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Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings

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Abstract

In the present paper, we prove that the modified implicit iteration sequence for a finite family of asymptotically quasi-nonexpansive mappings converges strongly to a common fixed point of the family in a uniformly convex Banach space, requiring one member T in the family to be semi-compact. Our results extend and improve some recent results.

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1. Introduction

Let E be a Banach space, C be a closed convex subset of E , T be a self-mapping on C . T is said to be nonexpansive provided $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; T is called asymptotically nonexpansive if there exists a sequence $\{u_n\}$ in $[0, +\infty)$ with $\lim_{n \rightarrow +\infty} u_n = 0$ such that $\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\|$ for all $x, y \in C$ and $n = 1, 2, \dots$. Denote by $F(T)$ the set of fixed points of T , that is, $F(T) = \{x \in C : Tx = x\}$. Throughout this paper, we always assume that $F(T) \neq \emptyset$. T is called asymptotically quasi-nonexpansive if there exists a sequence $\{u_n\}$ in $[0, +\infty)$ with $\lim_{n \rightarrow +\infty} u_n = 0$ such that $\|T^n x - q\| \leq (1 + u_n)\|x - q\|$ for all $x \in C$ and $q \in F(T)$ and $n = 1, 2, \dots$.

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From the above definitions, it follows that a nonexpansive mapping must be asymptotically nonexpansive, and an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive, but the converse does not hold.

The concept of quasi-nonexpansiveness was introduced by Diaz and Metcalf [1] in 1967, the concept of asymptotically nonexpansiveness was introduced by Goebel and Kirk [2] in 1972. The iterative approximation problems for nonexpansive mapping, asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mapping were studied extensively by Browder [3,4], Goebel and Kirk [2], Liu [5], Wittmann [6], Reich [7], Shoji and Takahashi [8], Chang et al. [9] in the settings of Hilbert spaces and uniformly convex Banach spaces.

Recently, Xu and Ori [10] have introduced an implicit iteration process for a finite family of nonexpansive mappings. Let $T_1, T_2, \dots, T_N$ be N self-mappings of C and suppose that $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$, the set of common fixed points of T_i , $i = 1, 2, \dots, N$. Hereafter, we will denote the index set $\{1, 2, \dots, N\}$ by I . An implicit iteration process for a finite family of nonexpansive mappings is defined as follows, with $\{t_n\}$ a real sequence in $(0, 1)$, $x_0 \in C$:

$$\begin{aligned} x_1 &= t_1 x_0 + (1 - t_1) T_1 x_1, \\ x_2 &= t_2 x_1 + (1 - t_2) T_2 x_2, \\ &\vdots \\ x_N &= t_N x_{N-1} + (1 - t_N) T_N x_N, \\ x_{N+1} &= t_{N+1} x_N + (1 - t_{N+1}) T_1 x_{N+1}, \\ &\vdots \end{aligned}$$

which can be written in the following compact form:

$$x_n = t_n x_{n-1} + (1 - t_n) T_n x_n, \quad n \geq 1, \quad (1.1)$$

where $T_k = T_{k \bmod N}$. (Here the $\bmod N$ function takes values in I .) And they proved the weak convergence of the process (1.1) to a common fixed point in the setting of a Hilbert space.

In this paper, we will extend the process (1.1) to a process for a finite family of asymptotically quasi-nonexpansive mappings, with $\{\alpha_n\}$ a real sequence in $(0, 1)$, and an initial point $x_0 \in C$, which is defined as follows:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1, \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1^2 x_{N+1}, \\ &\vdots \\ x_{2N} &= \alpha_{2N} x_{2N-1} + (1 - \alpha_{2N}) T_N^2 x_{2N}, \end{aligned}$$

$$\begin{aligned} x_{2N+1} &= \alpha_{2N+1}x_{2N} + (1 - \alpha_{2N+1})T_1^3x_{2N+1}, \\ &\vdots \end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_i^k x_n, \quad n \geq 1, \quad (1.2)$$

where $n = (k - 1)N + i, i \in I$.

Throughout this paper, we always suppose that the sequence $\{x_n\}$ generated by process (1.2) exists. Our purpose in this paper is to study the implicit iteration process (1.2) in the general setting of a uniformly convex Banach space and prove the strong convergence of the process to a common fixed point, requiring only one member T in the family $\{T_i, i \in I\}$ to be semi-compact. The results presented in this paper generalize and extend the corresponding main results of Wittmann [6], Xu and Ori [10].

2. Preliminaries

For the sake of convenience, we first recall some definitions and conclusions.

Definition 2.1 [2]. A Banach space E is said to be uniformly convex if the modulus of convexity of E

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{1}{2} \|x + y\| : \|x\| = \|y\| = 1, \|x - y\| = \epsilon \right\} > 0$$

for all $0 < \epsilon \leq 2$ (i.e., $\delta_E(\epsilon)$ is a function $(0, 2] \rightarrow (0, 1]$).

Definition 2.2 [9]. Let C be a closed subset of a Banach space E . A mapping $T : C \rightarrow C$ is said to be semi-compact if, for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \rightarrow 0$ ($n \rightarrow \infty$), there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightarrow x^* \in C$.

Definition 2.3. A mapping $T : C \rightarrow C$ is said to be uniformly L -Lipschitzian if

$$\|T^n x - T^n y\| \leq L \|x - y\|$$

for all $x, y \in C$ and $n = 1, 2, \dots$, where L is a positive constant.

Lemma 2.1 [11]. Let $p > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda \|x\|^p + (1 - \lambda) \|y\|^p - w_p(\lambda) g(\|x - y\|)$$

for all $x, y \in B_r$ and $\lambda \in [0, 1]$, where B_r is the closed ball of E with center zero and radius r and

$$w_p(\lambda) = \lambda^p(1 - \lambda) + \lambda(1 - \lambda)^p.$$

Lemma 2.2 [5]. Let the nonnegative number sequences $\{a_n\}$, $\{u_n\}$ satisfy that

$$a_{n+1} \leq (1 + u_n)a_n, \quad \forall n = 1, 2, \dots, \quad \sum_{n=1}^{+\infty} u_n < +\infty.$$

Then

- (1) $\lim_{n \rightarrow +\infty} a_n$ exists;
- (2) If $\liminf_{n \rightarrow +\infty} a_n = 0$, then $\lim_{n \rightarrow +\infty} a_n = 0$.

3. Main results

Now we give our theorems in this paper. First of all, we have the following

Theorem 3.1. Let C be a nonempty closed convex subset of Banach space. Let $\{T_i, i \in I\}$ be N asymptotically quasi-nonexpansive self-mappings of C , i.e., $\|T_i^n x - q_i\| \leq (1 + u_{in}) \times \|x - q_i\|$ for all $x \in C$, $q_i \in F(T_i)$, $i \in I$. Suppose that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and that $x_0 \in C$, $\{\alpha_n\} \subset (s, 1 - s)$ for some $s \in (0, 1)$, $\sum_{n=1}^{+\infty} u_{in} < +\infty$ for all $i \in I$. Then the implicitly iterative sequence $\{x_n\}$ generated by (1.2) converges to a common fixed point in F if and only if $\liminf_{n \rightarrow +\infty} d(x_n, F) = 0$, where $d(x, F)$ denotes the distance of x to set F , i.e., $d(x, F) = \inf_{y \in F} d(x, y)$.

Proof. The necessity of the conditions is obvious. Thus, we will only prove the sufficiency. For any $p \in F$, from $x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n^k x_n$, where $n = (k - 1)N + i$, $T_n = T_{n(\text{mod } N)} = T_i$, $i \in I$, it follows that

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n x_{n-1} + (1 - \alpha_n) T_n^k x_n - p\| \\ &\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) \|T_i^k x_n - p\| \\ &\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n)(1 + u_{ik}) \|x_n - p\| \\ &\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n + u_{ik}) \|x_n - p\|. \end{aligned}$$

Transposing and simplifying above inequality, and noticing that $0 < s < \alpha_n < 1 - s < 1$, we have

$$\alpha_n \|x_n - p\| \leq \alpha_n \|x_{n-1} - p\| + u_{ik} \|x_n - p\| \leq \alpha_n \|x_{n-1} - p\| + u_{ik} \frac{\alpha_n}{s} \|x_n - p\|$$

and

$$\frac{s - u_{ik}}{s} \|x_n - p\| \leq \|x_{n-1} - p\|. \quad (3.1)$$

Since $\sum_{k=1}^{+\infty} u_{ik} < +\infty$ for all $i \in I$, thus $\lim_{k \rightarrow +\infty} u_{ik} = 0$, there exists a natural number n_0 , as $k > n_0/N + 1$, i.e., $n > n_0$ such that $s - u_{ik} > 0$ and $u_{ik} < s/2$. Then (3.1) becomes

$$\|x_n - p\| \leq \frac{s}{s - u_{ik}} \|x_{n-1} - p\|. \quad (3.2)$$

Let

$$1 + v_{ik} = \frac{s}{s - u_{ik}} = 1 + \frac{u_{ik}}{s - u_{ik}}.$$

Then

$$v_{ik} = \left(\frac{1}{s - u_{ik}} \right) u_{ik} < \frac{2}{s} u_{ik},$$

therefore

$$\sum_{k=1}^{+\infty} v_{ik} < \frac{2}{s} \sum_{k=1}^{+\infty} u_{ik} < +\infty \quad \text{for all } i \in I$$

and (3.2) becomes

$$\|x_n - p\| \leq (1 + v_{ik}) \|x_{n-1} - p\|, \quad \forall p \in F. \quad (3.3)$$

This implies that $d(x_n, F) \leq (1 + v_{ik})d(x_{n-1}, F)$. From Lemma 2.2 we have $\lim_{n \rightarrow +\infty} d(x_n, F) = 0$. Hereafter, we will prove that $\{x_n\}$ is a Cauchy sequence. Notice that when $x > 0$, $1 + x \leq e^x$, from (3.3) we have

$$\|x_{n+m} - p\| \leq \exp \left\{ \sum_{i=1}^N \sum_{k=1}^{+\infty} v_{ik} \right\} \|x_n - p\| < M \|x_n - p\|, \quad \forall p \in F, \quad (3.4)$$

for all natural numbers m, n , where $M = \exp \{ \sum_{i=1}^N \sum_{k=1}^{+\infty} v_{ik} \} + 1 < +\infty$. For all $\epsilon > 0$, there exists a natural number n_1 such that when $n \geq n_1$, $d(x_n, F) \leq \epsilon/3M$ as $\lim_{n \rightarrow +\infty} d(x_n, F) = 0$; specifically, $d(x_{n_1}, F) \leq \epsilon/3M$. Thus there exists a point $p_1 \in F$ such that $\|x_{n_1} - p_1\| \leq d(x_{n_1}, p_1) \leq \epsilon/2M$ by the definition of $d(x_n, F)$. It follows, from (3.4), that when $n \geq n_1$, for all m ,

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p_1\| + \|x_n - p_1\| < M \|x_{n_1} - p_1\| + M \|x_{n_1} - p_1\| \leq \epsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Because the space is complete, the sequence $\{x_n\}$ is convergent. Let $\lim_{n \rightarrow +\infty} x_n = p$. Moreover, since the set of fixed points of an asymptotically quasi-nonexpansive mapping is closed, so is F , thus $p \in F$ from $\lim_{n \rightarrow +\infty} d(x_n, F) = 0$, i.e., p is a common point of $F(T_i)$, for all $i \in I$. This completes the proof. $\square$

Corollary 3.2. Suppose the conditions are as same as in Theorem 3.1. Then the implicitly iterative sequence $\{x_n\}$ generated by (1.2) converges to a common fixed point $p \in F$ if and only if there exists some infinite subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to p .

The main purpose of this paper is to prove the following convergent result for the process (1.2).

Theorem 3.3. Let E be a real uniformly convex Banach space, C be a bounded closed convex subset of E . Let T_i , $i \in I$, be N uniformly L -Lipschitzian asymptotically quasi-nonexpansive self-mappings of C such that $\sum_{n=1}^{+\infty} u_{in} < +\infty$ for all $i \in I$. Let $F =$

$\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and there exists one member T in $\{T_i, i \in I\}$ to be semi-compact. Let $x_0 \in C$, and $\{\alpha_n\} \subset (s, 1-s)$ for some $s \in (0, 1)$. Then the sequence $\{x_n\}$ defined by the implicit iteration process (1.2) strongly converges to a common fixed point of the mappings $\{T_i, i \in I\}$.

Proof. Since C is bounded, take $r > 0$ such that $C \subset B(0, r)$, where $B(0, r)$ is the closed ball of E with center zero and radius r . By Lemma 2.1 with $p = 2$, we get for any $q \in F$

$$\|x_n - q\|^2 \leq \alpha_n \|x_{n-1} - q\|^2 + (1 - \alpha_n) \|T_n^k x_n - q\|^2 - \alpha_n (1 - \alpha_n) g(\sigma_n),$$

where $\sigma_n = \|T_n^k x_n - x_{n-1}\| = \|T_i^k x_n - x_{n-1}\|$, $n = (k-1)N + i$, $i \in I$. Since T_n is asymptotically quasi-nonexpansive, it follows that

$$\begin{aligned} \|x_n - q\|^2 &\leq \alpha_n \|x_{n-1} - q\|^2 + (1 - \alpha_n)(1 + u_{ik})^2 \|x_n - q\|^2 - \alpha_n (1 - \alpha_n) g(\sigma_n) \\ &\leq \alpha_n \|x_{n-1} - q\|^2 + (1 - \alpha_n + v_{ik}) \|x_n - q\|^2 - \alpha_n (1 - \alpha_n) g(\sigma_n), \end{aligned}$$

where $v_{ik} = 2u_{ik} + u_{ik}^2$. Hence $\sum_{k=1}^{+\infty} v_{ik} < +\infty$ for all $i \in I$. Thus, from above inequality and $s < 1 - \alpha_n < 1 - s$, we have

$$\|x_n - q\|^2 \leq \|x_{n-1} - q\|^2 + \frac{v_{ik}}{s} \|x_n - q\|^2 - (1 - \alpha_n) g(\sigma_n). \quad (3.5)$$

Hence

$$\|x_n - q\|^2 \leq \|x_{n-1} - q\|^2 + \frac{v_{ik}}{s} \|x_n - q\|^2.$$

Therefore, as in Theorem 3.1, we can show that $\lim_{n \rightarrow +\infty} \|x_n - q\|^2$ exists and let $\lim_{n \rightarrow +\infty} \|x_n - q\|^2 = d$. From (3.5) it follows that

$$s g(\sigma_n) \leq (1 - \alpha_n) g(\sigma_n) \leq \|x_{n-1} - q\|^2 - \|x_n - q\|^2 + M v_{ik},$$

where $M = 2r/s < +\infty$, r is the ball radius. Consequently, $\lim_{n \rightarrow +\infty} g(\sigma_n) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n \rightarrow +\infty} \sigma_n = \lim_{n \rightarrow +\infty} \|T_n^k x_n - x_{n-1}\| = 0$. Hence

$$\|x_n - x_{n-1}\| = (1 - \alpha_n) \|T_n^k x_n - x_{n-1}\| \rightarrow 0,$$

as well as

$$\|x_n - x_{n+l}\| \rightarrow 0$$

for all $l < 2N$. Hence, when $n > N$, we have

$$\begin{aligned} \|x_{n-1} - T_n x_n\| &\leq \|x_{n-1} - T_n^k x_n\| + \|T_n^k x_n - T_n x_n\| \leq \sigma_n + L \|T_n^{k-1} x_n - x_n\| \\ &\leq \sigma_n + L [\|T_n^{k-1} x_n - T_{n-N}^{k-1} x_{n-N}\| + \|T_{n-N}^{k-1} x_{n-N} - x_{(n-N)-1}\| \\ &\quad + \|x_{(n-N)-1} - x_n\|]. \end{aligned}$$

Notice that $n \equiv (n - N) \pmod{N}$. Thus $T_n = T_{n-N}$ and above inequality becomes

$$\|x_{n-1} - T_n x_n\| \leq \sigma_n + L^2 \|x_n - x_{n-N}\| + L \sigma_{n-N} + L \|x_n - x_{(n-N)-1}\|,$$

which yields that $\lim_{n \rightarrow +\infty} \|x_{n-1} - T_n x_n\| = 0$. From $\|x_n - T_n x_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|$, it follows that

$$\lim_{n \rightarrow +\infty} \|x_n - T_n x_n\| = 0.$$

Hence for all $l \in I$

$$\begin{aligned} \|x_n - T_{n+l} x_n\| &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\| + \|T_{n+l} x_{n+l} - T_{n+l} x_n\| \\ &\leq (1+L)\|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\|, \end{aligned}$$

which implies that

$$\lim_{n \rightarrow +\infty} \|x_n - T_{n+l} x_n\| = 0 \quad (\forall l \in I).$$

Thus

$$\lim_{n \rightarrow +\infty} \|x_n - T_l x_n\| = 0 \quad (\forall l \in I), \quad (3.6)$$

which lies on the fact that any subsequence of a convergent number sequence converges to the same limit.

By hypothesis that there exists T in $\{T_i, i \in I\}$ to be semi-compact, we may assume that T_1 is semi-compact without loss of generality. Therefore by (3.6) it follows that $\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = 0$ and by the definition of semi-compact there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow x^* \in C$. By (3.6) again, we have

$$\|x^* - T_l x^*\| = \lim_{n_j \rightarrow \infty} \|x_{n_j} - T_l x_{n_j}\| = 0 \quad (\forall 1 \leq l \leq N). \quad (3.7)$$

It shows that $x^* \in F$ and $\liminf_{n \rightarrow +\infty} d(x_n, F) = 0$, therefore by Theorem 3.1 and Corollary 3.2 we have that $\{x_n\}$ converges to a common fixed point q in F . This completes the proof. $\square$

Theorem 3.4. Let E be a real uniformly convex Banach space, C be a bounded closed convex subset of E . Let $T_i, i \in I$, be N asymptotically nonexpansive self-mappings of C such that $\sum_{n=1}^{+\infty} u_{in} < +\infty$ for all $i \in I$. Let $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and there exists one member T in $\{T_i, i \in I\}$ to be semi-compact. Let $x_0 \in C$, and $\{\alpha_n\} \subset (s, 1-s)$ for some $s \in (0, 1)$. Then the sequence $\{x_n\}$ defined by the implicit iteration process (1.2) strongly converges to a common fixed point of the mappings $\{T_i, i \in I\}$.

Remark. It is pointed out by Xu and Ori [10] that it is yet unclear what assumptions on the mappings $\{T_i\}_i^N$ and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$. However, our theorems answered this question to some extent.

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References

- [1] J.B. Diaz, F.T. Metcalf, On the structure of the set of subsequential limit points of successive approximations, *Bull. Amer. Math. Soc.* 73 (1967) 516–519.
- [2] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* 35 (1972) 171–174.
- [3] F.E. Browder, Nonexpansive nonlinear operators in Banach spaces, *Proc. Natl. Acad. Sci. USA* 54 (1965) 1041–1044.
- [4] F.E. Browder, Convergence of approximates to fixed points of nonexpansive nonlinear mappings in Banach spaces, *Arch. Rational Mech. Anal.* 24 (1967) 82–90.
- [5] Q. Liu, Iterative sequence for asymptotically quasi-nonexpansive mappings, *J. Math. Anal. Appl.* 256 (2001) 1–7.
- [6] R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.* 58 (1992) 486–491.
- [7] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, *J. Math. Anal. Appl.* 75 (1980) 287–292.
- [8] N. Shoji, W. Takahashi, Strong convergence theorems for asymptotically nonexpansive semigroups in Hilbert spaces, *Nonlinear Anal.* 34 (1998) 87–99.
- [9] S.S. Chang, Y.J. Cho, J.K. Kim, K.H. Kim, Iterative approximation of fixed points for asymptotically non-expansive type mappings in Banach spaces, *Panamer. Math. J.* 11 (2001) 53–63.
- [10] H.K. Xu, R.G. Ori, An implicit iteration process for nonexpansive mappings, *Numer. Funct. Anal. Optim.* 22 (2001) 767–773.
- [11] H.K. Xu, Inequalities in Banach spaces with applications, *Nonlinear Anal.* 16 (1991) 1127–1138.