



Some reduction and transformation formulas for the Appell hypergeometric function F_2

Sheldon B. Opps^a, Nasser Saad^b, H.M. Srivastava^{c,*}

^a Department of Physics, University of Prince Edward Island, Charlottetown,
Prince Edward Island C1A 4P3, Canada

^b Department of Mathematics and Statistics, University of Prince Edward Island,
Charlottetown, Prince Edward Island C1A 4P3, Canada

^c Department of Mathematics and Statistics, University of Victoria,
Victoria, British Columbia V8W 3P4, Canada

Received 23 July 2004

Available online 2 October 2004

Submitted by William F. Ames

Abstract

An integral representation of the Appell series $F_2(\sigma, \alpha_1, \alpha_2; \beta_1, \beta_2; x, y)$ is used here to obtain several finite-sum expansions in terms of the less cumbersome hypergeometric functions ${}_2F_1$ and ${}_3F_2$. In the case when the parameters $\sigma, \alpha_1, \alpha_2, \beta_1$, and β_2 are all positive integers, some of our results may be seen as a generalization of the finite-sum expansions of the Appell series F_1 obtained by Cuyt et al. [J. Comput. Appl. Math. 5 (1999) 213–219]. Other important consequences (as well as potential applications) of our results are discussed and a listing of some useful reduction (and transformation) formulas is provided.

© 2004 Elsevier Inc. All rights reserved.

Keywords: Multiple hypergeometric functions; Appell series; Integral representations; Singular potentials; Gordon's integral; Gauss hypergeometric function; Clausen hypergeometric function; Kummer (or confluent) hypergeometric function; Reduction and transformation formulas; Singular operators; Dilogarithm function; Digamma (or Psi) function

* Corresponding author.

E-mail addresses: sopps@upei.ca (S.B. Opps), nsaad@upei.ca (N. Saad), harimsri@math.uvic.ca (H.M. Srivastava).

1. Introduction and definitions

The Appell hypergeometric series

$$F_2(\sigma, \alpha_1, \alpha_2; \beta_1, \beta_2; x, y)$$

arises frequently in various physical and chemical applications (cf. [2,3,5,6,8,11]). It is defined by

$$F_2(\sigma, \alpha_1, \alpha_2; \beta_1, \beta_2; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\sigma)_{m+n} (\alpha_1)_m (\alpha_2)_n}{(\beta_1)_m (\beta_2)_n} \frac{x^m}{m!} \frac{y^n}{n!} \quad (|x| + |y| < 1; \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- := \{0, -1, -2, \dots\}), \quad (1.1)$$

where $(\lambda)_k$ denotes the Pochhammer symbol defined, in terms of Gamma functions, by

$$(\lambda)_k := \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k = 0; \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda + 1)(\lambda + 2) \cdots (\lambda + k - 1) & (k \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases}$$

$\mathbb{N}$ being the set of positive integers.

Recently, Cuyt et al. [1] obtained a finite algebraic sum representation of the Appell hypergeometric series F_1 defined by

$$F_1(a, b, b'; c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad (\max\{|x|, |y|\} < 1) \quad (1.2)$$

in the case when the parameters a, b, b' , and c are positive integers with $c > a$. Their results can be summarized for the simplest case as follows [1, p. 215, Theorem 2.1a, b, and c]:

For any $s, t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $x \neq y$,

$$F_1(1, s + 1, t + 1; 2; x, y) = \frac{(-x)^t y^s}{(y - x)^{s+t+1}} \binom{s+t}{s} [\ln(1 - x) - \ln(1 - y)] \\ - \sum_{j=0}^{t-1} \binom{j+s}{s} \frac{(-x)^j y^s [1 - (1 - y)^{j-t}]}{(y - x)^{s+j+1} (t - j)} \\ - \sum_{k=0}^{s-1} \binom{k+t}{t} \frac{x^t (-y)^k [1 - (1 - x)^{k-s}]}{(x - y)^{t+k+1} (s - k)} \quad (\max\{|x|, |y|\} < 1). \quad (1.3)$$

Motivated by the aforementioned investigation by Cuyt et al. [1], we extend their results to obtain a finite algebraic sum representation of Appell's second series

$$F_2(\sigma, \alpha_1, \alpha_2; \beta_1, \beta_2; x, y)$$

in the case when the parameters $\sigma, \alpha_1, \alpha_2, \beta_1$, and β_2 are positive integers. We show that their results are indeed special cases of our representation for F_2 . As intermediate steps, we derive a number of reduction and transformation formulas for F_2 . The paper is organized as follows: In Section 2, we develop reduction formulas for F_2 in terms of the Gauss hypergeometric function ${}_2F_1$ and the Clausen hypergeometric function ${}_3F_2$. We then show that the derived form of F_2 reduces to the known expression (1.3) for F_1 . In Section 3, we extend these results to obtain finite-sum representations of F_2 with nonnegative parameters.

2. Reduction formulas involving the Gauss ${}_2F_1$ and the Clausen ${}_3F_2$ hypergeometric functions

Lemma 1 (cf., e.g., [10, pp. 20–21]). For $|z| < 1$, the Gauss hypergeometric function ${}_2F_1$ is given by

$${}_2F_1\left(\begin{matrix} a+1, 1 \\ 2 \end{matrix} \middle| z\right) = \begin{cases} -(az)^{-1}[1 - (1-z)^{-a}] & (a \neq 0), \\ -z^{-1}\ln(1-z) & (a = 0). \end{cases} \quad (2.1)$$

Theorem 1. For $|x| + |y| < 1$, the Appell hypergeometric function F_2 is given by

$$\begin{aligned} &F_2(a+1, \alpha_1, 1; \beta_1, 2; x, y) \\ &= -\frac{1}{ay} {}_2F_1\left(\begin{matrix} a, \alpha_1 \\ \beta_1 \end{matrix} \middle| x\right) + \frac{1}{ay(1-y)^a} {}_2F_1\left(\begin{matrix} a, \alpha_1 \\ \beta_1 \end{matrix} \middle| \frac{x}{1-y}\right) \\ &(a \neq 0; \alpha_1 \in \mathbb{C}; \beta_1 \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} &F_2(1, \alpha_1, 1; \beta_1, 2; x, y) \\ &= \frac{\alpha_1}{\beta_1 y} \left[\frac{x}{1-y} {}_3F_2\left(\begin{matrix} \alpha_1, 1, 1 \\ \beta_1 + 1, 2 \end{matrix} \middle| \frac{x}{1-y}\right) \right. \\ &\quad \left. - x {}_3F_2\left(\begin{matrix} \alpha_1, 1, 1 \\ \beta_1 + 1, 2 \end{matrix} \middle| x\right) \right] - \frac{\ln(1-y)}{y} \\ &(\alpha_1 \in \mathbb{C}; \beta_1 \in \mathbb{C} \setminus \mathbb{Z}_0^-). \end{aligned} \quad (2.3)$$

Proof. By means of Euler's integral representation of the Gauss hypergeometric function [9, p. 20, Eq. (1.6.6)]:

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| z\right) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \tau^{b-1} (1-\tau)^{c-b-1} (1-z\tau)^{-a} d\tau \\ &(\Re(c) > \Re(b) > 0; |\arg(1-z)| \leq \pi - \epsilon; 0 < \epsilon < \pi), \end{aligned} \quad (2.4)$$

we reduce the following double integral representation [9, p. 214, Eq. (8.2.3)]:

$$\begin{aligned} F_2(\sigma, \alpha_1, \alpha_2; \beta_1, \beta_2; x, y) &= \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_1)\Gamma(\beta_2-\alpha_2)} \int_0^1 \int_0^1 u^{\alpha_1-1} \tau^{\alpha_2-1} \\ &\quad \cdot (1-u)^{\beta_1-\alpha_1-1} (1-\tau)^{\beta_2-\alpha_2-1} (1-xu-y\tau)^{-\sigma} du d\tau \\ &(\Re(\beta_j) > \Re(\alpha_j) > 0; j = 1, 2; |x| + |y| < 1) \end{aligned} \quad (2.5)$$

into a single integral involving the Gauss hypergeometric function ${}_2F_1$ as follows:

$$\begin{aligned}
& F_2(\sigma, \alpha_1, \alpha_2; \beta_1, \beta_2; x, y) \\
&= \frac{\Gamma(\beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1 - \alpha_1)} \int_0^1 \frac{u^{\alpha_1-1}(1-u)^{\beta_1-\alpha_1-1}}{(1-xu)^\sigma} {}_2F_1\left(\begin{matrix} \sigma, \alpha_2 \\ \beta_2 \end{matrix} \middle| \frac{y}{1-xu}\right) du \\
& (\Re(\beta_1) > \Re(\alpha_1) > 0; |x| + |y| < 1). \tag{2.6}
\end{aligned}$$

The assertion (2.2) of Theorem 1 then follows by substituting the first expression of (2.1) into (2.6) with

$$\sigma = a + 1 \quad (a \neq 0) \quad \text{and} \quad \alpha_2 = \beta_2 - 1 = 1,$$

and using an integral representation analogous to (2.4). The proof of the assertion (2.3) of Theorem 1 is completed by first rewriting

$$-\ln\left(1 - \frac{y}{1-xu}\right) = -\ln(1-y) - \ln\left(1 - \frac{xu}{1-y}\right) + \ln(1-xu),$$

and then substituting from (2.1) into the integral representation (2.6), in view of the following (easily-derivable) identity:

$$\begin{aligned}
& \int_0^1 u^{\alpha_1-1}(1-u)^{\beta_1-\alpha_1-1} \ln(1-zu) du \\
&= -\frac{z\Gamma(\alpha_1+1)\Gamma(\beta_1-\alpha_1)}{\Gamma(\beta_1+1)} {}_3F_2\left(\begin{matrix} \alpha_1+1, 1, 1 \\ \beta_1+1, 2 \end{matrix} \middle| z\right) \\
& (\Re(\beta_1) > \Re(\alpha_1) > -1). \quad \square \tag{2.7}
\end{aligned}$$

Lemma 2 (cf. [7, p. 455 et seq.]). For $|z| < 1$ and $p, q \in \mathbb{N}_0$,

$${}_2F_1\left(\begin{matrix} q+1, p+1 \\ p+2 \end{matrix} \middle| z\right) = \begin{cases} \frac{p+1}{z^{p+1}} \sum_{k=0}^p \frac{(-1)^k}{k-q} \binom{p}{k} [1 - (1-z)^{k-q}] & (p < q), \\ \frac{p+1}{z^{p+1}} \left(\sum_{k=0 (k \neq q)}^p \frac{(-1)^k}{k-q} \binom{p}{k} [1 - (1-z)^{k-q}] \right. \\ \left. - (-1)^k \binom{p}{k} \ln(1-z) \right) & (p \geq q). \end{cases} \tag{2.8}$$

Theorem 2. For $\alpha_1 \in \mathbb{C}$, $\beta_1 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and $|x| + |y| < 1$,

$$\begin{aligned}
& F_2(q+1, \alpha_1, p+1; \beta_1, p+2; x, y) \\
&= -\frac{p!}{q(1-q)_p} \frac{p+1}{y^{p+1}} {}_2F_1\left(\begin{matrix} \alpha_1, q-p \\ \beta_1 \end{matrix} \middle| x\right) \\
&+ \frac{p+1}{y^{p+1}} \sum_{k=0}^p \frac{(-1)^k}{(q-k)(1-y)^{q-k}} \binom{p}{k} \sum_{m=0}^{p-k} (-x)^m \binom{p-k}{m} \frac{(\alpha_1)_m}{(\beta_1)_m} \\
&\cdot {}_2F_1\left(\begin{matrix} \alpha_1+m, q-k \\ \beta_1+m \end{matrix} \middle| \frac{x}{1-y}\right) \quad (p, q \in \mathbb{N}_0; p < q) \tag{2.9}
\end{aligned}$$

and

$$\begin{aligned}
& F_2(q+1, \alpha_1, p+1; \beta_1, p+2; x, y) \\
&= \frac{p+1}{y^{p+1}} {}_2F_1\left(\alpha_1, q-p \mid x\right) \left[\sum_{\substack{k=0 \\ (k \neq q)}}^p \frac{(-1)^k}{k-q} \binom{p}{k} - (-1)^q \binom{p}{q} \ln(1-y) \right] \\
&\quad - \frac{p+1}{y^{p+1}} \sum_{\substack{k=0 \\ (k \neq q)}}^p \frac{(-1)^k}{(k-q)(1-y)^{q-k}} \binom{p}{k} \\
&\quad \cdot \left[\sum_{m=0}^{p-k} (-x)^m \binom{p-k}{m} \frac{(\alpha_1)_m}{(\beta_1)_m} {}_2F_1\left(\alpha_1+m, q-k \mid \frac{x}{1-y}\right) \right] \\
&\quad - (-1)^q \frac{p+1}{y^{p+1}} \frac{\alpha_1}{\beta_1} \binom{p}{q} \sum_{m=0}^{p-q} (-x)^m \binom{p-q}{m} \frac{(\alpha_1+1)_m}{(\beta_1+1)_m} \\
&\quad \cdot \left[x {}_3F_2\left(\alpha_1+m+1, 1, 1 \mid x\right) \right. \\
&\quad \left. - \frac{x}{1-y} {}_3F_2\left(\alpha_1+m+1, 1, 1 \mid \frac{x}{1-y}\right) \right] \quad (p, q \in \mathbb{N}_0; p \geq q), \tag{2.10}
\end{aligned}$$

where, as usual, an empty sum is understood to be nil.

Proof. In view of (2.1), the integral representation (2.6) readily yields

$$\begin{aligned}
& F_2(q+1, \alpha_1, p+1; \beta_1, p+2; x, y) \\
&= \frac{p+1}{y^{p+1}} \sum_{k=0}^p \frac{(-p)_k}{(k-q)k!} \frac{\Gamma(\beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1-\alpha_1)} \\
&\quad \cdot \int_0^1 u^{\alpha_1-1} (1-u)^{\beta_1-\alpha_1-1} (1-xu)^{p-q} du \\
&\quad - \frac{p+1}{y^{p+1}} \sum_{k=0}^p \frac{(-p)_k}{(k-q)k!} \frac{\Gamma(\beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1-\alpha_1)} \\
&\quad \cdot \int_0^1 u^{\alpha_1-1} (1-u)^{\beta_1-\alpha_1-1} (1-xu)^{p-k} (1-y-xu)^{k-q} du \\
&= \frac{p+1}{y^{p+1}} \sum_{k=0}^p \frac{(-p)_k}{(k-q)k!} {}_2F_1\left(\alpha_1, q-p \mid x\right) \\
&\quad - \frac{p+1}{y^{p+1}} \sum_{k=0}^p \frac{(-p)_k}{(k-q)k!} \sum_{m=0}^{p-k} \frac{(-p+k)_m x^m}{(1-y)^{q-k} m!} \frac{\Gamma(\beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1-\alpha_1)}
\end{aligned}$$

$$\cdot \int_0^1 \frac{u^{\alpha_1+m-1}(1-u)^{\beta_1+m-\alpha_1-m-1}}{\left(1-\frac{xu}{1-y}\right)^{q-k}} du, \quad (2.11)$$

where we have also used each of the following elementary identities:

$$(1-xu)^{p-k} = \sum_{m=0}^{p-k} \frac{(-p+k)_m}{m!} (xu)^m \quad (p \geq k)$$

and

$$(1-y-xu)^{k-q} = (1-y)^{k-q} \left(1 - \frac{xu}{1-y}\right)^{k-q}.$$

The proof of the assertion (2.9) of Theorem 2 is completed by noticing that the last integral in (2.11) is the integral representation of

$${}_2F_1\left(\begin{matrix} \alpha_1 + m, q - k \\ \beta_1 + m \end{matrix} \middle| \frac{x}{1-y}\right)$$

as given by (2.4). The proof of the assertion (2.10) of Theorem 2 follows in a similar fashion and is, therefore, omitted. $\square$

Lemma 3 (cf. [7, p. 455 et seq.]). *Let $p, q, r \in \mathbb{N}_0$ and $|z| < 1$. Then*

$$\begin{aligned} & \frac{\Gamma(p+1)\Gamma(r+1)}{\Gamma(p+r+2)} {}_2F_1\left(\begin{matrix} q+1, p+1 \\ p+r+2 \end{matrix} \middle| z\right) \\ &= \begin{cases} \sum_{j=0}^r \frac{(-1)^j}{z^{p+j+1}} \binom{r}{j} \left(\sum_{k=0}^{p+j} \frac{(-1)^k}{k-q} \binom{p+j}{k} [1 - (1-z)^{k-q}] \right) & (p < q-r), \\ \sum_{j=0}^r \frac{(-1)^j}{z^{p+j+1}} \binom{r}{j} \left[\sum_{k=0 \atop (k \neq q)}^{p+j} \frac{(-1)^k}{k-q} \binom{p+j}{k} [1 - (1-z)^{k-q}] \right. \\ \quad \left. - (-1)^q \binom{p+j}{q} \ln(1-z) \right] & (p \geq q). \end{cases} \quad (2.12) \end{aligned}$$

Theorem 3. *For $p, q, r \in \mathbb{N}_0$ and $|x| + |y| < 1$,*

$$\begin{aligned} & \frac{\Gamma(p+1)\Gamma(r+1)}{\Gamma(p+r+2)} F_2(q+1, \alpha_1, p+1; \beta_1, p+r+2; x, y) \\ &= \sum_{j=0}^r \frac{(-1)^j}{y^{p+j+1}} \binom{r}{j} \left\{ -\frac{(p+j)!}{q(1-q)_{p+j}} {}_2F_1\left(\begin{matrix} \alpha_1, q-p-j \\ \beta_1 \end{matrix} \middle| x\right) \right. \\ & \quad \left. - \sum_{k=0}^{p+j} \frac{(-1)^k}{(k-q)(1-y)^{q-k}} \binom{p+j}{k} \right\} \end{aligned}$$

$$\cdot \left[\sum_{m=0}^{p+j-k} (-x)^m \binom{p+j-k}{m} \frac{(\alpha_1)_m}{(\beta_1)_m} {}_2F_1 \left(\begin{matrix} \alpha_1 + m, q - k \\ \beta_1 + m \end{matrix} \middle| \frac{x}{1-y} \right) \right] \Bigg\} \\ (p < q - r) \quad (2.13)$$

and

$$\begin{aligned} & \frac{\Gamma(p+1)\Gamma(r+1)}{\Gamma(p+r+2)} F_2(q+1, \alpha_1, p+1; \beta_1, p+r+2; x, y) \\ &= \sum_{j=0}^r \frac{(-1)^j}{y^{p+j+1}} \binom{r}{j} \left\{ {}_2F_1 \left(\begin{matrix} \alpha_1, q - p - j \\ \beta_1 \end{matrix} \middle| x \right) \right. \\ & \quad \cdot \left[\sum_{\substack{k=0 \\ (k \neq q)}}^{p+j} \frac{(-1)^k}{k-q} \binom{p+j}{k} - (-1)^q \binom{p+j}{q} \ln(1-y) \right] \\ & \quad - \sum_{\substack{k=0 \\ (k \neq q)}}^{p+j} \frac{(-1)^k}{(k-q)(1-y)^{p-k}} \binom{p+j}{k} \\ & \quad \cdot \left[\sum_{m=0}^{p+j-k} (-x)^m \binom{p+j-k}{m} \frac{(\alpha_1)_m}{(\beta_1)_m} {}_2F_1 \left(\begin{matrix} \alpha_1 + m, q - k \\ \beta_1 + m \end{matrix} \middle| \frac{x}{1-y} \right) \right] \\ & \quad + \frac{\alpha_1}{\beta_1} (-1)^k \binom{p+j}{q} \sum_{m=0}^{p+j-q} (-x)^m \binom{p+j-k}{m} \frac{(\alpha_1+1)_m}{(\beta_1+1)_m} \\ & \quad \cdot \left[\frac{x}{1-y} {}_3F_2 \left(\begin{matrix} \alpha_1 + m + 1, 1, 1 \\ \beta_1 + m + 1, 2 \end{matrix} \middle| \frac{x}{1-y} \right) - x {}_3F_2 \left(\begin{matrix} \alpha_1 + m + 1, 1, 1 \\ \beta_1 + m + 1, 2 \end{matrix} \middle| x \right) \right] \Bigg\} \\ & (p \geq q), \quad (2.14) \end{aligned}$$

where an empty sum is interpreted, as usual, to be nil.

Proof. First of all, by using (2.12), the integral representation (2.6) yields

$$\begin{aligned} & F_2(q+1, \alpha_1, p+1; \beta_1, p+r+2; x, y) \\ &= \frac{\Gamma(\beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1-\alpha_1)} \int_0^1 u^{\alpha_1-1} (1-u)^{\beta_1-\alpha_1-1} (1-xu)^{-q-1} \\ & \quad \cdot \left\{ \frac{\Gamma(p+r+2)}{\Gamma(p+1)\Gamma(r+1)} \sum_{j=0}^r \frac{(-r)_j}{j!} \frac{(1-xu)^{p+j+1}}{y^{p+j+1}} \left[\sum_{k=0}^{p+j} \frac{(-p-j)_k}{(k-q)k!} \right] \right\} du \\ & \quad - \frac{\Gamma(\beta_1)}{\Gamma(\alpha_1)\Gamma(\beta_1-\alpha_1)} \int_0^1 u^{\alpha_1-1} (1-u)^{\beta_1-\alpha_1-1} (1-xu)^{-q-1} \\ & \quad \cdot \left\{ \frac{\Gamma(p+r+2)}{\Gamma(p+1)\Gamma(r+1)} \sum_{j=0}^r \frac{(-r)_j}{j!} \frac{(1-xu)^{p+j+1}}{y^{p+j+1}} \right. \end{aligned}$$

$$\begin{aligned}
& \cdot \left[\sum_{k=0}^{p+j} \frac{(-p-j)_k}{(k-q)k!} (1-xu)^{q-k} (1-y)^{k-q} \left(1 - \frac{xu}{1-y}\right)^{k-q} \right] du \\
& = \frac{\Gamma(p+r+2)}{\Gamma(p+1)\Gamma(r+1)} \\
& \cdot \sum_{j=0}^r \left[\frac{(-r)_j}{j!} \frac{1}{y^{p+j+1}} {}_2F_1 \left(\begin{matrix} q-p-j, \alpha_1 \\ \beta_1 \end{matrix} \middle| x \right) \sum_{k=0}^{p+j} \frac{(-p-j)_k}{(k-q)k!} \right] \\
& - \frac{\Gamma(p+r+2)}{\Gamma(p+1)\Gamma(r+1)} \sum_{j=0}^r \left(\frac{(-r)_j}{j!} \frac{1}{y^{p+j+1}} \sum_{k=0}^{p+j} \left[\frac{(-p-j)_k}{(k-q)k!} (1-y)^{k-q} \right. \right. \\
& \cdot \left. \left. F_1 \left(\alpha_1, k-p-j, q-k; \beta_1; x, \frac{x}{1-y} \right) \right] \right), \tag{2.15}
\end{aligned}$$

where we have used the following known integral representation of the Appell F_1 series [8, p. 215, Eq. (8.2.5)]:

$$\begin{aligned}
& F_1(\sigma, \alpha_1, \alpha_2; \beta_1; x, y) \\
& = \frac{\Gamma(\beta_1)}{\Gamma(\sigma)\Gamma(\beta_1-\sigma)} \int_0^1 \tau^{\sigma-1} (1-\tau)^{\beta_1-\sigma-1} (1-x\tau)^{-\alpha_1} (1-y\tau)^{-\alpha_2} d\tau \\
& \quad (\Re(\beta_1) > \Re(\sigma) > 0; \max\{|\arg(1-x)|, |\arg(1-y)|\} \leq \pi - \epsilon; 0 < \epsilon < \pi).
\end{aligned}$$

Now the proof of (2.13) follows by using the identity [9, p. 211, Eq. (8.1.7)]:

$$F_1(a, b, b'; c; x, y) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} {}_2F_1 \left(\begin{matrix} a+m, b' \\ c+m \end{matrix} \middle| y \right) \frac{x^m}{m!} \tag{2.16}$$

and the fact that, since the second parameter of F_1 in (2.15) is a *nonpositive* integer ($0 \leq k \leq p+r$), the infinite sum collapses into a finite sum. The proof of (2.14) follows similarly. $\square$

Remark 1. By applying the symmetry relationship:

$$F_2(\sigma, \alpha_1, \alpha_2; \beta_1, \beta_2; x, y) = F_2(\sigma, \alpha_2, \alpha_1; \beta_2, \beta_1; y, x),$$

similar expressions for the Appell series

$$F_2(a+1, 1, \alpha_2; 2, \beta_2; y, x),$$

when $a \neq 0$ or when $a = 0$, can easily be developed from Theorem 1.

Remark 2. Although the derivations of (2.2) and (2.3) are straightforward, the results are quite powerful. Indeed, by using the comprehensive representation tables of

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| z \right) \quad \text{and} \quad {}_3F_2 \left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix} \middle| z \right),$$

provided by Prudnikov et al. [7, pp. 454–551], we can easily generate tables of the corresponding formulas for

$$F_2(a+1, \alpha_1, 1; \beta_1, 2; x, y)$$

by means of (2.2) and (2.3). A short collection of potentially useful formulas is given in Appendix A.

Remark 3. The result of Cuyt et al. [1, p. 215, Theorem 2.1a] follows as a special case of (2.2). Indeed, we notice first that

$$\begin{aligned} &F_1(1, s+1, t+1; 2; x, y) \\ &= \left(\frac{y}{x}\right)^{s+1} F_2\left(s+t+2, s+1, 1; s+t+2, 2; 1-\frac{y}{x}, y\right). \end{aligned} \quad (2.17)$$

Consequently, using (2.2), we have

$$\begin{aligned} &F_1(1, s+1, t+1; 2; x, y) \\ &= \left(\frac{y}{x}\right)^{s+1} \left[-\frac{1}{(s+t+1)y} {}_2F_1\left(\begin{matrix} s+t+1, s+1 \\ s+t+2 \end{matrix} \middle| 1-\frac{y}{x} \right) \right. \\ &\quad \left. + \frac{1}{(s+t+1)y(1-y)^{s+t+1}} {}_2F_1\left(\begin{matrix} s+t+1, s+1 \\ s+t+2 \end{matrix} \middle| \frac{x-y}{x(1-y)} \right) \right], \end{aligned}$$

which, in view of (2.8), yields

$$\begin{aligned} &F_1(1, s+1, t+1; 2; x, y) \\ &= \frac{(-x)^t y^s}{(y-x)^{s+t+1}} \binom{s+t}{s} [\ln(1-x) - \ln(1-y)] \\ &\quad - \sum_{k=0}^{s+t} (-1)^k \binom{s+t}{k} \frac{y^k x^{t+s-k} (1-y)^{s-k}}{(k-s)(x-y)^{s+t+1}} [(1-x)^{k-s} - (1-y)^{k-s}], \end{aligned} \quad (2.18)$$

where the equivalence of the sums follows by direct expansion.

3. Finite-sum representations of Appell's series with nonnegative parameters

Theorem 4. Each of the following finite-sum representations holds true for the Appell series F_2 :

$$\begin{aligned} &F_2(a+1, n, 1; m, 2; x, y) \\ &= \frac{1}{ay} \sum_{k=0}^{n-m} \binom{n-m}{k} \frac{(a)_k}{(m)_k} x^k [(1-x-y)^{-a-k} - (1-x)^{-a-k}] \\ &\quad (a \in \mathbb{C} \setminus \{0\}; m \in \mathbb{N}; n \in \mathbb{N}_0; n \geq m) \end{aligned} \quad (3.1)$$

and

$$\begin{aligned}
& F_2(1, n, 1; m, 2; x, y) \\
&= \frac{m-n}{my} \sum_{k=0}^{n-m-1} \frac{(-n+m+1)_k}{(k+1)(m+1)_k} \left[\left(\frac{x}{x+y-1} \right)^{k+1} - \left(\frac{x}{x-1} \right)^{k+1} \right] \\
&+ \frac{1}{y} \ln \left(\frac{1-x}{1-x-y} \right) \quad (m \in \mathbb{N}; n \in \mathbb{N}_0; n \geq m+1).
\end{aligned} \tag{3.2}$$

Proof. The proof of Theorem 4 follows directly from Theorem 1 by applying the Pfaff–Kummer transformation:

$$\begin{aligned}
{}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| z \right) &= (1-z)^{-a} {}_2F_1 \left(\begin{matrix} a, c-b \\ c \end{matrix} \middle| \frac{z}{z-1} \right) \\
&(a, b \in \mathbb{C}; c \in \mathbb{C} \setminus \mathbb{Z}_0^-; |\arg(1-z)| \leq \pi - \epsilon; 0 < \epsilon < \pi)
\end{aligned} \tag{3.3}$$

and the fact that

$${}_3F_2 \left(\begin{matrix} a, b, -n+m+1 \\ 2, c \end{matrix} \middle| x \right)$$

is a polynomial of degree $n-m-1$ in x ($n \geq m+1$). $\square$

Theorem 5. Suppose that $p, q, s, t \in \mathbb{N}_0$ with $t \geq s$. Then

$$\begin{aligned}
& F_2(q+1, t+1, p+1; s+1, p+2; x, y) \\
&= -\frac{(p+1) \cdot p!}{q(1-q)_p y^{p+1} (1-x)^{q-p}} \left[\sum_{l=0}^{t-s} \frac{(-1)^l (q-p)_l}{(s+1)_l} \binom{t-s}{l} \left(\frac{x}{x-1} \right)^l \right] \\
&- \frac{p+1}{y^{p+1}} \sum_{k=0}^p \frac{(-1)^k}{(1-x-y)^{q-k} (k-q)} \binom{p}{k} \left(\sum_{m=0}^{p-k} (-x)^m \binom{p-k}{m} \frac{(t+1)_m}{(s+1)_m} \right. \\
&\cdot \left. \left[\sum_{l=0}^{t-s} \frac{(-1)^l (q-k)_l}{(s+m+1)_l} \binom{t-s}{l} \left(\frac{x}{x+y-1} \right)^l \right] \right) \quad (p < q)
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
& F_2(q+1, t+1, p+1; s+1, p+2; x, y) \\
&= \frac{p+1}{y^{p+1}} \left(\sum_{\substack{k=0 \\ (k \neq q)}}^p \frac{(-1)^k}{k-q} \binom{p}{k} - \frac{(-p)_q}{q!} \ln(1-y) \right) \left(\sum_{l=0}^{p-q} \frac{(-p+q)_l (t+1)_l}{(s+1)_l} \frac{x^l}{l!} \right) \\
&- \frac{p+1}{y^{p+1}} \sum_{\substack{k=0 \\ (k \neq q)}}^p \frac{(-p)_k (1-x-y)^{k-q}}{(k-q) k!} \left(\sum_{j=0}^{p-k} \frac{(-p+k)_j (t+1)_j}{(s+1)_j} \frac{x^j}{j!} \right. \\
&\cdot \left. \left[\sum_{l=0}^{t-s} \frac{(-t+s)_l (q-k)_l}{(s+j+1)_l l!} \left(\frac{x}{x+y-1} \right)^l \right] \right) \\
&+ \frac{p+1}{y^{p+1}} \frac{(-p)_q}{q!} \sum_{m=0}^{p-q} \frac{(-p+q)_m (t+1)_m}{(s+1)_m} \frac{x^m}{m!}
\end{aligned}$$

$$\cdot \left[\frac{s-t}{s+m+1} \sum_{l=0}^{t-s-1} \frac{(-t+s+1)_l}{(l+1)(s+m+2)_l} \left\{ \left(\frac{x}{x+y-1} \right)^{l+1} - \left(\frac{x}{x-1} \right)^{l+1} \right\} + \ln \left(\frac{(1-x)(1-y)}{1-x-y} \right) \right] \quad (p \geq q). \quad (3.5)$$

Theorem 6. If $p, q, s, t \in \mathbb{N}_0$ with $t \geq s$, then

$$\begin{aligned} & \frac{\Gamma(p+1)\Gamma(r+1)}{\Gamma(p+r+2)} F_2(q+1, t+1, p+1; s+1, p+r+2; x, y) \\ &= -\frac{1}{q} \sum_{j=0}^r (-1)^j \binom{r}{j} \frac{(1-x)^{p+j-q} (p+j)!}{y^{p+j+1} (1-q)_{p+j}} \\ & \cdot \left[\sum_{l=0}^{t-s} \frac{(-t+s)_l (q-p-j)_l}{(s+1)_l l!} \left(\frac{x}{x-1} \right)^l \right] \\ & - \sum_{j=0}^r \frac{(-1)^j}{y^{p+j+1}} \binom{r}{j} \left[\sum_{k=0}^{p+j} \frac{(-1)^k \binom{p+j}{k}}{(k-q)(1-x-y)^{q-k}} \right. \\ & \cdot \left(\sum_{m=0}^{p+j-k} (-x)^m \binom{p+j-k}{m} \frac{(t+1)_m}{(s+1)_m} \right. \\ & \cdot \left. \left. \left[\sum_{l=0}^{t-s} (-1)^l \binom{t-s}{l} \frac{(q-k)_l}{(s+m+1)_l} \left(\frac{x}{x+y-1} \right)^l \right] \right] \right] \quad (p < q-r) \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} & \frac{\Gamma(p+1)\Gamma(r+1)}{\Gamma(p+r+2)} F_2(q+1, t+1, p+1; s+1, p+r+2; x, y) \\ &= \sum_{j=0}^r (-1)^j \binom{r}{j} \frac{(1-x)^{p+j-q}}{y^{p+j+1}} \\ & \cdot \left[\sum_{\substack{k=0 \\ (k \neq q)}}^{p+j} \frac{(-1)^k}{k-q} \binom{p+j}{k} - (-1)^q \binom{p+j}{q} \ln(1-y) \right] \\ & \cdot \left[\sum_{m=0}^{t-s} (-1)^m \binom{t-s}{m} \frac{(q-p-j)_m}{(s+1)_m} \left(\frac{x}{x-1} \right)^m \right] \\ & - \sum_{j=0}^r (-1)^j \binom{r}{j} \frac{1}{y^{p+j+1}} \\ & \cdot \left[\sum_{\substack{k=0 \\ (k \neq q)}}^{p+j} \frac{(-1)^k \binom{p+j}{k}}{(1-x-y)^{q-k} (k-q)} \left(\sum_{m=0}^{p+j-k} (-x)^m \binom{p+j-k}{m} \frac{(t+1)_m}{(s+1)_m} \right) \right] \end{aligned}$$

$$\begin{aligned}
& \cdot \left[\sum_{l=0}^{t-s} (-1)^l \binom{t-s}{l} \frac{(q-k)_l}{(s+m+1)_l} \left(\frac{x}{x+y-1} \right)^l \right] \Bigg] \\
& + \sum_{j=0}^r \frac{(-1)^{j+q}}{y^{p+j+1}} \binom{r}{j} \binom{p+j}{q} \left(\sum_{m=0}^{p+j-q} (-x)^m \binom{p+j-q}{m} \frac{(t+1)_m}{(s+1)_m} \right. \\
& \cdot \left[\frac{s-t}{s+m+1} \sum_{l=0}^{t-s-1} \frac{(-t+s+1)_l}{(l+1)(s+m+2)_l} \left\{ \left(\frac{x}{x+y-1} \right)^{l+1} - \left(\frac{x}{x-1} \right)^{l+1} \right\} \right. \\
& \left. \left. + \ln \left(\frac{(1-x)(1-y)}{1-x-y} \right) \right] \right) \quad (p \geq q). \tag{3.7}
\end{aligned}$$

Remark 4. The proofs of Theorems 5 and 6, although lengthy, are much akin to those of our results derived in this as well as the preceding sections, and we choose to omit the details here.

Remark 5. Our generalization of the above-mentioned more complicated result of Cuyt et al. [1, p. 215, Theorem 2.1c] can be seen by means of the following identity [9, p. 220, Eq. (8.3.1.7)]:

$$\begin{aligned}
F_2(\sigma, \alpha_1, \alpha_2; \sigma, \beta_2; x, y) &= (1-x)^{-\alpha_1} F_1 \left(\alpha_2, \alpha_1, \sigma - \alpha_1; \beta_2; \frac{y}{1-x}, y \right) \\
(\sigma &= s+t+2; \alpha_1 = s+1; \alpha_2 = a+1; \beta_2 = a+d+2). \tag{3.8}
\end{aligned}$$

Remark 6. By suitably applying the following linear transformations for the Appell series F_2 (cf., e.g., [7, p. 449]):

$$\begin{aligned}
& F_2(a, b, b'; c, c'; x, y) \\
&= (1-x)^{-a} F_2 \left(a, c-b, b'; c, c'; \frac{x}{x-1}, \frac{y}{1-x} \right) \\
&= (1-y)^{-a} F_2 \left(a, b, c'-b'; c, c'; \frac{x}{1-y}, \frac{y}{y-1} \right) \\
&= (1-x-y)^{-a} F_2 \left(a, c-b, c'-b'; c, c'; \frac{x}{x+y-1}, \frac{y}{x+y-1} \right), \tag{3.9}
\end{aligned}$$

it is not difficult to deduce several *further* results equivalent to those presented in this paper.

4. Concluding remarks and observations

Recently, there has been some interest in computing Gordon's integral (cf. [4, p. 662, Appendix f], [8,12]; see also [3,5,6,11]):

$$J_{\gamma}^{s,p}(\alpha, \alpha') := \int_0^{\infty} \tau^{\gamma-1+s} e^{-h\tau} {}_1F_1 \left(\alpha \middle| k\tau \right) {}_1F_1 \left(\alpha' \middle| k'\tau \right) d\tau \tag{4.1}$$

because of its relevance to the study of singular potentials. The integral for $J_y^{s,p}(\alpha, \alpha')$ is a *special* case of the following *well-known* integral formula:

$$\begin{aligned} & \int_0^\infty \tau^{\sigma-1} e^{-h\tau} {}_1F_1\left(\alpha_1 \middle| \beta_1; \kappa \tau\right) {}_1F_1\left(\alpha_2 \middle| \beta_2; \kappa' \tau\right) d\tau \\ &= h^{-\sigma} \Gamma(\sigma) F_2(\sigma, \alpha_1, \alpha_2; \beta_1, \beta_2; x, y) \\ & \quad (\Re(\sigma) > 0; |\kappa| + |\kappa'| < |h|). \end{aligned} \quad (4.2)$$

Direct application of the present results to the ongoing research on singular potentials and the computation of matrix elements of singular operators can be achieved by means of the known identity ([9, p. 219, Eq. (8.3.10)]; see also Eq. (3.9) above):

$$\begin{aligned} & F_2(\sigma, \alpha_1, \alpha_2; \beta_1, \beta_2; x, y) \\ &= (1-x-y)^{-\sigma} F_2\left(\sigma, \beta_1 - \alpha_1, \beta_2 - \alpha_2; \beta_1, \beta_2; -\frac{x}{1-x-y}, -\frac{y}{1-x-y}\right). \end{aligned} \quad (4.3)$$

No further computation shall be given here.

We have developed a finite-sum representation of the Appell function

$$F_2(\sigma, \alpha_1, \alpha_2; \beta_1, \beta_2; x, y)$$

in the case when the parameters $\sigma, \alpha_1, \alpha_2, \beta_1$, and β_2 are positive integers and have shown that one of our results is a generalization of the finite-sum representation of F_1 derived by Cuyt et al. [1]. Additionally, we have shown that an integral representation of Appell's F_2 series can be used to generate many reduction formulas in terms of the hypergeometric functions ${}_2F_1$ and ${}_3F_2$ that are very useful in applications. As a consequence of the present work, the integrals (4.1) and (4.2) containing the products of the Kummer (or confluent) hypergeometric functions can now be expressed in terms of elementary algebraic functions. The reduction formulas, as derived in Theorems 1, 2, and 3, expose the type of singularities of F_2 in the vicinity of the points $x + y = 1$ and at $(x, y) = (1, 1)$. Indeed our reduction formulas show two types of singularities near $x + y = 1$ and $(x, y) = (1, 1)$, one algebraic and the other logarithmic in nature.

Acknowledgments

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grants GP250050, GP249507, and OGP0007353.

Appendix A

Here we give a brief collection of reduction formulas that follow directly by using some of our results. It can be largely extended by using the work of Prudnikov et al. [7, pp. 454–546].

$$\begin{aligned}
& F_2\left(a+1, a+\frac{1}{2}, 1; 2a+1, 2; x, y\right) \\
&= \frac{1}{ay} \left[\left(\frac{2}{\sqrt{1-y} + \sqrt{1-x-y}} \right)^{2a} - \left(\frac{2}{1+\sqrt{1-x}} \right)^{2a} \right].
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
& F_2\left(a+1, a+\frac{1}{2}, 1; 2a, 2; x, y\right) \\
&= \frac{1}{ay} \left[\frac{1}{\sqrt{1-x-y}} \left(\frac{2}{\sqrt{1-y} + \sqrt{1-x-y}} \right)^{2a-1} \right. \\
&\quad \left. - \frac{1}{\sqrt{1-x}} \left(\frac{2}{1+\sqrt{1-x}} \right)^{2a-1} \right].
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
& F_2\left(a+1, a-\frac{1}{2}, 1; 2a, 2; x, y\right) \\
&= \frac{1}{ay} \left[\frac{1}{\sqrt{1-y}} \left(\frac{\sqrt{1-y} + \sqrt{1-x-y}}{2} \right)^{1-2a} - \left(\frac{1+\sqrt{1-x}}{2} \right)^{1-2a} \right].
\end{aligned} \tag{A.3}$$

$$F_2(2a+1, a+1, 1; a, 2; x, y) = \frac{1}{2ay} \left[\frac{1+x-y}{(1-x-y)^{2a+1}} - \frac{1+x}{(1-x)^{2a+1}} \right]. \tag{A.4}$$

$$F_2\left(a+1, 1+\frac{a}{2}, 1; \frac{a}{2}, 2; x, y\right) = -\frac{(1+x)}{ay(1-x)^{a+1}} + \frac{1+x-y}{ay(1-x-y)^{a+1}}. \tag{A.5}$$

$$F_2(1, \beta+1, 1; \beta, 2; x, y) = \frac{1}{\beta} \frac{x}{(x-1)(x+y-1)} + \frac{1}{y} \ln\left(\frac{1-x}{1-x-y}\right). \tag{A.6}$$

$$\begin{aligned}
& F_2(a+1, 1, 1; 2, 2; x, y) \\
&= -\frac{1}{a(a-1)y} [(1-x)^{1-a} - 1] \\
&\quad + \frac{1}{a(a-1)x(1-y)^{a-1}} \left[\left(1 - \frac{x}{1-y}\right)^{1-a} - 1 \right].
\end{aligned} \tag{A.7}$$

$$\begin{aligned}
& F_2\left(a+1, a+\frac{1}{2}, 1; \frac{1}{2}, 2; x, y\right) \\
&= \frac{(\sqrt{1-y} + \sqrt{x})^{-2a} + (\sqrt{1-y} - \sqrt{x})^{-2a} - (1+\sqrt{x})^{-2a} - (1-\sqrt{x})^{-2a}}{2ay}.
\end{aligned} \tag{A.8}$$

In particular, we obtain the following *further* consequences:

$$\begin{aligned}
& F_2(1, 1, 1; 2, 2; x, y) \\
&= \frac{1}{y} \left[\text{Li}_2\left(\frac{x}{1-y}\right) - \text{Li}_2(x) \right] - \frac{1-x-y}{xy} \ln\left(1 - \frac{x}{1-y}\right) \\
&\quad + \frac{1-x}{xy} \ln(1-x),
\end{aligned} \tag{A.9}$$

where the *Dilogarithmic function* $\text{Li}_2(z)$ is defined by

$$\begin{aligned}\text{Li}_2(z) &:= \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad (|z| \leq 1) \\ &= - \int_0^z \frac{\ln(1-\tau)}{\tau} d\tau. \\ F_2(3, 2, 1; 3, 2; x, y) &= \frac{1}{x(1-x)(1-x-y)} + \frac{1}{x^2 y} \ln\left(\frac{1-x-y}{(1-x)(1-y)}\right). \quad (\text{A.10})\end{aligned}$$

$$\begin{aligned}F_2(4, 2, 1; 4, 2; x, y) &= \frac{1}{x^2(1-y)} + \frac{1}{x^2(1-x)(1-x-y)} \\ &\quad - \frac{2}{x^3 y} \left[\ln(1-y) - \ln\left(1 - \frac{y}{1-x}\right) \right]. \quad (\text{A.11})\end{aligned}$$

$$\begin{aligned}F_2(4, 1, 2; 3, 3; x, y) &= - \left(2x[-4 + y(5+y)] + 2(1-y)^2 \ln(1-x) \right. \\ &\quad \left. + 2(1-y)(-5 + 2x + 7y) \ln\left(\frac{1-x-y}{1-y}\right) \right) (3x^2 y^2 (1-y)^2)^{-1}. \quad (\text{A.12})\end{aligned}$$

$$F_2\left(\frac{9}{2}, \frac{7}{2}, 1; \frac{5}{2}, 2; x, y\right) = -\frac{2(5-2x)}{35y(1-x)^{9/2}} - \frac{2(5-5y+2x)}{35y(1-x-y)^{9/2}}. \quad (\text{A.13})$$

$$\begin{aligned}F_2\left(1, -\frac{1}{2}, 1; \frac{1}{2}, 2; x, y\right) &= -\frac{x}{3y(1-y)} \left(1 + \frac{(1-y) \ln(\frac{1-x-y}{1-y})}{x} + \frac{(3-3y-x) \sqrt{\frac{1-y}{x}} \tanh^{-1} \sqrt{\frac{x}{1-y}}}{1-y} \right) \\ &\quad + \frac{x}{3y} \left(1 + \frac{\ln(1-x)}{x} + \frac{(3-x) \tanh^{-1} \sqrt{x}}{\sqrt{x}} \right) - \frac{\ln(1-y)}{y}. \quad (\text{A.14})\end{aligned}$$

$$\begin{aligned}F_2\left(1, \frac{1}{2}, 1; 1, 2; x, y\right) &= \frac{2}{y} \left[\sqrt{1-x} - \sqrt{1-\frac{x}{1-y}} \right] + \frac{2}{y} \ln\left(1 + \sqrt{\frac{1-x/(1-y)}{(1+\sqrt{1-x})}}\right) \\ &\quad - \ln\left(\frac{1-y}{y}\right). \quad (\text{A.15})\end{aligned}$$

$$F_2(6, 2, 5; 4, 6; x, y) = \frac{-x^5 + 5x^4y - 10x^3y^3 + 10x^2y^3 + y^5 + x(1 - 2y(2 + y)(1 + 2y(y - 1)))}{(1 - x)(1 - y)^4(1 - x - y)}. \quad (\text{A.16})$$

$$F_2\left(2, 1, 1; \frac{3}{2}, 2; x, y\right) = -\frac{1}{y} \frac{\arcsin(\sqrt{x})}{\sqrt{x(1-x)}} + \frac{1}{y} \frac{\arcsin\left(\sqrt{\frac{x}{1-y}}\right)}{\sqrt{x(1-x-y)}}. \quad (\text{A.17})$$

$$F_2\left(a + 1, a, 1; a + 1, 2; \frac{1}{2}, \frac{1}{2}\right) = -2^a \left[\psi\left(\frac{a+1}{2}\right) - \psi\left(\frac{a}{2}\right) \right] + \frac{2^{a+1}}{a} \Gamma(1-a) \quad (0 < a < 1), \quad (\text{A.18})$$

where the *Digamma* (or *Psi*) function $\psi(z)$ is defined by

$$\psi(z) := \frac{d}{dz} \{\ln \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \ln \Gamma(z) = \int_1^z \psi(\tau) d\tau.$$

References

- [1] A. Cuyt, K. Driver, J. Tan, B. Verdonk, A finite sum representation of the Appell series $F_1(a, b, b'; c; x, y)$, J. Comput. Appl. Math. 5 (1999) 213–219.
- [2] H. Exton, Multiple Hypergeometric Functions and Applications, Halsted Press, Ellis Horwood, Wiley, Chichester, New York, 1976.
- [3] E. Karule, Integrals of two confluent hypergeometric functions, J. Phys. A: Math. Gen. 23 (1990) 1969–1971.
- [4] L.D. Landau, E.M. Lifshitz, Quantum Mechanics: Non-Relativistic Theory, Pergamon Press, London, 1981.
- [5] P.-A. Lee, S.-H. Ong, H.M. Srivastava, Some integrals of the products of Laguerre polynomials, Internat. J. Comput. Math. 78 (2001) 303–321.
- [6] J.L. Madajczyk, M. Trippenbach, Singular part of the hydrogen dipole matrix element, J. Phys. A: Math. Gen. 22 (1989) 2369–2374.
- [7] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series, vol. 3: More Special Functions, Translated from the 1983 Russian Edition by G.G. Gould, Gordon and Breach, New York, 1990.
- [8] N. Saad, R.L. Hall, Integrals containing confluent hypergeometric functions with applications to perturbed singular potentials, J. Phys. A: Math. Gen. 36 (2003) 7771–7788.
- [9] L.J. Slater, Generalized Hypergeometric Functions, Cambridge Univ. Press, Cambridge, 1966.
- [10] H.M. Srivastava, P.W. Karlsson, Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood)/Wiley, Chichester/New York, 1985.
- [11] H.M. Srivastava, H.A. Mavromatis, R.S. Alassar, Remarks on some associated Laguerre integral results, Appl. Math. Lett. 16 (2003) 1131–1136.
- [12] V.F. Tarasov, W. Gordon's integral (1929) and its representations by means of Appell's functions F_2 , F_1 , and F_3 , J. Math. Phys. 44 (2003) 1449–1452.