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h-Stability for linear dynamic equations on time scales ☆

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Abstract

We study the h-stability for linear dynamic equations on time scales and their perturbations by using the Bihari type inequality on time scales and the unified time scale quadratic Lyapunov functions.

Keywords: Linear dynamic equation; h-Stability; Unified Gronwall's inequality; Quadratic Lyapunov function

1. Introduction

The theory of dynamic equations on time scales (aka measure chains) was introduced by Hilger [10] with the motivation of providing a unified approach to continuous and discrete analysis. The generalized derivative or Hilger derivative $f^{\Delta}(t)$ of a function $f:\mathbb{T}\to\mathbb{R}$, where $\mathbb{T}$ is a so-called "time scale" (an arbitrary closed nonempty subset of $\mathbb{R}$) becomes the usual derivative when $\mathbb{T}=\mathbb{R}$, that is $f^{\Delta}(t)=f'(t)$. On the other hand, if $\mathbb{T}=\mathbb{Z}$, then $f^{\Delta}(t)$ reduces to the usual forward difference, that is $f^{\Delta}(t)=\Delta f(t)$. This theory not only brought unification but also a wide generalization of the notions of time used in dynamic equations leading to new applications. Also, this theory allows one to get some insight into and better understanding of the subtle differences between discrete and continuous systems [2,3].

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This paper examines the h-stability for linear dynamic equations on time scales and their perturbations. The notion of h-stability was introduced by Pinto [13,14] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential stability and uniform Lipschitz stability) under some perturbations. In the study of stability properties of differential and difference systems, the notion of h-stability is very useful because, when we study the asymptotic stability, it is not easy to work with nonexponential types of stability. For the detailed results about h-stability for differential and difference systems, we refer to the papers [4–6] and [11–15].

Now, we mention without proof several foundational definitions and results from the calculus on time scales in an excellent introductory text by Bohner and Peterson [3].

Definition 1.1. A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$.

We assume throughout that $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$.

It is also assumed throughout that in $\mathbb{T}$ the interval $[a,b]$ means the set $\{t \in \mathbb{T}\colon a \leqslant t \leqslant b\}$ for the points $a < b$ in $\mathbb{T}$. Since a time scale may or may not be connected, we need the following concept of jump operators.

Definition 1.2. The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(t) = \inf\{s \in \mathbb{T}\colon s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T}\colon s < t\}$$

are called the *jump operators*.

The jump operators σ and ρ allow the classification of points in $\mathbb{T}$ in the following way:

Definition 1.3. A nonmaximal element $t \in \mathbb{T}$ is said to be *right-dense* if $\sigma(t) = t$, *right-scattered* if $\sigma(t) > t$, *left-dense* if $\rho(t) = t$, *left-scattered* if $\rho(t) < t$.

In the case $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, and if $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\sigma(t) = t + h$.

Definition 1.4. The mapping $\mu : \mathbb{T} \to \mathbb{R}_+$ defined by $\mu(t) = \sigma(t) - t$ is called the *graininess function*.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.

Definition 1.5. Let $f : \mathbb{T} \to \mathbb{R}^n$. f is called *differentiable* at $t \in \mathbb{T}^\kappa$, with (*delta*) *derivative* $f^\Delta(t) \in \mathbb{R}^n$ if given $\varepsilon > 0$ there exists a neighborhood U of t such that, for all $s \in U$,

$$\bigl|f^\sigma(t) - f(s) - f^\Delta(t)\bigl[\sigma(t) - s\bigr]\bigr| \leqslant \varepsilon\bigl|\sigma(t) - s\bigr|,$$

where $f^\sigma = f \circ \sigma$.

If $\mathbb{T} = \mathbb{R}$, then

$$f^\Delta(t) = \frac{df(t)}{dt},$$

and if $\mathbb{T} = \mathbb{Z}$, then

$$f^\Delta(t) = f(t+1) - f(t).$$

Some basic properties of delta derivatives are the following [3].

Theorem 1.6. *Assume that $f:\mathbb{T}\to\mathbb{R}^n$ and let $t\in\mathbb{T}^\kappa$.*

(i) *If f is differentiable at t, then f is continuous at t.*
(ii) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^\Delta(t)=\frac{f^\sigma(t)-f(t)}{\sigma(t)-t}.$$

(iii) *If f is differentiable at t and t is right-dense, then*

$$f^\Delta(t)=\lim_{t\to s}\frac{f(t)-f(s)}{t-s}.$$

(iv) *If f is differentiable at t, then*

$$f^\sigma(t)=f(t)+\mu(t)f^\Delta(t).$$

Definition 1.7. The function $f:\mathbb{T}\to\mathbb{R}^n$ is said to be *rd-continuous* (denoted by $f\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n)$) if, at all $t\in\mathbb{T}$,

(i) f is continuous at every right-dense point $t\in\mathbb{T}$,
(ii) $\lim_{s\to t^-}f(s)$ exists and is finite at every left-dense point $t\in\mathbb{T}$.

Definition 1.8. Let $f\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^n)$. Then $g:\mathbb{T}\to\mathbb{R}^n$ is called the *antiderivative* of f on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^\Delta(t)=f(t)$ for $t\in\mathbb{T}$. In this case, we define

$$\int_a^t f(s)\Delta s=g(t)-g(a),\qquad t\in\mathbb{T}.$$

Theorem 1.9. *Let $f:\mathbb{R}\to\mathbb{R}$ be continuously differentiable and suppose that $g:\mathbb{T}\to\mathbb{R}$ is delta differentiable. Then $f\circ g:\mathbb{T}\to\mathbb{R}$ is delta differentiable and the formula*

$$(f\circ g)^\Delta(t)=g^\Delta(t)\int_0^1 f'\big(g(t)+\delta\mu(t)g^\Delta(t)\big)\,d\delta$$

holds.

2. Main results

Consider the dynamic system

$$x^\Delta=f(t,x),\qquad x(t_0)=x_0,\tag{2.1}$$

where $f:\mathbb{T}\times\mathbb{R}^n\to\mathbb{R}^n$ and let f satisfies the following conditions:

(i) f is rd-continuous with respect to the first argument in $\mathbb{T}$.
(ii) f is locally Lipschitzian with respect to the second argument in $\mathbb{R}^n$.

(iii) f is *regressive* (denoted by $f \in \mathcal{R}(\mathbb{T}\times\mathbb{R}^n, \mathbb{R}^n)$), i.e., the function $\mathrm{id} + \mu(t) f(t,\cdot): \mathbb{R}^n \to \mathbb{R}^n$ is invertible.

Then the initial value problem (2.1) with $x(t_0) = x_0 \in \mathbb{R}^n$ admits exactly one solution, denoted by $x(t, t_0, x_0)$.

When $f(t,x) = A(t)x$, A is an $n \times n$ matrix-valued function on $\mathbb{T}$, (2.1) becomes the linear homogeneous dynamic system

$$x^{\Delta} = A(t)x, \tag{2.2}$$

where the norm of A is defined to be

$$|A| = \max_{|x| \leqslant 1} |Ax|.$$

Definition 2.1. Let $t_0 \in \mathbb{T}$ and assume that $A \in \mathcal{R}(\mathbb{T}\times\mathbb{R}^n, \mathbb{R}^n)$ is an $n \times n$ matrix-valued function. The unique matrix-valued solution of the initial value problem

$$Y^{\Delta} = A(t)Y, \qquad Y(t_0) = I \text{ (the identity matrix)}, \tag{2.3}$$

is called the *transition matrix* and it is denoted by $\Phi_A(t, t_0)$.

In this paper, we denote the solution of (2.3) as $\Phi_A(t,t_0)$ when $A(t)$ is time varying and denote the solution $e_A(t,t_0) = \Phi_A(t,t_0)$ (the matrix exponential as in [2]) only when $A(t) = A$ is a constant matrix. Thus, when $\mathbb{T} = \mathbb{R}$,

$$e_A(t,t_0) = e^{A(t-t_0)},$$

while if $\mathbb{T} = \mathbb{Z}$ and $I + A$ is invertible, then

$$e_A(t,t_0) = (I + A)^{(t-t_0)}.$$

Pinto [13] introduced the notion of h-stability which is an extension of the notions of exponential stability and uniform stability.

Definition 2.2. System (2.1) is called an *h-system* if there exist a positive function $h: \mathbb{T} \to \mathbb{R}$, a constant $c \geqslant 1$ and $\delta > 0$ such that

$$\bigl|x(t,t_0,x_0)\bigr| \leqslant c|x_0|h(t)h(t_0)^{-1}, \quad t \geqslant t_0,$$

if $|x_0| < \delta$ (here $h(t)^{-1} = 1/h(t)$). If h is bounded, then (2.1) is said to be h-stable.

For the various definitions of stability, we refer to [7] and we obtain the following possible implications for system (2.1) among the various types of stability:

$$\begin{aligned} h\text{-stability} \quad &\Rightarrow \quad \text{uniform exponential stability} \\ &\Rightarrow \quad \text{uniform Lipschitz stability} \\ &\Rightarrow \quad \text{uniform stability} \end{aligned}$$

as in [4]. The above implications can be proved by the following characterization due to Pinto [14, Lemma 1] in the case $\mathbb{T} = \mathbb{R}$, in terms of the transition matrix for system (2.2).

Lemma 2.3. *The time varying linear dynamic system* (2.2) *is an h-system if and only if there exist a positive function h defined on $\mathbb{T}$ and a constant $c \geqslant 1$ such that*

$$\left|\Phi_A(t,t_0)\right| \leqslant c h(t) h(t_0)^{-1}$$

for all $t \geqslant t_0$ with $t, t_0 \in \mathbb{T}$.

Proof. Suppose that (2.2) is an h-system. We note that for a linear system we have

$$x(t,t_0,x_0) = \Phi_A(t,t_0)x_0, \quad x_0 \in \mathbb{R}^n,$$

for $t \geqslant t_0$. Then we have

$$\left|\Phi_A(t,t_0)\right| = \max_{|u|\leqslant 1}\left|\Phi_A(t,t_0)u\right| = \max_{|u|\leqslant 1}\left|x(t,t_0,u)\right| \leqslant c h(t) h(t_0)^{-1}, \quad t \geqslant t_0.$$

Thus we obtain

$$\left|\Phi_A(t,t_0)\right| \leqslant c h(t) h(t_0)^{-1}$$

for all $t \geqslant t_0$ with $t, t_0 \in \mathbb{T}$.

Conversely, we have

$$\left|x(t,t_0,x_0)\right| = \left|\Phi_A(t,t_0)x_0\right| \leqslant \left|\Phi_A(t,t_0)\right||x_0| \leqslant c|x_0| h(t) h(t_0)^{-1}, \quad t \geqslant t_0.$$

Hence (2.2) is an h-system. □

It is widely known that the stability characteristics of a nonautonomous linear system of differential or difference equations can be characterized completely by a corresponding autonomous linear system by a Lyapunov transformation. DaCunha in [8] gave a definition of a Lyapunov transformation as follows.

Definition 2.4. A *Lyapunov transformation* is an invertible matrix $L(t) \in C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ with the property that, for some positive $\eta, \rho \in \mathbb{R}$,

$$\left|L(t)\right| \leqslant \rho \quad \text{and} \quad \det L(t) \geqslant \eta \tag{2.4}$$

for all $t \in \mathbb{T}$.

Note that an equivalent condition to (2.4) is that there exists $\rho > 0$ such that

$$\left|L(t)\right| \leqslant \rho \quad \text{and} \quad \left|L^{-1}(t)\right| \leqslant \rho \tag{2.5}$$

for all $t \in \mathbb{T}$ [8].

Suppose that $L(t) \in C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$, i.e., $L(t)$ is rd-continuous and its delta derivative exists, and $L(t)$ is invertible for all $t \in \mathbb{T}$. We consider the dynamic system

$$Z^{\Delta}(t) = G(t)Z(t), \quad Z(\tau) = I, \tag{2.6}$$

where

$$G(t) = L^{\sigma^{-1}}(t)A(t)L(t) - L^{\sigma^{-1}}(t)L^{\Delta}(t). \tag{2.7}$$

Then the transition matrix for systems (2.6) is given by

$$\Phi_G(t,\tau) = L^{-1}(t)\Phi_A(t,\tau)L(\tau) \tag{2.8}$$

for all $t, \tau \in \mathbb{T}$ [8, Theorem 3.8].

Lemma 2.5. *For systems* (2.2) *and* (2.6), $A(t)$ *is regressive if and only if* $G(t)$ *is also regressive.*

Proof. We see that $A(t)$ and $G(t)$ are regressive for each right-dense point $t \in \mathbb{T}^\kappa$.

Let $t \in \mathbb{T}^\kappa$ be any right-scattered point. Then we have

$$\begin{aligned} I + \mu(t)G(t) &= I + \mu(t)\big[L^{\sigma^{-1}}(t)A(t)L(t) - L^{\sigma^{-1}}(t)L^\Delta(t)\big] \\ &= L^{-1}\big(\sigma(t)\big)\big[I + \mu(t)A(t)\big]L(t) \end{aligned}$$

from the identity

$$A^\sigma(t) = A(t) + \mu(t)A^\Delta(t) \tag{2.9}$$

for any differentiable matrix-valued function $A(t)$ on $\mathbb{T}$ [2]. □

Now, system (2.2) with $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ can be transformed into the system

$$z^\Delta(t) = G(t)z(t), \qquad z(t_0) = z_0 \tag{2.10}$$

via the Lyapunov transformation $z(t) = L^{-1}(t)x(t)$. Then the notion of h-stability for (2.2) is preserved by the Lyapunov transformation:

Theorem 2.6. *System* (2.2) *is h-stable if and only if* (2.10) *is also h-stable.*

Proof. Suppose that (2.2) is h-stable. Then, in view of Lemma 2.3, there exists a constant $c > 0$ and a positive bounded function h defined on $\mathbb{T}$ such that

$$\big|\Phi_A(t, t_0)\big| \leqslant ch(t)h(t_0)^{-1}, \quad t \geqslant t_0 \in \mathbb{T},$$

where $\Phi_A(t, t_0)$ is the transition matrix of (2.2). By using Lemma 2.3 and (2.5), we have

$$\begin{aligned} \big|\Phi_G(t, t_0)\big| &= \big|L^{-1}(t)\Phi_A(t, t_0)L(t_0)\big| \leqslant \big|L^{-1}(t)\big|\big|\Phi_A(t, t_0)\big|\big|L(t_0)\big| \leqslant c\rho^2 h(t)h(t_0)^{-1} \\ &= dh(t)h(t_0)^{-1}, \quad t \geqslant t_0 \in \mathbb{T}, \end{aligned}$$

where $d = c\rho^2$ and $\Phi_G(t)$ is the transition matrix of (2.10). Hence (2.10) is h-stable.

The converse holds similarly. □

We consider the perturbed system

$$z^\Delta(t) = \big[G(t) + F(t)\big]z(t), \qquad z(t_0) = z_0, \tag{2.11}$$

where $G, F \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ and $G(t)$ satisfies (2.7).

DaCunha [7, Theorem 5.1] obtained the result about the uniform stability for the perturbed system (2.11) under the condition

$$\int_{t_0}^{\infty} \big|F(s)\big|\,\Delta s \leqslant \beta \tag{2.12}$$

for some $\beta \geqslant 0$. With a slight modification of (2.12), we have the following.

Theorem 2.7. *Suppose that* (2.2) *is* h*-stable. Then* (2.11) *is* h*-stable if there exists* $\beta \geqslant 0$ *such that for all* $t_0 \in \mathbb{T}$,

$$\int_{t_0}^{\infty} \frac{h(s)}{h(\sigma(s))} \left|F(s)\right| \Delta s \leqslant \beta.$$

Proof. Suppose (2.2) is h-stable. Then it follows from Theorem 2.6 that (2.10) is h-stable, i.e., there exist a constant $c > 0$ and a positive bounded function h defined on $\mathbb{T}$ such that

$$\left|\Phi_G(t, t_0)x_0\right| \leqslant ch(t)h(t_0)^{-1}$$

for all $t \geqslant t_0 \in \mathbb{T}$. For any t_0 and $z(t_0) = z_0$, by the variation of constants formula in [2], the solution $z(t)$ of (2.11) satisfies

$$z(t) = \Phi_G(t, t_0)z_0 + \int_{t_0}^{t} \Phi_G\big(t, \sigma(s)\big)F(s)z(s)\, \Delta s, \tag{2.13}$$

where $\Phi_G(t, t_0)$ is the transition matrix for the system (2.10).

By taking the norms of both sides of (2.13), we have

$$\left|z(t)\right| \leqslant ch(t)h(t_0)^{-1}|z_0| + c\int_{t_0}^{t} h(t)h\big(\sigma(s)\big)^{-1}\left|F(s)\right|\left|z(s)\right| \Delta s, \quad t \geqslant t_0.$$

By dividing by $h(t)$ on both sides,

$$\frac{|z(t)|}{h(t)} \leqslant c\frac{|z_0|}{h(t_0)} + c\int_{t_0}^{t} \frac{h(s)}{h(\sigma(s))}\left|F(s)\right|\frac{|z(s)|}{h(s)}\, \Delta s, \quad t \geqslant t_0.$$

In view of the Gronwall's inequality on time scale in [2], we obtain

$$\begin{aligned}
\frac{|z(t)|}{h(t)} &\leqslant c\frac{|z_0|}{h(t_0)} e_{c\frac{h(s)}{h(\sigma(s))}|F(s)|}(t, t_0) \\
&= c\frac{|z_0|}{h(t_0)} \exp\left(\int_{t_0}^{t} \xi_{\mu(t)}\left(c\frac{h(s)}{h(\sigma(s))}\left|F(s)\right|\right) \Delta s\right) \\
&= \begin{cases} c\frac{|z_0|}{h(t_0)} \exp\big(\int_{t_0}^{t} \frac{1}{\mu(s)} \operatorname{Log}\big(1 + \mu(s)c\frac{h(s)}{h(\sigma(s))}|F(s)|\big)\, \Delta s\big) & \text{if } \mu \neq 0, \\ c\frac{|z_0|}{h(t_0)} \exp\big(\int_{t_0}^{t} c\frac{h(s)}{h(\sigma(s))}|F(s)|\, ds\big) & \text{if } \mu = 0 \end{cases} \\
&\leqslant c\frac{|z_0|}{h(t_0)} \exp\left(\int_{t_0}^{t} c\frac{h(s)}{h(\sigma(s))}\left|F(s)\right| \Delta s\right) \\
&\leqslant c\frac{|z_0|}{h(t_0)} e^{c\beta},
\end{aligned}$$

where the cylinder transformation $\xi_\mu(z)$ is given by

$$\xi_\mu(z) = \begin{cases} \frac{1}{z}\operatorname{Log}(1+\mu z) & \text{if } \mu \neq 0 \text{ (for } z \neq -\frac{1}{\mu}), \\ z & \text{if } \mu = 0. \end{cases}$$

Thus

$$\left|z(t)\right| \leqslant d|z_0|h(t)h(t_0)^{-1}, \quad t \geqslant t_0,$$

where $d = ce^{c\beta}$. Hence (2.11) is h-stable. □

Corollary 2.8. *Suppose that* (2.2) *is h-stable with bounded function $\frac{h(t)}{h(\sigma(t))}$ on $\mathbb{T}$. Then* (2.11) *is h-stable if there exists a constant $\beta \geqslant 0$ such that for all $t_0 \in \mathbb{T}$,*

$$\int_{t_0}^{\infty} \left|F(s)\right| \Delta s \leqslant \beta.$$

Note that if $\mathbb{T} = \mathbb{R}$, then (2.11) is h-stable if there exists a constant $\beta \geqslant 0$ such that for all $t_0 \in \mathbb{R}$,

$$\int_{t_0}^{\infty} \left|F(s)\right| ds \leqslant \beta.$$

If $\mathbb{T} = \mathbb{Z}$, then (2.11) is h-stable if there exists a constant $\beta \geqslant 0$ such that for all $t_0 \in \mathbb{Z}$,

$$\sum_{s=t_0}^{\infty} \frac{h(s)}{h(s+1)} \left|F(s)\right| \leqslant \beta.$$

Now, we consider the nonlinear perturbed dynamic system

$$z^\Delta(t) = A(t)z(t) + g\big(t, z(t)\big), \tag{2.14}$$

where $g \in C_{\mathrm{rd}}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t,0) = 0$. We investigate the h-stability for (2.14) by using the concept of class $\hat{H}$ in [9] and the Bihari type inequality in [1, Theorem 5.8].

Definition 2.9. A function $w : \mathbb{R}^+ \to \mathbb{R}^+$ belongs to the *class* $\hat{H}$ if

(H_1) $w(u)$ is nondecreasing and continuous for $u \geqslant 0$ and positive for $u > 0$,
(H_2) there exists a continuous function ϕ on $\mathbb{R}^+$ with $w(\alpha u) \leqslant \phi(\alpha) w(u)$ for $\alpha > 0$, $u \geqslant 0$,
(H_3) $\lim_{u \to 0^+} \frac{w(u)}{u}$ exists.

Theorem 2.10. *Suppose that* (2.2) *is h-stable and*

$$\left|g(t,x)\right| \leqslant F(t)w\big(|x|\big), \quad t \geqslant t_0,$$

where F is positive and rd*-continuous, and $w \in \hat{H}$ with corresponding multiplier function ϕ. Let r be the solution of*

$$r^\Delta(t) = c\lambda(t)w\big(r(t)\big), \qquad r(t_0) = c$$

and assume that there is a bijective function W satisfying $(W \circ r)^\Delta = c\lambda$ with $\int_{t_0}^{\infty} \lambda(s)\,\Delta s < \infty$ for all $t_0 \in \mathbf{T}$. Then (2.14) *is h-stable.*

Proof. For any t_0 and $z_0 = z(t_0)$, the solution $z(t)$ of (2.14) with the initial value $z(t_0) = z_0$ is given by

$$z(t) = \Phi_A(t, t_0)z_0 + \int_{t_0}^{t} \Phi_A\big(t, \sigma(s)\big)g\big(s, z(s)\big)\,\Delta s, \tag{2.15}$$

where $\Phi_A(t, t_0)$ is the transition matrix for (2.2).

Since (2.2) is h-stable, there exists a positive function h defined on $\mathbb{T}$ such that

$$\big|\Phi_A(t, t_0)\big| \leqslant ch(t)h(t_0)^{-1}$$

for all $t \geqslant t_0 \in \mathbb{T}$.

By taking the norms of both sides, we obtain

$$\begin{aligned}
\big|z(t)\big| &\leqslant \big|\Phi(t, t_0)\big||z_0| + \int_{t_0}^{t} \big|\Phi_A\big(t, \sigma(s)\big)\big|\big|g\big(s, z(s)\big)\big|\,\Delta s \\
&\leqslant ch(t)h(t_0)^{-1}|z_0| + \int_{t_0}^{t} ch(t)h\big(\sigma(s)\big)^{-1}\big|g\big(s, z(s)\big)\big|\,\Delta s \\
&\leqslant ch(t)h(t_0)^{-1}|z_0| + \int_{t_0}^{t} ch(t)h\big(\sigma(s)\big)^{-1}F(s)w\big(|z(s)|\big)\,\Delta s.
\end{aligned}$$

Dividing by $h(t)h(t_0)^{-1}|z_0|$ ($z_0 \neq 0$) on both sides, we have

$$\frac{|z(t)|h(t_0)}{h(t)|z_0|} \leqslant c + c\int_{t_0}^{t} \frac{h(t_0)F(s)}{h(\sigma(s))|z_0|}w\left(\frac{|z_0|h(s)}{h(t_0)} \cdot \frac{h(t_0)|z(s)|}{h(s)|z_0|}\right)\Delta s.$$

Letting $u(t) = \frac{|z(t)|h(t_0)}{h(t)|z_0|}$, we rewrite

$$\begin{aligned}
\big|u(t)\big| &\leqslant c + c\int_{t_0}^{t} \frac{h(t_0)F(s)}{h(\sigma(s))|z_0|}w\left(\frac{|z_0|h(s)}{h(t_0)} \cdot u(s)\right)\Delta s \\
&\leqslant c + c\int_{t_0}^{t} \frac{h(t_0)F(s)}{h(\sigma(s))|z_0|}\phi\left(\frac{|z_0|h(s)}{h(t_0)}\right)w\big(u(s)\big)\,\Delta s.
\end{aligned}$$

Letting $\lambda(s) = \frac{h(t_0)F(s)}{h(\sigma(s))|z_0|}\phi(\frac{|z_0|h(s)}{h(t_0)})$, we have

$$u(t) \leqslant c + c\int_{t_0}^{t} \lambda(s)w\big(u(s)\big)\,\Delta s \quad \text{for all } t \in \mathbb{T}.$$

By the Bihari's inequality in [1], we obtain

$$u(t) \leqslant W^{-1}\left[W(c) + c\int_{t_0}^{t} \lambda(s)\,\Delta s\right] \leqslant W^{-1}\left[W(c) + c\int_{t_0}^{\infty} \lambda(s)\,\Delta s\right],$$

for all $t \in \mathbb{T}$.

Thus we have

$$\left|z(t)\right| \leqslant d|z_0|h(t)h(t_0)^{-1},$$

where $d = W^{-1}[W(c) + c\int_{t_0}^{\infty} \lambda(s)\Delta s]$. This implies that (2.14) is h-stable. □

Thus we obtain the following corollaries.

Corollary 2.11. *Suppose that* (2.2) *is h-stable and*

$$\left|g(t,x)\right| \leqslant \left|F(t)\right| w\big(|x|\big), \quad t \geqslant t_0 \in \mathbb{T} = \mathbb{R},$$

where $w \in \hat{H}$ with corresponding multiplier function ϕ and

$$\lambda(t) = \frac{h(t_0)|F(t)|}{h(t)|z_0|}\phi\left(\frac{|z_0|h(t)}{h(t_0)}\right),$$
$$d = W^{-1}\left[W(c) + c\int_{t_0}^{\infty} \lambda(s)\,ds\right].$$

Then (2.14) *is h-stable.*

Corollary 2.12. *Suppose that* (2.2) *is h-stable and*

$$\left|g(t,x)\right| \leqslant \left|F(t)\right| w\big(|x|\big), \quad t \geqslant t_0 \in \mathbb{T} = \mathbb{Z},$$

where $w \in \hat{H}$ with corresponding multiplier function ϕ and

$$\lambda(t) = \frac{h(t_0)|F(t)|}{h(t+1)|z_0|}\phi\left(\frac{|z_0|h(t)}{h(t_0)}\right),$$
$$d = W^{-1}\left[W(c) + c\sum_{s=t_0}^{\infty} \lambda(s)\right].$$

Then (2.14) *is h-stable.*

Finally, we study the h-stability for the linear dynamic system (2.2) by means of the quadratic Lyapunov functions on time scales.

DaCunha in [7] introduced a unified time scale quadratic Lyapunov function as follows:

Definition 2.13. Let $Q(t)$ be a symmetric matrix such that $Q(t) \in C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$. A *unified time scale quadratic Lyapunov function* is given by

$$x^T(t)Q(t)x(t), \quad t \geqslant t_0, \tag{2.16}$$

with the delta derivative

$$\begin{aligned}
\big[x^T(t)Q(t)x(t)\big]^\Delta &= x^T(t)\big[A^T(t)Q(t) \\
&\quad + \big(I+\mu(t)A^T(t)\big)\big(Q^\Delta(t)+Q(t)A(t)+\mu(t)Q^\Delta(t)A(t)\big)\big]x(t) \\
&= x^T(t)\big[A^T(t)Q(t)+Q(t)A(t)+\mu(t)A^T(t)Q(t)A(t) \\
&\quad + \big(I+\mu(t)A^T(t)\big)Q^\Delta(t)\big(I+\mu(t)A(t)\big)\big]x(t).
\end{aligned}$$

The matrix dynamic equation obtained by differentiating (2.16) with respect to t is given by

$$\begin{aligned}
&A^T(t)Q(t)+Q(t)A(t)+\mu(t)A^T(t)Q(t)A(t)+\big(I+\mu(t)A^T(t)\big)Q^\Delta(t)\big(I+\mu(t)A(t)\big) \\
&\quad = -M, \qquad M = M^T.
\end{aligned}$$

Lemma 2.14. *Suppose that $g:\mathbb{T}\to\mathbb{R}$ is positive delta differentiable and $\frac{g^\Delta(t)}{g(t)}$ is regressive. Then $\ln g(t)$ is delta differentiable and the formula*

$$\big(\ln g(t)\big)^\Delta = \xi_{\mu(t)}\left(\frac{g^\Delta(t)}{g(t)}\right)$$

holds, where

$$\xi_{\mu(t)}(z) = \begin{cases} \frac{1}{\mu(t)}\ln(1+\mu(t)z) & \text{if } \mu(t)\neq 0, \\ z & \text{if } \mu(t)=0. \end{cases}$$

Thus we have

$$\int_{t_0}^{t} \xi_{\mu(s)}\left(\frac{g^\Delta(s)}{g(s)}\right)\Delta s = \ln\left(\frac{g(t)}{g(t_0)}\right)$$

when $g^\Delta \in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$.

Proof. Letting $f(x)=\ln x$, we have $f'(x)=\frac{1}{x}$. Hence we obtain by the chain rule on time scale $\mathbb{T}$,

$$\begin{aligned}
\big(\ln g(t)\big)^\Delta &= \left(\int_0^1 f'\big(g(t)+\delta\mu(t)g^\Delta(t)\big)\,d\delta\right)g^\Delta(t) \\
&= \int_0^1 \frac{1}{g(t)+\delta\mu(t)g^\Delta(t)}\,d\delta g^\Delta(t) \\
&= \begin{cases} \frac{1}{\mu(t)}\ln(g(t)+\delta\mu(t)g^\Delta(t))\big|_{\delta=0}^{\delta=1} & \text{if } \mu(t)\neq 0, \\ \frac{g^\Delta(t)}{g(t)} & \text{if } \mu(t)=0 \end{cases} \\
&= \begin{cases} \frac{1}{\mu(t)}\ln(1+\mu(t)\frac{g^\Delta(t)}{g(t)}) & \text{if } \mu(t)\neq 0, \\ \frac{g^\Delta(t)}{g(t)} & \text{if } \mu(t)=0 \end{cases} \\
&= \xi_{\mu(t)}\left(\frac{g^\Delta(t)}{g(t)}\right).
\end{aligned}$$

This proof is complete. □

Note that if $\mathbb{T}=\mathbb{R}$, then we have

$$\big(\ln g(t)\big)^{\Delta}=\big(\ln g(t)\big)'=\frac{g'(t)}{g(t)},\quad t\in\mathbb{R},$$

and if $\mathbb{T}=\mathbb{Z}$, then we have

$$\big(\ln g(t)\big)^{\Delta}=\Delta\big(\ln g(t)\big)=\ln\left(1+\frac{\Delta g(t)}{g(t)}\right)=\ln\left(\frac{g(t+1)}{g(t)}\right),\quad t\in\mathbb{Z}.$$

Lemma 2.15. *If the delta differentiable function $h:\mathbb{T}\to\mathbb{R}$ is positive, then $\frac{h^{\Delta}(t)}{h(t)}$ is positively regressive, and $e_p(t,t_0)$ satisfies*

$$e_p(t,t_0)=\frac{h(t)}{h(t_0)},$$

where $p(t)=\frac{h^{\Delta}(t)}{h(t)}$.

Proof. In view of Lemma 2.14 we obtain

$$1+\mu(t)\frac{h^{\Delta}(t)}{h(t)}=1+\frac{h(\sigma(t))-h(t)}{h(t)}=\frac{h(\sigma(t))}{h(t)}>0,\quad t\in\mathbb{T}.\qquad\Box$$

DaCunha's result [7, Theorem 3.2] can be extended as the following theorem by using Lemma 2.15.

Theorem 2.16. *Suppose that there exist a symmetric matrix $Q(t)\in C^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^{n\times n})$ and a positive bounded differentiable function h defined on $\mathbb{T}$ satisfying the following properties for all $t\in\mathbb{T}$:*

(i) $\eta I\leqslant Q(t)\leqslant\rho I$,

(ii) $\big[A^T(t)Q(t)+\big(I+\mu(t)A^T(t)\big)\big(Q^{\Delta}(t)+Q(t)A(t)+\mu(t)Q^{\Delta}(t)A(t)\big)\big]\leqslant c\dfrac{h^{\Delta}(t)}{h(t)}I$,

where η and ρ are positive constants, and

$$c=\begin{cases}\eta, & \text{if } h^{\Delta}(t)\geqslant 0,\\ \rho, & \text{if } h^{\Delta}(t)<0.\end{cases}$$

Then system (2.2) *is h-stable.*

Proof. Let $x(t)=x(t,t_0,x_0)$ be any solution of (2.2). We see that for all $t\geqslant t_0$,

$$\begin{aligned}\big[x^T(t)Q(t)x(t)\big]^{\Delta}&\leqslant c\frac{h^{\Delta}(t)}{h(t)}\big|x(t)\big|^2\\ &\leqslant\begin{cases}\frac{c}{\eta}\frac{h^{\Delta}(t)}{h(t)}x^T(t)Q(t)x(t), & \text{if } h^{\Delta}(t)\geqslant 0,\\ \frac{c}{\rho}\frac{h^{\Delta}(t)}{h(t)}x^T(t)Q(t)x(t), & \text{if } h^{\Delta}(t)<0\end{cases}\\ &\leqslant\frac{h^{\Delta}(t)}{h(t)}x^T(t)Q(t)x(t).\end{aligned}$$

By using the time scale version of Gronwall's inequality in [1], we obtain

$$x^T(t)Q(t)x(t) \leqslant x^T(t_0)Q(t_0)x(t_0)e_{\frac{h^\Delta(t)}{h(t)}}(t,t_0), \quad t \geqslant t_0.$$

By (i) and Lemma 2.15, we obtain

$$\begin{aligned}\left|x(t)\right|^2 &\leqslant \frac{1}{\eta}x^T(t)Q(t)x(t) \leqslant \frac{1}{\eta}x^T(t_0)Q(t_0)x(t_0)e_{\frac{h^\Delta(t)}{h(t)}}(t,t_0)\\ &\leqslant \frac{\rho}{\eta}\left|x(t_0)\right|^2 e_{\frac{h^\Delta(t)}{h(t)}}(t,t_0) \leqslant \frac{\rho}{\eta}\left|x(t_0)\right|^2 h(t)h(t_0)^{-1}, \quad t \geqslant t_0.\end{aligned}$$

Hence we have

$$\left|x(t)\right| \leqslant d\left|x(t_0)\right|H(t)H(t_0)^{-1}, \quad t \geqslant t_0,$$

where $d = \sqrt{\frac{\rho}{\eta}}$ and $H(t) = \sqrt{h(t)}$. □

Corollary 2.17. *Assume the hypotheses of Theorem* 2.16*, then the following holds*:

(i) *If $h(t) = c$, for some constant c then system* (2.2) *is uniformly stable.*
(ii) *If $h(t) = e_\lambda(t,0)$, then system* (2.2) *is uniformly exponentially stable, that is, $|\Phi_A(t,t_0)| \leqslant \gamma e_{-\lambda}(t,t_0)$ for some positive constants λ and γ with $-\lambda \in \mathcal{R}^+$ and all $t \geqslant t_0$* [7, *Theorem* 2.2].

Corollary 2.18. *If $\mathbb{T} = \mathbb{R}$, then we have*

$$\left|x(t)\right|^2 \leqslant \frac{\rho}{\eta}\left|x(t_0)\right|^2 e_{\frac{h'(t)}{h(t)}}(t,t_0) \leqslant \frac{\rho}{\eta}\left|x(t_0)\right|^2 h(t)h(t_0)^{-1}.$$

Hence we have

$$\left|x(t)\right| \leqslant d\left|x(t_0)\right|H(t)H(t_0)^{-1}, \quad t \geqslant t_0,$$

where $d = \sqrt{\frac{\rho}{\eta}}$ and $H(t) = \sqrt{h(t)}$.

Corollary 2.19. *If $\mathbb{T} = \mathbb{Z}$ and $c = 1$, then we have*

$$\left|x(t)\right|^2 \leqslant \frac{\rho}{\eta}\left|x(t_0)\right|^2 e_{\frac{\Delta h(t)}{h(t)}}(t,t_0) \leqslant \frac{\rho}{\eta}\left|x(t_0)\right|^2 h(t)h(t_0)^{-1}.$$

Hence we have

$$\left|x(t)\right| \leqslant d\left|x(t_0)\right|H(t)H(t_0)^{-1}, \quad t \geqslant t_0,$$

where $d = \sqrt{\frac{\rho}{\eta}}$ and $H(t) = \sqrt{h(t)}$.

Example 2.20. To illustrate Theorem 2.16, we consider the linear dynamic system on time scale $\mathbb{T}$,

$$x^\Delta(t) = \begin{bmatrix} -2 & 1 \\ -1 & -a(t) \end{bmatrix} x(t), \tag{2.17}$$

where $a(t) = \sin t + 2$. If $\mu(t) \leqslant \frac{1}{2}$ for all $t \in \mathbb{T}$, then system (2.17) is h-stable.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (2.17). If we choose $Q(t) = I$ and $\eta = \rho = 1$, then we have

$$\left|x(t)\right| \leqslant \left|x(t_0)\right| \sqrt{e_{-\frac{1}{2}}(t, t_0)} = c\left|x(t_0)\right| h(t)h(t_0)^{-1}, \quad t \geqslant t_0,$$

where $h(t) = \sqrt{e_{-\frac{1}{2}}(t, 0)}$. Therefore system (2.17) is h-stable. □

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