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The role of non-linear diffusion in non-simultaneous blow-up

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Abstract

We study a parabolic system of two non-linear reaction–diffusion equations completely coupled through source terms and with power-like diffusivity. Under adequate hypotheses on the initial data, we prove that non-simultaneous blow-up is sometimes possible; i.e., one of the components blows up while the other remains bounded. The conditions for non-simultaneous blow-up rely strongly on the diffusivity parameters and significant differences appear between the fast-diffusion and the porous medium case. Surprisingly, flat (homogeneous in space) solutions are not always a good guide to determine whether non-simultaneous blow-up is possible.

Keywords: Blow-up; Parabolic system; Non-linear diffusion

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1. Introduction

We consider solutions (u, v) to the non-linear parabolic system

$$\begin{cases} u_t = (u^m)_{xx} + u^{p_{11}} v^{p_{12}}, \\ v_t = (v^n)_{xx} + u^{p_{21}} v^{p_{22}}, \end{cases} \quad (x, t) \in \mathbb{R} \times (0, T), \tag{1}$$

with continuous, bounded and symmetric initial data

$$\begin{cases} u(x, 0) = u_0(x), \\ v(x, 0) = v_0(x), \end{cases} \quad x \in \mathbb{R}. \tag{2}$$

We assume $p_{ij} \geqslant 0$ and $m, n > 0$. In this range of parameters the diffusivities may become degenerate or singular at the level zero. Moreover, the reaction terms may not be Lipschitz, leading to non-uniqueness phenomena. To avoid the technicalities to which these difficulties may lead, we will assume that $u_0, v_0 \geqslant \delta > 0$. Since we will be interested in the behaviour of the system for large values of the solutions, this is not a significant restriction. Solutions will be understood in a classical sense. We restrict ourselves to symmetric initial data u_0, v_0 non-increasing with $|x|$, such that $u_t, v_t \geqslant 0$. Monotonicity and symmetry assumptions are common for problems of this kind, see [1].

Systems of this kind are common in population dynamics. In this context u and v represent two different species with a symbiotic behaviour. The cooperation between them is represented by the coupled source terms.

The constant T denotes the maximal existence time for the solution. If it is infinite, we say that the solution is *global*. If it is finite, we have

$$\limsup_{t \nearrow T} \{ \|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty} \} = \infty,$$

and we say that the solution *blows up*. Solutions blow up if and only if the exponents p_{ij} verify any of the conditions

$$p_{11} > 1, \qquad p_{22} > 1 \quad \text{or} \quad (p_{11} - 1)(p_{22} - 1) - p_{12} p_{21} < 0.$$

This follows easily by comparison with global and blow-up *flat solutions*, that is, solutions of (1) that are independent of x. Thus, they satisfy the ordinary differential system

$$u'(t) = u^{p_{11}}(t) v^{p_{12}}(t), \qquad v'(t) = u^{p_{21}}(t) v^{p_{22}}(t), \tag{3}$$

with initial data $u(0) = u_0 > 0$, $v(0) = v_0 > 0$.

If a solution (u, v) blows up, a priori there is no reason why both components, u and v, should go to infinity simultaneously at the blow-up time T. Indeed, for certain choices of the parameters p_{ij} there are initial data for which one of the components of the system remains bounded while the other blows up. This phenomenon is known in the literature as *non-simultaneous blow-up* [2,3]. The aim of this paper is to characterize the range of parameters for which non-simultaneous blow-up occurs for problem (1)–(2).

The possibility of having non-simultaneous blow-up for (1)–(2) was first mentioned in [1]. However, the authors restrict themselves to flat solutions. System (3) has solutions with non-simultaneous blow-up such that u blows up and v remains bounded if and only if $p_{11} > p_{21} + 1$. However, in this case diffusion plays no role. A natural question arises: are there non-flat solutions with non-simultaneous blow-up (being u the blow-up component) out of this range?

Non-simultaneous blow-up for non-flat solutions of a parabolic system was first considered in [2], where the authors study (1)–(2) in the case $m = n = 1$. The necessary (under some restrictions on the initial data) and sufficient condition for the existence of non-simultaneous blow-up is again $p_{11} > p_{21} + 1$. Hence flat solutions are a good guide to determine the non-simultaneous blow-up range in the case of linear diffusion.

Our first result says that flat solutions still give the range for non-simultaneous blow-up when the blow-up component, u, is in the porous medium case.

Theorem 1. *Let $m \geqslant 1$. If u blows up while v remains bounded, then $p_{11} > p_{21} + 1$. Conversely, if $p_{11} > p_{21} + 1$, then there exist initial data (u_0, v_0) such that u blows up while v remains bounded.*

Since $p_{21} \geqslant 0$, in order to have non-simultaneous blow-up we need in particular that $p_{11} > 1$. Thus u can blow up by itself, without the help of v. Condition $p_{11} > p_{21} + 1$ says that p_{21} (which measures the influence of u in the equation for v) is small compared with p_{11} (which measures the capacity of u to blow up by itself); hence, when u blows up, it does not necessarily carry v along with it.

The surprising fact and the main novelty of this paper is that when the coefficient of non-linear diffusion of the blow-up component is less than one, $0 < m < 1$, the result for flat solutions is not a good guide any more, since diffusion plays a major role.

Theorem 2. *Let $0 < m < 1$. If u blows up while v remains bounded, then $p_{11} > \max\{1, p_{21} + (m + 1)/2\}$.*

We are not able to prove the converse in full generality, but we show a partial result that illustrates the general case.

Theorem 3. *Let $0 < m < 1$. If $p_{11} > \max\{1, p_{21} + (m + 1)/2\}$ and $p_{12} = 0$, then there exist initial data (u_0, v_0) such that u blows up while v remains bounded.*

Hence, for $0 < m < 1$ there is non-simultaneous blow-up for a range of parameters for which this phenomenon is not possible in the case of flat solutions. We believe that the result remains true without the extra hypothesis $p_{12} = 0$, but the proof of this fact seems delicate, see Section 3.

Organization of the paper. The key to obtain the conditions for non-simultaneous blow-up is a detailed knowledge of the blow-up behaviour of u when v is a bounded function. This is done in Section 2, where in addition we find the blow-up set of u. We postpone the proof of the main results to Section 3. In Appendix A we prove that our results are valid for the same system of equations, but now defined in a bounded interval with zero flux at the boundaries.

Throughout the paper $\mathbb{R}_+ = (0, \infty)$, and C and c denote positive constants that may change from one line to another, or even in the same line.

2. Non-simultaneous blow-up behaviour

Our purpose in this section is to establish the blow-up behaviour of u when blow-up is non-simultaneous. To this aim we consider v as a frozen coefficient. Since we are dealing with symmetric solutions, we regard u as a blow-up solution to

$$\begin{cases} u_t = (u^m)_{xx} + u^{p_{11}} h, & (x,t) \in \mathbb{R}_+ \times (0,T), \\ u_x(0,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}_+, \end{cases} \tag{4}$$

with $u_x \leqslant 0$, $u_t \geqslant 0$. The function $h = h(x,t) \geqslant c > 0$ is bounded, continuous and satisfies $h_x \leqslant 0$, $h_t \geqslant 0$. The behaviour of solutions to problem (4) has been widely studied when $h = 1$, see [1]. In the general case, since h is bounded both from above and from below, we expect u to behave in a similar way. Therefore, we introduce the following numbers:

$$\alpha = \frac{1}{p_{11} - 1}, \qquad \beta = \frac{p_{11} - m}{2(p_{11} - 1)},$$

which are determined from the self-similar structure of the problem with $h = 1$. In this special case there is a self-similar solution when $\alpha > 0$ that takes the form

$$U(x,t) = (T-t)^{-\alpha} F\big(x(T-t)^{-\beta}\big), \tag{5}$$

and satisfies

$$U(x,t) \leqslant C x^{-\alpha/\beta}, \quad (x,t) \in \mathbb{R}_+ \times (0,T), \tag{6}$$

see [1]. Observe that in the blow-up range for (4) ($p_{11} > 1$), we have $\alpha > 0$.

In the next two lemmas we show that, even when $h \neq 1$, the blow-up rate is self-similar.

Lemma 4. *Let $p_{11} > 1$ and u a solution of* (4). *Then there exists a constant $C > 0$ such that*

$$u(0,t) \leqslant C(T-t)^{-\alpha}. \tag{7}$$

Proof. Let us define $M(t) = \|u(\cdot,t)\|_\infty = u(0,t)$. Following ideas from [4], we set

$$\phi_M(y,s) = \frac{1}{M(t)} u(ay, bs + t), \quad y \geqslant 0,\ -t/b \leqslant s \leqslant 0,$$

where $a = M^{(m-p_{11})/2}$, $b = M^{1-p_{11}}$. Since u blows up, $M \nearrow \infty$ as $t \nearrow T$. On the other hand, since $p_{11} > 1$, $b \searrow 0$.

We claim that there exists a positive constant C such that for every M large enough

$$(\phi_M)_s(0,0) \geqslant C > 0. \tag{8}$$

The blow-up rate follows from this inequality. Indeed, writing it in terms of M, we get $M^{-p_{11}} M' \geqslant C$, which, after integration from t to T yields (7).

The proof of (8) relies strongly on $\{\phi_M\}$ being a family of uniformly bounded solutions of

$$(\phi_M)_s = \left(\phi_M^m\right)_{yy} + \phi_M^{p_{11}} h_M, \tag{9}$$

where $h_M(y,s) = h(ay, bs+t)$. The uniform bound, $0 \leqslant \phi_M \leqslant 1$, is a consequence of $u_t \geqslant 0$. Uniformly bounded solutions of (9) turn out to be equicontinuous in compact subsets of their common domain, cf. [5]. Observe that for any $S < 0$, the domain contains the compact set $[0,L] \times [-S,0]$ if M is large enough. Therefore, given $\{\phi_{M_j}\}$, there is a continuous function Φ and a subsequence, which we denote again by $\{\phi_{M_j}\}$, such that $\phi_{M_j} \to \Phi$ as $M_j \to \infty$, uniformly on $[0,L] \times [-S,0]$. Moreover, $\Phi(0,0) = 1$. Therefore, there exists a neighbourhood of $(0,0)$, U, such that $\Phi > 1/2$ in U. Since we have uniform convergence in $\bar{U}$ (we can assume that $\bar{U}$ is compact), for j large enough we have that $1/4 \leqslant \phi_{M_j} \leqslant 1$ in $\bar{U}$. Thus, the functions ϕ_{M_j} are solutions of uniformly parabolic equations in $\bar{U}$. Since they are uniformly bounded, we get using well-known Schauder estimates [6],

$$\|\phi_{M_j}\|_{C^{2+\alpha,1+\alpha/2}} \leqslant C \quad \text{in } \bar{U}. \tag{10}$$

Now we proceed arguing by contradiction. Assume that there exists a sequence $\{\phi_{M_j}\}$ such that $(\phi_{M_j})_s(0,0) \to 0$. Estimates (10) imply that $\Phi_s = (\Phi^m)_{yy} + K\Phi^{p_{11}}$, with $K = \lim_{t \nearrow T} h_M(y,s)$ and $\Phi_s(0,0) = 0$. But, since $\Phi_s \geqslant 0$ and $(\Phi_s)_y(0,0) = 0$, we have, by Hopf's lemma, $\Phi_s \equiv 0$, so that Φ does not depend on s. We get that Φ is a nonnegative solution of $(\Phi^m)_{yy} + K\Phi^{p_{11}} = 0$ with $\Phi_y \leqslant 0$ and $\Phi(0) = 1$. Hence Φ is concave. Moreover, since $K > 0$, there must be at least a point where Φ is strictly concave; otherwise $\Phi^{p_{11}} \equiv 0$, which is impossible. This implies that Φ has to cross the y-axis, which is a contradiction. □

Lemma 5. *Let $p_{11} > 1$ and u a solution of* (4). *Then there exist constants $C, c > 0$ such that*

$$u(x,t) \geqslant C(T-t)^{-\alpha} \quad \text{if } x \leqslant c(T-t)^{\beta}. \tag{11}$$

Proof. Since u is a subsolution of $u_t = (u^m)_{xx} + Cu^{p_{11}}$, by comparison with a flat solution of this latter problem with the same blow-up time, we have that

$$u(0,t) > C(T-t)^{-\alpha}.$$

Otherwise, u would be below the flat solution at a certain time, which would imply that both solutions would have different blow-up times, a contradiction.

To extend this estimate to sets of the form $x \leqslant c(T-t)^{\beta}$, we observe that u is a supersolution of

$$\begin{cases} u_t = (u^m)_{xx}, & (x,t) \in \mathbb{R}_+ \times (0,T), \\ u(0,t) = C(T-t)^{-\alpha}, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}_+. \end{cases} \tag{12}$$

Problem (12) has a self-similar solution, U, with finite blow-up time T, that takes the form (5) (see [7] for $m = 1$, [8,9] for $m > 1$ and [10] for $0 < m < 1$). We introduce the rescaled function $\tilde{u}(x,t) = AU(Bx,t)$. If $A = B^{\gamma}$, with $\gamma = 2/(1-m)$, then $\tilde{u}$ satisfies the following problem:

$$\begin{cases} \tilde{u}_t = (\tilde{u}^m)_{xx}, & (x,t) \in \mathbb{R}_+ \times (0,T), \\ \tilde{u}(0,t) = AU(0,t) = AC(T-t)^{-\alpha}, & t \in (0,T), \\ \tilde{u}(x,0) = AU(Bx,0), & x \in \mathbb{R}_+. \end{cases}$$

Choosing A small enough, by comparison, $\tilde{u}(x,t) \leqslant u(x,t)$. Thus,

$$u(x,t) \geqslant A(T-t)^{-\alpha} F\big(A^{1/\gamma} x (T-t)^{-\beta}\big) \geqslant CA(T-t)^{-\alpha}$$

for $x \leqslant \xi_0 A^{-1/\gamma}(T-t)^{\beta}$, where $C = \min_{\xi \in [0,\xi_0]} F(\xi)$. If we take ξ_0 small, $C > 0$. □

Lemmas 4 and 5 and the existence of self-similar solutions are the results needed in the proofs of our non-simultaneous theorems. For the sake of completeness, we carry on our study of the blow-up behaviour of u by describing its spatial structure near the origin close to the blow-up time. As a byproduct, we obtain the blow-up set.

Let us define

$$\phi(x) = \begin{cases} x, & x \in \big(0, \frac{1}{3}\big), \\ -3x^2 + 3x - \frac{1}{3}, & x \in \big(\frac{1}{3}, \frac{2}{3}\big), \\ 1 - x, & x \in \big(\frac{2}{3}, 1\big). \end{cases}$$

For $\varepsilon > 0$ we set $\phi_\varepsilon(x) = \varepsilon \phi(\varepsilon^2 x)$.

Lemma 6. *Let $p_{11} > 1$ and u be a solution to* (4) *with $u_x < 0$. If $p_{11} > m$, then there exists a constant $C > 0$ such that*

$$u(x,t) \leqslant \left(C \int_0^x \phi_\varepsilon(s)\, ds \right)^{-1/(m(\gamma-1))} \tag{13}$$

for $1 < \gamma < (p_{11} - (1-m))/m$ and $(x,t) \in [0, 1/\varepsilon^2] \times [0,T)$.

Proof. We follow ideas from [11,12]. The function $w = u^m$ verifies

$$g'(w) w_t = w_{xx} + w^{p_{11}/m} h, \qquad g(w) = w^{1/m}.$$

We introduce

$$J(x,t) = w_x(x,t) + \phi_\varepsilon(x) w^\gamma(x,t),$$

and claim that $J \leqslant 0$ in $[0, 1/\varepsilon^2] \times [0,T)$. Assume it is true, then

$$\int_0^x \frac{w_s}{w^\gamma}\, ds \leqslant -\int_0^x \phi_\varepsilon(s)\, ds,$$

which implies that

$$\frac{1}{1-\gamma} w(x,t)^{1-\gamma} \leqslant -\int_0^x \phi_\varepsilon(s)\, ds,$$

from where we get (13) if $\gamma > 1$.

We are therefore confronted with the proof of the claim. Using that $w_x = J - \phi_\varepsilon w^\gamma$, we compute

$$
\begin{aligned}
g' J_t - J_{xx} &+ \left(\frac{1-m}{m}\right)\frac{w_x}{w} J_x \\
&= bJ - \phi_\varepsilon'' w^\gamma + w^{p_{11}/m} h_x + \frac{p_{11}-1-m\gamma+m}{m} w^{p_{11}/m+\gamma-1} h\phi_\varepsilon \\
&\quad + \frac{2m\gamma+m-1}{m} w^{2\gamma-1}\phi_\varepsilon\phi_\varepsilon' - \gamma\frac{m\gamma-1}{m} w^{3\gamma-2}\phi_\varepsilon^3,
\end{aligned}
$$

where b is a bounded function for $0<x<1/\varepsilon^2$ and $0<t<T$. Therefore

$$g' J_t - J_{xx} + \left(\frac{1-m}{m}\right)\frac{w_x}{w} J_x - bJ \leqslant 0$$

if

$$
\begin{aligned}
&\frac{\phi_\varepsilon'' w^\gamma - h_x w^{p_{11}/m}}{\phi_\varepsilon} + \frac{p_{11}-1-m\gamma+m}{m} w^{p_{11}/m+\gamma-1} h \\
&\quad - \frac{2m\gamma+m-1}{m} w^{2\gamma-1}\phi_\varepsilon' + \gamma\frac{m\gamma-1}{m} w^{3\gamma-2}\phi_\varepsilon^2 \geqslant 0.
\end{aligned}
$$

Hence, since $h_x \leqslant 0$ and $m\gamma < p_{11} - (1-m)$, we need

$$\left(\frac{p_{11}-m\gamma-1+m}{m}\right) w^{p_{11}/m} h \geqslant \left(\frac{2m\gamma+m-1}{m}\right)\varepsilon^3\phi'\left(\varepsilon^2 x\right) - \frac{\varepsilon^4\phi''(\varepsilon^2 x) w}{\phi(\varepsilon^2 x)}, \tag{14}$$

which is true if ε is small enough.

On the other hand, since $w_x(0,t)=0$ and $w_x(x,0)<0$, we have

$$J(0,t)=0, \qquad J\left(1/\varepsilon^2, t\right) \leqslant 0 \quad \text{and} \quad J(x,0)<0,$$

and the claim follows from the maximum principle. □

Corollary 7. *Let $p_{11} > 1$ and u be a solution to* (4), *with $u_x < 0$. If $p_{11} > m$, then there exists a constant $C>0$, depending only on ε, p_{11} and m, such that*

$$u(x,t) \leqslant C x^{-2/m(\gamma-1)} \quad \textit{for } (x,t) \in \left[0, 1/\left(3\varepsilon^2\right)\right] \times [0,T). \tag{15}$$

Remark 8. When $0<m<1$ we have $\gamma < (p_{11}-(1-m))/m < p_{11}/m$. Hence we are not able to obtain the self-similar decay (6) for solutions of (4). As we will see, this forces us to assume $p_{12}=0$ in Theorem 3.

Remark 9. If $m \geqslant 1$ we have $p_{11} \leqslant p_{11} - (1-m)$. Hence, we can take $\gamma = p_{11}/m$ and get

$$u(x,t) \leqslant C x^{-\alpha/\beta} \quad \text{for } (x,t) \in \left[0, 1/\left(3\varepsilon^2\right)\right] \times [0,T).$$

Next lemma determines the blow-up set of u, $B(u)$, when v is bounded; i.e., when blow-up is non-simultaneous. As expected, the sign of β, which depends on the relation between p_{11} and m, determines the blow-up set, even when $h(x,t) \neq 1$.

Lemma 10. *Let u be a solution of* (4). *The blow-up set of u is given by*

$$B(u) = \begin{cases} \{0\}, & \beta > 0, \\ [0, L], & \beta = 0, \\ \bar{\mathbb{R}}_+, & \beta < 0. \end{cases}$$

Remark 11. Our proof does not exclude the possibility of having $L = \infty$.

Proof. If $\beta > 0$, the blow-up set follows directly from estimate (15).

If $\beta \leqslant 0$, we regard u as a supersolution of

$$\begin{cases} u_t = (u^m)_{xx}, & (x,t) \in \mathbb{R}_+ \times (0,T), \\ u(0,t) = C(T-t)^{-\alpha}, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}_+. \end{cases}$$

If $\tilde{u}$ is a solution to this problem, $B(\tilde{u}) \subseteq B(u)$. It is known (see [13,14]) that $B(\tilde{u}) = \bar{\mathbb{R}}_+$ if $\beta < 0$ and $B(\tilde{u}) = [0, \tilde{L}]$ if $\beta = 0$, and the result follows. □

3. Proofs of the main results

We now have the tools to prove the main theorems of the paper.

Proof of Theorem 1. If v is bounded, the function u is a subsolution of

$$u_t = \left(u^m\right)_{xx} + Cu^{p_{11}} \tag{16}$$

that has finite time blow-up. Solutions of (16) are global in time if $p_{11} \leqslant 1$. To see this we can compare with a flat solution of (16) with initial data $u(x,0) = \|u_0\|_{L^\infty}$. Hence we must have $p_{11} > 1$.

Next we prove that $p_{11} > p_{21} + 1$. If we plug the blow-up rate (11) into the equation for v, we have

$$v_t \geqslant \left(v^n\right)_{xx} + v^{p_{22}} \frac{C}{(T-t)^{\frac{p_{21}}{p_{11}-1}}} \chi_{\{x \leqslant c(T-t)^\beta\}}.$$

Set $w = v^n$, which is bounded, strictly positive and verifies $w_t \geqslant 0$ and $w_x \leqslant 0$. We get,

$$cw_t \geqslant \frac{1}{n} w^{(1-n)/n} w_t \geqslant w_{xx} + w^{p_{22}/n} \frac{C}{(T-t)^{\frac{p_{21}}{p_{11}-1}}} \chi_{\{x \leqslant c(T-t)^\beta\}}.$$

The constant that appears in front of the time derivative does not play any fundamental role. Hence we drop it in the sequel.

Now consider the following problem, whose solution is below w:

$$\begin{cases} z_t = z_{xx} + C(T-t)^{-\frac{p_{21}}{p_{11}-1}} \chi_{\{x \leqslant c(T-t)^\beta\}}, & (x,t) \in \mathbb{R}_+ \times (0,T), \\ z_x(0,t) = 0, & t \in (0,T), \\ z(x,0) = z_0(x) = w_0(x), & x \in \mathbb{R}_+, \end{cases} \tag{17}$$

with $z_x \leqslant 0$ and $z_t \geqslant 0$. We use the representation formula for solutions of the heat equation to compute the solution of (17), cf. [15]. Let Γ be the fundamental solution of the heat equation, namely

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{1/2}} \exp\left(-\frac{x^2}{4t}\right).$$

We get

$$\begin{aligned} z(0,t) &= \int_{\mathbb{R}_+} \Gamma(y,t) z_0(y)\,dy \\ &\quad + \int_0^t \int_{\mathbb{R}_+} \Gamma(y,t-\tau) \frac{C}{(T-\tau)^{\frac{p_{21}}{p_{11}-1}}} \chi_{\{y \leqslant c(T-\tau)^\beta\}}\,dy\,d\tau \\ &\geqslant \int_0^t \int_{\mathbb{R}_+} \Gamma(y,t-\tau) \frac{C}{(T-\tau)^{\frac{p_{21}}{p_{11}-1}}} \chi_{\{y \leqslant c(T-\tau)^\beta\}}\,dy\,d\tau. \end{aligned} \tag{18}$$

If we do the change of variables

$$y = s\sqrt{t-\tau}, \qquad dy = \sqrt{t-\tau}\,ds, \tag{19}$$

the last integral in (18) can be bounded from below, using that $\beta \leqslant 1/2$, by

$$\int_0^t \frac{C}{(T-\tau)^{\frac{p_{21}}{p_{11}-1}}} \int_0^{c(T-\tau)^{\beta-1/2}} e^{-s^2/4}\,ds\,d\tau \geqslant \int_0^t \frac{C}{(T-\tau)^{\frac{p_{21}}{p_{11}-1}}}\,d\tau.$$

If $p_{11} \leqslant p_{21} + 1$, the last integral diverges as $t \nearrow T$. Hence, z blows up and so does v, a contradiction.

The proof of the converse follows directly by considering flat solutions. □

Proof of Theorem 2. We follow the same technique as in the proof of Theorem 1. In particular, condition $p_{11} > 1$ holds.

To prove that $2p_{11} > 2p_{21} + m + 1$, consider z a solution of (17). We obtain

$$\begin{aligned} z(0,t) &\geqslant \int_0^t \int_{\mathbb{R}_+} \frac{C}{(T-\tau)^{\frac{p_{21}}{p_{11}-1}}} e^{-s^2/4} \chi_{\{s \leqslant (T-\tau)^\beta/(t-\tau)^{1/2}\}}\,ds\,d\tau \\ &\geqslant \int_0^t \frac{C}{(T-\tau)^{\frac{p_{21}}{p_{11}-1}}} \int_0^{c(T-\tau)^{\beta-1/2}} e^{-s^2/4}\,ds\,d\tau \geqslant \int_0^t \frac{C}{(T-\tau)^{\frac{p_{21}}{p_{11}-1}-\beta+\frac{1}{2}}}\,d\tau. \end{aligned}$$

If $p_{21}/(p_{11}-1) \geqslant \beta + 1/2$, the last integral diverges as $t \nearrow T$. This leads to a contradiction with the fact that v is bounded. □

To prove Theorem 3 we cannot follow the ideas of the proof of Theorem 1 anymore. Indeed, if we want to construct non-simultaneous blow-up solutions outside the range

$p_{11} > p_{21} + 1$, we have to look among non-flat ones. It is at this point where we need to impose $p_{12} = 0$. Under this condition we may use a self-similar solution U of (4) with $h = 1$ as the u component. The spatial shape of U plays a fundamental role. The main difficulty in order to remove the restriction $p_{12} = 0$ is to prove that solutions of (4) with $h \neq 1$ have an approximately self-similar spatial shape.

Apart from the self-similar behaviour of U, we want to use the representation formula to handle the v component. Hence we prove first an auxiliary result that establishes the non-simultaneous blow-up condition when one of the components has linear diffusion and the other one is self-similar.

Let (U, z) be a solution of

$$\begin{cases} U_t = (U^m)_{xx} + U^{p_{11}}, \\ z_t = z_{xx} + z^{p_{22}/n} U^{p_{21}}, \end{cases} \quad (x,t) \in \mathbb{R}_+ \times (0,T), \tag{20}$$

with

$$\begin{cases} U_x = 0, \\ z_x = 0, \end{cases} \quad t \in (0,T), \tag{21}$$

and decreasing initial data

$$\begin{cases} U(x,0) = U_0(x), \\ z(x,0) = z_0(x), \end{cases} \quad x \in \mathbb{R}_+, \tag{22}$$

such that U is self-similar and $z_t \geqslant 0$. The assumption on the initial data excludes the possibility of having flat solutions. We will also assume $z_0 \geqslant \delta > 0$.

Lemma 12. *Let $0 < m < 1$. If $p_{11} > \max\{1, p_{21} + (m+1)/2\}$, there exist initial data (U_0, z_0) such that U blows up while z remains bounded.*

Proof. Since U verifies (5) and (6), we have

$$U(x,t) \leqslant \begin{cases} C(T-t)^{-\alpha} & \text{if } x \leqslant c(T-t)^{\beta}, \\ Cx^{-\alpha/\beta} & \text{if } x \geqslant c(T-t)^{\beta}. \end{cases}$$

Observe that in the range of parameters involved, the exponents verify $\alpha, \beta > 0$.

If $p_{22} > n$, we fix z_0 such that $\|z_0\|_\infty < 1/4$. We claim that $z(x,t) \leqslant z(0,t) \leqslant 1$ for all $0 \leqslant t < T$. Assume not and let $t_0 < T$ be the first time such that $z(0, t_0) = 1$. Using the representation formula, we get

$$\begin{aligned} z(0,t) &= \int_{\mathbb{R}_+} \Gamma(y,t) z_0(y)\, dy + \int_0^t \int_{\mathbb{R}_+} \Gamma(y, t-\tau) z^{p_{22}/n}(y,\tau) U^{p_{21}}(y,\tau)\, dy\, d\tau \\ &\leqslant z_0(0) + z^{p_{22}/n}(0,t) \int_0^t \int_{\mathbb{R}_+} \Gamma(y,t-\tau) \frac{C}{(T-\tau)^{\frac{p_{21}}{p_{11}-1}}} \chi_{\{y \leqslant c(T-\tau)^{\beta}\}}\, dy\, d\tau \\ &\quad + z^{p_{22}/n}(0,t) \int_0^t \int_{\mathbb{R}_+} \Gamma(y,t-\tau) C y^{-\frac{2p_{21}}{p_{11}-m}} \chi_{\{y \geqslant c(T-\tau)^{\beta}\}}\, dy\, d\tau \end{aligned}$$

$$= z_0(0) + z^{p_{22}/n}(0,t)I_1(t) + z^{p_{22}/n}(0,t)I_2(t).$$

If we do the change of variables given by (19), we obtain

$$I_1(t) \leqslant \int_0^t \frac{C}{c(T-\tau)^{\frac{p_{21}}{p_{11}-1}}} \int_0^{(T-\tau)^\beta/(t-\tau)^{1/2}} e^{-s^2/4}\,ds\,d\tau \leqslant \int_0^t \frac{C}{(t-\tau)^{\frac{p_{21}}{p_{11}-1}-\beta+\frac{1}{2}}}\,d\tau$$

and

$$I_2(t) \leqslant C\int_0^t (T-\tau)^{-\frac{p_{21}}{p_{11}-m}} \int_{c(T-\tau)^\beta/(t-\tau)^{1/2}}^{\infty} e^{-s^2/4} s^{-\frac{2p_{21}}{p_{11}-m}}\,ds\,d\tau$$
$$\leqslant \int_0^t C(T-\tau)^{-\frac{p_{21}}{p_{11}-m}}\,d\tau.$$

Since $m < 1$, we have $\beta > 1/2$. This and the condition $2p_{21} < 2p_{11} - m - 1$ imply that I_1 and I_2 are uniformly bounded in $[0, t_0]$. Moreover, they can be made as small as we please by taking T small enough. Thus, if we choose U_0 large such that the blow-up time T of $U(x,t) = (T-t)^{-\alpha}F(x(T-t)^{-\beta})$ is small,

$$z(0,t) \leqslant z_0(0) + \frac{1}{2}z^{p_{22}/n}(0,t). \tag{23}$$

Since $z(0,t) \leqslant 1$ for $0 \leqslant t \leqslant t_0$, $z(0,t)^{p_{22}/n} \leqslant z(0,t)$ and hence

$$z(0,t_0) \leqslant z_0(0) + \frac{1}{2}z^{p_{22}/n}(0,t_0) \leqslant z_0(0) + \frac{1}{2}z(0,t_0).$$

We conclude that $z(0,t_0) \leqslant 1/2$, a contradiction.

If $p_{22} \leqslant n$, we take z_0 such that $1 \leqslant \|z_0\|_\infty \leqslant 2$. Hence $z(0,t) \geqslant 1$ for all t. Let $t_0 < T$ be the first time such that $z(0,t_0) = 6$. If such a time does not exist, then z is bounded for all $0 \leqslant t < T$, and the result follows. Arguing as in the previous case, we conclude that $z(0,t)$ verifies (23) for $0 \leqslant t \leqslant t_0$ and $z(0,t)^{p_{22}/n} \leqslant z(0,t)$. But this means that

$$3 = \frac{1}{2}z(0,t_0) \leqslant z_0(0) \leqslant 2,$$

a contradiction. □

Proof of Theorem 3. Lemma 12 guarantees that we can take U_0, z_0 such that the z component of the solution to (20)–(22) is bounded. Hence $\bar{v} = z^{1/n}$ is bounded. Moreover, it verifies

$$\begin{cases} n\bar{v}^{n-1}v_t = (\bar{v}^n)_{xx} + \bar{v}^{p_{22}}U^{p_{21}}, & (x,t) \in \mathbb{R}_+ \times (0,T), \\ \bar{v}_x(0,t) = 0, & t \in (0,T), \\ \bar{v}(x,0) = z_0^{1/n}(x), & x \in \mathbb{R}_+. \end{cases}$$

Thus, $\bar{v}$ is a bounded supersolution of $cv_t = (v^n)_{xx} + v^{p_{22}}U^{p_{21}}$. The constant in front of v_t can be dropped with a change of variables. By a comparison argument, we conclude that any solution v of (1) with initial data small is bounded. □

Appendix A. The Neumann problem

We study blow-up solutions of the same parabolic system considered in the previous sections, but now in a bounded interval,

$$\begin{cases} u_t = (u^m)_{xx} + u^{p_{11}} v^{p_{12}}, \\ v_t = (v^n)_{xx} + u^{p_{21}} v^{p_{22}}, \end{cases} \quad (x,t) \in (-L, L) \times (0, T), \tag{A.1}$$

with Neumann boundary conditions

$$\begin{cases} (u^m)_x(-L,t) = (u^m)_x(L,t) = 0, \\ (v^n)_x(-L,t) = (v^n)_x(L,t) = 0, \end{cases} \quad t \in (0, T), \tag{A.2}$$

and initial data

$$\begin{cases} u(x,0) = u_0(x), \\ v(x,0) = v_0(x), \end{cases} \quad x \in (-L, L). \tag{A.3}$$

As before, we will assume that u_0 and v_0 are strictly positive, continuous, bounded, symmetric and non-increasing for $x \in \mathbb{R}_+$.

Theorem A. *Theorems* 1–3 *hold true for* (A.1)–(A.3).

First of all, let us briefly describe how to adapt the tools used in the proofs of the main theorems when we deal with problem (A.1)–(A.3).

The proof of Lemma 4 still holds if $\beta \geqslant 0$. However, if $\beta < 0$, the spatial interval of definition of ϕ_M, $[0, L/a]$, contracts to zero as $t \nearrow T$. To avoid this contraction, we perform an even periodic extension to the positive real line. This extension is possible, since $(u^m)_x(0,t) = (u^m)_x(L,t) = 0$. In this situation we apply Lemma 4 of Section 2.

Concerning Lemma 5, the result holds true if we redefine problem (12) to the bounded interval $[0, L]$ by adding the boundary condition, $(u^m)_x(L,t) = 0$. Since $U_x \leqslant 0$, the self-similar solution of (12) in $\mathbb{R}_+$ is a subsolution of the problem restricted to the interval.

Proof of Theorem A. The proof follows the same ideas as the proofs of Theorems 1–3. Nevertheless, there is a slight difference when obtaining the condition for non-simultaneous blow-up $p_{11} > p_{21} + 1$ (respectively $2p_{11} > 2p_{21} + m + 1$). Indeed, mimicking the previous proofs, we consider the following boundary value problem:

$$\begin{cases} z_t = z_{xx} + C(T-t)^{\frac{-p_{21}}{p_{11}-1}} \chi_{\{x \leqslant c(T-t)^\beta\}}, & (x,t) \in (0, L) \times (0, T), \\ z_x(0,t) = z_x(L,t) = 0, & t \in (0, T), \\ z(x,0) = z_0(x) = v_0^n(x), & x \in (0, L). \end{cases} \tag{A.4}$$

We extend z_0 to the positive real line by defining a continuous, positive and non-increasing function $\hat{z}_0$, verifying $\hat{z}_0(x) = z_0(x)$ in $[0, L]$. If z is the solution of (A.4) with initial data z_0 and $\hat{z}$ the solution extended to $\mathbb{R}_+$ with initial data $\hat{z}_0$, a comparison argument in $[0, L] \times [0, T)$ yields $z(x,t) \geqslant \hat{z}(x,t)$. The function $\hat{z}$ fulfills the hypotheses of problem (17). □

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