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Boundary controllability of Sobolev-type abstract nonlinear integrodifferential systems

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Abstract

Sufficient conditions are established for boundary controllability of various classes of Sobolev-type nonlinear systems including integrodifferential systems in Banach spaces. The results are obtained using the strongly continuous semigroup of operators and the Banach contraction principle. Examples are provided to illustrate the theory.

Keywords: Boundary controllability; Nonlinear system; Integrodifferential system; Delay system; Semigroup theory; Fixed point theorem

1. Introduction

Controllability of Sobolev-type nonlinear integrodifferential systems in Banach spaces has been discussed by Balachandran and Dauer [3] with the help of the Schauder fixed point theorem. In [5], Balachandran and Sakthivel studied the controllability of Sobolev-type semilinear functional integrodifferential systems in Banach spaces by using the Schaefer fixed point theorem. These types of equations occur in thermodynamics, in the flow of fluid through fissured rocks and in the shear in second order fluids. Kwun et al. [14] studied approximate controllability for delay Volterra systems with bounded linear operators, and in [4] Balachandran and Sakthivel discussed this problem for delay integrodifferential systems in Banach spaces.

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Several abstract settings have been developed to describe the distributed control systems in which the control is exercised through the boundary. Balakrishnan [6] first constructed a solution for a parabolic boundary control equation with L^2 controls that can be expressed as a mild solution to an operator equation using semigroup theory. Fattorini [11] developed a semigroup approach for boundary control systems. Lasiecka [15] established the regularity of optimal boundary controls for parabolic equations. In [7–9] Barbu discussed the general theory of boundary control systems and the existence of solutions for boundary control problems governed by parabolic equations with nonlinear boundary value conditions. In [10] Cirina studied the existence of boundary controls for quasilinear systems of hyperbolic equations.

The formulation of boundary control problems in terms of semigroup theory offers the following advantage over a variational approach. The semigroup approach can treat a problem where the spatial domain does not have C^∞ boundary, such as for an n-dimensional parallelepiped. Related abstract descriptions of boundary control systems and their applications to various fields of study can be found in [13,16–18,24].

Han and Park [12] studied the boundary controllability of semilinear systems with nonlocal condition. Recently the problem of boundary controllability of delay integrodifferential systems in Banach spaces has been investigated by Balachandran and Anandhi [1,2]. The purpose of this paper is to establish sufficient conditions for the boundary controllability of various types of nonlinear Sobolev-type systems including integrodifferential systems in Banach spaces. The approach will use semigroup theory and the Banach fixed point theorem.

2. Preliminaries

Let Y and Z be Banach spaces with norms $|\cdot|$ and $\|\cdot\|$, respectively. Let σ be a linear, closed and densely defined operator with domain $D(\sigma) \subseteq Y$ and range $R(\sigma) \subseteq Z$, and let θ be a linear operator with $D(\theta) \subseteq Y$ and $R(\theta) \subseteq X$, a Banach space.

Consider the boundary control nonlinear system

$$\begin{aligned} &\big(Ex(t)\big)' = \sigma x(t) + f\big(t, x(t)\big), \quad t \in J = [0, b], \\ &\theta x(t) = B_1 u(t), \\ &x(0) = x_0, \end{aligned} \tag{1}$$

where $E : D(E) \subset Y \to R(E) \subset Z$ is a linear operator, the control function $u \in L^1(J, U)$, a Banach space of admissible control functions with U as a Banach space, $B_1 : U \to X$ is a linear continuous operator, and the nonlinear operator $f : J \times Y \to Z$ is given.

Let $y(t) = Ex(t)$ for $x \in Y$, then (1) can be written as

$$\begin{aligned} &y'(t) = \sigma E^{-1} y(t) + f\big(t, E^{-1} y(t)\big), \quad t \in J, \\ &\tilde{\theta} y(t) = B_1 u(t), \\ &y(0) = y_0, \end{aligned} \tag{2}$$

where $\tilde{\theta} = \theta E^{-1} : Z \to X$ is a linear operator. Let $A : Y \to Z$ be a linear operator defined by

$$D\bigl(AE^{-1}\bigr) = \bigl\{w \in D\bigl(\sigma E^{-1}\bigr)\colon \tilde{\theta} w = 0\bigr\},$$
$$AE^{-1}w = \sigma E^{-1}w, \quad \text{for } w \in D\bigl(AE^{-1}\bigr).$$

The operators $A : D(A) \subset Y \to Z$ and $E : D(E) \subset Y \to Z$ satisfy the following hypotheses.

(H_1) A and E are closed linear operators.
(H_2) $D(E) \subset D(A)$ and E is bijective.
(H_3) $E^{-1} : Z \to D(E)$ is continuous.
(H_4) The resolvent $R(\lambda, AE^{-1})$ is a compact operator for some $\lambda \in \rho(AE^{-1})$, the resolvent set of AE^{-1}.

The hypotheses (H_1), (H_2) and the Closed Graph Theorem imply the boundedness of the linear operator $AE^{-1} : Z \to Z$.

Lemma 2.1 [21]. *Let $S(t)$ be a uniformly continuous semigroup and let A be its infinitesimal generator. If the resolvent $R(\lambda\colon A)$ of A is compact for every $\lambda \in \rho(A)$, then $S(t)$ is a compact semigroup.*

Let $B_r = \{y \in Y\colon |y| \leqslant r\}$, for some $r > 0$. We shall make the following hypotheses.

(i) $D(\sigma) \subset D(\theta)$ and the restriction of θ to $D(\sigma)$ is continuous relative to graph norm of $D(\sigma)$.
(ii) The operator AE^{-1} is the infinitesimal generator of a C_0 semigroup $T(t)$ on Z and there exists a constant $M > 0$ such that $\|T(t)\| \leqslant M$.
(iii) There exists a linear continuous operator $B : U \to Z$ such that $\sigma E^{-1}B \in L(U, Z)$, $\tilde{\theta}(Bu) = B_1 u$, for all $u \in U$. Also, $Bu(t)$ is continuously differentiable and $\|Bu\| \leqslant C\|B_1 u\|$ for all $u \in U$, where C is a constant.
(iv) For all $t \in (0, b]$ and $u \in U$, $T(t)Bu \in D(AE^{-1})$. Moreover, there exists a positive function $\nu \in L^1(0, b)$ such that $\|AE^{-1}T(t)B\| \leqslant \nu(t)$, a.e. $t \in (0, b)$.

Let $y(t)$ be the solution of (2). Then define the function $z(t) = y(t) - Bu(t)$. From the assumptions, it follows that $z(t) \in D(AE^{-1})$. Hence (2) can be written in terms of A and B as

$$y'(t) = AE^{-1}z(t) + \sigma E^{-1}Bu(t) + f\bigl(t, E^{-1}y(t)\bigr), \quad t \in J,$$
$$y(t) = z(t) + Bu(t),$$
$$y(0) = y_0.$$

If u is continuously differentiable on $[0, b]$, then z can be defined as a mild solution to the Cauchy problem

$$z'(t) = AE^{-1}z(t) + \sigma E^{-1}Bu(t) - Bu'(t) + f\big(t, E^{-1}y(t)\big),$$
$$z(0) = y(0) - Bu(0),$$

and the solution of (2) is given by

$$y(t) = T(t)\big[y(0) - Bu(0)\big] + Bu(t) + \int_0^t T(t-s)\big[\sigma E^{-1}Bu(s) - Bu'(s) + f\big(s, E^{-1}y(s)\big)\big]\,ds. \tag{3}$$

Since the differentiability of the control u represents an unrealistic and severe requirement, it is necessary to extend the concept of a solution for general inputs $u \in L^1(J, U)$. Integrating (3) by parts, yields

$$y(t) = T(t)y(0) + \int_0^t \big[T(t-s)\sigma E^{-1}B - AE^{-1}T(t-s)B\big]u(s)\,ds + \int_0^t T(t-s)f\big(s, E^{-1}y(s)\big)\,ds,$$

which is well defined. Hence the mild solution of system (1) is given by

$$x(t) = E^{-1}T(t)Ex(0) + \int_0^t E^{-1}\big[T(t-s)\sigma E^{-1}B - AE^{-1}T(t-s)B\big]u(s)\,ds + \int_0^t E^{-1}T(t-s)f\big(s, x(s)\big)\,ds. \tag{4}$$

Definition 2.2. System (1) is said to be *controllable* on interval J if for every $x_0, x_1 \in Y$, there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of (1) satisfies $x(b) = x_1$.

Further, assume the following conditions.

(v) There exist constants $N,\ K > 0$ such that $\int_0^b \nu(t)\,dt \leqslant K$ and $|E^{-1}| \leqslant N$.

(vi) The linear operator W from $L^2(J, U)$ into Y defined by

$$Wu = \int_0^b E^{-1}\big[T(b-s)\sigma E^{-1}B - AE^{-1}T(b-s)B\big]u(s)\,ds$$

induces an invertible operator $\widetilde{W}$ defined on $L^2(J, U)/\ker W$, and there exists a constant $K_1 > 0$ such that $\|\widetilde{W}^{-1}\| \leqslant K_1$. The construction of $\widetilde{W}^{-1}$ in general Banach spaces is outlined in [22].

(vii) $f : J \times Y \to Z$ is continuous and there exist constants $M_1, M_2 > 0$ such that for all $y_1, y_2 \in B_r$

$$\big\| f(t, y_1) - f(t, y_2)\big\| \leqslant M_1 |y_1 - y_2|$$

and

$$M_2 = \max_{t \in J} \big\| f(t, 0)\big\|.$$

(viii) $NM\|Ex_0\| + N[bM\|\sigma E^{-1}B\| + K]K_1[|x_1| + NM\|Ex_0\| + L] + L \leqslant r$, where $L = bNM[M_1 r + M_2]$.

(ix) Let $q = bNMM_1[NK_1(bM\|\sigma E^{-1}B\| + K) + 1]$ be such that $0 \leqslant q < 1$.

3. Controllability of nonlinear system

Theorem 3.1. *If the hypotheses* (i)–(ix) *are satisfied, then the boundary control nonlinear system* (1) *is controllable on* J.

Proof. Using hypothesis (vi), for an arbitrary function $x(\cdot)$, define the control

$$u(t) = \widetilde{W}^{-1}\left[x_1 - E^{-1}T(b)Ex_0 - \int_0^b E^{-1}T(b-s) f\big(s, x(s)\big)\, ds\right](t).$$

Let $V = C(J, B_r)$. Using this control, it will now be shown that the operator Φ defined by

$$\begin{aligned}
\Phi x(t) = {} & E^{-1}T(t)Ex_0 + \int_0^t E^{-1}\big[T(t-s)\sigma E^{-1}B - AE^{-1}T(t-s)B\big]\widetilde{W}^{-1} \\
& \times \left[x_1 - E^{-1}T(b)Ex_0 - \int_0^b E^{-1}T(b-\tau) f\big(\tau, x(\tau)\big)\, d\tau\right](s)\, ds \\
& + \int_0^t E^{-1}T(t-s) f\big(s, x(s)\big)\, ds
\end{aligned}$$

has a fixed point. This fixed point is then a solution of (1).

Clearly $\Phi x(b) = x_1$, which means that the control u steers the system from the initial state x_0 to x_1 in time b provided the operator Φ has a fixed point.

First to see that Φ maps V into itself, let $x \in V$ then

$$\begin{aligned}
\big|\Phi x(t)\big| \leqslant {} & \big|E^{-1}T(t)Ex_0\big| + \Bigg|\int_0^t E^{-1}\big[T(t-s)\sigma E^{-1}B - AE^{-1}T(t-s)B\big]\widetilde{W}^{-1} \\
& \times \left[x_1 - E^{-1}T(b)Ex_0 - \int_0^b E^{-1}T(b-\tau) f\big(\tau, x(\tau)\big)\, d\tau\right](s)\, ds\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_0^t E^{-1}T(t-s)f\big(s,x(s)\big)\,ds\right|\\
&\leqslant \left|E^{-1}\right|\left\|T(t)Ex_0\right\| + \int_0^t \left|E^{-1}\right|\big[\left\|T(t-s)\right\|\left\|\sigma E^{-1}B\right\|\\
&\quad + \left\|AE^{-1}T(t-s)B\right\|\big]\left\|\widetilde{W}^{-1}\right\|\Bigg[|x_1| + \left|E^{-1}\right|\left\|T(b)Ex_0\right\|\\
&\quad + \int_0^b \left|E^{-1}\right|\left\|T(b-\tau)\right\|\big[\left\|f\big(\tau,x(\tau)\big)-f(\tau,0)\right\| + \left\|f(\tau,0)\right\|\big]\,d\tau\Bigg]\,ds\\
&\quad + \int_0^t \left|E^{-1}\right|\left\|T(t-s)\right\|\big[\left\|f\big(s,x(s)\big)-f(s,0)\right\| + \left\|f(s,0)\right\|\big]\,ds\\
&\leqslant NM\|Ex_0\| + N\big[bM\left\|\sigma E^{-1}B\right\| + K\big]K_1\big[|x_1| + NM\|Ex_0\| + L\big] + L\\
&\leqslant r.
\end{aligned}
$$

Thus, Φ maps V into itself.

Now, for $x_1, x_2 \in V$

$$
\begin{aligned}
\left|\Phi x_1(t) - \Phi x_2(t)\right| &\leqslant \int_0^t \left|E^{-1}\right|\big[\left\|T(t-s)\right\|\left\|\sigma E^{-1}B\right\| + \left\|AE^{-1}T(t-s)B\right\|\big]\left\|\widetilde{W}^{-1}\right\|\\
&\quad \times\left[\int_0^b \left|E^{-1}\right|\left\|T(b-\tau)\right\|\left\|f\big(\tau,x_1(\tau)\big)-f\big(\tau,x_2(\tau)\big)\right\|\,d\tau\right]ds\\
&\quad + \int_0^t \left|E^{-1}\right|\left\|T(t-s)\right\|\left\|f\big(s,x_1(s)\big)-f\big(s,x_2(s)\big)\right\|\,ds\\
&\leqslant bNMM_1\big[NK_1\big(bM\left\|\sigma E^{-1}B\right\| + K\big)+1\big]\left|x_1(t)-x_2(t)\right|\\
&\leqslant q\left|x_1(t)-x_2(t)\right|.
\end{aligned}
$$

Therefore, Φ is a contraction mapping.

Hence there exists a unique fixed point $x \in Y$ such that $\Phi x(t) = x(t)$. Any fixed point of Φ is a mild solution of (1) on J satisfying $x(b) = x_1$. Thus, system (1) is controllable on J. □

4. Controllability of integrodifferential system

Consider the boundary control integrodifferential system of the form

$$\begin{aligned}
&\big(Ex(t)\big)' = \sigma x(t) + \int_0^t k(t,s) f\big(s, x(s)\big)\,ds, \quad t \in J,\\
&\theta x(t) = B_1 u(t),\\
&x(0) = x_0,
\end{aligned} \tag{5}$$

where $k : J \times J \to R$ is a continuous function and $f : J \times Y \to Z$ is given. Using the similar argument as in the previous section, the mild solution of the system (5) is given by

$$\begin{aligned}
x(t) = E^{-1}T(t)Ex(0) &+ \int_0^t E^{-1}\big[T(t-s)\sigma E^{-1}B - AE^{-1}T(t-s)B\big]u(s)\,ds\\
&+ \int_0^b E^{-1}T(t-s)\left(\int_0^s k(s,\tau) f\big(\tau, x(\tau)\big)\,d\tau\right)ds.
\end{aligned}$$

Consider the following conditions:

(A_1) There exists a constant $N_1 > 0$ such that $|k(t,s)| \leqslant N_1$.
(A_2) $NM\|Ex_0\| + NK_1[bM\|\sigma E^{-1}B\| + K][|x_1| + NM\|Ex_0\| + L] + L \leqslant r$, where $L = b^2 NMN_1[M_1 r + M_2]$.
(A_3) Let $q = b^2 NMN_1M_1[NK_1(bM\|\sigma E^{-1}B\| + K) + 1]$ be such that $0 \leqslant q < 1$.

Theorem 4.1. *If the hypotheses* (i)–(vii) *and* (A_1)–(A_3) *are satisfied, then the boundary control integrodifferential system* (5) *is controllable on* J.

Proof. Using the hypothesis (vi), for an arbitrary function $x(\cdot)$, define the control

$$\begin{aligned}
u(t) = \widetilde{W}^{-1}\Bigg[x_1 &- E^{-1}T(b)Ex_0\\
&- \int_0^b E^{-1}T(b-s)\left(\int_0^s k(s,\tau) f\big(\tau, x(\tau)\big)\,d\tau\right)ds\Bigg](t).
\end{aligned}$$

Using this control, the operator Φ defined by

$$\begin{aligned}
\Phi x(t) = E^{-1}&T(t)Ex_0\\
&+ \int_0^t E^{-1}\big[T(t-s)\sigma E^{-1}B - AE^{-1}T(t-s)B\big]\widetilde{W}^{-1}\Bigg[x_1 - E^{-1}T(b)Ex_0\\
&- \int_0^b E^{-1}T(b-\tau)\left(\int_0^\tau k(\tau,\eta) f\big(\eta, x(\eta)\big)\,d\eta\right)d\tau\Bigg](s)\,ds
\end{aligned}$$

$$+\int_0^t E^{-1}T(t-s)\left(\int_0^s k(s,\tau)f\big(\tau,x(\tau)\big)\,d\tau\right)ds$$

has a fixed point. To see this, first note that Φ maps V into itself. For $x \in V$,

$$\begin{aligned}
\big|\Phi x(t)\big| &\leqslant \big|E^{-1}\big|\big\|T(t)Ex_0\big\| + \int_0^t \big|E^{-1}\big|\big[\big\|T(t-s)\big\|\big\|\sigma E^{-1}B\big\| \\
&\quad + \big\|AE^{-1}T(t-s)B\big\|\big]\big\|\widetilde{W}^{-1}\big\|\Bigg[|x_1| + \big|E^{-1}\big|\big\|T(b)Ex_0\big\| \\
&\quad + \int_0^b \big|E^{-1}\big|\big\|T(b-\tau)\big\|\left(\int_0^\tau \big|k(\tau,\eta)\big|\big[\big\|f\big(\eta,x(\eta)\big) \right. \\
&\quad \left. - f(\eta,0)\big\| + \big\|f(\eta,0)\big\|\big]\,d\eta\right)d\tau\Bigg]ds \\
&\quad + \int_0^t \big|E^{-1}\big|\big\|T(t-s)\big\|\left(\int_0^s \big|k(s,\tau)\big|\big[\big\|f\big(\tau,x(\tau)\big) \right. \\
&\quad \left. - f(\tau,0)\big\| + \big\|f(\tau,0)\big\|\big]\,d\tau\right)ds \\
&\leqslant NM\|Ex_0\| + NK_1\big[bM\big\|\sigma E^{-1}B\big\| + K\big]\big[|x_1| + NM\|Ex_0\| + L\big] + L \\
&\leqslant r.
\end{aligned}$$

Thus, Φ maps V into itself.

Now, for $x_1, x_2 \in V$

$$\begin{aligned}
&\big|\Phi x_1(t) - \Phi x_2(t)\big| \\
&\quad\leqslant \int_0^t \big|E^{-1}\big|\big[\big\|T(t-s)\big\|\big\|\sigma E^{-1}B\big\| + \big\|AE^{-1}T(t-s)B\big\|\big]\big\|\widetilde{W}^{-1}\big\| \\
&\qquad \times \left[\int_0^b \big|E^{-1}\big|\big\|T(b-\tau)\big\|\left(\int_0^\tau \big|k(\tau,\eta)\big|\big\|f\big(\eta,x_1(\eta)\big)\right.\right. \\
&\qquad\quad \left.\left. - f\big(\eta,x_2(\eta)\big)\big\|\,d\eta\right)d\tau\right]ds \\
&\qquad + \int_0^t \big|E^{-1}\big|\big\|T(t-s)\big\|\left(\int_0^s \big|k(s,\tau)\big|\big\|f\big(\tau,x_1(\tau)\big) - f\big(\tau,x_2(\tau)\big)\big\|\,d\tau\right)ds \\
&\quad\leqslant b^2NMN_1M_1\big[NK_1\big(bM\big\|\sigma E^{-1}B\big\| + K\big) + 1\big]\big|x_1(t) - x_2(t)\big| \\
&\quad\leqslant q\big|x_1(t) - x_2(t)\big|.
\end{aligned}$$

Hence, by the Banach fixed point theorem, there exists a unique fixed point $x \in Y$ which is a mild solution of (5) on J satisfying $x(b) = x_1$. Thus, system (5) is controllable on J. □

5. Controllability of nonlinear delay system

Consider the boundary control nonlinear delay system of the form

$$\begin{aligned}
&\big(Ex(t)\big)' = \sigma x(t) + f\big(t, x\big(\gamma_1(t)\big), x\big(\gamma_2(t)\big), \dots, x\big(\gamma_n(t)\big)\big), \quad t \in J,\\
&\theta x(t) = B_1 u(t),\\
&x(0) = x_0,
\end{aligned} \tag{6}$$

where $\gamma_i(t) : J \to J$, $i = 1, 2, \dots, n$, are continuous functions and the nonlinear operator $f : J \times Y^n \to Z$ is continuous. The mild solution of the system (6) is given by

$$\begin{aligned}
x(t) = {}& E^{-1}T(t)Ex_0 + \int_0^t E^{-1}\big[T(t-s)E^{-1}\sigma B - AE^{-1}T(t-s)B\big]u(s)\,ds\\
&+ \int_0^t E^{-1}T(t-s) f\big(s, x\big(\gamma_1(s)\big), x\big(\gamma_2(s)\big), \dots, x\big(\gamma_n(s)\big)\big)\,ds.
\end{aligned}$$

In addition to the above assumptions, assume the following conditions.

(C$_1$) $f : J \times Y^n \to Z$ is continuous and there exist constants M_3 and M_4 such that for all $v_i, w_i \in B_r$, $i = 1, 2, \dots, n$,

$$\big\| f(t, v_1, v_2, \dots, v_n) - f(t, w_1, w_2, \dots, w_n)\big\| \leqslant M_3 \sum_{i=1}^{n} |v_i - w_i|$$

and

$$M_4 = \max_{t \in J}\big\| f(t, 0, \dots, 0)\big\|.$$

(C$_2$) There exists a constant p such that for all $x_1, x_2 \in Y$

$$\big|x_1\big(\gamma_i(t)\big) - x_2\big(\gamma_i(t)\big)\big| \leqslant p\big|x_1(t) - x_2(t)\big|, \quad \text{for } i = 1, 2, \dots n.$$

(C$_3$) $NM\|Ex_0\| + N[bM\|\sigma E^{-1}B\| + K]K_1[|x_1| + NM\|Ex_0\| + L] + L \leqslant r$, where $L = bNM(M_3nr + M_4)$.

(C$_4$) Let $q = bnpNMM_3[NK_1(bM\|\sigma E^{-1}B\| + K) + 1]$.

Theorem 5.1. *If the hypotheses* (i)–(vi) *and* (C$_1$)–(C$_4$) *are satisfied, then the boundary control nonlinear delay system* (6) *is controllable on* J.

Proof. Using the hypothesis (vi), for an arbitrary function $x(\cdot)$, define the control

$$u(t) = \widetilde{W}^{-1}\Bigg[x_1 - E^{-1}T(b)Ex_0 - \int_0^b E^{-1}T(t-s)f\big(s, x\big(\gamma_1(s)\big), x\big(\gamma_2(s)\big), \ldots, x\big(\gamma_n(s)\big)\big)\,ds\Bigg](t).$$

We shall show that, when using this control, the operator Φ defined on Y by

$$\begin{aligned}\Phi x(t) &= E^{-1}T(t)Ex_0\\ &\quad + \int_0^t E^{-1}\big[T(t-s)\sigma E^{-1}B - AE^{-1}T(t-s)B\big]\widetilde{W}^{-1}\Bigg[x_1 - E^{-1}T(b)Ex_0\\ &\quad - \int_0^b E^{-1}T(b-\tau)f\big(\tau, x\big(\gamma_1(\tau)\big), x\big(\gamma_2(\tau)\big), \ldots, x\big(\gamma_n(\tau)\big)\big)\,d\tau\Bigg](s)\,ds\\ &\quad + \int_0^t E^{-1}T(t-s)f\big(s, x\big(\gamma_1(s)\big), x\big(\gamma_2(s)\big), \ldots, x\big(\gamma_n(s)\big)\big)\,ds\end{aligned}$$

has a fixed point.

First, we show that Φ maps V into itself. For $x \in V$,

$$\begin{aligned}\big|\Phi x(t)\big| &\leqslant \big|E^{-1}\big|\big\|T(t)Ex_0\big\| + \int_0^t \big|E^{-1}\big|\big[\big\|T(t-s)\big\|\big\|\sigma E^{-1}B\big\|\\ &\quad + \big\|AE^{-1}T(t-s)B\big\|\big]\big\|\widetilde{W}^{-1}\big\|\Bigg[|x_1| + \big|E^{-1}\big|\big\|T(b)Ex_0\big\|\\ &\quad + \int_0^b \big|E^{-1}\big|\big\|T(b-\tau)\big\|\big[\big\|f\big(\tau, x\big(\gamma_1(\tau)\big), x\big(\gamma_2(\tau)\big), \ldots, x\big(\gamma_n(\tau)\big)\big)\\ &\quad - f(\tau, 0, \ldots, 0)\big\| + \big\|f(\tau, 0, \ldots, 0)\big\|\big]\,d\tau\Bigg]\,ds\\ &\quad + \int_0^t \big|E^{-1}\big|\big\|T(t-s)\big\|\big[\big\|f\big(s, x\big(\gamma_1(s)\big), x\big(\gamma_2(s)\big), \ldots, x\big(\gamma_n(s)\big)\big)\\ &\quad - f(s, 0, \ldots, 0)\big\| + \big\|f(s, 0, \ldots, 0)\big\|\big]\,ds\\ &\leqslant NM\|Ex_0\| + N\big[bM\big\|\sigma E^{-1}B\big\| + K\big]K_1\big[|x_1| + NM\|Ex_0\| + L\big] + L\\ &\leqslant r.\end{aligned}$$

Thus, Φ maps V into itself.

Now, for $x_1, x_2 \in V$

$$
\begin{aligned}
&\bigl|\Phi x_1(t) - \Phi x_2(t)\bigr| \\
&\quad \leqslant \int_0^t \bigl|E^{-1}\bigr|\bigl[\bigl\|T(t-s)\bigr\|\bigl\|\sigma E^{-1}B\bigr\| + \bigl\|AE^{-1}T(t-s)B\bigr\|\bigr]\bigl\|\widetilde{W}^{-1}\bigr\| \\
&\qquad \times \Biggl[\int_0^b \bigl|E^{-1}\bigr|\bigl\|T(b-\tau)\bigr\|\bigl\|f\bigl(\tau, x_1\bigl(\gamma_1(\tau)\bigr), x_1\bigl(\gamma_2(\tau)\bigr), \ldots, x_1\bigl(\gamma_n(\tau)\bigr)\bigr) \\
&\qquad - f\bigl(\tau, x_2\bigl(\gamma_1(\tau)\bigr), x_2\bigl(\gamma_2(\tau)\bigr), \ldots, x_2\bigl(\gamma_n(\tau)\bigr)\bigr)\bigr\|\,d\tau\Biggr]\,ds \\
&\qquad + \int_0^t \bigl|E^{-1}\bigr|\bigl\|T(t-s)\bigr\|\bigl\|f\bigl(s, x_1\bigl(\gamma_1(s)\bigr), x_1\bigl(\gamma_2(s)\bigr), \ldots, x_1\bigl(\gamma_n(s)\bigr)\bigr) \\
&\qquad - f\bigl(s, x_2\bigl(\gamma_1(s)\bigr), x_2\bigl(\gamma_2(s)\bigr), \ldots, x_2\bigl(\gamma_n(s)\bigr)\bigr)\bigr\|\,ds \\
&\quad \leqslant \bigl[\bigl(bM\bigl\|\sigma E^{-1}B\bigr\| + K\bigr)K_1 bN^2 MM_3 + bNMM_3\bigr]\bigl[\bigl|x_1\bigl(\gamma_1(\tau)\bigr) - x_2\bigl(\gamma_1(\tau)\bigr)\bigr| \\
&\qquad + \bigl|x_1\bigl(\gamma_2(\tau)\bigr) - x_2\bigl(\gamma_2(\tau)\bigr)\bigr| + \cdots + \bigl|x_1\bigl(\gamma_n(\tau)\bigr) - x_1\bigl(\gamma_n(\tau)\bigr)\bigr|\bigr] \\
&\quad \leqslant bnqNMM_3\bigl[NK_1\bigl(bM\bigl\|\sigma E^{-1}B\bigr\| + K\bigr) + 1\bigr]\bigl|x_1(t) - x_2(t)\bigr| \\
&\quad \leqslant p\bigl|x_1(t) - x_2(t)\bigr|.
\end{aligned}
$$

Hence, Φ is a contraction mapping and has a unique fixed point $x \in Y$. This fixed point is a mild solution of (6) on J satisfying $x(b) = x_1$. Thus, system (6) is controllable on J. □

6. Controllability of delay integrodifferential system

Consider the boundary control delay integrodifferential system of the form

$$
\begin{aligned}
&\bigl(Ex(t)\bigr)' = \sigma x(t) + f\left(t, x\bigl(\gamma_1(t)\bigr), \int_0^t k(t,s)g\bigl(s, x\bigl(\gamma_2(s)\bigr)\bigr)\,ds\right), \quad t \in J, \\
&\theta x(t) = B_1 u(t), \\
&x(0) = x_0, \qquad\qquad (7)
\end{aligned}
$$

where $k : J \times J \to R$ is a continuous function and the nonlinear operators $f : J \times Y \times Y \to Z$ and $g : J \times Y \to Y$ are given.

To establish the results we shall assume the following conditions.

(a) $f : J \times Y \times Y \to Z$ is continuous and there exist constants $M_5, M_6 > 0$ such that for all $v_1, v_2 \in B_r$ and $w_1, w_2 \in Y$ we have

$$\bigl\|f(t, v_1, w_1) - f(t, v_2, w_2)\bigr\| \leqslant M_5\bigl[|v_1 - v_2| + |w_1 - w_2|\bigr]$$

and

$$M_6 = \max_{t \in J} \| f(t, 0, 0) \|.$$

(b) $g : J \times Y \to Y$ is continuous and there exist constants $L_1, L_2 > 0$ such that for all $v_1, v_2 \in B_r$

$$\| g(t, v_1) - g(t, v_2) \| \leqslant L_1 |v_1 - v_2|$$

and

$$L_2 = \max_{t \in J} \| g(t, 0) \|.$$

(c) There exists a constant N_1 such that

$$|k(t, s)| \leqslant N_1 \quad \text{for } (t, s) \in J \times J.$$

(d) There exists a constant p such that for all $x_1, x_2 \in Y$

$$\left| x_1(\gamma_i(t)) - x_2(\gamma_i(t)) \right| \leqslant p \left| x_1(t) - x_2(t) \right|, \quad \text{for } i = 1, 2.$$

(e) $NM\|Ex_0\| + NK_1[bM\|\sigma E^{-1}B\| + K][|x_1| + NM\|Ex_0\| + L] + L \leqslant r$, where $L = bNM[M_5(r + bN_1(L_1 r + L_2)) + M_6]$.

(f) Let $q = bpNMM_5[1 + bN_1L_1][NK_1(Mb\|\sigma E^{-1}B\| + K) + 1]$ be such that $0 \leqslant q < 1$.

The mild solution of the system (7) is given by

$$x(t) = E^{-1}T(t)Ex_0 + \int_0^t E^{-1}\big[T(t-s)\sigma E^{-1}B - AE^{-1}T(t-s)B\big]u(s)\,ds$$
$$+ \int_0^t E^{-1}T(t-s) f\left(s, x(\gamma_1(s)), \int_0^s k(s,\tau) g(\tau, x(\gamma_2(\tau)))\,d\tau\right) ds.$$

Theorem 6.1. *If the hypotheses* (i)–(vi) *and* (a)–(f) *are satisfied, then the boundary control delay integrodifferential system* (7) *is controllable on* J.

Proof. Using the hypothesis (vi), for an arbitrary function $x(\cdot)$, define the control

$$u(t) = \widetilde{W}^{-1}\left[x_1 - E^{-1}T(b)Ex_0 - \int_0^b E^{-1}T(b-s) f\left(s, x(\gamma_1(s)), \int_0^s k(s,\tau) g(\tau, x(\gamma_2(\tau)))\,d\tau\right) ds\right](t).$$

We shall show that, when using this control, the operator Φ defined on Y by

$$\begin{aligned}\Phi x(t) = {} & E^{-1}T(t)Ex_0 \\ & + \int_0^t E^{-1}\big[T(t-s)\sigma E^{-1}B - AE^{-1}T(t-s)B\big]\widetilde{W}^{-1}\Bigg[x_1 - E^{-1}T(b)Ex_0 \\ & + \int_0^b E^{-1}T(b-\tau)f\Bigg(\tau, x\big(\gamma_1(\tau)\big), \int_0^\tau k(\tau,\eta)g\big(\eta, x\big(\gamma_2(\eta)\big)\big)\,d\eta\Bigg)\Bigg](s)\,ds \\ & + \int_0^t E^{-1}T(t-s)f\Bigg(s, x\big(\gamma_1(s)\big), \int_0^s k(s,\tau)g\big(\tau, x\big(\gamma_2(\tau)\big)\big)\,d\tau\Bigg)\,ds\end{aligned}$$

has a fixed point.

First it is shown that Φ maps V into itself. For $x \in V$,

$$\begin{aligned}&\big|\Phi x(t)\big| \\ &\quad \leqslant \big|E^{-1}\big|\big\|T(t)Ex_0\big\| + \int_0^t \big|E^{-1}\big|\big[\big\|T(t-s)\big\|\big\|\sigma E^{-1}B\big\| + \big\|AE^{-1}T(t-s)B\big\|\big] \\ &\quad\quad \times \big\|\widetilde{W}^{-1}\big\|\Bigg[|x_1| + \big|E^{-1}\big|\big\|T(b)Ex_0\big\| + \int_0^b \big|E^{-1}\big|\big\|T(b-\tau)\big\| \\ &\quad\quad \times \Bigg[\Bigg\|f\Bigg(\tau, x\big(\gamma_1(\tau)\big), \int_0^\tau k(\tau,\eta)g\big(\eta, x\big(\gamma_2(\eta)\big)\big)\,d\eta\Bigg) - f(\tau,0,0)\Bigg\| \\ &\quad\quad\quad + \big\|f(\tau,0,0)\big\|\Bigg]\,d\tau\Bigg]\,ds \\ &\quad\quad + \int_0^t \big|E^{-1}\big|\big\|T(t-s)\big\|\Bigg[\Bigg\|f\Bigg(s, x\big(\gamma_1(s)\big), \int_0^s k(s,\tau)g\big(\tau, x\big(\gamma_2(\tau)\big)\big)\,d\tau\Bigg) \\ &\quad\quad - f(s,0,0)\Bigg\| + \big\|f(s,0,0)\big\|\Bigg]\,ds \\ &\quad \leqslant NM\|Ex_0\| + N\big[bM\big\|\sigma E^{-1}B\big\| + K\big]K_1\big[|x_1| + NM\|Ex_0\| + L\big] + L \\ &\quad \leqslant r.\end{aligned}$$

Thus, Φ maps V into itself. Now, for $x_1, x_2 \in V$,

$$\begin{aligned}&\big|\Phi x_1(t) - \Phi x_2(t)\big| \\ &\quad \leqslant \int_0^t \big|E^{-1}\big|\big[\big\|T(t-s)\big\|\big\|\sigma E^{-1}B\big\| + \big\|AE^{-1}T(t-s)B\big\|\big]\big\|\widetilde{W}^{-1}\big\|\end{aligned}$$

$$\times\left[\int_0^b |E^{-1}|\|T(b-\tau)\|\left\|f\left(\tau, x_1(\gamma_1(\tau)), \int_0^\tau k(\tau,\eta)g(\eta, x_1(\gamma_2(\eta)))\,d\eta\right)\right.\right.$$
$$\left.\left.- f\left(\tau, x_2(\gamma_1(\tau)), \int_0^\tau k(\tau,\eta)g(\eta, x_2(\gamma_2(\eta)))\,d\eta\right)\right\|d\tau\right]ds$$
$$+\int_0^t |E^{-1}|\|T(t-s)\|\left\|f\left(s, x_1(\gamma_1(s)), \int_0^s k(s,\tau)g(\tau, x_1(\gamma_2(\tau)))\,d\tau\right)\right.$$
$$\left.- f\left(s, x_2(\gamma_1(s)), \int_0^s k(s,\tau)g(\tau, x_2(\gamma_2(\tau)))\,d\tau\right)\right\|ds$$
$$\leqslant pbNMM_5(1+bN_1L_1)\big[NK_1\big(Mb\|\sigma E^{-1}B\|+K\big)+1\big]\big|x_1(t)-x_2(t)\big|$$
$$\leqslant q\big|x_1(t)-x_2(t)\big|.$$

Therefore, Φ is a contraction mapping. Hence there exists a unique fixed point $x \in Y$ which is a mild solution of (7) on J satisfying $x(b) = x_1$. Thus, system (7) is controllable on J. □

7. Applications

Theorem 7.1. *Let Ω be a bounded, open subset of R^n, and let Γ be a sufficiently smooth boundary of Ω. Consider the following boundary control system*

$$\frac{\partial}{\partial t}\big(z(t,y)-\Delta z(t,y)\big)-\Delta z(t,y)=\mu\big(t,z(t,y)\big), \quad \text{in } Q=(0,b)\times\Omega,$$
$$z(t,0)=u(t,0), \quad \text{on } \Sigma=(0,b)\times\Gamma,\ t\in[0,b],$$
$$z(t,y)=0, \quad z(0,y)=z_0(y), \quad \text{for } y\in\Omega, \tag{8}$$

where $u \in L^2(\Sigma)$, $z_0 \in L^2(\Omega)$ and $\mu \in L^2(Q)$. If conditions (i)–(ix) *of Theorem* 3.1 *are satisfied, then system* (8) *is controllable.*

Proof. The above problem can be formulated abstractly into the boundary control system (1) by suitably choosing $Y = Z = L^2(\Omega)$, $X = H^{-1/2}(\Gamma)$, $U = L^2(\Gamma)$, $B_1 = I$, the identity operator, the operator $E : D(E) \subset Y \to Z$ defined by $Ew = w - \Delta w$ with $D(E) = H^2(\Omega)$ and

$$D(\sigma)=\big\{z\in L^2(\Omega);\ \Delta z\in L^2(\Omega)\big\}, \quad \sigma z=\Delta z.$$

The operator θ is the "trace" operator such that $\theta z = z|_\Gamma$ is well defined and belongs to $H^{-1/2}(\Gamma)$ for each $z \in D(\sigma)$ (see [20]).

Define the operator $A : D(A) \subset Y \to Z$ by

$$AE^{-1}w=\Delta E^{-1}w \quad \text{with } D\big(AE^{-1}\big)=H_0^1(\Omega)\cup H^2(\Omega).$$

Here $H^k(\Omega)$, $H^s(\Gamma)$ are the usual Sobolev spaces on Ω, Γ. Then A and E can be written, respectively, as

$$Aw = \sum_{n=1}^{\infty} n^2(w, w_n)w_n, \quad w \in D(A),$$
$$Ew = \sum_{n=1}^{\infty} \big(1+n^2\big)(w, w_n)w_n, \quad w \in D(E),$$

where $w_n(y) = \sqrt{2}\sin ny$, $n = 1, 2, 3, \ldots,$ is the orthogonal set of eigenvectors of A. Furthermore, for $w \in Y$

$$E^{-1}w = \sum_{n=1}^{\infty} \frac{1}{1+n^2}(w, w_n)w_n,$$
$$AE^{-1}w = \sum_{n=1}^{\infty} \frac{n^2}{1+n^2}(w, w_n)w_n,$$
$$T(t)w = \sum_{n=1}^{\infty} e^{\frac{n^2}{1+n^2}t}(w, w_n)w_n.$$

It is easy to see that AE^{-1} generates a strongly continuous semigroup $T(t)$ on Z. Hence, assumptions (i) and (ii) are satisfied.

To verify (iii) and (iv) define the linear operator $B : L^2(\Gamma) \to L^2(\Omega)$ by $Bu = v_u$, where v_u is the unique solution to the Dirichlet boundary value problem

$$\Delta v_u = 0 \quad \text{in } \Omega,$$
$$v_u = u \quad \text{in } \Gamma.$$

In other words (see [19])

$$\int_{\Omega} v_u \Delta\psi \, dx = \int_{\Gamma} u \frac{\partial \psi}{\partial n} \, dx, \quad \text{for all } \psi \in H_0^1(\Omega) \cup H^2(\Omega), \tag{9}$$

where $\frac{\partial\psi}{\partial n}$ denotes the outward normal derivative of ψ. This outward normal is well defined as an element of $H^{1/2}(\Gamma)$. From (9), it follows that

$$\|v_u\|_{L^2(\Omega)} \leqslant C_1\|u\|_{H^{-1/2}(\Gamma)}, \quad \text{for all } u \in H^{-1/2}(\Gamma),$$

and

$$\|v_u\|_{H^1(\Omega)} \leqslant C_2\|u\|_{H^{1/2}(\Gamma)}, \quad \text{for all } u \in H^{1/2}(\Gamma).$$

From the above estimates it follows by an interpolation argument [23] that

$$\big\|AE^{-1}T(t)B\big\|_{L(L^2(\Gamma), L^2(\Gamma))} \leqslant C_3 t^{-3/4}, \quad \text{for all } t > 0 \text{ with } \nu(t) = C_3 t^{-3/4},$$

where C_i, $i = 1, 2, 3,$ are positive constants independent of u.

Assume the nonlinear function μ satisfies

$$\big\|\mu(t, v_1) - \mu(t, v_2)\big\| \leqslant K_1\|v_1 - v_2\|, \quad v_1 \in B_r,\ K_1 > 0,$$

and the bounded invertible operator $\widetilde{W}$ exists. Choose b and other constants such that the conditions (viii) and (ix) are satisfied. Hence all the conditions stated in Theorem 3.1 are satisfied and so the system (8) is controllable on $[0, b]$. □

Theorem 7.2. *Consider the boundary control system*

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(z(t,y) - \Delta z(t,y)\big) - \Delta z(t,y) = \mu\big(t, z(t,y)\big), \quad \text{in } Q,\\
&\frac{\partial z(t,0)}{\partial n} + \beta z(t,0) = u(t,0), \quad \text{on } \Sigma,\ t \in J,\\
&z(t,y) = 0, \quad z(0,y) = z_0(y), \quad \text{for } y \in \Omega,
\end{aligned} \tag{10}$$

where $z_0 \in L^2(\Omega)$, $u \in L^2(\Gamma)$, $\mu \in L^2(Q)$ *and* β *is a nonnegative constant. Then system* (10) *is controllable provided the conditions of Theorem* 3.1 *are satisfied.*

Proof. To formulate this as a boundary control problem (1), suitably choose the spaces Y, Z, U, X and the operators E, B_1, σ and θ as follows. Let $Y = Z = L^2(\Omega)$, $U = X = L^2(\Gamma)$, $B_1 = I$, the identity operator, and $\theta z = \beta z + \frac{\partial z}{\partial n}$. The operator $E : D(E) \subset Y \to Z$ is defined by $Ez = z - \Delta z$ with domain $D(E) = H^2(\Omega)$ and $\sigma z = \Delta z$ with $D(\sigma) = H^2(\Omega)$. The operator A is given by

$$AE^{-1}z = \Delta E^{-1}z \quad \text{with } D\big(AE^{-1}\big) = \big\{z \in H^2(\Omega);\ \theta E^{-1}z = 0\big\}.$$

Then A and E can be written as in the previous example, and it can be easily seen that AE^{-1} is the infinitesimal generator of a strongly continuous semigroup $T(t)$. Define the linear operator $B : L^2(\Gamma) \to L^2(\Omega)$ by $Bu = v_u$, where $v_u \in H^1(\Omega)$ is the unique solution to the Neumann boundary value problem,

$$\begin{aligned}
&v_u - \Delta v_u = 0 \quad \text{in } \Omega,\\
&\beta v_u + \frac{\partial v_u}{\partial n} = u \quad \text{in } \Gamma.
\end{aligned} \tag{11}$$

Consider on the product space $H^1(\Omega) \times H^1(\Omega)$ the bilinear functional

$$h(y,\psi) = \int_\Omega (y\psi + \operatorname{grad} y \operatorname{grad} \psi)\,dx - \int_\Gamma (u - \beta y)\psi\,d\sigma, \tag{12}$$

where $u \in H^{-1/2}(\Gamma)$. Here $\int_\Gamma u\psi\,d\sigma$ is the value of u at $\psi \in H^{1/2}(\Gamma)$. Since h is coercive, there is a $v_u \in H^1(\Omega)$ satisfying $h(v_u, \psi) = 0$ for all $\psi \in H^1(\Omega)$. Hence, $v_u = Bu$ is the solution to (11). From (12) it follows that

$$\|v_u\|_{H^1(\Omega)} \leqslant C\|u\|_{H^{-\frac{1}{2}}(\Gamma)}.$$

Since the operator $-AE^{-1}$ is self-adjoint and positive

$$\int_0^b \big\|AE^{-1}T(t)y_0\big\|^2_{L^2(\Omega)}\,dt \leqslant C\|y_0\|^2_{D((-AE^{-1})^{1/2})}, \tag{13}$$

for all $y_0 \in D((-AE^{-1})^{1/2}) = H^1(\Omega)$.

Let δ be the scalar function defined by

$$\delta(t) = \lim_{n\to\infty} \inf \left\| A_n T(t) \right\|_{L(H^1(\Omega), L^2(\Omega))}, \quad t \in [0, b],$$

where $A_n = AE^{-1}(I + n^{-1}AE^{-1})^{-1}$, for $n = 1, 2, \ldots$. Obviously,

$$\left\| AE^{-1}T(t) \right\|_{L(H^1(\Omega), L^2(\Omega))} \leqslant \delta(t), \quad \text{for } t \in (0, b]. \tag{14}$$

Also, (13) implies

$$\int_0^b \left\| A_n T(t) \right\|^2_{L(H^1(\Omega), L^2(\Omega))} dt \leqslant C, \quad \text{for all } n.$$

Therefore, by Fatou's lemma it follows that $\delta \in L^2(0, b)$ and hence from (13) and (14)

$$\left\| AE^{-1}T(t)Bu \right\|_{L^2(\Omega)} \leqslant C\delta(t)\|u\|_{L^2(\Gamma)}, \quad \text{for all } t \in (0, b),\ u \in L^2(\Gamma),$$

with $\nu(t) = C\delta(t) \in L^2(0, b)$. Thus, assumptions (i)–(iv) are satisfied. Further, the nonlinear function μ satisfies

$$\left\| \mu(t, v_1) - \mu(t, v_2) \right\| \leqslant K_1 \|v_1 - v_2\|, \quad v_1 \in B_r,\ K_1 > 0.$$

Assume the bounded invertible operator $\widetilde{W}$ exists and choose b and other constants in such a way that the conditions (viii) and (ix) are satisfied. Hence, all of the conditions stated in Theorem 3.1 are satisfied, and system (10) is controllable on $[0, b]$. □

Example 7.3. Consider the partial delay integrodifferential equation of the form

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(z(t, y) - \Delta z(t, y)\big) - \Delta z(t, y) = z(t - h, y) + \int_0^t \sin z(s - h, y)\, ds, \quad \text{in } Q, \\
&\frac{\partial z(t, 0)}{\partial n} + \beta z(t, 0) = u(t, 0), \quad \text{on } \Sigma,\ t \in J, \\
&z(t, y) = 0, \quad z(0, y) = z_0(y), \quad \text{for } y \in \Omega,
\end{aligned} \tag{15}$$

where $z_0 \in L^2(\Omega)$, $u \in L^2(\Gamma)$ and β is a nonnegative constant.

Let $Y = Z = L^2(\Omega)$, $U = X = L^2(\Gamma)$, $B_1 = I$, the identity operator, $\theta z = \beta z + \frac{\partial z}{\partial n}$ and $\sigma z = \Delta z$ with domain $D(\sigma) = H^2(\Omega)$. Define the operators $E : D(E) \subset Y \to Z$, and A by

$$\begin{aligned}
&Ez = z - \Delta z \quad \text{with domain } D(E) = H^2(\Omega), \\
&AE^{-1}z = \Delta E^{-1}z \quad \text{with } D(AE^{-1}) = \big\{ z \in H^2(\Omega):\ \theta E^{-1}z = 0 \big\},
\end{aligned}$$

respectively, where A and E are as in Theorem 7.1. It can be seen that AE^{-1} generates a strongly continuous semigroup $T(t)$, $t \geqslant 0$.

Let us take

$$\int_0^t k(t,s)g\big(s,z(s-h)\big)(y)\,ds = \int_0^t \sin z(s-h,y)\,ds,$$

$$f\left(t, z(t-h), \int_0^t k(t,s)g\big(s,z(s-h)\big)\,ds\right)(y) = z(t-h,y) + \int_0^t \sin z(s-h,y)\,ds,$$

where $k(t,s) = 1$. Obviously

$$\left\| \left[z(t-h,y) + \int_0^t \sin z(s-h,y)\,ds \right] - \left[x(t-h,y) + \int_0^t \sin x(s-h,y)\,ds \right] \right\| \leqslant (1+b)\big\| z(s-h,y) - x(s-h,y) \big\|.$$

Using the similar argument as in Theorem 7.2, we see that the conditions (i)–(iv) are satisfied. Assume that the bounded invertible operator $\widetilde{W}$ exists. Choose b and other constants such that the conditions (e) and (f) of Theorem 6.1 are satisfied. Hence the system (15) is controllable on $[0, b]$.

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