase in $z_m(t)$ leads to $z_m(t) \ge z(h)/2 > 0$, $t \in [h, 1]$, $m \ge M$. $Az(t) = \int_0^t z(s)\,ds > 0$, $t \in [h, 1]$, $\{f(s, Az_m(s) + m^{-1}, z_m(s))\}$ converges to $f(s, Az(s), z(s))$, $s \in [h, 1]$. By the dominated convergence theorem (the dominated function $F(s) = k(s) \sup p(0, R_0 + 1] \max q[z(h)/2, R_0])$, we obtain

$$\begin{aligned} z(t) - z(h) &= \lim_{m\to+\infty} z_m(t) - \lim_{m\to+\infty} z_m(h) \\ &= \lim_{m\to+\infty} \int_h^t f(s, Az_m(s) + m^{-1}, z_m(s))\,ds \\ &= \int_h^t f(s, Az(s), z(s))\,ds, \end{aligned}$$

i.e.,

$$z(t) - z(h) = \int_h^t f(s, Az(s), z(s))\,ds. \tag{3.10}$$

Next, we prove that $z(0^+) = \lim_{t\to 0^+} z(t) = 0$.

In fact, if $z(0^+) > 0$, then $z(t) \ge z(0^+)$, $t \in (0, 1]$, from the increase in $z(t)$. From

$$z_m(t) = \int_0^t f(s, Az_m(s) + m^{-1}, z_m(s))\,ds + m^{-1},$$

we have

$$\int_0^t \frac{z_m'(s)}{q(z_m(s))}\,ds \le \int_0^t k(s)\,ds \sup p(0, R_0 + 1],$$

i.e.,

$$\int_{1/m}^{z_m(t)} \frac{dz}{q(z)} \leq \int_0^t k(s)\,ds \sup p(0, R_0+1].$$

Letting $m \to +\infty$, we obtain

$$\int_0^{z(t)} \frac{dz}{q(z)} \leq \int_0^t k(s)\,ds \sup p(0, R_0+1]. \tag{3.11}$$

Let $t \to 0^+$ in (3.11). Then $\int_0^{z(0^+)} (1/q(z))\,dz = 0$. This is a contradiction.

Put $h \to 0^+$ in (3.10) to obtain

$$z(t) = \int_0^t f(s, Az(s), z(s))\,ds, \qquad t \in [0,1]. \tag{3.12}$$

Equation (3.12) shows that $z(t)$ is a positive solution of (3.9). Put $x(t) = \int_0^t z(s)\,ds$. Then $x(t)$ is a $C^1[0,1]$ positive solution of (1.3).

The proof of the existence of a minimal $C^1[0,1]$ positive solution of (1.3) is very similar to that of Theorem 3.3 and is omitted. The proof is complete. ∎

EXAMPLE 3.1. $f(t, x, x') = t^{-\alpha}(1-t)^{-\beta}(2+\sin x^{-\gamma})(x')^{-\delta}$, $0 < \alpha < 1$, $0 < \beta < 1$, $\gamma > 0$, $\delta > 0$.

Put $k(t) = t^{-\alpha}(1-t)^{-\beta}$, $p(x) = 2+\sin x^{-\gamma}$, $q(y) = y^{-\delta}$. Obviously, (H1) holds.

$$F(r) = \int_0^r \frac{1}{q(y)}\,dy = \frac{r^{1+\delta}}{1+\delta}.$$

There must be $R_0 > 0$ such that $F(R_0) > \int_0^1 k(t)\,dt \sup p(0, R_0+1]$ and $F(r) \to +\infty$ $(r \to +\infty)$. (H2) and (H3) hold. Hence (1.1) and (1.3) have a minimal $C^1[0,1]$ positive solution by Theorems 3.3 and 3.4.

Remark 3.1. The theorems in [2, 4, 7, 8, 10] are not applied to the existence of the solution of Example 3.1.

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