

# The effect of oscillations in the dynamics of differential equations with delay

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## Abstract

The purpose of this paper is to study the dynamic behavior of delay differential equations of the form

$$\dot{x}(t) = f(x(t-1); a(\varepsilon \sin(\nu t), \varepsilon \cos(\nu t)); \alpha), \quad \varepsilon, \nu, \alpha \in \mathbb{R},$$

provided that  $a$  and  $f$  meet some hypotheses. By augmenting the above equation, the explicit time-dependent terms are replaced by state-dependent terms. The augmented system is autonomous and has a pair of purely imaginary and simple zero eigenvalues. Applying the center manifold reduction, the existence of an attractive integral manifold with periodic structure for the original equation is shown. Furthermore, we give a description of the flow on the obtained manifold. This allows us to determine the sufficient conditions for existence of saddle-node bifurcation. To illustrate our results, we consider an autonomous equation perturbed by a periodic function.

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## 1. Introduction

Let us consider the nonautonomous scalar delay differential equations of the form

$$\dot{x}(t) = f(x(t-1); a(\varepsilon \sin(\nu t), \varepsilon \cos(\nu t)); \alpha), \quad \varepsilon, \nu, \alpha \in \mathbb{R}, \quad (1.1)$$

under the following hypotheses:

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(H1) The functions  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$  and  $a: \mathbb{R}^2 \rightarrow \mathbb{R}$  are  $C^\infty$ -smooth, such that

$$f(0, a(0, 0), \alpha) = 0, \quad \frac{\partial f(0, a(0, 0), \alpha)}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial f(0, a(0, 0), \alpha)}{\partial x_2} = 0$$

$\frac{\partial}{\partial x_i}$  denoting the derivative with respect to the  $i$ th variable.

The delay equation (1.1) can be viewed as a perturbation, in a special way, of equation

$$\dot{x}(t) = f_0(x(t-1), \alpha),$$

with  $f_0(x, \alpha) := f(x; a(0, 0); \alpha)$ . It is well known that this class of perturbations is successfully used in control theory to stabilize systems of differential equations.

The aim of this paper is to study the impact of parameters  $(\varepsilon, \nu, \alpha)$  on the local behavior of solutions of Eq. (1.1). In order to do this, we present a theorem which ensures the existence of a local integral manifold. By second-order averaging method [14], we examine, when  $\alpha = 0$  is a parameter of bifurcation, whether  $\frac{2\pi}{\nu}$ -periodic solutions survive under small periodic perturbations. Namely, we obtain the criterion of existence of saddle-node bifurcation, at  $\alpha = 0$ , of the Poincaré map of Eq. (1.1).

In the case where the function  $f$  is independent of  $t$ , i.e.  $f(x, \cdot, \alpha) = \tilde{f}(x, \alpha)$ , Eq. (1.1) is an autonomous equation which is extensively studied in [1,3,4,8,9]. When the delay is a multiple of the period, the Floquet theory have been studied in [2,7], and then we can describe how the Floquet multipliers depend on parameter  $\alpha$ . When the delay is not a multiple of the period, but commensurable with it, then, in some cases, the Floquet multipliers can be deduced from the explicit solution of a system of ordinary differential equation. By linearizing along a periodic solution of period 3 of a retarded autonomous equation of delay 1, Walther and Skubachevskii [13] have obtained some information on Floquet multipliers of this problem, and they formulated, under some conditions, a criterion for the hyperbolicity of the 3-periodic solution. The most difficult case arises if the delay is incommensurable with the period. In [5], the method of normal forms is presented for periodic functional differential equations, but that works only for equations with autonomous linear part.

Our work is motivated by the results obtained in [11]. In fact, we associates to Eq. (1.1) an autonomous system which represents the “fold-Hopf” singularity (i.e., the set of critical spectral values of the linearized equation at zero is given by  $\{\pm iw, 0\}$ ,  $w \neq 0$ ). Having obtained the reduced equation on center manifolds, which is a three-dimensional of autonomous ordinary differential equations, we show how the method developed in [11] for the computation of the terms of center manifolds can be useful to proved the existence of an integral manifold associated with Eq. (1.1) and to calculate its Taylor approximation. Moreover, we show that Eq. (1.1) on this manifold is a nonautonomous scalar ordinary differential equation which inherits the local properties of existence and stability of  $\frac{2\pi}{\nu}$ -periodic solutions. By using the averaging method (see [14]), we give a sufficient conditions for Eq. (1.1), which yield the saddle-node bifurcation (i.e., where on one side of the parameter value  $\alpha = 0$  there are no fixed points (respectively no  $\frac{2\pi}{\nu}$ -periodic solution) and on the other side there are two fixed points (respectively two  $\frac{2\pi}{\nu}$ -periodic solutions) of the Poincaré map of (1.1) (respectively of Eq. (1.1)).

The paper is organized as follows: The preliminary Section 2 provides the elementary notions and background needed in the rest of the paper. Section 3 presents the autonomous system associated with Eq. (1.1), and the study of the existence and properties of the associated center manifolds. With the help of the results in Section 3, we give an explicit form of the integral manifold of Eq. (1.1) in Section 4. In Section 5, we give sufficient conditions of saddle-node

bifurcation for the studied equation. The paper ends with an application of the obtained result to a special example.

## 2. Definitions and preliminary results

In this section we introduce some of the basic theory of retarded functional differential equations (RFDEs) and the results needed in the remainder of this paper, for more details we refer to [7] and [14]. Let  $r \geq 0$ , a given real number and  $C$  the space of continuous functions mapping  $[-r, 0]$  into  $\mathbb{R}^n$  with the uniform norm. If  $x$  is a continuous function taking  $[\sigma - r, \sigma + \alpha]$ ,  $\alpha \geq 0$ , into  $\mathbb{R}^n$ , then, for each  $t \in [\sigma, \sigma + \alpha]$ , we let  $x_t \in C$  be defined by  $x_t(\theta) = x(t + \theta)$ ,  $-r \leq \theta \leq 0$ . We will need to consider the following (RFDE)

$$\frac{d}{dt}x(t) = f(t, x_t), \quad (2.1)$$

where  $f : D \subseteq \mathbb{R} \times C \rightarrow \mathbb{R}^n$  is a continuous function and  $D$  is an open subset on  $\mathbb{R} \times C$ . Then for any  $(\sigma, \phi) \in D$ , there exists a solution of Eq. (2.1) through  $(\sigma, \phi)$ , that is, for  $(\sigma, \phi) \in D$ , there exists  $A > 0$  and  $x \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$  such that  $x$  satisfies (2.1) for  $t \in [\sigma, \sigma + A]$  and  $x_\sigma(\sigma, \phi) = \phi$ .

**Definition 2.1.** (See [7].) Let us suppose  $f(t, 0) = 0$  for all  $t \in \mathbb{R}$ . The solution  $x = 0$  of Eq. (2.1) is said to be stable if for any  $\sigma \in \mathbb{R}$ ,  $\varepsilon > 0$ , there is a  $\delta = \delta(\varepsilon, \sigma)$  such that  $\phi \in B(0, \delta)$  implies  $x_t(\sigma, \phi) \in B(0, \varepsilon)$  for  $t \geq \sigma$ . The solution  $x = 0$  of Eq. (2.1) is said to be asymptotically stable if it is stable and there is a  $b_0 = b_0(\sigma) > 0$  such that  $\phi \in B(0, b_0)$  implies  $x(\sigma, \phi)(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

In the following we introduce the definition of a local integral manifolds for a general evolutionary process.

**Definition 2.2.** (See [10].) Let  $\{M_t, t \in \mathbb{R}\}$  be a family of a closed subspaces of  $C$  parameterized by  $t$  which are uniformly isomorphic to each other (i.e. there exists a constant  $a > 0$  so that for each pair  $t, s \in \mathbb{R}$  with  $0 \leq t - s \leq 1$  there is a linear invertible operator  $S : M_t \rightarrow M_s$  such that  $\max\{\|S\|, \|S^{-1}\|\} < a$ ). A family of (possibly nonlinear) operators  $X(t, s) : M_s \rightarrow M_t$ ,  $(t, s) \in \Delta := \{(t, s) \in \mathbb{R} \times \mathbb{R} : t \geq s\}$ , is said to be an evolutionary process in  $C$  if the following conditions hold:

- (i)  $X(t, t) = I_t \forall t \in \mathbb{R}$ , where the  $I_t$  is the identity on  $M_t$ ;
- (ii)  $X(t, s)X(s, r) = X(t, r) \forall (t, r), (r, s) \in \Delta$ ;
- (iii)  $\|X(t, s)x - X(t, s)y\| \leq Ke^{\omega(t-s)}\|x - y\| \forall x, y \in M_s$ , where  $K, \omega$  are positive constants.

We now give the definition of integral manifolds for evolutionary processes.

**Definition 2.3.** (See [10].) For an evolutionary processes  $(X(t, s))_{t \geq s}$  in  $C$ , a set  $N \subseteq \bigcup_{t \in \mathbb{R}} \{t\} \times X_t$  is said to be a local integral manifold if for every  $t \in \mathbb{R}$  the space phase  $X_t$  is split into a direct sum  $X_t = X_t^1 \oplus X_t^2$  with projections  $P_1(t)$  and  $P_2(t)$  such that

$$\sup_{t \in \mathbb{R}} \|P_j(t)\| < \infty, \quad j = 1, 2,$$

and there exists a family of continuously Fréchet differentiable mapping  $g_t : X_t^1 \rightarrow X_t^2$ ,  $t \in \mathbb{R}$ , and an open convex  $V$  neighborhood of 0 so that

$$N = \{(t, x + g_t(x)) \in \mathbb{R} \times X_t : x \in X_t^1 \cap V\}$$

and

$$X(t, s)Gr(g_s) = Gr(g_t), \quad (t, s) \in \Delta,$$

where  $Gr(g_s)$  denotes the graph of the mapping  $g_s$ .

In the sequel, we will introduce the concept of the method of averaging, which will be used in connection with equations of the following form

$$\dot{x}(t) = \varepsilon f(x, t) + \varepsilon^2 g(x, t, \varepsilon) \quad \text{for } x \in \mathbb{R}^n, \quad (2.2)$$

where

$$\begin{aligned} f &: U \times \mathbb{R} \rightarrow \mathbb{R}^n, \\ g &: U \times \mathbb{R} \times [0, \varepsilon_0) \rightarrow \mathbb{R}^n \end{aligned}$$

are  $C^r$  ( $r \geq 1$ ) on their respective domains of definition with  $U$  some open set in  $\mathbb{R}^n$ , and periodic in  $t$  with same period  $T > 0$ . The associated averaged equation is given by

$$\dot{y} = \varepsilon \bar{f}(y), \quad y \in \mathbb{R}^n, \quad (2.3)$$

where

$$\bar{f}(y) = \frac{1}{T} \int_0^T f(y, t) dt.$$

We state the averaging theorem.

**Theorem 2.4.** (See [14].) *There exists a  $C^r$  change of coordinates  $x = y + \varepsilon w(y, t)$  under which (2.2) becomes*

$$\dot{y} = \varepsilon \bar{f}(y) + \varepsilon^2 f_1(y, t, \varepsilon),$$

where  $f_1$  is of period  $T$  in  $t$ . Moreover,

- (i) if  $x$  and  $y$  are solutions of (2.2) and (2.3), respectively, with  $x(t_0) = x_0$ ,  $y(t_0) = y_0$ , and  $|x_0 - y_0| = O(\varepsilon)$ , then  $|x(t) - y(t)| = O(\varepsilon)$  on a time scale  $O(1/\varepsilon)$  provided  $y(t) \in U$  on a time scale  $O(1/\varepsilon)$ ;
- (ii) if  $p_0$  is a hyperbolic fixed point of (2.3), then there exists  $\varepsilon_0 > 0$  such that, for all  $0 < \varepsilon \leq \varepsilon_0$ , (2.2) possesses an isolated hyperbolic period orbit  $\gamma_\varepsilon(t) = p_0 + O(\varepsilon)$  of the same stability type as  $p_0$ ;
- (iii) if  $x^s \in W^s(\gamma^\varepsilon)$  is a solution of (2.2) lying in the stable manifold of the hyperbolic period orbit  $\gamma_\varepsilon(t) = p_0 + O(\varepsilon)$ ,  $y^s \in W^s(p_0)$  is a solution of (2.3) lying in the stable manifold of the hyperbolic fixed point  $p_0$ , and if  $|x(0) - y(0)| = O(\varepsilon)$ , then  $|x^s(t) - y^s(t)| = O(\varepsilon)$  for  $t \in [0, \infty)$ . A similar statement holds for solutions lying in the unstable manifold on the time interval  $(-\infty, 0]$ .

Let a one-parameter family of systems similar to (2.2):

$$\dot{x} = \varepsilon f_{\mu}(x, t, \varepsilon), \quad \mu \in \mathbb{R}, \quad (2.4)$$

with the associated family of averaged systems

$$\dot{y} = \varepsilon \overline{f_{\mu}}(y). \quad (2.5)$$

The following theorem can be found in [14].

**Theorem 2.5.** (See [14].) When  $\mu = \mu_0$ , assume that there is an equilibrium  $y_0$  of system (2.5). Then a saddle-node bifurcation occurs at  $(y, \mu) = (y_0, \mu_0)$  if and only if the following hypothesis are satisfied:

- (i)  $\overline{f_{\mu_0}}(y_0) = \frac{\partial \overline{f_{\mu_0}}(y_0)}{\partial y} = 0.$
- (ii)  $\frac{\partial \overline{f_{\mu_0}}(y_0)}{\partial \mu} \frac{\partial^2 \overline{f_{\mu_0}}(y_0)}{\partial y^2} \neq 0.$

The following theorem will be useful in Section 5.

**Theorem 2.6.** (See [6].) If at  $(y, \mu) = (p_0, \mu_0)$  (2.5) undergoes a saddle-node or a Hopf bifurcation, then, for  $\mu$  near  $\mu_0$  and  $\varepsilon$  small enough, the Poincaré map of (2.4) also undergoes a saddle-node or a Hopf bifurcation

### 3. The associated autonomous system to Eq. (1.1)

Consider the autonomous differential equation

$$\dot{X}(t) = AX(t) + G(X_t, \alpha), \quad (3.1)$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \nu \\ 0 & -\nu & 0 \end{pmatrix}$$

and

$$G(\phi, \alpha) = \begin{pmatrix} f(\phi_1(-1), a(\phi_2(0), \phi_3(0)), \alpha) \\ 0 \\ 0 \end{pmatrix} \quad \forall \phi \in C := C([-1, 0], \mathbb{R}^3).$$

From hypothesis (H1), the function  $G$  is  $C^\infty$ -smooth on the phase space  $C$  with  $G(0, \alpha) = 0$  and  $D_\phi G(0, \alpha) = 0$ . Moreover, if  $x$  is  $\frac{2\pi}{\nu}$ -periodic solution of Eq. (1.1) with initial data  $x_0 = \phi \in C$ , then the vector  $X = \text{col}(x, y, z)$  where

$$y(t) = \varepsilon \sin(\nu t) \quad \text{and} \quad z(t) = \varepsilon \cos(\nu t)$$

is  $\frac{2\pi}{\nu}$ -periodic solution of system (3.1) with initial data

$$X_0(\theta) = \text{col}(\phi(\theta), \sin(\nu\theta), \cos(\nu\theta)).$$

**Remark 3.1.** (1) One can study the existence of periodic solutions of period  $\frac{2\pi}{\nu}$  by applying the center manifold reduction method, unfortunately, one can remark that existence of a nontrivial

$\frac{2\pi}{\nu}$ -periodic solution of Eq. (3.1), does not yield the existence of a nontrivial  $\frac{2\pi}{\nu}$ -periodic solution of Eq. (1.1). That is the converse of the precedent implication is not true.

(2) The auxiliary variables  $y(t) = \epsilon \sin(\nu t)$ ,  $z(t) = \epsilon \cos(\nu t)$  are introduced here, so that (1.1) is transformed into the autonomous 3-dimensional equation (3.1). This technique is very effective, yet not new, and this fact should be acknowledge to the work in [12].

The linearized equation of (3.1) at  $x = 0$  has the form  $\dot{X}(t) = LX_t$  where  $L\phi = A\phi(0)$ . Then  $\Lambda = \{\pm i\nu, 0\}$  are the only values of the punctual spectrum. In other words, Eq. (3.1) presents a fold-Hopf singularity.

The basis of the generalized subspace  $P$  associated to  $\Lambda$  is given by

$$\phi_1(\theta) = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} e^{-i\nu\theta}, \quad \phi_2(\theta) = \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} e^{i\nu\theta} \quad \text{and} \quad \phi_3(\theta) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The associated complementary subspace is given by  $Q = \{\phi \in C: \phi(0) = 0\}$  and the basis  $\Psi = \text{col}(\psi_1, \psi_2, \psi_3)$  of the dual subspace  $X_c^*$  is given by

$$\psi_1(\theta) = (0, -i/2, 1/2)e^{i\nu\theta}, \quad \psi_2(\theta) = \overline{\psi_1(\theta)} \quad \text{and} \quad \psi_3(\theta) = (1, 0, 0).$$

According to the local center manifold theorem (see [11]), for all  $\alpha \in \mathbb{R}$  there exists a  $C^\infty$ -map  $h_\alpha(\cdot)$  from  $\mathbb{R}^3$  into  $Q$ , such that  $h_\alpha(0) = Dh_\alpha(0) = 0$ , and there exists a neighborhood  $V$  of zero in  $\mathbb{R}^3$  such that, for each  $\xi \in V$ , there exists  $\delta = \delta(\xi) > 0$  and a function  $X$  defined on  $]-\delta - r, \delta[$  such that  $X_0 = \Phi\xi + h_\alpha(\xi)$  and  $X$  verifies Eq. (3.1) on  $]-\delta, \delta[$  and satisfies the identity

$$X_t = \Phi Y(t) + h_\alpha(Y(t)), \quad \text{for } t \in [0, \delta[.$$

where  $Y = \text{col}(Y_1, Y_2, Y_3)$  is the unique solution of the ordinary differential equation

$$\begin{cases} \frac{d}{dt}Y(t) = BY(t) + \Psi(0)G(\Phi Y(t) + h_\alpha(Y(t))), \\ Y(0) = \xi, \quad \xi \in \mathbb{R}^3, \end{cases} \quad (3.2)$$

$B$  being the following matrix

$$\begin{pmatrix} -i\nu & 0 & 0 \\ 0 & i\nu & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Before beginning the main result, we need to give some properties of function

$$h_\alpha = \text{col}(h_1^\alpha, h_2^\alpha, h_3^\alpha).$$

The following result was obtained and proved in a more general case in [11].

**Theorem 3.2.** (See [11].) Given a  $C^1$  map  $h_\alpha$  from  $\mathbb{R}^3$  into  $Q$  with  $h_\alpha(0) = 0$  and  $Dh_\alpha(0) = 0$ , a necessary condition for the graph of  $h_\alpha$  to be a local center manifold of Eq. (3.1) is that there exists a neighborhood  $V$  of zero in  $\mathbb{R}^3$  such that, for each  $Y \in V$ ,

$$\begin{aligned} \frac{d}{d\theta}(h_\alpha(Y))(\theta) &= -i\nu Y_1 \frac{\partial h_\alpha(Y)}{\partial Y_1}(\theta) + i\nu Y_2 \frac{\partial h_\alpha(Y)}{\partial Y_2}(\theta) \\ &\quad + Dh_\alpha(Y)(\theta)\Psi(0)G(\Phi Y + h_\alpha(Y), \alpha) \\ &\quad + \Phi(\theta)\Psi(0)G(\Phi Y + h_\alpha(Y), \alpha), \end{aligned} \quad (3.3)$$

$$\frac{d}{d\theta}(h_\alpha(Y))(0) = G(\Phi Y + h_\alpha(Y), \alpha). \quad (3.4)$$

We have the following proposition.

**Proposition 3.3.** *The function  $h_\alpha$  associated with Eq. (3.1) satisfies:*

- (1)  $h_1^\alpha(0, 0, Y_3)(\theta) = 0$ , for  $Y_3 \in \mathbb{R}$  such that  $(0, 0, Y_3) \in V$  and  $\theta \in [-1, 0]$ .
- (2)  $h_2^\alpha(Y_1, Y_2, Y_3)(\theta) = h_3^\alpha(Y_1, Y_2, Y_3)(\theta) = 0$ , for  $(Y_1, Y_2, Y_3) \in V$  and  $\theta \in [-1, 0]$ .

**Proof.** From relation (3.3) of Theorem 3.2 we have:

$$\begin{aligned} \frac{d}{d\theta}(h_1^\alpha(Y))(\theta) &= -i\nu Y_1 \frac{\partial h_1^\alpha(Y)(\theta)}{\partial Y_1} + i\nu Y_2 \frac{\partial h_1^\alpha(Y)(\theta)}{\partial Y_2} \\ &\quad + \left[ \frac{\partial h_1^\alpha(Y)(\theta)}{\partial Y_3} + 1 \right] f(Y_3 + h_1^\alpha(Y)(-1), a(iY_1 - iY_2, Y_1 + Y_2), \alpha), \\ \frac{d}{d\theta}(h_2^\alpha(Y, \alpha))(\theta) &= -i\nu Y_1 \frac{\partial h_2^\alpha(Y)(\theta)}{\partial Y_1} + i\nu Y_2 \frac{\partial h_2^\alpha(Y)(\theta)}{\partial Y_2} \\ &\quad + \frac{\partial h_2^\alpha(Y)(\theta)}{\partial Y_3} f(Y_3 + h_1^\alpha(Y)(-1), a(iY_1 - iY_2, Y_1 + Y_2), \alpha), \end{aligned} \quad (3.5)$$

$$\begin{aligned} \frac{d}{d\theta}(h_3^\alpha(Y))(\theta) &= -i\nu Y_1 \frac{\partial h_3^\alpha(Y)(\theta)}{\partial Y_1} + i\nu Y_2 \frac{\partial h_3^\alpha(Y)(\theta)}{\partial Y_2} \\ &\quad + \frac{\partial h_3^\alpha(Y)(\theta)}{\partial Y_3} f(Y_3 + h_1^\alpha(Y)(-1), a(iY_1 - iY_2, Y_1 + Y_2), \alpha). \end{aligned} \quad (3.6)$$

To prove the properties (2) it suffices to show that all homogeneous part of  $h_2^\alpha$  and  $h_3^\alpha$  vanish. Let  $j \in \{2, 3\}$  and  $m \geq 2$ . The regularity of function  $h$  allows us to write

$$h_j^\alpha(Y) = \sum_{l=2}^m h_{j,l}^\alpha(Y) + o(|(Y)|^m), \quad \text{for } Y \in V,$$

where, for all  $\alpha \in \mathbb{R}$ ,  $h_{j,l}^\alpha(Y)$  is the homogeneous part of order  $l$  of the function  $h_j^\alpha(Y)$ ,  $j = 2, 3$ . Then, the relations (3.5) and (3.6) imply that for all  $m \geq 3$

$$\begin{aligned} \frac{d}{d\theta}(h_{j,m}^\alpha(Y)) &= -i\nu Y_1 \frac{\partial h_{j,m}^\alpha(Y)}{\partial Y_1} + i\nu Y_2 \frac{\partial h_{j,m}^\alpha(Y)}{\partial Y_2} \\ &\quad + \sum_{l=2}^{m-1} \frac{\partial h_{j,m-l+1}^\alpha(Y)}{\partial Y_3} [f(Y_3 + h_1^\alpha(Y)(-1), a(iY_1 - iY_2, Y_1 + Y_2), \alpha)]_l \end{aligned} \quad (3.7)$$

and

$$\frac{d}{d\theta}(h_{j,2}^\alpha(Y))(\theta) = -i\nu Y_1 \frac{\partial h_{j,2}^\alpha(Y)(\theta)}{\partial Y_1} + i\nu Y_2 \frac{\partial h_{j,2}^\alpha(Y)(\theta)}{\partial Y_2}, \quad (3.8)$$

where  $[U(Y)]_l$  is the homogeneous part of order  $l$  of the function  $U(\cdot)$ . Moreover, writing

$$h_{j,m}^\alpha(Y)(\theta) = \sum_{q=0}^m \sum_{p=0}^q c_{p,q}^{m,j}(\theta) Y_1^{m-p} Y_2^{p-q} Y_3^q$$

we see that relation (3.8) is equivalent to

$$\frac{d}{d\theta}(c_{p,q}^{2,j}(\theta)) = -i\nu(m-2p+q)c_{p,q}^{2,j}(\theta) \quad \text{and} \quad c_{p,q}^{2,j}(0) = 0.$$

This yields that  $c_{p,q}^{2,j}(\theta) = 0$  for all  $\theta \in [-1, 0]$ , which proves that  $h_j^2(Y)(\theta) = 0$ .

Suppose that  $h_{j,n}^\alpha(Y)(\theta) = 0$  for all  $n \in \{2, \dots, m-1\}$ , then relation (3.7) implies that

$$\frac{d}{d\theta}(h_{j,m}^\alpha(Y)) = -i\nu Y_1 \frac{\partial h_{j,m}^\alpha(Y)}{\partial Y_1} + i\nu Y_2 \frac{\partial h_{j,m}^\alpha(Y)}{\partial Y_2},$$

and so we obtain from  $h_{j,m}^\alpha(Y)(0) = 0$  that  $h_{j,m}^\alpha(Y)(\theta) = 0$ .

From the hypothesis (H1) we have

$$\begin{aligned} \frac{d}{d\theta}(h_{1,m}^\alpha(0, 0, Y_3)) &= \left[ f\left(Y_3 + \sum_{l=2}^{m-1} h_{1,m-l+1}^\alpha(0, 0, Y_3)(-1), a(0, 0), \alpha\right) \right]_m \\ &\quad + \sum_{l=2}^{m-1} \frac{\partial h_{1,m-l+1}^\alpha(0, 0, Y_3)}{\partial Y_3} \\ &\quad \times \left[ f\left(Y_3 + \sum_{l=2}^{m-1} h_{1,m-l+1}^\alpha(0, 0, Y_3)(-1), a(0, 0), \alpha\right) \right]_l. \end{aligned}$$

Then by applying the process as above, we obtain

$$\frac{d}{d\theta}h_1^\alpha(0, 0, Y_3)(\theta) = 0, \quad \text{i.e.} \quad h_1^\alpha(0, 0, Y_3)(\theta) = h_1^\alpha(0, 0, Y_3)(0) \quad \forall \theta \in [-1, 0].$$

Moreover, we have  $h_1^\alpha(Y) \in \mathcal{Q}$ , which implies that  $h_1^\alpha(Y)(0) = 0$  and then  $h_1^\alpha(0, 0, Y_3)(\theta) = 0$   $\forall \theta \in [-1, 0]$ .

Finally, we conclude that  $h_j^\alpha(Y)(\theta) = 0$ .  $\square$

#### 4. Integral manifolds

In the sequel, we state a result of existence of a local integral manifold of Eq. (1.1).

For  $t, \alpha, \epsilon \in \mathbb{R}$  and the subset  $V_0$  defined by

$$V_0 := \{y \in \mathbb{R}^2: (0, 0, y) \in V\},$$

we consider the application defined on  $V_0$  by

$$g_t^{\epsilon, \alpha}(y) = h_1^\alpha\left(\frac{\epsilon}{2}e^{-i\nu t}, \frac{\epsilon}{2}e^{i\nu t}, y\right) \quad \forall y \in V_0.$$

The graph of  $g_t^{\epsilon, \alpha}$  is given by

$$M_t^{\epsilon, \alpha} = \left\{ y + h_1^\alpha\left(\frac{\epsilon}{2}e^{-i\nu t}, \frac{\epsilon}{2}e^{i\nu t}, y\right), y \in V_0 \right\}.$$

We have the following theorem.

**Theorem 4.1.** *There exists  $\varepsilon_0 > 0$  such that for all  $0 < \varepsilon < \varepsilon_0$ , the set  $N_{\varepsilon, \alpha}$  given by*

$$N_{\varepsilon, \alpha} = \{(t, \phi): t \in \mathbb{R}, \phi \in M_t^{\varepsilon, \alpha}\}$$

*is an attractive local integral manifold for Eq. (1.1).*



**Proof.** First, we show the invariance of the set  $N_{\varepsilon, \alpha}$ .

Let  $x$  be a solution of Eq. (1.1), with  $\phi = x_s \in M_s^{\varepsilon, \alpha}$ ,

$$\phi(\theta) = \phi(0) + h_1^\alpha \left( \frac{\varepsilon}{2} e^{-i\nu s}, \frac{\varepsilon}{2} e^{i\nu s}, \phi(0) \right), \quad \phi(0) \in V_0,$$

then the function  $X$  defined on  $[-r, \infty[$  by

$$X(t) = \begin{cases} \begin{pmatrix} x(t) \\ \varepsilon \sin(\nu t) \\ \varepsilon \cos(\nu t) \end{pmatrix} & \text{for } t \in [0, \infty[, \\ \begin{pmatrix} \phi(t) \\ \varepsilon \sin(\nu t) \\ \varepsilon \cos(\nu t) \end{pmatrix} & \text{for } t \in [-1, 0], \end{cases} \quad (4.1)$$

satisfies

$$\dot{X}(t) = \begin{pmatrix} f(x(t-1), a(\varepsilon \sin(\nu t), \varepsilon \cos(\nu t)), \alpha) \\ \varepsilon \cos(\nu t) \\ -\varepsilon \sin(\nu t) \end{pmatrix} = AX(t) + G(X_t, \alpha)$$

for all  $t \in [0, \infty[$ .

It follows that  $X$  is solution of Eq. (3.1). For  $\theta \in [-r, 0]$ ,

$$\begin{aligned} X_s(\theta) &= \begin{pmatrix} \phi(\theta) \\ \varepsilon \sin(\nu s + \nu \theta) \\ \varepsilon \cos(\nu s + \nu \theta) \end{pmatrix} \\ &= \begin{pmatrix} \phi(0) \\ \frac{i\varepsilon}{2} e^{-i\nu s} e^{-i\nu \theta} - \frac{i\varepsilon}{2} e^{i\nu s} e^{i\nu \theta} \\ \frac{\varepsilon}{2} e^{-i\nu s} e^{-i\nu \theta} + \frac{\varepsilon}{2} e^{i\nu s} e^{i\nu \theta} \end{pmatrix} + \begin{pmatrix} h_1^\alpha \left( \frac{\varepsilon}{2} e^{-i\nu s}, \frac{\varepsilon}{2} e^{i\nu s}, \phi(0) \right) \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

and by properties (2) of Proposition 3.3 we have

$$X_s(\theta) = \Phi Y^s + h_\alpha(Y^s), \quad \text{where } Y^s = \begin{pmatrix} \frac{\varepsilon}{2} e^{-i\nu s} \\ \frac{\varepsilon}{2} e^{i\nu s} \\ \phi(0) \end{pmatrix}.$$

Then for  $\varepsilon$  small enough such that  $Y^s \in V$ , the vector  $X_s$  belongs to the local center manifold  $M$ . This implies that  $X_t \in M_\alpha$ , so  $x_t \in M_t^{\varepsilon, \alpha}$  for all  $t > s$ , which shows the invariance of  $N_{\varepsilon, \alpha}$ .

To prove the attractiveness, let  $x(\sigma, \phi)$  be the solution of (1.1) through  $(\sigma, \phi) \in \mathbb{R} \times C$ , such that  $|x_t(\sigma, \phi)| < \delta$  for very small  $\delta > 0$ , and let

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

be the corresponding solution of Eq. (3.1) as defined in (4.1). The attractiveness property of center manifold associated with Eq. (3.1), yields that

$$|X_t^Q - h_\alpha(X_t^P)| \leq K_1 e^{-\beta_1 t} \quad \forall t \geq 0. \quad (4.2)$$

with  $X_t^P = \langle \Psi, X_t \rangle$  and  $X_t^Q = X_t - \Phi X_t^P$ . On the other hand, we have that

$$X_t^P = \begin{pmatrix} -\frac{i}{2} y(t) + \frac{1}{2} z(t) \\ \frac{i}{2} y(t) + \frac{1}{2} z(t) \\ x(t) \end{pmatrix} = \begin{pmatrix} -\frac{i}{2} \varepsilon \sin(\nu t) + \frac{1}{2} \varepsilon \cos(\nu t) \\ \frac{i}{2} \varepsilon \sin(\nu t) + \frac{1}{2} \varepsilon \cos(\nu t) \\ x(t) \end{pmatrix},$$

and so

$$\Phi(\theta)X_t^p = \begin{pmatrix} x(t) \\ \varepsilon \sin(v(t+\theta)) \\ \varepsilon \cos(v(t+\theta)) \end{pmatrix}.$$

It follows that

$$X_t^Q(\theta) = \begin{pmatrix} x_t(\theta) - x(t) \\ 0 \\ 0 \end{pmatrix},$$

which implies, by relation (4.2), that there exists constants  $K_1 > 0$  and  $\beta_1 > 0$  with  $K_1$  depending on  $\phi$  such that

$$\left| x_t - x(t) - h_1^\alpha \left( \frac{\varepsilon}{2} e^{-ivt}, \frac{\varepsilon}{2} e^{ivt}, x(t) \right) \right| \leq K_1 e^{-\beta_1 t}, \quad t \geq 0.$$

This completes the proof of the theorem.  $\square$

**Remark 4.2.** (i) Notice that the function  $y$  defined by

$$y(t) = x(t) + h_1^\alpha \left( \frac{\varepsilon}{2} e^{-ivt}, \frac{\varepsilon}{2} e^{ivt}, x(t) \right)$$

belongs to  $M_t^{\varepsilon, \alpha} \forall t \geq 0$ . Therefore, the above theorem shows the attractiveness

$$\text{dis}(x_t(\sigma, \phi), M_t) \rightarrow 0$$

as  $t \rightarrow +\infty$ .

(ii) According to the invariance properties established in the proof of Theorem 4.1, we conclude that the following ordinary differential equation

$$\dot{y}(t) = f \left( y(t) + h_1^\alpha \left( \frac{\varepsilon}{2} e^{-ivt}, \frac{\varepsilon}{2} e^{ivt}, y(t) \right) (-1), a(\varepsilon \sin(vt), \varepsilon \cos(vt)), \alpha \right) \quad (4.3)$$

is the reduced system of the periodic retarded equation (1.1) on the local integral manifold  $N^{\varepsilon, \alpha}$ .

The following proposition shows the relation between instability of periodic solutions, which remain small, of Eq. (1.1) and those of its reduced system (4.3).

**Proposition 4.3.** Assume that Eq. (4.3) has an unstable periodic solution  $p$  such that  $p(t) \in V_0$ . Then the associated solution  $\gamma$ , given by  $\gamma_t = p(t) + h_1^\alpha(\frac{\varepsilon}{2} e^{-ivt}, \frac{\varepsilon}{2} e^{ivt}, p(t))$ , of Eq. (1.1) is also unstable.

**Proof.** Let  $\delta > 0$ ,  $c := \sup\{Dh_1^\alpha(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, Y) : |Y - p(0)| < \delta\}$  and  $\delta_1 = \frac{\delta}{2(1+c)}$ . The instability of  $p(t)$  implies the existence of  $b_1 > 0$ ,  $\sigma_1 > 0$ ,  $t_0 > \sigma_1$  and  $a_{\delta_1} \in V_0$  such that  $|a_{\delta_1} - p(0)| < \delta_1$  and  $|y(t_0) - p(t_0)| > b_1$ , with  $y$  solution of Eq. (4.3) through  $(\sigma_1, a_{\delta_1})$ .

Let  $\phi_\delta = a_{\delta_1} + h_1^\alpha(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, a_{\delta_1})$ , then

$$|\phi_\delta - \gamma_0| \leq |a_{\delta_1} - p(0)| + \left| h_1^\alpha \left( \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, a_{\delta_1} \right) - h_1^\alpha \left( \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, p(0) \right) \right| \leq \delta_1 + c\delta_1 = \frac{\delta}{2} < \delta.$$

On the other hand, since  $\phi_\delta \in M_0^\varepsilon$ , the invariance properties imply that the solution  $x(\sigma_1, \phi_\delta)$  of Eq. (1.1) through  $(\sigma_1, \phi_\delta)$  is in  $M_t^{\varepsilon, \alpha}$ , which yields that

$$x_t(\sigma_1, \phi_\delta) = x(t) + h_1^\alpha \left( \frac{\varepsilon}{2} e^{-ivt}, \frac{\varepsilon}{2} e^{ivt}, x(t) \right),$$

and consequently

$$|x_{t_0} - \gamma_{t_0}| \geq |x(t_0) - p(t_0)| > b_1,$$

which proves the instability of  $\gamma$  as solution of Eq. (1.1).  $\square$

## 5. Application to saddle-node bifurcation

In this section, we will give a sufficient condition for a system (1.1), which yield the saddle-node bifurcation.

Let  $(z, \alpha) \in V_0 \times \mathbb{R}$  and define the coefficients  $C_1, C_2(z, \alpha), C_3$  as

$$\begin{aligned} C_1 &:= \frac{1}{2} \left[ \frac{\partial^2}{\partial Y_1^2} a(0, 0) + \frac{\partial^2}{\partial Y_2^2} a(0, 0) \right], \\ C_2(z, \alpha) &:= \frac{1}{2} \left( -i \frac{\partial}{\partial Y_1} h_1^\alpha(0, 0, z)(-1) + i \frac{\partial}{\partial Y_2} h_1^\alpha(0, 0, z)(-1) \right) \frac{\partial}{\partial Y_1} a(0, 0) \\ &\quad + \frac{1}{2} \left( \frac{\partial}{\partial Y_1} h_1^\alpha(0, 0, z)(-1) + \frac{\partial}{\partial Y_2} h_1^\alpha(0, 0, z)(-1) \right) \frac{\partial}{\partial Y_2} a(0, 0) \end{aligned}$$

and

$$C_3 := \frac{1}{4} \left[ \left( \frac{\partial}{\partial Y_1} a(0, 0) \right)^2 + \left( \frac{\partial}{\partial Y_2} a(0, 0) \right)^2 \right].$$

Recall that  $h_1^\alpha$  is the first component of the function  $h^\alpha$  which represent the center manifold associated to Eq. (3.1).

In the sequel, we will suppose that the function  $f$  satisfies

$$f(x, a(0, 0), \alpha) = 0$$

for  $|x|$  sufficiently small.

We need the following lemma.

**Lemma 5.1.** *The averaged equation of the periodic ordinary differential equation (4.3) is given by*

$$\dot{z}(t) = \varepsilon^2 \overline{f_1}(z(t), \alpha), \quad (5.1)$$

where

$$\begin{aligned} \overline{f_1}(z, \alpha) &= C_1 \frac{\partial f}{\partial x_2}(z, a(0, 0), \alpha) + C_2(z, \alpha) \frac{\partial^2 f}{\partial x_1 \partial x_2}(z, a(0, 0), \alpha) \\ &\quad + C_3 \frac{\partial^2 f}{\partial x_2^2}(z, a(0, 0), \alpha). \end{aligned}$$

**Proof.** For  $\varepsilon > 0$  and  $\alpha \in \mathbb{R}$ , we define

$$V(\varepsilon, t, x, \alpha) := h_1^\alpha \left( \frac{\varepsilon}{2} e^{-i\nu t}, \frac{\varepsilon}{2} e^{i\nu t}, x \right) (-1).$$

According to the first properties given in Proposition 3.3, the Taylor expansion of the maps  $\varepsilon \rightarrow V(\varepsilon, t, x, \alpha)$  and  $\varepsilon \rightarrow a(\varepsilon \sin(\nu t), \varepsilon \cos(\nu t))$  are given by

$$V(\varepsilon, t, x, \alpha) = \varepsilon V_1(t, x, \alpha) + \varepsilon^2 V_2(t, x, \alpha) + o(\varepsilon^2)$$

and

$$a(\varepsilon \sin(vt), \varepsilon \cos(vt)) = a(0, 0) + \varepsilon a_1(t) + \varepsilon^2 a_2(t) + o(\varepsilon^2),$$

where

$$\begin{aligned} V_1(t, x, \alpha) &= \frac{1}{2} \frac{\partial}{\partial Y_1} h_1^\alpha(0, 0, x)(-1)e^{-ivt} + \frac{1}{2} \frac{\partial}{\partial Y_2} h_1^\alpha(0, 0, x)(-1)e^{ivt}, \\ V_2(t, x, \alpha) &= \frac{1}{4} \frac{\partial^2}{\partial Y_1^2} h_1^\alpha(0, 0, x)(-1)e^{-2ivt} + \frac{1}{2} \frac{\partial^2}{\partial Y_1 \partial Y_2} h_1^\alpha(0, 0, x)(-1) \\ &\quad + \frac{1}{4} \frac{\partial^2}{\partial Y_2^2} h_1^\alpha(0, 0, x)(-1)e^{2ivt}, \\ a_1(t) &= \frac{\partial}{\partial Y_1} a(0, 0) \sin(vt) + \frac{\partial}{\partial Y_2} a(0, 0) \cos(vt) \end{aligned}$$

and

$$\begin{aligned} a_2(t) &:= \frac{\partial^2}{\partial Y_1^2} a(0, 0) (\sin(vt))^2 + 2 \frac{\partial^2}{\partial Y_1 \partial Y_2} a(0, 0) \sin(vt) \cos(vt) \\ &\quad + \frac{\partial^2}{\partial Y_2^2} a(0, 0) (\cos(vt))^2. \end{aligned}$$

Moreover, if we define

$$W(x, t, \varepsilon, \alpha) := f\left(x + h_1^\alpha\left(\frac{\varepsilon}{2}e^{-ivt}, \frac{\varepsilon}{2}e^{ivt}, x\right)(-1), a(\varepsilon \sin(vt), \varepsilon \cos(vt)), \alpha\right),$$

then we can write

$$\begin{aligned} W(x, t, \varepsilon, \alpha) &= Df(x, a(0, 0), \alpha)(V(\varepsilon, t, x, \alpha), a(\varepsilon \sin(vt), \varepsilon \cos(vt))) \\ &\quad + D^2 f(x, a(0, 0), \alpha)(V(\varepsilon, t, x, \alpha), a(\varepsilon \sin(vt), \varepsilon \cos(vt)))^2 + \dots, \end{aligned}$$

where  $D^j f(x, y, \alpha)$  represents the  $j$ th derivative of the function  $f$  with respect to  $(x, y)$ , and  $(x, y)^{\overline{m}} := ((x, y), \dots, (x, y))$   $m$  times. From hypothesis (H1) and after some calculations we obtain

$$W(x, t, \varepsilon, \alpha) = \varepsilon g_1(t, x, \alpha) + \varepsilon^2 g_2(t, x, \alpha) + o(\varepsilon^2),$$

with

$$g_1(t, x, \alpha) = \frac{\partial f}{\partial x_2}(x, a(0, 0), \alpha) a_1(t)$$

and

$$\begin{aligned} g_2(t, x, \alpha) &= \frac{\partial f}{\partial x_2}(x, a(0, 0), \alpha) a_2(t) + \frac{\partial^2 f}{\partial x_1 \partial x_2}(x, a(0, 0), \alpha) V_1(t, x, \alpha) a_2(t) \\ &\quad + \frac{1}{2} \frac{\partial^2 f}{\partial x_2^2}(x, a(0, 0), \alpha) (a_1(t))^2. \end{aligned}$$

Consequently, Eq. (4.3) can be written

$$\dot{x}(t) = \varepsilon g_1(t, x(t), \alpha) + \varepsilon^2 g_2(t, x(t), \alpha) + o(\varepsilon^2). \quad (5.2)$$

We remark that the first term of the vector field of Eq. (5.2) have null average, and the averaging Theorem 2.4 (see [14]) implies that there exists a change of coordinate  $x = y + \varepsilon w_1(t, y, \alpha)$  such that Eq. (5.2) becomes

$$\dot{y}(t) = \varepsilon^2 f_1(t, y(t), \alpha) + o(\varepsilon^2), \quad (5.3)$$

where

$$f_1(t, y, \alpha) = D_y g_1(t, y, \alpha) w_1(t, y, \alpha) + g_2(t, y, \alpha), \quad (5.4)$$

and  $w_1$  verifies

$$\frac{\partial w_1(t, y, \alpha)}{\partial t} = g_1(t, y, \alpha). \quad (5.5)$$

It follows that the results of first order averaging are inconclusive, and so second order one is required, i.e., there exists a change of coordinates  $y = z + \varepsilon^2 w_2(t, z, \alpha)$  such that Eq. (5.3) becomes

$$\dot{z}(t) = \varepsilon^2 \overline{f_1}(z(t), \alpha) + o(\varepsilon^2), \quad (5.6)$$

where

$$\overline{f_1}(z, \alpha) = \frac{\nu}{2\pi} \int_0^{2\pi/\nu} f_1(t, z, \alpha) dt.$$

However, from (5.5),  $w_1$  can be chosen to be

$$w_1(t, y) = \int_0^t g_1(s, y, \alpha) ds = \frac{\partial f}{\partial x_2}(y, a(0, 0), \alpha) \int_0^t a_1(s) ds.$$

Then, from formula (5.4), we have

$$\begin{aligned} \overline{f_1}(z, \alpha) &= \frac{\nu}{2\pi} \int_0^{2\pi/\nu} (D_y g_1(t, y, \alpha) w_1(t, y, \alpha) + g_2(t, y, \alpha)) dt \\ &= \frac{\nu}{2\pi} \frac{\partial f}{\partial x_2}(z, a(0, 0), \alpha) \frac{\partial^2 f}{\partial x_1 \partial x_2}(z, a(0, 0), \alpha) \int_0^{2\pi/\nu} \left( a_1(s) \int_0^s a_1(t) dt \right) ds \\ &\quad + \frac{\nu}{2\pi} \frac{\partial f}{\partial x_2}(z, a(0, 0), \alpha) \int_0^{2\pi/\nu} a_2(s) ds \\ &\quad + \frac{\nu}{2\pi} \frac{\partial^2 f}{\partial x_1 \partial x_2}(z, a(0, 0), \alpha) \int_0^{2\pi/\nu} V_1(s, z, \alpha) a_1(s) ds \\ &\quad + \frac{\nu}{4\pi} \frac{\partial^2 f}{\partial x_2^2}(z, a(0, 0), \alpha) \int_0^{2\pi/\nu} (a_1(s))^2 ds. \end{aligned}$$

On the other hand we have

$$\begin{aligned} \frac{v}{2\pi} \int_0^{2\pi/v} \left( a_1(s) \int_0^s a_1(t) dt \right) ds &= \frac{v}{4\pi} \left[ \left( \int_0^s a_1(t) dt \right)^2 \right]_{s=0}^{s=2\pi/v} \\ &= \frac{v}{4\pi} \left( \int_0^{2\pi/v} a_1(t) dt \right)^2 = 0, \end{aligned}$$

$$\int_0^{2\pi/v} a_2(s) ds = \frac{\pi}{v} \left[ \frac{\partial^2}{\partial Y_1^2} a(0, 0) + \frac{\partial^2}{\partial Y_2^2} a(0, 0) \right],$$

$$\int_0^{2\pi/v} a_1(s) ds = \frac{\pi}{v} \left[ \left( \frac{\partial}{\partial Y_1} a(0, 0) \right)^2 + \left( \frac{\partial}{\partial Y_2} a(0, 0) \right)^2 \right],$$

and

$$\begin{aligned} \int_0^{2\pi/v} V_1(s, z, \alpha) a_1(s) ds &= \frac{\pi}{v} \left( -i \frac{\partial}{\partial Y_1} h_1^\alpha(0, 0, z)(-1) + i \frac{\partial}{\partial Y_2} h_1^\alpha(0, 0, z)(-1) \right) \frac{\partial}{\partial Y_1} a(0, 0) \\ &\quad + \frac{\pi}{v} \left( \frac{\partial}{\partial Y_1} h_1^\alpha(0, 0, z)(-1) + \frac{\partial}{\partial Y_2} h_1^\alpha(0, 0, z)(-1) \right) \frac{\partial}{\partial Y_2} a(0, 0). \end{aligned}$$

Finally, the averaged equation associated with Eq. (5.6) is given by

$$\dot{z}(t) = \varepsilon^2 \overline{f_1}(z(t), \alpha),$$

where

$$\begin{aligned} \overline{f_1}(z, \alpha) &= C_1 \frac{\partial f}{\partial x_2}(z, a(0, 0), \alpha) + C_2(z, \alpha) \frac{\partial^2 f}{\partial x_1 \partial x_2}(z, a(0, 0), \alpha) \\ &\quad + C_3 \frac{\partial^2 f}{\partial x_2^2}(z, a(0, 0), \alpha), \end{aligned}$$

which achieve the proof.  $\square$

As consequence we have.

**Proposition 5.2.** Assume that the following conditions are satisfied:

- (i)  $C_3 \frac{\partial^2 f}{\partial x_2^2}(e) = 0,$
- (ii)  $C_1 \frac{\partial^2 f}{\partial x_1 \partial x_2}(e) + \frac{\partial}{\partial z} C_2(0, 0) \frac{\partial^2 f}{\partial x_1 \partial x_2}(e) + C_3 \frac{\partial^3 f}{\partial x_1 \partial x_2^2}(e) = 0,$
- (iii)  $\left[ C_1 + 2 \frac{\partial}{\partial z} C_2(0, 0) \right] \frac{\partial^3 f}{\partial x_1^2 \partial x_2}(e) + \frac{\partial^2}{\partial z^2} C_2(0, 0) \frac{\partial^2 f}{\partial x_1 \partial x_2}(e) + C_3 \frac{\partial^4 f}{\partial x_1^2 \partial x_2^2}(e) \neq 0,$

and

$$(iv) \quad C_1 \frac{\partial^2 f}{\partial \alpha \partial x_2}(e) + C_3 \frac{\partial^3 f}{\partial \alpha \partial x_2^2}(e) \neq 0,$$

with  $e := (0, a(0, 0), 0)$ .

Then for  $\varepsilon$  sufficiently small, a saddle node bifurcation occurs for the system (4.3) at  $(x, \alpha) = (0, 0)$ .

**Proof.** Conditions (i)–(iv) of proposition imply that the function  $\bar{f}$  given in Lemma 5.1 satisfies

$$\bar{f}_1(0, 0) = \frac{\partial \bar{f}_1(0, 0)}{\partial z} = 0$$

and

$$\frac{\partial \bar{f}_1(0, 0)}{\partial \alpha} \frac{\partial \bar{f}_1(0, 0)}{\partial z^2} \neq 0,$$

which are the conditions in Theorem 2.5 to have a saddle-node bifurcation of the averaged system (5.1) at the point  $(0, 0)$ . Using Theorem 2.6, we conclude that the Poincaré map of (4.3) undergoes a saddle-node bifurcation for  $\alpha$  near 0 and  $\varepsilon$  sufficiently small.  $\square$

Using the foregoing proposition we can now prove the main result of this section.

**Theorem 5.3.** *Under the same conditions (i)–(iv), there exists  $\varepsilon_0 > 0$  such that for  $0 < \varepsilon < \varepsilon_0$ , the dynamical system generated by the time-one map related to Eq. (1.1) on the local integral manifold given by Theorem 4.1 undergoes a saddle-node bifurcation at the point  $(x, \alpha) = (0, 0)$ .*

**Proof.** The proof of the theorem follows from Proposition 5.2 and from the fact that (4.3) is the reduced equation of the periodic delay differential equation (1.1) on the local integral manifold  $N_{\varepsilon, \alpha}$  presented in Section 4.  $\square$

**Remark.** The averaging method is applied to the ODE (4.3) for the flow on the local integral manifold of the origin for (1.1), rather than applying the usual normal form technique to the autonomous FDE (3.1). This used technique is especially useful for studying the behavior of a such periodic systems, for more details on this choice we refer the reader to the work W. Zhang and K. Huseyin [15].

## 6. Example

In this section we discuss a special application of the obtained results.  
Consider the following periodic retarded equation

$$\dot{x}(t) = r(\varepsilon \sin(\nu t), \varepsilon \cos(\nu t))g(x(t-1), \alpha), \quad \varepsilon, \alpha, \nu \in \mathbb{R}, \quad (6.1)$$

where  $r, g: \mathbb{R}^2 \rightarrow \mathbb{R}$  are  $C^\infty$ -smooth functions, satisfying

$$g(0, 0) = \frac{\partial}{\partial x_1} g(0, 0) = 0 \quad \text{and} \quad r(0, 0) = 0.$$

The system (6.1) is a particular case of Eq. (1.1) with  $f(x, y, \alpha) = yg(x, \alpha)$  and  $a(\cdot, \cdot) = r(\cdot, \cdot)$ . Moreover, Eq. (6.1) can be viewed as a special perturbation of the retarded autonomous differential equation

$$\dot{x}(t) = g(x(t-1), \alpha). \quad (6.2)$$

When the function  $r$  is a constant function, (6.1) is the autonomous equation (6.2). A comprehensive study of that equation can be found in [1,3,4] and [8,9]. A more realistic model can be the periodic RFDE (6.1).

Put

$$C_1 := \frac{1}{2} \left[ \frac{\partial^2}{\partial x_1^2} r(0, 0) + \frac{\partial^2}{\partial x_2^2} r(0, 0) \right].$$

**Theorem 6.1.** Assume that

$$\frac{\partial^2}{\partial x_1^2} g(0, 0) \frac{\partial}{\partial \alpha} g(0, 0) \neq 0$$

and  $C_1 \neq 0$ . Then the system (6.1) undergoes also a saddle node bifurcation at  $(x, \alpha) = (0, 0)$  for  $\varepsilon$  sufficiently small.

**Proof.** We have  $\frac{\partial}{\partial x_2} f(x, y, \alpha) = g(x, \alpha)$ . Then from the hypothesis in theorem we have the following

$$\frac{\partial^2 f}{\partial x_2^2}(0) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(0) = \frac{\partial^3 f}{\partial x_1 \partial x_2^2}(0) = \frac{\partial^3 f}{\partial \alpha \partial x_2^2}(0) = \frac{\partial^4 f}{\partial x_1^2 \partial x_2^2}(0) = 0 \quad (6.3)$$

and

$$\frac{\partial^3 f}{\partial x_1^2 \partial x_2}(0) \frac{\partial^2 f}{\partial \alpha \partial x_2}(0) \neq 0. \quad (6.4)$$

On the other hand, by applying the formulas in Theorem 3.2 and after some calculations, we obtain

$$\frac{\partial^2}{\partial Y_1 \partial Y_3} h_1^0(0, 0, 0)(-1) = \frac{\partial^2}{\partial Y_2 \partial Y_3} h_1^0(0, 0, 0)(-1) = 0$$

with  $h_1^\alpha$  is the first component of the function  $h^\alpha$  which represent the center manifold associated to Eq. (3.1). It follows that the function  $C_2$  defined in Section 5 satisfies

$$\frac{\partial}{\partial z} C_2(0, 0) = 0. \quad (6.5)$$

The conditions (6.3)–(6.5) and  $C_1 \neq 0$  imply that (i)–(iv) of Proposition 5.2 are satisfied. Then by applying Theorem 5.3, we conclude that there exists  $\varepsilon_0 > 0$  such that for  $0 < \varepsilon < \varepsilon_0$ , the system (6.1) undergoes a saddle-node bifurcation at the point  $(x, \alpha) = (0, 0)$ .  $\square$

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