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Multiple positive solutions for multipoint boundary value problems with one-dimensional p-Laplacian ☆

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Abstract

In this paper we consider the multiplicity of positive solutions for the one-dimensional p-Laplacian differential equation $(\phi_p(u'))' + q(t)f(t,u,u') = 0$, $t \in (0,1)$, subject to some boundary conditions. By means of a fixed point theorem due to Avery and Peterson, we provide sufficient conditions for the existence of multiple positive solutions to some multipoint boundary value problems.

Keywords: Positive solutions; Boundary value problems; One-dimensional p-Laplacian; Fixed point theorem

1. Introduction

In this paper we study the existence of multiple positive solutions to the boundary value problem (BVP) for the one-dimensional p-Laplacian

$$\big(\phi_p(u')\big)' + q(t)f(t,u,u') = 0, \quad t \in (0,1), \tag{1.1}$$

$$u(0) = \sum_{i=1}^{n-2} \alpha_i u(\xi_i), \qquad u'(1) = \sum_{i=1}^{n-2} \beta_i u'(\xi_i), \tag{1.2}$$

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$$u'(0)=\sum_{i=1}^{n-2}\alpha_i u'(\xi_i), \qquad u(1)=\sum_{i=1}^{n-2}\beta_i u(\xi_i), \tag{1.3}$$

where $\phi_p(s)=|s|^{p-2}s$, $p>1$, $\xi_i\in(0,1)$ with $0<\xi_1<\xi_2<\cdots<\xi_{n-2}<1$ and α_i, β_i, f satisfy

(H$_1$) $\alpha_i,\beta_i\in[0,\infty)$ satisfy $0<\sum_{i=1}^{n-2}\alpha_i<1$ and $\sum_{i=1}^{n-2}\beta_i<1$;
(H$_2$) $f\in C([0,1]\times[0,\infty)\times(-\infty,+\infty),[0,\infty))$;
(H$_3$) $q(t)$ is a nonnegative continuous function defined in $(0,1)$, $q(t)$ is not identically zero on any subinterval of $(0,1)$.

The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1,2]. Since then there has been much current attention focused on the study of nonlinear multipoint boundary value problems, see, for example, [3–6].

Equations of the above form occur in the study of the n-dimensional p-Laplace equation, non-Newtonian fluid theory and the turbulent flow of a gas in a porous medium [7]. When the nonlinear term f does not depend on the first-order derivative, Eq. (1.1) together with some multipoint boundary conditions has been studied by several researchers, for example, see [8–10].

Recently, D. Ma, Z. Du and W. Ge [8] have obtained the existence of monotone positive solutions for the following BVP:

$$\big(\phi_p(u')\big)'+q(t)f(t,u)=0, \quad t\in(0,1), \tag{1.4}$$

$$u'(0)=\sum_{i=1}^{n-2}\alpha_i u'(\xi_i), \qquad u(1)=\sum_{i=1}^{n-2}\beta_i u(\xi_i). \tag{1.5}$$

The main tool is the monotone iterative technique.

The authors in [9,10] considered the multipoint BVP for one-dimensional p-Laplacian

$$\big(\phi_p(u')\big)'+f(t,u)=0, \quad t\in(0,1), \tag{1.6}$$

$$\phi_p\big(u'(0)\big)=\sum_{i=1}^{n-2}\alpha_i\phi_p\big(u'(\xi_i)\big), \qquad u(1)=\sum_{i=1}^{n-2}\beta_i u(\xi_i), \tag{1.7}$$

and BVP (1.4), (1.5). Using a fixed point theorem in a cone, we provide sufficient conditions for the existence of multiple positive solutions to the above BVPs.

However, multiplicity is not available for the case when the nonlinear term is involved in first-order derivative explicitly. This paper will fill this gap in the literature. The purpose of this paper is to improve and generalize the results in the above mentioned references. We shall prove that (1.1), (1.2) and (1.1), (1.3) possess at least three positive solutions.

Our main results will depend on an application of a fixed point theorem due to Avery and Peterson which deals with fixed points of a cone-preserving operator defined on an ordered Banach space. The emphasis here is that the nonlinear term is involved explicitly with the first-order derivative.

2. Background material and definitions

For the convenience of the reader, we present here the necessary definitions from the theory of cones in Banach spaces. We also state in this section the Avery–Peterson fixed point theorem.

Definition 2.1. Let E be a real Banach space. A nonempty convex closed set $P \subset E$ is said to be a cone provided that

(i) $au \in P$ for all $u \in P$ and all $a \geqslant 0$ and
(ii) $u, -u \in P$ implies $u = 0$.

Note that every cone $P \subset X$ induces an ordering in E given by $x \leqslant y$ if $y - x \in P$.

Definition 2.2. The map α is said to be a nonnegative continuous *concave* functional on a cone P of a real Banach space E provided that $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha\big(tx + (1-t)y\big) \geqslant t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $0 \leqslant t \leqslant 1$.

Similarly, we say the map β is a nonnegative continuous *convex* functional on a cone P of a real Banach space E provided that $\beta : P \to [0, \infty)$ is continuous and

$$\beta\big(tx + (1-t)y\big) \leqslant t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $0 \leqslant t \leqslant 1$.

Definition 2.3. An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then for positive real numbers a, b, c, and d, we define the following convex sets

$$P(\gamma, d) = \big\{x \in P \mid \gamma(x) < d\big\},$$
$$P(\gamma, \alpha, b, d) = \big\{x \in P \mid b \leqslant \alpha(x),\ \gamma(x) \leqslant d\big\},$$
$$P(\gamma, \theta, \alpha, b, c, d) = \big\{x \in P \mid b \leqslant \alpha(x), \theta(x) \leqslant c,\ \gamma(x) \leqslant d\big\},$$

and a closed set

$$R(\gamma, \psi, a, d) = \big\{x \in P \mid a \leqslant \psi(x),\ \gamma(x) \leqslant d\big\}.$$

The following fixed point theorem due to Avery and Peterson is fundamental in the proofs of our main results.

Theorem 2.1. [11] *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leqslant \lambda\psi(x)$ for $0 \leqslant \lambda \leqslant 1$, such that for some positive numbers M and d,*

$$\alpha(x) \leqslant \psi(x) \quad \text{and} \quad \|x\| \leqslant M\gamma(x), \tag{2.1}$$

for all $x \in \overline{P(\gamma, d)}$. Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b, and c with $a < b$ such that

(S1) *$\{x \in P(\gamma, \theta, \alpha, b, c, d) \mid \alpha(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma, \theta, \alpha, b, c, d)$;*

(S2) $\alpha(Tx) > b$ *for* $x \in P(\gamma, \alpha, b, d)$ *with* $\theta(Tx) > c$;
(S3) $0 \notin R(\gamma, \psi, a, d)$ *and* $\psi(Tx) < a$ *for* $x \in R(\gamma, \psi, a, d)$ *with* $\psi(x) = a$.

Then T *has at least three fixed points* $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ *such that*

$$\begin{aligned}
&\gamma(x_i) \leqslant d \quad \text{for } i = 1, 2, 3,\\
&b < \alpha(x_1),\\
&a < \psi(x_2) \quad \text{with } \alpha(x_2) < b,\\
&\psi(x_3) < a.
\end{aligned}$$

3. Existence of triple positive solutions to (1.1), (1.2)

We consider the Banach space $E = (C^1[0, 1], \|\cdot\|)$ with the maximum norm

$$\|x\| = \max\Big\{\max_{0\leqslant t\leqslant 1}\big|x(t)\big|, \max_{0\leqslant t\leqslant 1}\big|x'(t)\big|\Big\}.$$

Denote $C^{1+}[0, 1] = \{\omega \in C^1[0, 1]\colon\ \omega(t) \geqslant 0,\ t \in [0, 1]\}$. Define the cone $P_1 \subset E$ by

$$P_1 = \Bigg\{x \in E\colon\ x(t) \geqslant 0,\ x(0) = \sum_{i=1}^{n-2}\alpha_i x(\xi_i),\ x'(1) = \sum_{i=1}^{n-2}\beta_i x'(\xi_i),\ x \text{ is concave on } [0, 1]\Bigg\}.$$

Let the nonnegative continuous concave functional α_1, the nonnegative continuous convex functionals θ_1, γ_1, and the nonnegative continuous functional ψ_1 be defined on the cone P_1 by

$$\gamma_1(x) = \max_{0\leqslant t\leqslant 1}\big|x'(t)\big|, \qquad \psi_1(x) = \theta_1(x) = \max_{0\leqslant t\leqslant 1}\big|x(t)\big|, \qquad \alpha_1(x) = \min_{\delta\leqslant t\leqslant 1-\delta}\big|x(t)\big|,$$

where $\delta \in (0, 1/2)$.

Lemma 3.1. *If* $x \in P_1$, *then*

$$\max_{0\leqslant t\leqslant 1}\big|x(t)\big| \leqslant \left(1 + \frac{\sum_{i=1}^{n-2}\alpha_i\xi_i}{1 - \sum_{i=1}^{n-2}\alpha_i}\right)\max_{0\leqslant t\leqslant 1}\big|x'(t)\big|.$$

Proof. Since $x(t) = x(0) + \int_0^t x'(s)\,ds$, so we have

$$\max_{0\leqslant t\leqslant 1}\big|x(t)\big| \leqslant \big|x(0)\big| + \max_{0\leqslant t\leqslant 1}\big|x'(t)\big|.$$

On the other hand,

$$\begin{aligned}
\left(1 - \sum_{i=1}^{n-2}\alpha_i\right)x(0) &= x(0) - \sum_{i=1}^{n-2}\alpha_i x(0)\\
&= \sum_{i=1}^{n-2}\alpha_i x(\xi_i) - \sum_{i=1}^{n-2}\alpha_i x(0)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-2} \alpha_i \big[x(\xi_i) - x(0)\big] \\
&= \sum_{i=1}^{n-2} \alpha_i \xi_i x'(\eta_i),
\end{aligned}
$$

where $\eta_i \in (0, \xi_i)$, so

$$
\big|x(0)\big| = \left| \frac{\sum_{i=1}^{n-2} \alpha_i \xi_i x'(\eta_i)}{1 - \sum_{i=1}^{n-2} \alpha_i} \right| \leqslant \frac{\sum_{i=1}^{n-2} \alpha_i \xi_i |x'(\eta_i)|}{1 - \sum_{i=1}^{n-2} \alpha_i} \leqslant \frac{\sum_{i=1}^{n-2} \alpha_i \xi_i}{1 - \sum_{i=1}^{n-2} \alpha_i} \max_{0 \leqslant t \leqslant 1} \big|x'(t)\big|.
$$

Therefore, the result holds. □

Lemma 3.2. [12] *If $x \in P_1$, then $x(t) \geqslant t(1-t) \max_{0 \leqslant t \leqslant 1} |x(t)|$.*

Lemma 3.3. *Let* (H_1)–(H_3) *hold. Then for $x \in C^{1+}[0,1]$, the problem*

$$
\big(\phi_p(u')\big)' + q(t) f\big(t, x(t), x'(t)\big) = 0, \quad t \in (0,1), \tag{3.1}
$$

$$
u(0) = \sum_{i=1}^{n-2} \alpha_i u(\xi_i), \qquad u'(1) = \sum_{i=1}^{n-2} \beta_i u'(\xi_i), \tag{3.2}
$$

has a unique solution

$$
u(t) = B_x + \int_0^t \phi_p^{-1}\left(A_x + \int_s^1 q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\, d\tau \right) ds, \tag{3.3}
$$

where A_x, B_x satisfy

$$
\phi_p^{-1}(A_x) = \sum_{i=1}^{n-2} \beta_i \phi_p^{-1}\left(A_x + \int_{\xi_i}^1 q(s) f\big(s, x(s), x'(s)\big)\, ds \right), \tag{3.4}
$$

$$
B_x = \frac{1}{1 - \sum_{i=1}^{n-2} \alpha_i} \sum_{i=1}^{n-2} \alpha_i \int_0^{\xi_i} \phi_p^{-1}\left(A_x + \int_s^1 q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\, d\tau \right) ds.
$$

Denote $k = \frac{\phi_p(\sum_{i=1}^{n-2} \beta_i)}{1 - \phi_p(\sum_{i=1}^{n-2} \beta_i)}$, then there exists a unique $A_x \in [0, k \int_0^1 q(s) f(s, x(s), x'(s))\, ds]$ satisfying (3.4).

Proof. The proof is similar to [8, Lemma 2.2], so we omit the details. □

Lemma 3.4. *Let* (H_1)–(H_3) *hold. If $x \in C^{1+}[0,1]$, then the unique solution of problem* (3.1)–(3.2) *satisfies $u(t) \geqslant 0$, $t \in [0,1]$.*

Proof. It is easy to check. □

By Lemmas 3.1, 3.2 and the concavity of x, the functionals defined above satisfy

$$\begin{aligned} &\delta(1-\delta)\theta_1(x) \leqslant \alpha_1(x) \leqslant \theta_1(x) = \psi_1(x), \\ &\|x\| = \max\{\theta_1(x), \gamma_1(x)\} \leqslant \left(1 + \frac{\sum_{i=1}^{n-2} \alpha_i \xi_i}{1 - \sum_{i=1}^{n-2} \alpha_i}\right)\gamma_1(x), \end{aligned} \tag{3.5}$$

for all $x \in \overline{P_1(\gamma_1, d)} \subset P_1$. Therefore, condition (2.1) is satisfied.

For any $x \in P_1$, define the operator

$$\begin{aligned} (T_1 x)(t) = {}& \frac{1}{1 - \sum_{i=1}^{n-2} \alpha_i} \sum_{i=1}^{n-2} \alpha_i \int_0^{\xi_i} \phi_p^{-1}\left(A_x + \int_s^1 q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\, d\tau\right) ds \\ &+ \int_0^t \phi_p^{-1}\left(A_x + \int_s^1 q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\, d\tau\right) ds. \end{aligned} \tag{3.6}$$

By Lemma 3.3, we know $T_1 x$ is well defined. Furthermore, we have the following result.

Lemma 3.5. *$T_1 : P_1 \to P_1$ is completely continuous.*

Proof. It is easy to check that $T_1 P_1 \subset P_1$. By similar arguments in [13,14], $T_1 : P_1 \to P_1$ is completely continuous. □

We are now ready to apply the Avery–Peterson fixed point theorem to the operator T_1 to give sufficient conditions for the existence of at least three positive solutions to problem (1.1), (1.2).

Let

$$\begin{aligned} M_1 &= \int_0^1 q(t)\, dt, \\ C_1 &= \int_\delta^{1-\delta} \phi_p^{-1}\left(\int_s^1 q(\tau)\, d\tau\right) ds, \\ b_1 &= \frac{1 - \sum_{i=1}^{n-2} \beta_i}{\delta^2(1-\delta)[2(1 - \sum_{i=1}^{n-2} \beta_i \xi_i) - \delta(1 - \sum_{i=1}^{n-2} \beta_i)]}, \\ N_1 &= -b_1 \frac{1 - \sum_{i=1}^{n-2} \alpha_i (1 - \xi_i^2)}{1 - \sum_{i=1}^{n-2} \alpha_i} + 2b_1 \frac{(1 - \sum_{i=1}^{n-2} \beta_i \xi_i)(1 - \sum_{i=1}^{n-2} \alpha_i (1 - \xi_i))}{(1 - \sum_{i=1}^{n-2} \alpha_i)(1 - \sum_{i=1}^{n-2} \beta_i)}. \end{aligned}$$

Theorem 3.1. *Assume* (H_1)–(H_3) *hold. Let*

$$0 < a < b \leqslant \frac{\delta^2(1-\delta)[2(1 - \sum_{i=1}^{n-2} \beta_i \xi_i) - \delta(1 - \sum_{i=1}^{n-2} \beta_i)]}{2(1 - \sum_{i=1}^{n-2} \beta_i \xi_i)} d,$$

and suppose that f satisfies the following conditions:

$$\text{(A1)}\quad f(t,u,v) \leqslant \frac{1}{M_1(k+1)}\phi_p(d),$$
$$\textit{for } (t,u,v) \in [0,1] \times \left[0, d + \frac{\sum_{i=1}^{n-2}\alpha_i\xi_i}{1-\sum_{i=1}^{n-2}\alpha_i}d\right] \times [-d,d],$$

$$\text{(A2)}\quad f(t,u,v) \geqslant \phi_p\left(\frac{b}{\delta(1-\delta)C_1}\right), \quad \textit{for } (t,u,v) \in [\delta, 1-\delta] \times [b, N_1 b] \times [-d,d],$$

$$\text{(A3)}\quad f(t,u,v) < \frac{1}{M_1(k+1)}\phi_p\left(\left(1-\sum_{i=1}^{n-2}\alpha_i\right)a\right),$$
$$\textit{for } (t,u,v) \in [0,1] \times [0,a] \times [-d,d].$$

Then the boundary value problem (1.1), (1.2) *has at least three positive solutions* x_1, x_2, *and* x_3 *satisfying*

$$\max_{0\leqslant t\leqslant 1}\left|x_i'(t)\right| \leqslant d, \quad \textit{for } i = 1,2,3,$$
$$b < \min_{\delta\leqslant t\leqslant 1-\delta}\left|x_1(t)\right|,$$
$$a < \max_{0\leqslant t\leqslant 1}\left|x_2(t)\right|, \quad \textit{with} \min_{\delta\leqslant t\leqslant 1-\delta}\left|x_2(t)\right| < b,$$
$$\max_{0\leqslant t\leqslant 1}\left|x_3(t)\right| < a. \tag{3.7}$$

Proof. Problem (1.1), (1.2) has a solution $x = x(t)$ if and only if x solves the operator equation $x = T_1 x$. Thus we set out to verify that the operator T_1 satisfies the Avery–Peterson fixed point theorem which will prove the existence of three fixed points of T_1.

If $x \in \overline{P_1(\gamma, d)}$, then $\gamma_1(x) = \max_{0\leqslant t\leqslant 1}|x'(t)| \leqslant d$. With Lemma 3.1 we have

$$\max_{0\leqslant t\leqslant 1}\left|x(t)\right| \leqslant d + \frac{\sum_{i=1}^{n-2}\alpha_i\xi_i}{1-\sum_{i=1}^{n-2}\alpha_i}d,$$

then assumption (A1) implies $f(t,x(t),x'(t)) \leqslant \frac{1}{M_1(k+1)}\phi_p(d)$. On the other hand, for $x \in P_1$, there is $T_1 x \in P_1$, then $T_1 x$ is concave on [0, 1], and $\max_{t\in[0,1]}|(T_1x)'(t)| = \max\{|(T_1x)'(0)|, |(T_1x)'(1)|\}$, so

$$\begin{aligned}
\gamma(T_1x) &= \max_{t\in[0,1]}\left|(T_1x)'(t)\right| \\
&= \max_{t\in[0,1]}\left\{\left|(T_1x)'(0)\right|, \left|(T_1x)'(1)\right|\right\} \\
&= \phi_p^{-1}\left(A_x + \int_0^1 q(r) f\big(r, x(r), x'(r)\big)\,dr\right) \\
&\leqslant \phi_p^{-1}\left((k+1)\int_0^1 q(r) f\big(r, x(r), x'(r)\big)\,dr\right)
\end{aligned}$$

$$\leqslant \phi_p^{-1}\left((k+1)\frac{1}{M_1(k+1)}\phi_p(d)\int_0^1 q(r)\,dr\right)$$
$$= d.$$

Hence, $T_1 : \overline{P_1(\gamma, d)} \to \overline{P_1(\gamma, d)}$.

To check condition (S1) of Theorem 2.1, we choose

$$x_0(t) = -b't^2 + 2b'\frac{1-\sum_{i=1}^{n-2}\beta_i\xi_i}{1-\sum_{i=1}^{n-2}\beta_i}t - b'\frac{\sum_{i=1}^{n-2}\alpha_i\xi_i^2}{1-\sum_{i=1}^{n-2}\alpha_i} + 2b'\frac{(1-\sum_{i=1}^{n-2}\beta_i\xi_i)(\sum_{i=1}^{n-2}\alpha_i\xi_i)}{(1-\sum_{i=1}^{n-2}\alpha_i)(1-\sum_{i=1}^{n-2}\beta_i)},$$

where

$$b' = \frac{b(1-\sum_{i=1}^{n-2}\beta_i)}{\delta^2(1-\delta)[2(1-\sum_{i=1}^{n-2}\beta_i\xi_i) - \delta(1-\sum_{i=1}^{n-2}\beta_i)]}.$$

Obviously,

$$x_0(0) = \sum_{i=1}^{n-2}\alpha_i x_0(\xi_i), \qquad x_0'(1) = \sum_{i=1}^{n-2}\beta_i x_0'(\xi_i),$$

$x_0(t) \geqslant 0$ and is concave on [0, 1], so $x_0 \in P_1$. Again,

$$\alpha_1(x_0) = \min_{\delta\leqslant t\leqslant 1-\delta}\left|x_0(t)\right| = x_0(\delta) > \frac{b}{\delta(1-\delta)} > b,$$

$$\theta_1(x_0) = \max_{0\leqslant t\leqslant 1}\left|x_0(t)\right| = x_0(1) = N_1 b > x_0(\delta) > \frac{b}{\delta(1-\delta)} > b,$$

$$\gamma_1(x_0) = \max_{0\leqslant t\leqslant 1}\left|x_0'(t)\right| = x_0'(0) = \frac{2b(1-\sum_{i=1}^{n-2}\beta_i\xi_i)}{\delta^2(1-\delta)[2(1-\sum_{i=1}^{n-2}\beta_i\xi_i) - \delta(1-\sum_{i=1}^{n-2}\beta_i)]} \leqslant d.$$

So $x_0 \in P_1(\gamma_1, \theta_1, \alpha_1, b, N_1 b, d)$ and $\{x \in P_1(\gamma_1, \theta_1, \alpha_1, b, N_1 b, d) \mid \alpha_1(x) > b\} \neq \emptyset$. Hence, if $x \in P_1(\gamma_1, \theta_1, \alpha_1, b, N_1 b, d)$, then $b \leqslant x(t) \leqslant N_1 b$, $|x'(t)| \leqslant d$ for $\delta \leqslant t \leqslant 1-\delta$. As $(T_1 x)'$ is nonnegative on [0, 1], from assumption (A2) and Lemma 3.2, we have

$$\begin{aligned}
\alpha(T_1 x) &= \min_{\delta\leqslant t\leqslant 1-\delta}\left|(T_1 x)(t)\right| \\
&\geqslant \delta(1-\delta)\max_{0\leqslant t\leqslant 1}\left|(T_1 x)(t)\right| \\
&= \delta(1-\delta)(T_1 x)(1) \\
&= \frac{\delta(1-\delta)}{1-\sum_{i=1}^{n-2}\alpha_i}\sum_{i=1}^{n-2}\alpha_i\int_0^{\xi_i}\phi_p^{-1}\left(A_x + \int_s^1 q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau\right)ds \\
&\quad + \delta(1-\delta)\int_0^1 \phi_p^{-1}\left(A_x + \int_s^1 q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau\right)ds
\end{aligned}$$

$$\begin{aligned}
&\geqslant \delta(1-\delta)\int_0^1 \phi_p^{-1}\left(\int_s^1 q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau\right)ds \\
&\geqslant \delta(1-\delta)\int_\delta^{1-\delta} \phi_p^{-1}\left(\int_s^1 q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau\right)ds \\
&\geqslant \delta(1-\delta)\cdot\frac{b}{\delta(1-\delta)C_1}\int_\delta^{1-\delta} \phi_p^{-1}\left(\int_s^1 q(\tau)\,d\tau\right)ds \\
&= b.
\end{aligned}$$

This shows that condition (S1) of Theorem 2.1 is satisfied.

Secondly, with (3.6), we have

$$\alpha(T_1x) \geqslant \delta(1-\delta)\theta(T_1x) > \delta(1-\delta)N_1 b > \delta(1-\delta)\frac{b}{\delta(1-\delta)} > b,$$

for all $x \in P_1(\gamma_1, \alpha_1, b, d)$ with $\theta(T_1x) > N_1 b$. Thus, condition (S2) of Theorem 2.1 is satisfied.

We finally show that (S3) of Theorem 2.1 also holds. Clearly, as $\psi_1(0) = 0 < a$, there holds that $0 \notin R(\gamma_1, \psi_1, a, d)$. Suppose that $x \in R(\gamma_1, \psi_1, a, d)$ with $\psi_1(x) = a$. Then, by the assumption (A3), we have

$$\begin{aligned}
\psi_1(T_1x) &= \max_{0\leqslant t\leqslant 1}\big|(T_1x)(t)\big| \\
&= (T_1x)(1) \\
&= \frac{1}{1-\sum_{i=1}^{n-2}\alpha_i}\sum_{i=1}^{n-2}\alpha_i\int_0^{\xi_i}\phi_p^{-1}\left(A_x + \int_s^1 q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau\right)ds \\
&\quad + \int_0^1 \phi_p^{-1}\left(A_x + \int_s^1 q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau\right)ds \\
&\leqslant \frac{1}{1-\sum_{i=1}^{n-2}\alpha_i}\sum_{i=1}^{n-2}\alpha_i\int_0^{1}\phi_p^{-1}\left(A_x + \int_s^1 q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau\right)ds \\
&\quad + \int_0^1 \phi_p^{-1}\left(A_x + \int_s^1 q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau\right)ds \\
&= \frac{1}{1-\sum_{i=1}^{n-2}\alpha_i}\int_0^1 \phi_p^{-1}\left(A_x + \int_s^1 q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau\right)ds \\
&\leqslant \frac{1}{1-\sum_{i=1}^{n-2}\alpha_i}\phi_p^{-1}\left((k+1)\int_0^1 q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\,d\tau\right)
\end{aligned}$$

$$< \frac{1}{1-\sum_{i=1}^{n-2}\alpha_i}\phi_p^{-1}\left((k+1)\frac{1}{M_1(k+1)}\phi_p\left(\left(1-\sum_{i=1}^{n-2}\alpha_i\right)a\right)\int_0^1 q(\tau)\,d\tau\right)$$

$$= \frac{1}{1-\sum_{i=1}^{n-2}\alpha_i}\left(1-\sum_{i=1}^{n-2}\alpha_i\right)a = a.$$

So, condition (S3) of Theorem 2.1 is satisfied. Therefore, an application of Theorem 2.1 implies the boundary value problem (1.1), (1.2) has at least three positive solutions x_1, x_2, and x_3 satisfying (3.7). The proof is complete. □

Example 3.1. Consider the differential equation

$$\left(|u'|u'\right)' + f(t,u,u') = 0, \quad t \in (0,1), \tag{3.8}$$

with boundary condition

$$\begin{cases} u(0) = \frac{1}{2}u\left(\frac{1}{4}\right) + \frac{1}{4}u\left(\frac{1}{2}\right) + \frac{1}{6}u\left(\frac{3}{4}\right), \\ u'(1) = \frac{1}{4}u'\left(\frac{1}{4}\right) + \frac{1}{3}u'\left(\frac{1}{2}\right) + \frac{1}{3}u'\left(\frac{3}{4}\right), \end{cases} \tag{3.9}$$

where

$$f(t,u,v) = \begin{cases} 2\times 10^{-5}\sin t + 5184u^{20} + 2\times 10^{-5}\left(\frac{v}{181\times 119^{10}}\right)^3, & \text{for } u \leqslant \frac{6419}{54}, \\ 2\times 10^{-5}\sin t + 5184\cdot\left(\frac{6419}{54}\right)^{20} + 2\times 10^{-5}\left(\frac{v}{181\times 119^{10}}\right)^3, & \text{for } u > \frac{6419}{54}. \end{cases}$$

Choose $a = \frac{1}{2}$, $b = 1$, $\delta = \frac{1}{4}$, $d = 181\times 119^{10}$, we note that $M_1 = 1$, $C_1 = \frac{3\sqrt{3}-1}{12}$, $b_1 = \frac{49}{27}$, $k = \frac{121}{23}$, $N_1 = \frac{6419}{54}$. Consequently, $f(t,u,v)$ satisfies

$$f(t,u,v) < \frac{1}{M_1(k+1)}\phi_3\left(\left(1-\sum_{i=1}^{3}\alpha_i\right)a\right) = \frac{23}{82944},$$

$$\text{for } (t,u,v) \in [0,1]\times\left[0,\frac{1}{2}\right]\times[-d,d],$$

$$f(t,u,v) > \phi_3\left(\frac{b}{\delta(1-\delta)C_1}\right) = \left(\frac{144}{3\sqrt{3}-1}\right)^2,$$

$$\text{for } (t,u,v) \in \left[\frac{1}{4},\frac{3}{4}\right]\times[1,N_1]\times[-d,d],$$

$$f(t,u,v) < \frac{1}{M_1(k+1)}\phi_3(d) = \frac{23}{144}d^2, \quad \text{for } (t,u,v) \in [0,1]\times\left[0,\frac{11}{2}d\right]\times[-d,d].$$

Then all conditions of Theorem 3.1 hold. Thus, with Theorem 3.1, problem (3.8), (3.9) has at least three positive solutions.

Remark 3.1. We can also consider the following BVP similarly:

$$\left(\phi_p(u')\right)' + q(t)f(t,u,u') = 0, \quad t \in (0,1), \tag{3.10}$$

$$u(0) = 0, \qquad u'(1) = \sum_{i=1}^{n-2}\beta_i u'(\xi_i). \tag{3.11}$$

4. Existence of triple positive solutions to (1.1), (1.3)

Now we deal with problem (1.1), (1.3). The method is just similar to what we have done in Section 3, so we omit the proof of the main result of this section.

Define the cone $P_2 \subset E$ by

$$P_2 = \left\{ x \in E\colon\ x(t) \geqslant 0,\ x'(0) = \sum_{i=1}^{n-2} \alpha_i x'(\xi_i),\ x(1) = \sum_{i=1}^{n-2} \beta_i x(\xi_i),\right.$$
$$\left. x \text{ is concave on } [0,1] \right\}.$$

Let the nonnegative continuous concave functional α_2, the nonnegative continuous convex functionals θ_2, γ_2, and the nonnegative continuous functional ψ_2 be defined on the cone P_2 by

$$\gamma_2(x) = \max_{0\leqslant t\leqslant 1} \left|x'(t)\right|, \qquad \psi_2(x) = \theta_2(x) = \max_{0\leqslant t\leqslant 1} \left|x(t)\right|, \qquad \alpha_2(x) = \min_{\delta\leqslant t\leqslant 1-\delta} \left|x(t)\right|.$$

Lemma 4.1. *If $x \in P_2$, then*

$$\max_{0\leqslant t\leqslant 1} \left|x(t)\right| \leqslant \frac{1 - \sum_{i=1}^{n-2} \beta_i \xi_i}{1 - \sum_{i=1}^{n-2} \beta_i} \max_{0\leqslant t\leqslant 1} \left|x'(t)\right|.$$

Proof. Since $x(t) = x(1) - \int_t^1 x'(s)\,ds$, so we have

$$\max_{0\leqslant t\leqslant 1} \left|x(t)\right| \leqslant \left|x(1)\right| + \max_{0\leqslant t\leqslant 1} \left|x'(t)\right|.$$

On the other hand,

$$\begin{aligned}
\left(1 - \sum_{i=1}^{n-2} \beta_i\right) x(1) &= x(1) - \sum_{i=1}^{n-2} \beta_i x(1) \\
&= \sum_{i=1}^{n-2} \beta_i x(\xi_i) - \sum_{i=1}^{n-2} \beta_i x(1) \\
&= \sum_{i=1}^{n-2} \beta_i \left[x(\xi_i) - x(1)\right] \\
&= -\sum_{i=1}^{n-2} \beta_i (1-\xi_i) x'(\tau_i),
\end{aligned}$$

where $\tau_i \in (1, \xi_i)$, so

$$\begin{aligned}
\left|x(1)\right| &= \left|\frac{\sum_{i=1}^{n-2} \beta_i (1-\xi_i) x'(\tau_i)}{1 - \sum_{i=1}^{n-2} \beta_i}\right| \leqslant \frac{\sum_{i=1}^{n-2} \beta_i (1-\xi_i) |x'(\tau_i)|}{1 - \sum_{i=1}^{n-2} \beta_i} \\
&\leqslant \frac{\sum_{i=1}^{n-2} \beta_i (1-\xi_i)}{1 - \sum_{i=1}^{n-2} \beta_i} \max_{0\leqslant t\leqslant 1} \left|x'(t)\right|.
\end{aligned}$$

Therefore, the result holds. □

Lemma 4.2. [8] *Let* (H_1)–(H_3) *hold. Then for* $x \in C^{1+}[0,1]$, *the problem*

$$\big(\phi_p(u')\big)' + q(t) f\big(t, x(t), x'(t)\big) = 0, \quad t \in (0,1), \tag{4.1}$$

$$u'(0) = \sum_{i=1}^{n-2} \alpha_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{n-2} \beta_i u(\xi_i), \tag{4.2}$$

has a unique solution

$$u(t) = F_x - \int_t^1 \phi_p^{-1}\left(E_x - \int_0^s q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\, d\tau\right) ds, \tag{4.3}$$

where E_x, F_x *satisfy*

$$\phi_p^{-1}(E_x) = \sum_{i=1}^{n-2} \alpha_i \phi_p^{-1}\left(E_x - \int_0^{\xi_i} q(s) f\big(s, x(s), x'(s)\big)\, ds\right),$$

$$F_x = -\frac{1}{1 - \sum_{i=1}^{n-2} \beta_i} \sum_{i=1}^{n-2} \beta_i \int_{\xi_i}^1 \phi_p^{-1}\left(E_x - \int_0^s q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\, d\tau\right) ds. \tag{4.4}$$

Denote $l = \frac{\phi_p(\sum_{i=1}^{n-2} \alpha_i)}{1 - \phi_p(\sum_{i=1}^{n-2} \alpha_i)}$, *then there exists a unique* $E_x \in [-l \int_0^1 q(s) f(s, x(s), x'(s))\, ds, 0]$ *satisfying* (4.4).

Lemma 4.3. *Let* (H_1)–(H_3) *hold. If* $x \in C^{1+}[0,1]$, *then the unique solution of problem* (4.1)–(4.2) *satisfies* $u(t) \geqslant 0$, $t \in [0,1]$.

By Lemmas 3.1, 3.2 and the concavity of x, the functionals defined above satisfy

$$\delta(1-\delta)\theta_2(x) \leqslant \alpha_2(x) \leqslant \theta_2(x) = \psi_1(x),$$
$$\|x\| = \max\big\{\theta_2(x), \gamma_2(x)\big\} \leqslant \frac{1 - \sum_{i=1}^{n-2} \beta_i \xi_i}{1 - \sum_{i=1}^{n-2} \beta_i} \gamma_2(x), \tag{4.5}$$

for all $x \in \overline{P_2(\gamma_2, d)} \subset P_2$. Therefore, condition (2.1) is satisfied.

For any $x \in P_2$, define the operator

$$(T_2 x)(t) = -\frac{1}{1 - \sum_{i=1}^{n-2} \beta_i} \sum_{i=1}^{n-2} \beta_i \int_{\xi_i}^1 \phi_p^{-1}\left(E_x - \int_0^s q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\, d\tau\right) ds$$
$$- \int_t^1 \phi_p^{-1}\left(E_x - \int_0^s q(\tau) f\big(\tau, x(\tau), x'(\tau)\big)\, d\tau\right) ds. \tag{4.6}$$

By Lemma 4.2, we know $T_2 x$ is well defined.

Lemma 4.4. $T_2 : P_2 \to P_2$ *is completely continuous.*

Let

$$C_2 = \int_\delta^{1-\delta} \phi_p^{-1}\left(\int_0^s q(\tau)\,d\tau\right)ds,$$

$$b_2 = \frac{1-\sum_{i=1}^{n-2}\beta_i}{\delta(1-\delta)[1-\sum_{i=1}^{n-2}\beta_i\xi_i^2-(1-\delta)^2(1-\sum_{i=1}^{n-2}\beta_i)]},$$

$$N_2 = b_2\frac{1-\sum_{i=1}^{n-2}\beta_i\xi_i^2}{1-\sum_{i=1}^{n-2}\beta_i} + 2b_2\frac{\sum_{i=1}^{n-2}\alpha_i\xi_i(1-\sum_{i=1}^{n-2}\beta_i\xi_i)}{(1-\sum_{i=1}^{n-2}\alpha_i)(1-\sum_{i=1}^{n-2}\beta_i)}.$$

Theorem 4.1. *Assume* (H_1)–(H_3) *hold. Let*

$$0 < a < b \leqslant \frac{\delta(1-\delta)(1-\sum_{i=1}^{n-2}\alpha_i)[1-\sum_{i=1}^{n-2}\beta_i\xi_i^2-(1-\delta)^2(1-\sum_{i=1}^{n-2}\beta_i)]}{2(1-\sum_{i=1}^{n-2}\beta_i)(1-\sum_{i=1}^{n-2}\alpha_i(1-\xi_i))}d,$$

and suppose that f *satisfies the following conditions*:

(B1) $f(t,u,v) \leqslant \frac{1}{M_1(l+1)}\phi_p(d),$

for $(t,u,v) \in [0,1] \times \left[0, \frac{1-\sum_{i=1}^{n-2}\beta_i\xi_i}{1-\sum_{i=1}^{n-2}\beta_i}d\right] \times [-d,d],$

(B2) $f(t,u,v) \geqslant \phi_p\left(\frac{b}{\delta(1-\delta)C_2}\right),$ *for* $(t,u,v) \in [\delta, 1-\delta] \times [b, N_2 b] \times [-d,d],$

(B3) $f(t,u,v) < \frac{1}{M_1(l+1)}\phi_p\left(\left(1-\sum_{i=1}^{n-2}\beta_i\right)a\right),$

for $(t,u,v) \in [0,1] \times [0,a] \times [-d,d].$

Then the boundary value problem (1.1), (1.3) *has at least three positive solutions* x_1, x_2, *and* x_3 *satisfying*

$$\max_{0\leqslant t\leqslant 1}\left|x_i'(t)\right| \leqslant d, \quad \text{for } i = 1,2,3,$$

$$b < \min_{\delta\leqslant t\leqslant 1-\delta}\left|x_1(t)\right|,$$

$$a < \max_{0\leqslant t\leqslant 1}\left|x_2(t)\right|, \quad \text{with } \min_{\delta\leqslant t\leqslant 1-\delta}\left|x_2(t)\right| < b,$$

$$\max_{0\leqslant t\leqslant 1}\left|x_3(t)\right| < a.$$

Example 4.1. Consider the differential equation

$$\left(|u'|u'\right)' + f(t,u,u') = 0, \quad t \in (0,1), \tag{4.7}$$

with boundary condition

$$\begin{cases} u'(0) = \frac{1}{2}u'\left(\frac{1}{4}\right) + \frac{1}{4}u'\left(\frac{1}{2}\right) + \frac{1}{6}u'\left(\frac{3}{4}\right), \\ u(1) = \frac{1}{4}u\left(\frac{1}{4}\right) + \frac{1}{3}u\left(\frac{1}{2}\right) + \frac{1}{3}u\left(\frac{3}{4}\right), \end{cases} \tag{4.8}$$

where

$$f(t,u,v)=\begin{cases}2\times 10^{-5}\sin t+5184u^{20}+2\times 10^{-5}(\frac{v}{181\times 44^{10}})^3, & \text{for } u\leqslant \frac{1037}{24},\\ 2\times 10^{-5}\sin t+5184\cdot(\frac{1037}{24})^{20}+2\times 10^{-5}(\frac{v}{181\times 44^{10}})^3, & \text{for } u>\frac{1037}{24}.\end{cases}$$

Choose $a=\frac{1}{2}$, $b=1$, $\delta=\frac{1}{4}$, $d=181\times 44^{10}$, we note that $M_1=1$, $C_2=\frac{3\sqrt{3}-1}{12}$, $b_2=\frac{2}{3}$, $l=\frac{121}{23}$, $N_2=\frac{1037}{24}$. Consequently, $f(t,u,v)$ satisfies

$$f(t,u,v)<\frac{1}{M_1(l+1)}\phi_3\left(\left(1-\sum_{i=1}^{3}\beta_i\right)a\right)=\frac{23}{82944},$$
$$\text{for } (t,u,v)\in[0,1]\times\left[0,\frac{1}{2}\right]\times[-d,d]$$
$$f(t,u,v)>\phi_3\left(\frac{b}{\delta(1-\delta)C_2}\right)=\left(\frac{144}{3\sqrt{3}-1}\right)^2,$$
$$\text{for } (t,u,v)\in\left[\frac{1}{4},\frac{3}{4}\right]\times[1,N_2]\times[-d,d],$$
$$f(t,u,v)<\frac{1}{M_1(k+1)}\phi_3(d)=\frac{23}{144}d^2,\quad \text{for } (t,u,v)\in[0,1]\times\left[0,\frac{25}{4}d\right]\times[-d,d].$$

Then all conditions of Theorem 4.1 hold. Thus, with Theorem 4.1, problem (4.7), (4.8) has at least three positive solutions.

Remark 4.1. We can also consider the following BVP similarly:

$$\left(\phi_p(u')\right)'+q(t)f(t,u,u')=0,\quad t\in(0,1),\tag{4.9}$$

$$u'(0)=\sum_{i=1}^{n-2}\alpha_i u'(\xi_i),\qquad u(1)=0.\tag{4.10}$$

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