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Uniqueness of entire functions

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Abstract

Let f be a nonconstant entire function and let a be a meromorphic function satisfying $T(r,a) = S(r,f)$ and $a \not\equiv a'$. If $f(z) = a(z) \Leftrightarrow f'(z) = a(z)$ and $f(z) = a(z) \Rightarrow f''(z) = a(z)$, then $f \equiv f'$, and $a \not\equiv a'$ is necessary. This extended a result due to Jank, Mues and Volkmann.

Keywords: Meromorphic function; Entire function; Unicity

1. Introduction

Let f be a nonconstant meromorphic function in the whole complex plane. We use the following standard notation of value distribution theory:

$$T(r,f),\ m(r,f),\ N(r,f),\ \bar{N}(r,f),\ \ldots$$

(see [2,5]). We denote by $S(r,f)$ any function satisfying

$$S(r,f) = o\{T(r,f)\},$$

as $r \to +\infty$, possibly outside of a set with finite measure.

Let a be a meromorphic function. If the function a satisfies $T(r,a) = S(r,f)$, then a is called a small function related to f.

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Let f, g be two meromorphic functions, and let a, b be two complex numbers. If $g(z) = b$ whenever $f(z) = a$, then we write $f = a \Rightarrow g = b$. If $f = a$ if and only if $g = b$, we write $f = a \Leftrightarrow g = b$. If $f = a \Leftrightarrow g = a$, then we say that f and g share a.

Mues and Steinmetz [4] obtained

Theorem A. *Let a, b be two distinct finite numbers, and let f be a nonconstant entire function. If $f(z) = a \Leftrightarrow f'(z) = a$, $f(z) = b \Leftrightarrow f'(z) = b$, then $f \equiv f'$.*

In 1986, Jank et al. [3] proved

Theorem B. *Let f be a nonconstant entire function, and a be a nonzero value. If $f(z) = a \Leftrightarrow f'(z) = a$ and $f(z) = a \Rightarrow f''(z) = a$, then $f \equiv f'$.*

We pose the following question.

Question *A*. If the value a is replaced by a small function a related to f, is it still true that $f \equiv f'$?

In this paper, we give an affirmative answer to Question A.

Theorem 1. *Let f be a nonconstant entire function; let a be a small function related to f and satisfy $a \not\equiv a'$. If $f(z) = a(z) \Leftrightarrow f'(z) = a(z)$ and $f(z) = a(z) \Rightarrow f''(z) = a(z)$, then $f \equiv f'$.*

Remark 1. Let $f(z) = e^{e^z} + e^z$, and $a(z) = e^z$. Then by Theorem 2.7 in [1], we know that $T(r, a) = S(r, f)$. Obviously, $f - a \neq 0$ and $f' - a \neq 0$. Hence, $f(z) = a(z) \Leftrightarrow f'(z) = a(z)$ and $f(z) = a(z) \Rightarrow f''(z) = a(z)$. But $f \not\equiv f'$. This shows that $a \not\equiv a'$ is necessary in Theorem 1.

2. Proof of Theorem 1

By Theorem A, we consider only the case that a is not a constant. For the sake of simplicity, we set

$$A(z) = a(z) - a'(z) \quad \text{and} \quad F(z) = f(z) - a(z). \tag{2.1}$$

Then

$$T(r, A) = S(r, f) \quad \text{and} \quad T(r, F) = T(r, f) + S(r, f). \tag{2.2}$$

We prove the theorem by contradiction. In the following we suppose that $f \not\equiv f'$. Next we proceed in the proof step by step as follows.

Step 1. We prove that

$$\bar{N}_{(2}\left(r, \frac{1}{F}\right) = \bar{N}_{(2}\left(r, \frac{1}{f' - a}\right) = S(r, F), \tag{2.3}$$

where $\bar{N}_{(2}(r, 1/F)$ is the counting function for multiple zero points of F and each zero point is counted one time.

Let z_0 be a multiple zero point of $F = f - a$ with multiplicity $k \geqslant 2$. Then by $f(z) = a(z) \Rightarrow f'(z) = a(z)$, we have (near z_0)

$$f(z) - a(z) = (z - z_0)^k f_1(z), \quad f_1(z_0) \neq 0, \tag{2.4}$$

$$f'(z) - a(z) = (z - z_0)^l f_2(z), \quad f_2(z_0) \neq 0. \tag{2.5}$$

By (2.4), we have

$$f'(z) - a'(z) = (z - z_0)^{k-1} f_3(z), \quad f_3(z_0) \neq 0. \tag{2.6}$$

Thus by (2.5) and (2.6),

$$A(z) = a(z) - a'(z) = (z - z_0)^m A_1(z), \quad A_1(z_0) \neq 0,$$

where $m \geqslant \min\{l, k-1\} \geqslant 1$. Thus

$$\bar{N}_{(2}\left(r, \frac{1}{F}\right) \leqslant \bar{N}\left(r, \frac{1}{A}\right) = S(r, F).$$

Similarly, we have

$$\bar{N}_{(2}\left(r, \frac{1}{f' - a}\right) = S(r, F).$$

Thus (2.3) is proved.

Step 2. We prove that

$$T(r, F) = 2N_{1)}\left(r, \frac{1}{F}\right) + S(r, F), \tag{2.7}$$

where $N_{1)}(r, 1/F)$ is the counting function for simple zero points of F.

By the second fundamental theorem with three small functions (see [2,5]) and the first fundamental theorem we have

$$\begin{aligned} T(r, F) &= T\left(r, \frac{1}{F}\right) + O(1) = N\left(r, \frac{1}{F}\right) + m\left(r, \frac{1}{F}\right) + O(1) \\ &\leqslant N\left(r, \frac{1}{F}\right) + m\left(r, \frac{1}{F'}\right) + S(r, F) \\ &\leqslant N\left(r, \frac{1}{F}\right) + T(r, f') - N\left(r, \frac{1}{f' - a'}\right) + S(r, F) \\ &\leqslant \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{f' - a}\right) + S(r, F) \leqslant 2\bar{N}\left(r, \frac{1}{F}\right) + S(r, F). \end{aligned} \tag{2.8}$$

Now let z_0 be a simple zero point of F but not a zero point of a. Then by $f(z) = a(z) \Rightarrow f'(z) = a(z)$ and $f'(z) = a(z) \Rightarrow f''(z) = a(z)$, we know that $a(z_0) \neq 0$, $A(z_0) \neq 0$ and

$$f(z_0) = f'(z_0) = f''(z_0) = a(z_0). \tag{2.9}$$

Thus we assume that, near z_0,

$$f(z) = a(z_0) + a(z_0)(z - z_0) + \frac{1}{2}a(z_0)(z - z_0)^2 + \frac{1}{6}f'''(z_0)(z - z_0)^3 + \cdots. \tag{2.10}$$

Hence we have

$$f'(z) = a(z_0) + a(z_0)(z - z_0) + \frac{1}{2}f'''(z_0)(z - z_0)^2 + \cdots, \tag{2.11}$$

$$f''(z) = a(z_0) + f'''(z_0)(z - z_0) + \cdots. \tag{2.12}$$

Thus we get

$$f(z) - f'(z) = \frac{1}{2}\big[a(z_0) - f'''(z_0)\big](z - z_0)^2 + \cdots.$$

Hence we have

$$\begin{aligned} 2N_{1)}\left(r, \frac{1}{f-a}\right) &\leqslant N\left(r, \frac{1}{f-f'}\right) + 2N\left(r, \frac{1}{a}\right) \leqslant T(r, f - f') + S(r, f) \\ &= m\left(r, f\left(1 - \frac{f'}{f}\right)\right) + S(r, f) \leqslant m(r, f) + S(r, f) \\ &= T(r, f) + S(r, f) = T(r, F) + S(r, F). \end{aligned} \tag{2.13}$$

Thus by (2.3), (2.8) and (2.13) we get (2.7).

Step 3. We prove that

$$\frac{F''}{F'} \not\equiv \frac{A + A'}{A}. \tag{2.14}$$

If

$$\frac{F''}{F'} \equiv \frac{A + A'}{A}, \tag{2.15}$$

then integrating the two sides of (2.15) we obtain that

$$F' = CAe^z, \tag{2.16}$$

where C is a nonzero constant. Thus by (2.1) we get

$$f' = a' + CAe^z, \tag{2.17}$$

$$f'' = a'' + C(A + A')e^z. \tag{2.18}$$

If $A' \equiv 0$, then $A \equiv C_0$ ($\neq 0$), where C_0 is a nonzero constant. Thus by (2.1), we have

$$a = C_0 + C_1 e^z, \tag{2.19}$$

where C_1 is a constant.

By (2.17) and (2.19) we get

$$f' = C_3 e^z \quad \text{and} \quad f = C_3 e^z + C_4,$$

where C_2, C_3 and C_4 are constants.

Since a is a small function of f, by (2.19) we get $C_1 = 0$, i.e., a is a nonzero constant, a contradiction.

Thus we deduce that $A' \not\equiv 0$. Set

$$R(z) = \frac{F(z)}{Ce^z - 1}. \tag{2.20}$$

By (2.7) there exist simple zero points of F which are not zero points of A'. Let z_0 be a simple zero point of F but not a zero point of A'. Then by $f(z) = a(z) \Rightarrow f'(z) = a(z)$ and $f'(z) = a(z) \Rightarrow f''(z) = a(z)$, we have

$$f(z_0) = f'(z_0) = f''(z_0) = a(z_0).$$

Thus by (2.17) and (2.18),

$$a(z_0) = a'(z_0) + CA(z_0)e^{z_0}, \qquad a(z_0) = a''(z_0) + C\big[A(z_0) + A'(z_0)\big]e^{z_0}.$$

Thus we get

$$Ce^{z_0} = 1,$$

since $A'(z_0) \neq 0$.

On the other hand, by (2.17) it is easily to see that $Ce^z = 1 \Rightarrow f'(z) = a(z)$. Thus, by $f'(z) = a(z) \Rightarrow f(z) = a(z)$ we deduce that $Ce^z = 1 \Rightarrow F(z) = 0$. Hence by (2.2), (2.16) and (2.20) we have

$$N(r, R) \leqslant N(r, F) \leqslant N(r, a) = S(r, F), \tag{2.21}$$

$$\begin{aligned} N\left(r, \frac{1}{R}\right) &\leqslant N_{(2}\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{A'}\right) \leqslant 2N\left(r, \frac{1}{F'}\right) + N\left(r, \frac{1}{A'}\right) \\ &\leqslant 2N\left(r, \frac{1}{A}\right) + N\left(r, \frac{1}{A'}\right) \leqslant 2T(r, A) + T(r, A') + O(1) \\ &= S(r, f). \end{aligned} \tag{2.22}$$

By (2.20) we have

$$f = F + a = a + R(Ce^z - 1). \tag{2.23}$$

Thus by (2.17), (2.18) and (2.23) we obtain that

$$(R' + R - A)Ce^z = R', \tag{2.24}$$

$$(R'' + 2R' + R - A - A')Ce^z = R''. \tag{2.25}$$

Thus by (2.24) and (2.25) we obtain that

$$R''R - R'R - 2(R')^2 = AR'' - (A + A')R'. \tag{2.26}$$

If

$$R''R - R'R - 2(R')^2 \equiv 0, \tag{2.27}$$

then

$$AR'' - (A + A')R' \equiv 0. \tag{2.28}$$

If $R' \equiv 0$, then $R \equiv C_5$, where C_5 is a constant. Thus by (2.20) we get $F(z) = C_5 \times (Ce^z - 1)$. Hence by (2.17) we deduce that $A \equiv C_5$, which contradicts $A' \not\equiv 0$.

So $R' \not\equiv 0$. Thus by (2.28),

$$\frac{R''}{R'} = 1 + \frac{A'}{A}. \tag{2.29}$$

Integrating the two sides of (2.29), we get

$$R' = C_6 A e^z, \tag{2.30}$$

where C_6 is a constant. Thus

$$R'' = C_6(A + A')e^z. \tag{2.31}$$

By (2.30), (2.31) and (2.27) we obtain

$$R = \frac{2C_6 A^2 e^z}{A'}. \tag{2.32}$$

By (2.30) and (2.32) we have

$$\frac{2R'}{R} = \frac{A'}{A}.$$

Thus we get

$$R^2 = C_7 A,$$

where C_7 is a constant.

Hence we have

$$T(r, R) = O\big(T(r, A)\big) = S(r, F). \tag{2.33}$$

Thus by (2.20) we know

$$T(r, F) = T(r, e^z) + S(r, F).$$

Thus we get $S(r, F) = S(r, e^z)$. By (2.24) we get that $R' \equiv 0$, which contradicts $R' \not\equiv 0$.

Now we consider the remain case that

$$R''R - R'R - 2(R')^2 \not\equiv 0.$$

In this case we deduce from (2.26) that

$$R = \frac{A(R''/R) - (A + A')(R'/R)}{R''/R - R'/R - 2(R')^2/R^2}. \tag{2.34}$$

Thus by (2.21), (2.33), (2.34) and Nevanlinna's first fundamental theorem, we get

$$\begin{aligned} m(r, R) &\leqslant O\left\{N(r, R) + N\left(r, \frac{1}{R}\right)\right\} + S(r, R) + S(r, F) \\ &\leqslant S(r, R) + S(r, F). \end{aligned} \tag{2.35}$$

Hence

$$T(r,R)=m(r,R)+N(r,R)\leqslant S(r,R)+S(r,F)=S(r,F).$$

Next we obtain a contradiction as the case of $R'\not\equiv 0$.

Step 4. We prove that

$$T(r,F)\leqslant 2\bar{N}\left(r,\frac{1}{F'}\right)+S(r,F). \tag{2.36}$$

Indeed, by $f(z)=a(z)\Rightarrow f'(z)=a(z)$ and $f'(z)=a(z)\Rightarrow f''(z)=a(z)$, we know that at every simple zero point z_0 of F, $A(z_0)\neq 0$ and $f'(z_0)=f''(z_0)=a(z_0)$, that is $F'(z_0)=A(z_0)$ and $F''(z_0)=A(z_0)+A'(z_0)$. Hence we get from (2.14) that

$$\begin{aligned}N_{1)}\left(r,\frac{1}{F}\right)&\leqslant N\left(r,\frac{1}{F''/F'-(A+A')/A}\right)\leqslant T\left(r,\frac{F''}{F'}\right)+S(r,F)\\&\leqslant N\left(r,\frac{F''}{F'}\right)+S(r,F)\leqslant \bar{N}\left(r,\frac{1}{F'}\right)+S(r,F).\end{aligned}$$

Thus by (2.7), we obtain (2.36).

Step 5. Set

$$\phi=\frac{F'}{F}-\frac{F''-A'}{F'-A}, \tag{2.37}$$

$$\psi=\frac{AF''-(A+A')F'}{F}. \tag{2.38}$$

Obviously, by the logarithmic derivative lemma (see [2,5]) we have

$$m(r,\phi)=S(r,F),\qquad m(r,\psi)=S(r,F).$$

We rewrite (2.37) and (2.38) in the following form:

$$\phi=\frac{f'-a'}{f-a}-\frac{f''-a'}{f'-a}, \tag{2.39}$$

$$\psi=\frac{A(f''-a)-(A+A')(f'-a)}{f-a}. \tag{2.40}$$

It is easy to see that all the possible poles of ϕ and ψ come from the multiple zero points of F, the multiple zero points of $f'-a$, and the poles of A. Thus by (2.3) we get

$$N(r,\phi)\leqslant \bar{N}_{(2}\left(r,\frac{1}{F}\right)+\bar{N}_{(2}\left(r,\frac{1}{f'-a}\right)+N(r,A)=S(r,F),$$

$$N(r,\psi)\leqslant 2\bar{N}_{(2}\left(r,\frac{1}{F}\right)+2N(r,A)=S(r,F).$$

So

$$T(r,\phi)=S(r,F)\quad\text{and}\quad T(r,\psi)=S(r,F). \tag{2.41}$$

Now let z_0 be a simple zero point of F but not a zero point of a. Then by (2.10)–(2.12), (2.39) and (2.40), we get

$$\phi(z_0) = -\frac{f'''(z_0) - a(z_0)}{2A(z_0)}$$

and

$$\psi(z_0) = f'''(z_0) - a(z_0) - A'(z_0).$$

Thus we get

$$2A(z_0)\phi(z_0) + \psi(z_0) + A'(z_0) = 0. \tag{2.42}$$

We claim that

$$2A\phi + \psi + A' \equiv 0. \tag{2.43}$$

If $2A\phi + \psi + A' \not\equiv 0$, then by (2.41) and (2.42) we get

$$N_{1)}\left(r, \frac{1}{F}\right) \leqslant N\left(r, \frac{1}{2A\phi + \psi + A'}\right) + N\left(r, \frac{1}{a}\right)$$
$$\leqslant T(r, \phi) + T(r, \psi) + 3T(r, a) + S(r, F) = S(r, F). \tag{2.44}$$

Thus by (2.7) and (2.44) we get a contradiction, $T(r, F) \leqslant S(r, F)$. So, (2.43) is proved.

Step 6. We prove that

$$2AA'\psi' + \psi^3 + 2A\psi^2 + \left[2AA' - 2AA'' - (A')^2\right]\psi \equiv 0. \tag{2.45}$$

Let z_1 satisfy $F'(z_1) = 0$ and $A(z_1) \neq 0$, $A'(z_1) \neq 0$ (such points must exist by (2.36)). Then $F(z_1) \neq 0$. Otherwise, we get $f(z_1) = f'(z_1) = a(z_1)$ by $f(z) = a(z) \Rightarrow f'(z) = a(z)$. Thus, $a(z_1) = a'(z_1)$, which contradicts $A(z_1) \neq 0$.

Hence by (2.37) and (2.38) we have

$$\phi(z_1) = \frac{F''(z_1) - A'(z_1)}{A(z_1)}, \tag{2.46}$$

$$\psi(z_1) = \frac{A(z_1)F''(z_1)}{F(z_1)}. \tag{2.47}$$

Hence by (2.43), (2.46) and (2.47) we obtain

$$\left[2F(z_1) + A(z_1)\right]F''(z_1) = A'(z_1)F(z_1).$$

If $2F(z_1) + A(z_1) = 0$, then $F(z_1) = 0$, a contradiction. Hence $2F(z_1) + A(z_1) \neq 0$. Thus we have

$$F''(z_1) = \frac{A'(z_1)F(z_1)}{2F(z_1) + A(z_1)}. \tag{2.48}$$

Thus by (2.46)–(2.48) we get

$$\phi(z_1) = -\frac{A'(z_1)[F(z_1) + A(z_1)]}{A(z_1)[2F(z_1) + A(z_1)]}, \tag{2.49}$$

$$\psi(z_1) = \frac{A(z_1)A'(z_1)}{2F(z_1) + A(z_1)}. \tag{2.50}$$

Now we compute the derivative of ϕ, ψ at z_1. By (2.37), (2.38) and (2.48) we get

$$\phi'(z_1) = \frac{F'''(z_1)}{A(z_1)} - \frac{A''(z_1)}{A(z_1)} + \frac{A'(z_1)}{2F(z_1)+A(z_1)} + \frac{(A')^2(z_1)[F(z_1)+A(z_1)]^2}{A^2(z_1)[2F(z_1)+A(z_1)]^2}, \tag{2.51}$$

$$\psi'(z_1) = \frac{A(z_1)F'''(z_1)}{F(z_1)} - \frac{A(z_1)A'(z_1)}{2F(z_1)+A(z_1)}. \tag{2.52}$$

By (2.43) we get

$$2A'\phi + 2A\phi' + \psi' + A'' \equiv 0.$$

Thus we have

$$2A'(z_1)\phi(z_1) + 2A(z_1)\phi'(z_1) + \psi'(z_1) + A''(z_1) = 0. \tag{2.53}$$

By (2.49) and (2.51)–(2.53) we get

$$F'''(z_1) = \frac{F(z_1)}{2F(z_1)+A(z_1)}\left\{A''(z_1) - \frac{A(z_1)A'(z_1)}{2F(z_1)+A(z_1)} + \frac{2(A')^2(z_1)[F(z_1)+A(z_1)]}{A(z_1)[2F(z_1)+A(z_1)]} - \frac{2(A')^2(z_1)[F(z_1)+A(z_1)]^2}{A(z_1)[2F(z_1)+A(z_1)]^2}\right\}. \tag{2.54}$$

By (2.50) we get

$$F(z_1) = -\frac{A(z_1)}{2} + \frac{A(z_1)A'(z_1)}{2\psi(z_1)}. \tag{2.55}$$

Hence by (2.52), (2.54) and (2.55) and some computation, we obtain

$$2A(z_1)A'(z_1)\psi'(z_1) + \psi^3(z_1) + 2A(z_1)\psi^2(z_1) + \left[2A(z_1)A'(z_1) - 2A(z_1)A''(z_1) - (A')^2(z_1)\right]\psi(z_1) = 0. \tag{2.56}$$

Set

$$\Delta = 2AA'\psi' + \psi^3 + 2A\psi^2 + \left[2AA' - 2AA'' - (A')^2\right]\psi. \tag{2.57}$$

If $\Delta \not\equiv 0$, then by (2.56) and (2.41) we get

$$\bar{N}\left(r, \frac{1}{F'}\right) \leqslant N\left(r, \frac{1}{\Delta}\right) + N\left(r, \frac{1}{A}\right) + N\left(r, \frac{1}{A'}\right) \leqslant T(r, \Delta) + S(r, F) \leqslant S(r, F). \tag{2.58}$$

Thus by (2.36) and (2.58) we get a contradiction, $T(r, F) = S(r, F)$. Hence $\Delta \equiv 0$. Thus (2.45) is proved.

Step 7. Set

$$H = \frac{F' - A}{F} = \frac{f' - a}{f - a}. \tag{2.59}$$

Then by (2.3) we have

$$\bar{N}(r, H) + \bar{N}\left(r, \frac{1}{H}\right) \leqslant \bar{N}_{(2}\left(r, \frac{1}{F}\right) + \bar{N}_{(2}\left(r, \frac{1}{f' - a}\right) = S(r, F). \tag{2.60}$$

By (2.59) we get

$$F' = A + HF, \tag{2.61}$$

$$F'' = A' + H'F + HF' = A' + HA + (H' + H^2)F. \tag{2.62}$$

By (2.38) we have

$$AF'' - (A + A')F' - \psi F = 0. \tag{2.63}$$

Thus by (2.61)–(2.63) we have

$$\big[(H' + H^2)A - (A + A')H - \psi\big]F = A^2(1 - H). \tag{2.64}$$

If $(H' + H^2)A - (A + A')H - \psi \equiv 0$, then by (2.64) and $A \not\equiv 0$ we deduce that $H \equiv 1$. Thus by (2.59), $f \equiv f'$, a contradiction.

Hence $(H' + H^2)A - (A + A')H - \psi \not\equiv 0$. Thus by (2.64) we have

$$F = \frac{A^2(1 - H)}{(H' + H^2)A - (A + A')H - \psi}. \tag{2.65}$$

By (2.64) we get

$$\begin{aligned}
&\big[A'H' + AH'' + A'H^2 + 2AHH' - (A' + A'')H - (A + A')H' - \psi'\big]F \\
&\qquad + \big[AH' + AH^2 - (A + A')H - \psi\big]F' \\
&\quad = 2AA'(1 - H) - A^2H'.
\end{aligned} \tag{2.66}$$

Thus by (2.61) and (2.66) we have

$$\begin{aligned}
&\big[-AH' + AH'' - AH^2 + 3AHH' - (A' + A'' + \psi)H + AH^3 - \psi'\big]F \\
&\quad = 2AA' + A\psi - (AA' - A^2)H - 2A^2H' - A^2H^2.
\end{aligned} \tag{2.67}$$

By (2.67) and (2.65) we get

$$\begin{aligned}
&\big[-AH' + AH'' - AH^2 + 3AHH' - (A' + A'' + \psi)H + AH^3 - \psi'\big]A^2(1 - H) \\
&\quad = \big[2AA' + A\psi - (AA' - A^2)H - 2A^2H' - A^2H^2\big] \\
&\qquad \times \big[(H' + H^2)A - (A + A')H - \psi\big].
\end{aligned}$$

Thus we have

$$P_1H^2 + P_2HH' + P_3HH'' + P_4(H')^2 = P_5H + P_6H' + P_7H'' + P_8, \tag{2.68}$$

where

$$\begin{aligned}
&P_1 = A^2A' + A(A')^2 - A^2A'' + A^2\psi, \qquad P_2 = A^2A' - A^3, \\
&P_3 = A^3, \qquad P_4 = -2A^3, \\
&P_5 = A^2A' - A^2A'' + 2A(A')^2 + A^2\psi + A^2\psi', \\
&P_6 = -A^3 - 2A^2A' - 3A^2\psi, \\
&P_7 = A^3, \qquad P_8 = A\psi^2 + 2AA'\psi - A^2\psi'.
\end{aligned}$$

If

$$P_1H^2 + P_2HH' + P_3HH'' + P_4(H')^2 \equiv 0,$$

then by (2.68) we have

$$P_5H + P_6H' + P_7H'' + P_8 \equiv 0. \tag{2.69}$$

If $P_8 \equiv 0$, that is

$$A\psi^2 + 2AA'\psi - A^2\psi' \equiv 0,$$

then we have

$$\left(\frac{A^2}{\psi}\right)' = -A.$$

Set

$$K = -\frac{A^2}{\psi}. \tag{2.70}$$

Then by (2.70), K is meromorphic and

$$K' = A, \qquad K'' = A', \qquad K''' = A''. \tag{2.71}$$

Thus by (2.70) we get

$$\psi = -\frac{(K')^2}{K}. \tag{2.72}$$

By (2.71), (2.72), (2.45), and some computation we obtain that

$$\begin{aligned} &2K(K')^2K'' - (K')^4 + 2K(K')^3 - 2K^2K'K'' \\ &\quad + 2K^2K'K''' - 3K^2(K'')^2 = 0. \end{aligned} \tag{2.73}$$

Claim. K has no poles and zeros.

Let z_0 be a zero of K with multiple q $(\geqslant 1)$. Then by (2.73), z_0 is also a zero of K'. Thus $q \geqslant 2$. Next we consider two cases.

Case 1: $q = 2$. We assume that (near z_0),

$$K(z) = a_2(z - z_0)^2 + a_3(z - z_0)^3 + a_4(z - z_0)^4 + \cdots, \quad a_2 \neq 0. \tag{2.74}$$

Thus we get

$$K'(z) = 2a_2(z - z_0) + 3a_3(z - z_0)^2 + 4a_4(z - z_0)^3 + \cdots, \tag{2.75}$$

$$K''(z) = 2a_2 + 6a_3(z - z_0) + 12a_4(z - z_0)^2 + \cdots, \tag{2.76}$$

$$K'''(z) = 6a_3 + 24a_4(z - z_0) + \cdots. \tag{2.77}$$

Hence by (2.74)–(2.77) and (2.73) we get

$$
\begin{aligned}
&16a_2^4(z-z_0)^4\big[1+b_1(z-z_0)+b_2(z-z_0)^2+\cdots\big]\\
&\quad-16a_2^4(z-z_0)^4\big[1+c_1(z-z_0)+c_2(z-z_0)^2+\cdots\big]\\
&\quad+16a_2^4(z-z_0)^5\big[1+d_1(z-z_0)+d_2(z-z_0)^2+\cdots\big]\\
&\quad-8a_2^4(z-z_0)^5\big[1+e_1(z-z_0)+e_2(z-z_0)^2+\cdots\big]\\
&\quad+24a_2^3a_3(z-z_0)^5\big[1+f_1(z-z_0)+f_2(z-z_0)^2+\cdots\big]\\
&\quad-12a_2^4(z-z_0)^4\big[1+g_1(z-z_0)+g_2(z-z_0)^2+\cdots\big]\equiv 0. \qquad (2.78)
\end{aligned}
$$

Thus by (2.78) we get $16a_2^4-16a_2^4-12a_2^4=0$. Hence, $a_2=0$, a contradiction.

Case 2: $q\geqslant 3$. We assume that (near z_0),

$$K(z)=a_q(z-z_0)^q+a_{q+1}(z-z_0)^{q+1}+a_{q+2}(z-z_0)^{q+2}+\cdots,\quad a_q\neq 0. \qquad (2.79)$$

Thus we get

$$
\begin{aligned}
K'(z)&=qa_q(z-z_0)^{q-1}+(q+1)a_{q+1}(z-z_0)^q\\
&\quad+(q+2)a_{q+2}(z-z_0)^{q+1}+\cdots, \qquad (2.80)\\
K''(z)&=q(q-1)a_q(z-z_0)^{q-2}+(q+1)qa_{q+1}(z-z_0)^{q-1}+\cdots, \qquad (2.81)\\
K'''(z)&=q(q-1)(q-2)a_q(z-z_0)^{q-3}\\
&\quad+(q+1)q(q-1)a_{q+1}(z-z_0)^{q-2}+\cdots. \qquad (2.82)
\end{aligned}
$$

Thus by (2.79)–(2.82) and (2.73) we get

$$
\begin{aligned}
&2a_q^4q^3(q-1)(z-z_0)^{4q-4}\big[1+b_1(z-z_0)+b_2(z-z_0)^2+\cdots\big]\\
&\quad-a_q^4q^4(z-z_0)^{4q-4}\big[1+c_1(z-z_0)+c_2(z-z_0)^2+\cdots\big]\\
&\quad+2a_q^4(z-z_0)^{4q-3}\big[1+d_1(z-z_0)+d_2(z-z_0)^2+\cdots\big]\\
&\quad-2a_q^4q^2(q-1)(z-z_0)^{4q-3}\big[1+e_1(z-z_0)+e_2(z-z_0)^2+\cdots\big]\\
&\quad+2a_q^4q^2(q-1)(q-2)(z-z_0)^{4q-4}\big[1+f_1(z-z_0)+f_2(z-z_0)^2+\cdots\big]\\
&\quad-3a_3^4q^2(q-1)^2(z-z_0)^{4q-4}\big[1+g_1(z-z_0)+g_2(z-z_0)^2+\cdots\big]\equiv 0. \qquad (2.83)
\end{aligned}
$$

Thus by (2.83) we get $2q^3(q-1)a_q^4-q^4a_q^4+2q^2(q-1)(q-2)a_q^4-3q^2(q-1)^2a_q^4=0$. Hence, $a_q=0$, a contradiction.

Hence we prove that $K\neq 0$. Similarly, we can prove $K\neq\infty$. Thus the claim is proved.

Thus there exists an entire function $\alpha=\alpha(z)$ such that

$$K=e^{\alpha}. \qquad (2.84)$$

By (2.71) and (2.84) we have

$$2(\alpha')^2\alpha''=2\alpha'\alpha''-2\alpha'\alpha'''+3(\alpha'')^2. \qquad (2.85)$$

If $\alpha''\not\equiv 0$, then by (2.85) and the same argument as used in proving $K\neq 0$ we get $\alpha'\neq 0$. Thus by (2.85) we have

$$\alpha'=\frac{\alpha''/\alpha'-\alpha'''/\alpha'+(3/2)(\alpha''/\alpha')^2}{\alpha''/\alpha'}.$$

Therefore we obtain a contradiction,

$$\begin{aligned}
T(r,\alpha') = m(r,\alpha') &\leqslant m\left(r, \frac{\alpha''}{\alpha'} - \frac{\alpha'''}{\alpha'} + \frac{3}{2}\cdot\left(\frac{\alpha''}{\alpha'}\right)^2\right) + m\left(r, \frac{\alpha'}{\alpha''}\right) + S(r,\alpha') \\
&\leqslant T\left(r, \frac{\alpha''}{\alpha'}\right) + S(r,\alpha') = m\left(r, \frac{\alpha''}{\alpha'}\right) + N\left(r, \frac{\alpha''}{\alpha'}\right) + S(r,\alpha') \\
&\leqslant \bar{N}\left(r, \frac{1}{\alpha'}\right) + \bar{N}(r,\alpha') + S(r,\alpha') = S(r,\alpha').
\end{aligned}$$

Hence $\alpha'' \equiv 0$. That is,

$$\alpha = \lambda z + \mu,$$

where λ, μ are two constants.

Thus by (2.71), (2.72) and (2.84) we get

$$A = \lambda e^{\lambda z+\mu}, \tag{2.86}$$

$$\psi = -\lambda^2 e^{\lambda z+\mu}. \tag{2.87}$$

Hence by (2.63) we have

$$F'' - (1+\lambda)F' + \lambda F = 0. \tag{2.88}$$

Solving Eq. (2.88) we obtain

$$F(z) = \begin{cases} c_1 e^z + c_2 e^{\lambda z}, & \lambda \neq 1, \\ (c_1 + c_2 z)e^z, & \lambda = 1. \end{cases} \tag{2.89}$$

Here c_1, c_2 are two constants.

Since A is a small function of related to F, we deduce from (2.86) and (2.89) that $\lambda = 0$. But by (2.86), $A \equiv 0$, which is a contradiction.

Hence

$$P_8 \not\equiv 0.$$

Thus by (2.69) we have

$$\frac{1}{H} = -\frac{1}{P_8}\left(P_5 + P_6\frac{H'}{H} + P_7\frac{H''}{H}\right). \tag{2.90}$$

Hence we get

$$m\left(r, \frac{1}{H}\right) = S(r,H) + S(r,F). \tag{2.91}$$

By (2.90), (2.60), (2.68), (2.41) and (2.2) we get

$$\begin{aligned}
N\left(r, \frac{1}{H}\right) &\leqslant N\left(r, \frac{1}{P_8}\right) + N(r,P_5) + N(r,P_6) + N(r,P_7) \\
&\quad + N\left(r, \frac{H'}{H}\right) + N\left(r, \frac{H''}{H}\right) + O(1) \\
&\leqslant 3\bar{N}(r,H) + 3\bar{N}\left(r, \frac{1}{H}\right) + S(r,F) = S(r,F).
\end{aligned} \tag{2.92}$$

Thus by (2.91) and (2.92) we obtain

$$T(r,H)=m\left(r,\frac{1}{H}\right)+N\left(r,\frac{1}{H}\right)=S(r,F). \tag{2.93}$$

Then by (2.59) we have

$$\frac{1}{F}=\frac{1}{A}\frac{F'}{F}-\frac{H}{A}.$$

Thus by (2.93), (2.2) and (2.3) we get

$$m\left(r,\frac{1}{F}\right)=m\left(r,\frac{1}{A}\frac{F'}{F}-\frac{H}{A}\right)=S(r,F) \tag{2.94}$$

and

$$\begin{aligned}N\left(r,\frac{1}{F}\right)&\leqslant 2N\left(r,\frac{1}{A}\right)+N\left(r,\frac{F'}{F}\right)+N(r,H)\\&\leqslant \bar{N}(r,F)+\bar{N}\left(r,\frac{1}{F}\right)+S(r,F)\\&\leqslant N_1\left(r,\frac{1}{F}\right)+S(r,F)\leqslant N\left(r,\frac{1}{F}\right)+S(r,F).\end{aligned}$$

Thus we have

$$N\left(r,\frac{1}{F}\right)=N_1\left(r,\frac{1}{F}\right)+S(r,F). \tag{2.95}$$

By (2.94) we get

$$T(r,F)=N\left(r,\frac{1}{F}\right)+S(r,F). \tag{2.96}$$

Thus by (2.3), (2.7), (2.60), (2.95) and (2.96) we get a contradiction, $T(r,F)=S(r,F)$. Therefore

$$P_1H^2+P_2HH'+P_3HH''+P_4(H')^2\not\equiv 0.$$

Thus by (2.68) we have

$$H^2=\frac{H(P_5+P_6(H'/H)+P_7(H''/H))+P_8}{P_1+P_2(H'/H)+P_3(H''/H)+P_4(H'/H)^2}.$$

Thus we have

$$\begin{aligned}2m(r,H)=m(r,H^2)&\leqslant m\left(r,H\left(P_5+P_6\frac{H'}{H}+P_7\frac{H''}{H}\right)+P_8\right)\\&\quad+T\left(r,P_1+P_2\frac{H'}{H}+P_3\frac{H''}{H}+P_4\left(\frac{H'}{H}\right)^2\right)\\&\leqslant m(r,H)+S(r,H)+S(r,F).\end{aligned}$$

Hence we get

$$m(r,H)\leqslant S(r,H)+S(r,F).$$

Since $N(r, H) = S(r, F)$, we get

$$T(r, H) = S(r, F). \tag{2.97}$$

Next we get a contradiction as the case of $P_8 \not\equiv 0$. Thus the proof of Theorem 1 is complete. □

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