

Semistable extremal ground states for nonlinear evolution equations in unbounded domains[☆]

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Abstract

In this paper we show that dissipative reaction–diffusion equations in unbounded domains possess extremal semistable ground states equilibria, which bound asymptotically the global dynamics. Uniqueness of such positive ground state and their approximation by extremal equilibria in bounded domains is also studied. The results are then applied to the important case of logistic equations.

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1. Introduction

In this paper we study the asymptotic behaviour of some nonlinear parabolic equations in unbounded domains. More precisely we consider the following model problem

$$\begin{cases} u_t - \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(0) = u_0 \end{cases} \quad (1.1)$$

posed in some space of functions defined in an unbounded domain $\Omega \subset \mathbb{R}^N$, denoted X and that will be made precise below.

In case of bounded domains it has been recently shown in [6,7,9] that these type of problems, under suitable dissipativity assumptions on the nonlinear term, have a remarkable dynamical behaviour given by the following theorem.

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Theorem 1.1. *There exist two ordered extremal equilibria for problem (1.1), φ_m and φ_M , minimal and maximal, respectively, in the sense that any other equilibrium, ψ , satisfies $\varphi_m \leq \psi \leq \varphi_M$. Furthermore, the ordered set $\{v \in X: \varphi_m \leq v \leq \varphi_M\}$ uniformly attracts the dynamics of the systems, i.e.,*

$$\varphi_m(x) \leq \liminf_{t \rightarrow \infty} u(t, x; u_0) \leq \limsup_{t \rightarrow \infty} u(t, x; u_0) \leq \varphi_M(x) \quad (1.2)$$

uniformly in $x \in \Omega$ for bounded sets of initial data. Moreover, the minimal equilibrium is stable from below and the maximal one is stable from above.

Finally, there exists a global attractor \mathcal{A} for problem (1.1) which satisfies

$$\varphi_m \leq \mathcal{A} \leq \varphi_M$$

and $\varphi_m, \varphi_M \in \mathcal{A}$.

Hence the extremal equilibria are the caps of the attractor. Note that here X is a suitable space of functions defined on the bounded set Ω . Typically $X = C(\overline{\Omega})$ but many other standard spaces are possible such as some subspaces of Sobolev spaces $W^{s,q}(\Omega)$, depending on the boundary conditions considered. However, due to the smoothing effect of (1.1) solutions typically enter $X = C(\overline{\Omega})$ and then this is a natural space for Theorem 1.1.

Our goal in this paper is to extend Theorem 1.1 to hold in the case of unbounded domains. For this we follow the dynamical strategy in [6,7,9]. However it is important to note that the fact the domain is unbounded introduces several important technicalities that makes the result nontrivial. Moreover several new restrictions appear now, in contrast with the case of a bounded domain.

First, one should note that in the case of unbounded domains the choice of the space of initial data, X , is not a naive question. As mentioned above, in the case of a bounded domain, the smoothing of the equation is enough to show that the nonlinear semigroup defined by the equation (which we assumed now to have globally defined and bounded solutions) is asymptotically compact and then trajectories are relatively compact. On the other hand, in the case of unbounded domains the asymptotic compactness of the nonlinear semigroup is strongly related to the suitable spatial decay of solutions as $|x| \rightarrow \infty$ and not only to smoothing.

Second, but strongly related to the point above, one has to note that both diffusion and reaction in the equation have to cooperate together to achieve such spatial decay of solutions. Otherwise global unbounded solutions can exist and/or the nonlinear semigroup is not asymptotically compact. It is also worth noting that, now, linear terms in the equation play a much more significant role than in the case of bounded domains. For example for the nonlinear term

$$f(x, u) = u - u^3,$$

which is a prototypical example for which things go wrong, global unbounded solutions do exist in standard spaces. This is originated by the linear term, which is the bad term in the equation. This behaviour does not take place in bounded domains. See [4] for an exhaustive discussion of dissipative mechanisms for (1.1) and the interplay between diffusion and reaction for the semigroup to be asymptotically compact in spaces of the type $X = H^{2\alpha,q}(\Omega)$ or $X = BUC(\Omega)$.

In fact, the results in [4] are our starting point to prove our results here. In particular, in any of the cases in [4] in which an attractor is shown to exist, we will prove Theorem 1.1. As the functions φ_m and φ_M will converge to zero as $|x| \rightarrow \infty$, they are commonly denoted as ground states.

Also, note that the analysis here carries out for other than the model problem (1.1). In particular, we can consider other diffusion operator in divergence form with smooth bounded coefficients.

The paper is organised as follows. In Section 2 we quote the results in [4] that we take as a starting point in this paper. Some other results in [4] will be quoted in other sections as needed. In Section 3 we prove Theorem 1.1 in different situations for the growth and structure conditions of the nonlinear term and in different functions spaces X . In Section 4 we address the question of the existence of a minimal semistable positive ground state. Then in Section 5 we give conditions under which there exists a unique positive equilibria for (1.1), which must be then a globally stable ground state for positive solution. Finally, in Section 6 the particularly important case of logistic nonlinearities are considered to illustrate the scope of our previous results.

As mentioned above, several new restrictions will appear in the results, in comparison with the case of bounded domains. Such differences will be pointed out at suitable places of the paper.

2. Previous results

Here we quote the results in [4] that will be needed further below. First, we take $L^q(\Omega)$, $1 < q < \infty$, as a base space. Then we can consider the scale of spaces of Bessel potentials associated to $A = -\Delta$ with Dirichlet boundary conditions which we denote by $H_D^{2\alpha,q}(\Omega)$, $-1 \leq \alpha \leq 1$. Moreover, the semigroup generated by $-A$ satisfies the smoothing estimate

$$\|e^{-At}u_0\|_{H_D^{2\alpha,q}(\Omega)} \leq \frac{M(t)}{t^{\alpha-\beta}} \|u_0\|_{H_D^{2\beta,q}(\Omega)} \quad (2.1)$$

for all $-1 \leq \beta \leq \alpha \leq 1$, with $M(t) = M_{\alpha,\beta}e^{\mu t}$ for certain $\mu \in \mathbb{R}$ and $M_{\alpha,\beta} \geq 1$. In particular, for $1 < q < r < \infty$, we have the estimate

$$\|e^{-At}u_0\|_{L^r(\Omega)} \leq M \frac{e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{L^q(\Omega)}, \quad t > 0. \quad (2.2)$$

We can consider certain classes of perturbations of the Laplacian on that scale. More precisely, let $L_U^p(\Omega)$ be the space of functions in Ω such that

$$\sup_{x \in \Omega} \int_{B(x,1) \cap \Omega} |V(y)|^p dy < \infty$$

with norm

$$\|V\|_{L_U^p(\Omega)} = \sup_{x \in \Omega} \|V\|_{L^p(B(x,1) \cap \Omega)}.$$

If $V \in L_U^p(\Omega)$ for some $p > N/2$ then $S(t)$, the semigroup generated by $\Delta - V$ in $L^q(\Omega)$, $1 < q < \infty$, is analytic, order-preserving and satisfies an estimate as in (2.2), see [8].

If $p > N/2$ and $p \geq q$ then $\Delta - VI$ generates an (order-preserving) analytic semigroup in any $H_D^{2\alpha,q}(\Omega)$, with $\alpha \in [0, 1)$, and satisfies (2.1), for all $0 \leq \beta \leq \alpha \leq 1$ (see Lemma 2.3 in [4]). Moreover, the operator $-\Delta + V$ has the same domain that Laplacian. On the other hand, if $p \geq q'$ then (2.1) holds for $-1 \leq \beta \leq \alpha \leq 0$. Therefore, if $p > N/2$ and $p \geq q, q'$, then (2.1) holds for any $-1 \leq \beta \leq \alpha \leq 1$.

Notice that in (2.2) we can take any number μ such that

$$-\mu < \inf_{\varphi \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla \varphi|^2 + \int_\Omega V(x)\varphi^2}{\int_\Omega \varphi^2} = \inf_{\varphi \in H^1(\Omega)} \frac{\int_\Omega |\nabla \varphi|^2 + \int_\Omega V(x)\varphi^2}{\int_\Omega \varphi^2}. \quad (2.3)$$

In particular, we say that $-\Delta + V$ has exponential decay if we can take $\mu < 0$ above.

Now for nonlinear equations, suppose that there exists a decomposition of f as

$$f(x, s) = g(x) + m(x)s + f_0(x, s) \quad (2.4)$$

where $f_0: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz function in $s \in \mathbb{R}$ uniformly respect to $x \in \Omega$ and

$$f_0(x, 0) = 0, \quad \frac{\partial}{\partial s} f_0(x, 0) = 0. \quad (2.5)$$

Depending on the space where we pose the problem we will sometimes need to impose certain growth restrictions on f . More precisely, assume that f satisfies (2.4) and (2.5), where f_0 is a locally Lipschitz function in $s \in \mathbb{R}$, uniformly in $x \in \Omega$. Also assume that the following growth restriction holds:

$$|f_0(x, s) - f_0(x, r)| \leq c(1 + |s|^{\rho-1} + |r|^{\rho-1})|s - r| \quad (2.6)$$

for all $x \in \Omega$, $s, r \in \mathbb{R}$ and $\rho \geq 1$. In such a case, some restrictions on ρ are needed in order to obtain local existence for the nonlinear problem (1.1) as the following theorem shows (see [4] for a proof).

Theorem 2.1. Suppose that $m \in L_U^{p_0}(\Omega)$ for some $p_0 > N/2$, $p_0 \geq q$, and $g \in L^q(\Omega)$.

Then, problem (1.1) is well-posed in $H_D^{2\alpha,q}(\Omega)$, $0 \leq \alpha \leq 1$,

(1) if $2\alpha - \frac{N}{q} < 0$, provided (2.6) holds with

$$1 \leq \rho \leq \rho_C = 1 + \frac{2q}{N - 2\alpha q};$$

(2) if $2\alpha - \frac{N}{q} = 0$, provided (2.6) holds with

$$1 \leq \rho < \rho_C = \infty;$$

(3) if $2\alpha - \frac{N}{q} > 0$ no restrictions on the growth of f_0 are needed.

Remark 2.2.

- (i) Notice that ρ_C above depends on the space we pose the problem, and is known as the *critical exponent* for $H_D^{2\alpha,q}(\Omega)$.
- (ii) Notice that posing the problem in $X = H_D^{2\alpha,q}(\Omega)$ with $2\alpha - N/q > 0$ we will work with functions vanishing as $|x| \rightarrow \infty$ since $H_D^{2\alpha,q}(\Omega) \subset L^q(\Omega) \cap BUC_D(\Omega)$ (the subset of functions in $BUC(\Omega)$ that vanish on the boundary) and this is included in $BUC_{0,D}(\Omega)$, the space of bounded uniformly continuous functions tending to 0 as $|x| \rightarrow \infty$ which also vanish on $\partial\Omega$.

We recall some results about global existence and uniform bounds for solutions of (1.1) in [4]. For this, suppose that there exist suitable functions $C(x)$ and $D(x) \geq 0$ in Ω such that

$$f(x, s)s \leq C(x)s^2 + D(x)|s| \quad \text{for all } s \in \mathbb{R}, x \in \Omega. \quad (2.7)$$

Notice that $|g(x)| \leq D(x)$.

The following result about global existence of solutions holds (see Theorem 4.1 in [4]).

Theorem 2.3. Suppose f as in Theorem 2.1. Also assume that f satisfies (2.7) with

$$D \in L^r(\Omega) \cap L^s(\Omega) \quad \text{for some } r > N/2, q \geq s \geq \frac{qN}{N + 2q} \quad (2.8)$$

and

$$C \in L_U^p(\Omega) \quad \text{for some } p > N/2.$$

Then, the unique solution of (1.1) with initial data $u_0 \in H_D^{2\alpha,q}(\Omega)$ is globally defined and remains in a bounded set of $L^q(\Omega) \cap L^\infty(\Omega)$ for compact intervals bounded away from 0.

Moreover, given a bounded set of initial data in $H_D^{2\alpha,q}(\Omega)$, the solution at time $t > 0$ remains in a bounded set of $H_D^{2\beta,q}(\Omega) \cap L^\infty(\Omega)$ for any $\beta < 1$.

Under these assumptions, we can then define a nonlinear semigroup

$$S(t) : X \rightarrow X$$

by

$$S(t)u_0 = u(t, x; u_0).$$

If we assume in addition the exponential decay of the semigroup generated by $\Delta + C$ we have the following results giving an estimate on the asymptotic behaviour of solutions of (1.1) (see Theorem 5.1 in [4]).

Theorem 2.4. *Let f be as in Theorem 2.1. Suppose that f satisfies (2.7) with C and D as in Theorem 2.3. Assume in addition that*

the semigroup generated by $\Delta + C$ has exponential decay. (2.9)

Then, there exists ϕ , the unique solution of

$$\begin{cases} -\Delta\phi = C(x)\phi + D(x) & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.10)$$

Furthermore, $0 \leq \phi \in L^q(\Omega) \cap L^\infty(\Omega)$ and the solutions of (1.1) satisfy

$$\limsup_{t \rightarrow \infty} |u(t, x; u_0)| \leq \phi(x) \quad (2.11)$$

uniformly in $x \in \Omega$ and for u_0 in bounded set of initial data $X = H_D^{2\alpha, q}(\Omega)$. Also, if $|u_0(x)| \leq \phi(x)$ then $|u(t, x; u_0)| \leq \phi(x)$ for all times.

If we assume in addition that $p \geq r$, then $0 \leq \phi \in H_D^{2, r}(\Omega) \cap L^q(\Omega) \subset BUC_{0, D}(\Omega)$.

Remark 2.5.

- (i) Note that existence and uniqueness of ϕ for (2.10) follows from the exponential decay of the semigroup generated by $\Delta + C$ and the hypotheses on $D(x)$.
In particular, the assumption $0 \leq D \in L^s(\Omega)$ implies $0 \leq \phi \in L^q(\Omega)$ while the assumption $D \in L^r(\Omega)$ implies $\phi \in L^\infty(\Omega)$. If in addition, $p \geq s$ then $\phi \in H_D^{2, s}(\Omega)$ while if $p \geq r$ then $\phi \in H_D^{2, r}(\Omega) \cap L^q(\Omega) \subset BUC_{0, D}(\Omega)$, as in Remark 2.2, since $r > N/2$.
- (ii) The previous theorem gives, in particular, an $L^q(\Omega) \cap L^\infty(\Omega)$ bound of solutions of (1.1) for large times. Namely, given a bounded set B of $H_D^{2\alpha, q}(\Omega)$, $q \leq \sigma \leq \infty$ and $\varepsilon > 0$, there exists a time $T = T(B, \varepsilon) > 0$ such that for all $t \geq T$,

$$|u(t, x; u_0)| \leq \phi(x) + \psi(t, x)$$

with

$$\|\psi(t)\|_{L^\sigma(\Omega)} \leq \varepsilon,$$

see (3.3) below. In particular, we obtain the smallness of the tails of the solutions. That is, given $\varepsilon > 0$, there exists $R = R(B, \varepsilon) > 0$ and a time $T = T(B, \varepsilon) > 0$ such that for all $t \geq T$ and $u_0 \in B$,

$$\int_{\Omega \cap \{|x| \geq R\}} |u(t, x, u_0)|^\sigma dx \leq \varepsilon \quad (2.12)$$

for $q \leq \sigma < \infty$. Moreover, if $p \geq r$, then for all $t \geq T$ and $u_0 \in B$,

$$\sup_{\Omega \cap \{|x| \geq R\}} |u(t, x, u_0)| \leq \varepsilon. \quad (2.13)$$

Furthermore, from Theorem 5.5 in [4] there exists a global attractor for (1.1). The key of the proof is (2.12). From here, the semigroup can be shown to be asymptotically compact. First, asymptotic compactness in $L^q(\Omega)$ is obtained and then, by means of the variation of constants formula, asymptotic compactness in $H_D^{2\alpha, q}(\Omega)$ follows. More precisely,

Theorem 2.6. *Under the assumptions of Theorem 2.4 the nonlinear semigroup in $H_D^{2\alpha, q}(\Omega)$ has a compact global attractor $\mathcal{A} \subset H_D^{2\alpha, q}(\Omega) \cap L^\infty(\Omega)$. Moreover, for all $u \in \mathcal{A}$, $|u(x)| \leq \phi(x)$ for any $x \in \Omega$.*

Even more, if $g \in L^\sigma(\Omega)$ for some $p_0 \geq \sigma \geq q$, then, for any $\beta < 1$ this attractor is a bounded set in $H_D^{2, \sigma}(\Omega)$ and a compact set in $H_D^{2\beta, \sigma}(\Omega)$ and attracts bounded sets of $H_D^{2\alpha, q}(\Omega)$ in the norm of $H_D^{2\beta, \sigma}(\Omega)$. In particular, we can always take $\sigma = q$.

Remark 2.7. Notice that for all $u_0 \in H_D^{2\alpha,q}(\Omega)$, the solution $u(t; u_0)$ is relatively compact in $H_D^{2\beta,\sigma}(\Omega)$, for any $\beta < 1$. In particular, if $\sigma > N/2$, $u(t; u_0)$ is relatively compact with the uniform convergence in compact sets of Ω since $H_D^{2\beta,\sigma}(\Omega) \subset C_{\text{loc}}^\theta(\Omega)$ for β close to 1, and some $\theta = \theta(\beta) > 0$.

Even more, as mentioned in Remark 2.2, for β close to one, $H_D^{2\beta,\sigma}(\Omega) \subset BUC_{0,D}(\Omega)$. Therefore, if (2.13) is satisfied (for example if $p \geq r$) then $u(t; u_0)$ is relatively compact with the uniform convergence in Ω . More generally, a convergent sequence in $H_D^{2\beta,\sigma}(\Omega)$ with small tails in $L^\infty(\Omega)$, converges uniformly in Ω .

3. Extremal equilibria

We now prove the main theorem concerning existence and properties of extremal equilibria.

Theorem 3.1. Suppose f is as in Theorem 2.1 and problem (1.1) is posed in $X = H_D^{2\alpha,q}(\Omega)$. Also assume that Theorem 2.4 (and hence Theorems 2.3 and 2.6) holds.

Then,

- (i) if $p \geq \min\{q, r\}$ then there exist two ordered extremal equilibria $\varphi_m \leq \varphi_M$, $\varphi_m, \varphi_M \in H_D^{2\alpha,q}(\Omega)$, and

$$\varphi_m(x) \leq \liminf_{t \rightarrow \infty} u(t, x; u_0) \leq \limsup_{t \rightarrow \infty} u(t, x; u_0) \leq \varphi_M(x) \quad (3.1)$$

for $x \in \Omega$ and uniformly for bounded sets of initial data in $H_D^{2\alpha,q}(\Omega)$. The global attractor for (1.1) satisfies $\mathcal{A} \subset [\varphi_m, \varphi_M]$, $\varphi_m, \varphi_M \in \mathcal{A}$. Furthermore, φ_M is globally asymptotically stable in $H_D^{2\alpha,q}(\Omega)$ from above and φ_m is so from below.

- (ii) Assume in addition to (i) that $g \in L^\sigma(\Omega)$ for some $\sigma > N/2$, $p_0 \geq \sigma \geq q$ (in particular we can take $\sigma = q$ if $q > N/2$).

Then (3.1) holds uniformly in compact sets of Ω and for bounded sets of initial data in $H_D^{2\alpha,q}(\Omega)$. Furthermore, φ_M is globally asymptotically stable in $H_D^{2\alpha,q}(\Omega)$ from above and φ_m is so from below, with uniform convergence in compact sets of Ω .

- (iii) Finally, also assume $p \geq r$ then (3.1) holds uniformly in Ω and for bounded sets of initial data in $H_D^{2\alpha,q}(\Omega)$.

Furthermore, φ_M is globally asymptotically stable in $H_D^{2\alpha,q}(\Omega)$ from above and φ_m is so from below, with uniform convergence in Ω .

Remark 3.2. Notice that for bounded domains no additional conditions on p, q, r are needed to obtain the result (see [6]). The conclusions holds under the assumptions in Theorem 2.4.

Proof. (i) We start by building the candidate to maximal equilibrium. The argument for the minimal equilibrium is analogous. From (2.7), ϕ , the unique solution of (2.10), is formally a supersolution for (1.1) since

$$-\Delta\phi = C(x)\phi + D(x) \geq f(x, \phi).$$

Thus, $S(t)\phi \leq \phi$ is decreasing and we expect it to converge to φ_M , i.e., we have

$$\lim_{t \rightarrow \infty} S(t)\phi = \varphi_M \quad \text{in } H_D^{2\alpha,q}(\Omega). \quad (3.2)$$

Step 1. To make this argument precise, first assume $r \geq q$. Then, from (2.8), we have $D \in L^q(\Omega)$ and so, in (2.10), $\phi \in H_D^{2,q}(\Omega)$ from the assumption $p \geq \min\{q, r\} = q$. Thus, we can take ϕ as initial data for (1.1). Moreover, since ϕ is a supersolution of the nonlinear problem, $S(t)\phi$ is decreasing.

Furthermore, $S(t)\phi$ is also relatively compact in $H_D^{2\alpha,q}(\Omega)$ (see Theorem 2.6). So, the ω -limit set of $S(t)\phi$ exists. But, the pointwise monotonic convergence implies that the ω -limit set is just one point which we denote by φ_M . Then, (3.2) is proved and so φ_M is an equilibrium point.

We now show that φ_M is the maximal equilibrium. For this, we prove that the asymptotic dynamics enters below φ_M . More precisely, we prove the attraction given by (3.1).

For this, notice that, by (2.7), we have that for every $u_0 \in H_D^{2\alpha,q}(\Omega)$

$$|u(t, x; u_0)| \leq v(t, x; |u_0|) \quad (3.3)$$

for all $t > 0$ and almost every $x \in \Omega$, where v solves

$$\begin{cases} v_t - \Delta v = C(x)v + D(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ v(0) = |u_0| \end{cases}$$

that is $v(t) = \phi + S_{\Delta+C}(t)(|u_0| - \phi)$, see [4]. We also have that $v(t, x; |u_0|) \rightarrow \phi(x)$ in $L^q(\Omega)$ as $t \rightarrow \infty$, uniformly in Ω and for u_0 in bounded sets of $X = H_D^{2\alpha,q}(\Omega)$. Even more, $v(t, x; |u_0|) \rightarrow \phi$ in $H_D^{2\alpha,q}(\Omega)$ since $C \in L_U^p(\Omega)$, and $p \geq q$, and $D \in L^q(\Omega)$, see Lemma 2.6 in [4].

Let $s > 0$. Letting the nonlinear semigroup act at time s in (3.3), by monotonicity, we have

$$u(t + s, x; u_0) = S(s)u(t, x; u_0) \leq S(s)v(t, x; |u_0|). \quad (3.4)$$

Now, since $v(t, x; |u_0|) \rightarrow \phi(x)$ as $t \rightarrow \infty$ in $H_D^{2\alpha,q}(\Omega)$ we have, by the continuity of the nonlinear semigroup,

$$\lim_{t \rightarrow \infty} S(s)v(t, x; |u_0|) = S(s) \lim_{t \rightarrow \infty} v(t, x; |u_0|) = S(s)\phi(x). \quad (3.5)$$

Thus, taking limits as t to infinity in (3.4) we have, by (3.5)

$$\limsup_{t \rightarrow \infty} u(t + s, x; u_0) \leq S(s) \lim_{t \rightarrow \infty} v(t, x; |u_0|) = S(s)\phi(x) \quad (3.6)$$

for $x \in \Omega$ and for u_0 in bounded sets of X , i.e.,

$$\limsup_{t \rightarrow \infty} u(t, x; u_0) \leq S(s)\phi(x)$$

for all s . And taking limits as s to ∞ ,

$$\limsup_{t \rightarrow \infty} u(t, x; u_0) \leq \varphi_M(x) \quad (3.7)$$

for $x \in \Omega$ and for u_0 in bounded sets of $X = H_D^{2\alpha,q}(\Omega)$.

The result for the minimal equilibrium is analogous. As they are equilibria, $\varphi_m, \varphi_M \in \mathcal{A}$, the global attractor, and clearly, $\mathcal{A} \subset [\varphi_m, \varphi_M]$.

Step 2. Assume now $r < q$. Hence, in principle, we cannot guarantee $\phi \in H_D^{2\alpha,q}(\Omega)$ since we only know $p \geq \min\{q, r\} = r$ which gives $\phi \in H_D^{2,r}(\Omega)$. We now show that the nonlinear problem in $H_D^{2,r}(\Omega)$ is well-posed. For this, notice that $|g(x)| \leq D(x)$. Thus, in particular, $g \in L^r(\Omega)$. Then, since $p_0 \geq q > r$ and $2 - N/r > 0$, from Theorem 2.1, part (3), we have that $S(t)$ is locally well-posed in $H_D^{2\beta,r}(\Omega)$ with some $\beta > 0$ such that $2\beta - N/r > 0$.

Also, from Theorem 2.3 (with $q = r$ and $\alpha = \beta$, using $r > N/2$ and $r \geq s > Nq/(N + 2q) > Nr/(N + 2r)$) solutions in $H_D^{2\beta,r}(\Omega)$ are globally defined. Even more, from Theorem 2.6 in $H_D^{2\beta,r}(\Omega)$, there exists a global attractor, \mathcal{A} , of the nonlinear problem which is a bounded set of $H_D^{2,r}(\Omega)$ and a compact set of $H_D^{2\sigma,r}(\Omega)$, for every $\sigma < 1$. Even more, since $p_0 \geq q > r$ we have, from Theorem 2.6 in $H_D^{2\beta,r}(\Omega)$ with $\sigma = q$, that $S(t)\phi \in H_D^{2,q}(\Omega)$, for $t > 0$. Furthermore, the positive orbit $S(t)\phi$ is relatively compact in $H_D^{2\alpha,q}(\Omega)$, monotonically decreasing and so (3.2) is also proved in this case.

Now, (3.3) is valid for $u_0 \in H_D^{2\alpha,q}(\Omega)$ and we have $v(t; |u_0|) \rightarrow \phi$ in $H_D^{2\beta,r}(\Omega)$, uniformly in Ω and for bounded sets of initial data $u_0 \in H_D^{2\alpha,q}(\Omega)$ (see Lemma 2.6 in [4]). Then, arguments from (3.3) to (3.7) hold using the continuity of the nonlinear semigroup in $H_D^{2\beta,r}(\Omega)$.

Therefore, (3.7) is also proved in this case.

Step 3. For the asymptotic stability of φ_M notice that given any $\varphi_M \leq u_0 \in H_D^{2\alpha,q}(\Omega)$ we have

$$\varphi_M(x) = u(t, x; \varphi_M) \leq u(t, x; u_0), \quad \text{in } \Omega, \quad t \geq 0. \quad (3.8)$$

Also we have that $u(t, x; u_0)$ is relatively compact in $H_D^{2\alpha, q}(\Omega)$, see Theorem 2.6. Thus, the ω -limit set $\omega(u_0)$ exists. Now, from (3.8), $\omega(u_0) \geq \varphi_M$ and from (3.7), $\omega(u_0) \leq \varphi_M$. Thus, we must have $\omega(u_0) = \{\varphi_M\}$, that is, $u(t; u_0) \rightarrow \varphi_M$ in $H_D^{2\alpha, q}(\Omega)$ as $t \rightarrow \infty$. Even more, the convergence is valid also in $H_D^{2\beta, \sigma}(\Omega)$ with β and σ as in Theorem 2.6.

(ii) From (i) we have (3.2). Now, from the assumptions and Theorem 2.6 we also have $S(t)\phi \rightarrow \varphi_M$ in $H_D^{2\beta, \sigma}(\Omega)$. Thus, arguments from (3.3) to (3.8) can be carried out with convergence in $H_D^{2\beta, \sigma}(\Omega)$ and hence, uniform convergence in compact sets of Ω since $\sigma > N/2$; see Remark 2.7.

(iii) Finally, since we also have $p \geq r$ then we have (2.13) and this with the uniform convergence in compact sets of Ω obtained in (ii) gives the result; see Remark 2.7. \square

In what follows we will prove Theorem 3.1 under several alternative conditions on the nonlinear term.

Assume first that f satisfies growth condition (1) or (2) in Theorem 2.1. Then taking advantage of this extra structure on the nonlinear term, we will obtain a better result than in Theorem 3.1 since we can weaken the regularity assumptions on $D(x)$ in Theorem 2.3.

Theorem 3.3. *Let $X = H_D^{2\alpha, q}(\Omega)$, $1 < q < \infty$. Suppose f is as in Theorem 2.1 and satisfies (2.6) and*

$$g \in L^a(\Omega) \cap L^b(\Omega)$$

with $a = \max\{N(\rho - 1)/2, 1\}$, $b = \max\{N\rho/2, 1\}$.

Also assume that f satisfies (2.7) with $C \in L_U^p(\Omega)$ for some $p > N/2$ such that the semigroup generated by $\Delta + C$ has exponential decay and

$$D \in L^r(\Omega) \cap L^s(\Omega) \quad \text{with } r > \frac{N}{2} \left(1 - \frac{1}{\rho}\right), \quad q \geq s > \frac{qN}{N + 2q}.$$

Then, the conclusions of Theorem 3.1 hold true. Even more, the uniform convergence in Ω holds with only the assumptions in point (ii).

Proof. First, notice that, from Theorems 5.2 and 5.5 in [4], results in Theorem 2.6 apply. Moreover we also have (2.11)–(2.13).

(i) Assume first $p \geq \min\{q, r\}$ and let ϕ be as in (2.10). In this case we cannot guarantee that ϕ belongs to $L^\infty(\Omega)$ since $D(x)$ might not be in $L^r(\Omega)$ with $r > N/2$.

First assume, $r \geq q$ then $D \in L^q(\Omega)$. Thus, $\phi \in H_D^{2\alpha, q}(\Omega)$, since $p \geq \min\{q, r\} = q$, and $S(t)\phi \rightarrow \varphi_M$ in $H_D^{2\alpha, q}(\Omega)$ as in the proof of Theorem 3.1.

Assume now that $r < q$. Then, $\phi \in L^q(\Omega) \cap H_D^{2, r}(\Omega)$ since $p \geq \min\{q, r\} = r$, and the nonlinear problem is well-posed in $H_D^{2\beta, r}(\Omega)$, for some $0 < \beta < 1$, since the hypothesis on r implies that $1 < \rho < \rho_C(H_D^{2\beta, r}) = 1 + \frac{2r}{N-2\beta r}$, see Theorem 2.1. Now, as in the proof of Theorem 3.1 we have that solution starting at ϕ enters in $H_D^{2\alpha, q}(\Omega)$ for $t > 0$.

As Theorem 2.6 applies, there exists a global attractor $\mathcal{A} \subset H_D^{2\beta, r}(\Omega)$. Now, since $S(t)\phi$ is decreasing and $S(t)$ is asymptotically compact in $H_D^{2\beta, r}(\Omega)$ we have $S(t)\phi \rightarrow \varphi_M$ in $H_D^{2\beta, r}(\Omega)$ as $t \rightarrow \infty$. Since $p_0 \geq q$, again we have that $S(t)\phi$ enters in $H_D^{2\alpha, q}(\Omega)$ and the convergence in (3.2) holds in $H_D^{2\alpha, q}(\Omega)$. The result now follows as in the proof of Theorem 3.1(i).

(ii) Since in addition to (i) we have that $g \in L^\sigma(\Omega)$ for some $\sigma > N/2$, $p_0 \geq \sigma \geq q$ (in particular we can take $\sigma = q$ if $q > N/2$), then the uniform convergence in compact sets in (3.2) follows from Theorem 2.6; see Remark 2.7.

Finally, since (2.13) holds, we get the uniform convergence in Ω . \square

Now we give some other structure conditions on $f(x, u)$ that allows to obtain alternative asymptotic L^∞ bounds on the solutions without requiring conditions (2.7), (2.9).

Suppose now that f is as in Theorem 2.1. Assume that assumption in Theorem 2.3 holds. Also assume that $m, g \in L^\infty(\Omega)$ and

$$f(x, s)s \leq h(s)|s| \quad \text{for all } x \in \Omega, \quad |s| \geq M \tag{3.9}$$

where h is a continuous function such that $h(s) < 0$ for all $|s| \geq M$. Note that this condition will replace assumption (2.9) in Theorems 3.1 and 3.3.

Note then that from Theorem 5.3 in [4] we get the existence of a bounded absorbing set in $L^\infty(\Omega)$. More precisely, we have

Theorem 3.4. Suppose f is as in Theorem 2.1 and let $X = H_D^{2\alpha,q}(\Omega)$. Suppose that solutions of problem (1.1) are globally defined. Also assume that $m, g \in L^\infty(\Omega)$ and f satisfies (3.9).

Then, for any bounded set $B \subset H_D^{2\alpha,q}(\Omega)$ there exists $T(B) > 0$ such that

$$\|u(t; u_0)\|_{L^\infty(\Omega)} \leq M, \quad t \geq T(B),$$

for all $u_0 \in B$, with M as in (3.9).

Suppose now that $C(x)$ admits a decomposition of the form $C(x) = C_0(x) - C_1(x)$ such that $C_0 \geq 0$, $C_1 \in L_U^p(\Omega)$ and the semigroup generated by $\Delta - C_1$ has exponential decay. Also assume that $C_0 \in L^r(\Omega) \cap L^s(\Omega)$ with r and s as in Theorem 3.1 or Theorem 3.3. Then, Theorem 5.4 in [4] gives an estimate on the asymptotic behaviour of the form (2.11) for solutions in terms of $\phi(x) \geq 0$ where ϕ is now the unique solution of

$$\begin{cases} -\Delta\phi + C_1(x)\phi = C_0(x)M + D(x) & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega \end{cases} \quad (3.10)$$

with M as in (3.9). Namely, we have,

$$\limsup_{t \rightarrow \infty} |u(t, x; u_0)| \leq \phi(x) \quad (3.11)$$

uniformly in $x \in \Omega$ and in bounded sets of $u_0 \in H_D^{2\alpha,q}(\Omega)$. Also, (2.12) and (2.13) hold true. From here, the nonlinear semigroup is asymptotically compact and the existence of a global attractor follows (see [4]). Furthermore, if in addition $p \geq r$ then $\phi \in L^q(\Omega) \cap BUC_D(\Omega)$. Hence $\phi(x) \rightarrow 0$ for $|x| \rightarrow \infty$.

Now, we have

Theorem 3.5. Suppose that f is as in Theorem 2.3. Also assume that $m, g \in L^\infty(\Omega)$ and

$$f(x, s)s \leq h(s)|s| \quad \text{for all } x \in \Omega, |s| \geq M$$

where $h(s)$ is a continuous function such that $h(s) < 0$ for all $|s| \geq M$.

Finally, assume that $C(x)$ admits a decomposition of the form $C(x) = C_0(x) - C_1(x)$ with $0 \leq C_0 \in L^r(\Omega) \cap L^s(\Omega)$ and $C_1 \in L_V^p(\Omega)$ with r, s as in Theorem 3.1 or Theorem 3.3 respectively, and the semigroup generated by $\Delta - C_1$ has exponential decay.

Then, the results in Theorem 3.1 or Theorem 3.3, respectively, hold.

Proof. Note again that, from Theorems 5.4 and 5.5 in [4], results in Theorem 2.6 apply. Moreover, in case of the assumptions of Theorem 3.3 or Theorem 3.1, with $p \geq r$, respectively, we also have (2.13).

Note that thanks to Theorem 3.4 we can truncate the nonlinear term in (1.1) and we can take initial data in $L^q(\Omega) \cap L^\infty(\Omega)$.

Also, note that now $\phi(x)$ is not a supersolution of (1.1), but at a point such that $\phi(x) \leq M$ then

$$f(x, \phi(x)) \leq (C_0(x) - C_1(x))\phi(x) + D(x) \leq C_0(x)M - C_1(x)\phi(x) + D(x).$$

Therefore, we set $\hat{\phi}(x) = \min\{\phi(x), M\}$ which is a supersolution for (1.1). Then, $u(t, x; \hat{\phi})$ is nonincreasing and relatively compact, as in Theorem 2.6. Therefore this solution converges, to a limit that we denote $\varphi_M(x)$, in $H_D^{2\alpha,q}(\Omega)$, uniformly in compact sets of Ω or uniform in Ω , according to the cases in Theorem 3.1 or Theorem 3.3 respectively.

Let now $u_0 \in H_D^{2\alpha,q}(\Omega)$. Then, since problem (1.1) is asymptotically compact, the ω -limit of u_0 , $\omega(u_0)$, exists and it is a nonempty, compact, invariant set. Thus, on the one hand, from Theorem 3.4, $\omega(u_0)$ is below M . On the other hand, from (3.11), any function in $\omega(u_0)$ is below ϕ . As a consequence,

$$\omega(u_0) \leq \hat{\phi} = \min\{\phi, M\}. \quad (3.12)$$

Letting the semigroup act in both sides of the inequality and using the comparison principle and the invariance property of the ω -limits we have that $\omega(u_0) \leq \varphi_M$ for all $u_0 \in H_D^{2\alpha,q}(\Omega)$. In particular, φ_M is the maximum equilibrium for (1.1).

Note now that following the cases in Theorem 3.1 or Theorem 3.3 we can show that the solution starting at ϕ , $u(t; \phi)$, converges to φ_M since $\phi \geq \hat{\phi} \geq \varphi_M$ and $\omega(\phi) \leq \varphi_M$. Note that this is the equivalent to (3.2) in Theorem 3.1 although we cannot ensure that $u(t; \phi)$ is decreasing.

On the other hand, note that given a bounded set, B , of initial data in $H_D^{2\alpha,q}(\Omega)$, after Theorem 3.4, we can assume B is also bounded in the sup norm by M . Hence, (3.3) is satisfied, where now v stands for the solution the linear parabolic problem associated to (3.10).

Therefore, we can now follow the arguments in the proof of Theorem 3.1 or Theorem 3.3, respectively and the results of the theorem follow. \square

We now consider the case of quasi-monotone nonlinear terms, that is, we assume that f is of the form (2.4) and (2.5) and f_0 satisfies

$$\frac{\partial}{\partial s} f_0(x, s) \leq L(x) \quad \text{for all } x \in \Omega, s \in \mathbb{R}, \quad (3.13)$$

with $L \in L_U^{p_0}(\Omega)$, $p_0 > N/2$. This implies, in particular, that $-f_0(x, s) + L(x)s$ is a monotone function in s and

$$f_0(x, s)s \leq L(x)s^2.$$

Thus,

$$f(x, s)s \leq (m(x) + L(x))s^2 + |g(x)||s|,$$

or in other words, f satisfies (2.7) with $C(x) = m(x) + L(x)$ and $D(x) = |g(x)|$. Hence, $C \in L_U^p(\Omega)$ with $p = p_0 > N/2$ and then if we assume that $g \in L^{r_0}(\Omega) \cap L^q(\Omega)$ for some $r_0 > N/2$ then Theorem 2.3 is satisfied, with $p = p_0$, $r = r_0$, $s = q$, and solutions are global.

Furthermore, if $p_0 \geq r_0$, then from Theorem 7.1 in [4], problem (1.1) is well-posed in $X = L^q(\Omega)$ and solutions are globally defined. Observe that results in [4] are obtained assuming that L is a constant. But, the same arguments allow to obtain the results for $L \in L_U^{p_0}(\Omega)$, with $p_0 > N/2$.

Suppose now that f satisfies (2.7) with $C(x)$ and $D(x)$ as in Theorem 2.4 or Theorem 3.3. Then, from Theorem 7.2 in [4] we have (2.11), with $\phi(x)$ as in (2.10). Also, Theorem 2.6 applies with $\sigma = r_0 > N/2$.

These and the arguments above lead to

Theorem 3.6. Assume that f satisfies (2.4), f_0 satisfies (3.13) and $m \in L_U^{p_0}(\Omega)$, with $p_0 > N/2$, $g \in L^{r_0}(\Omega) \cap L^q(\Omega)$, with $p_0 \geq r_0 > N/2$. Moreover assume f satisfies (2.7), with $C \in L_U^p(\Omega)$ for some $p > N/2$. Also, assume

$$D \in L^r(\Omega) \cap L^s(\Omega) \quad \text{for some } q \geq s \geq \frac{qN}{N+2q}.$$

Assume either

- (i) $r > N/2$, or
- (ii) f satisfies (2.6) and $r > \frac{N}{2}(1 - \frac{1}{p})$.

Then there exist two ordered extremal equilibria $\varphi_m \leq \varphi_M$, $\varphi_m, \varphi_M \in L^q(\Omega)$, and

$$\varphi_m(x) \leq \liminf_{t \rightarrow \infty} u(t, x; u_0) \leq \limsup_{t \rightarrow \infty} u(t, x; u_0) \leq \varphi_M(x)$$

uniformly in compact sets of Ω and for bounded sets of initial data in $L^q(\Omega)$. The global attractor for (1.1) satisfies $\mathcal{A} \subset [\varphi_m, \varphi_M]$, $\varphi_m, \varphi_M \in \mathcal{A}$. Furthermore, φ_M is globally asymptotically stable in $L^q(\Omega)$ from above and φ_m is so from below.

Even more the \limsup and \liminf above are uniform in $x \in \Omega$ in cases (i) and (ii), provided $p \geq r$ in the former.

The same result holds if f is as in Theorem 3.5.

Proof. The proof follows as in the proofs of Theorem 3.1 or Theorem 3.3, with $\alpha = 0$. Since from Theorems 7.2 and 5.5 in [4], the nonlinear semigroup is asymptotically compact in $L^q(\Omega)$, and $\phi \in L^q(\Omega)$, we have that convergence (3.2) holds in $L^q(\Omega)$. Moreover, again from Theorem 5.5 in [4], Theorem 2.6 applies and then convergence (3.2) holds in $H_D^{2\alpha,\sigma}(\Omega)$ for all $\alpha < 1$ and $\sigma = r_0$. Hence, uniform convergence in compact sets of Ω in (3.2) follows; see Remark 2.7. The rest of the proof follows as in Theorem 3.1 or Theorem 3.3.

The uniform convergence in Ω follows in case (i) if $p \geq r$ since in such a case we have (2.13). In case (ii) this follows as in Theorem 3.3.

The arguments above can be slightly modified in case f is as in Theorem 3.5. \square

Now we consider the problem (1.1) in the space of bounded and uniformly continuous functions, $BUC(\Omega)$. For simplicity we assume here that $\Omega = \mathbb{R}^N$, so the boundary is empty.

Suppose then that f is of the form (2.4) with

$$m \in BUC^\mu(\mathbb{R}^N), \quad 0 < \mu \leq 1, \quad \text{and} \quad g \in BUC(\mathbb{R}^N).$$

Then, problem (1.1) is well-posed in $BUC(\mathbb{R}^N)$ (see [4,5]).

Suppose in addition that f also satisfies (2.7) with $C \in L_U^p(\mathbb{R}^N)$ for some $p > N/2$ and either

- (i) $D \in L^r(\mathbb{R}^N)$ for some $r > N/2$, or
- (ii) f satisfies (2.6) and $D \in L^r(\mathbb{R}^N)$, $r > \frac{N}{2}(1 - \frac{1}{\rho})$, and $g \in L^a(\mathbb{R}^N)$ with $a = \max\{N(\rho - 1)/2, 1\}$.

Then, the solution of (1.1) starting at $BUC(\mathbb{R}^N)$ is globally defined, see Theorem 7.3 in [4].

If in addition $C \in L_U^p(\mathbb{R}^N)$, $p \geq r$, and the semigroup generated by $\Delta + C$ has exponential decay, and $m \in L_U^{p_0}(\mathbb{R}^N)$ for $p_0 > N/2$ then from Theorem 7.4 in [4] we have (2.10) with $0 \leq \phi \in BUC_0(\mathbb{R}^N)$ as in (2.11) or (3.10), depending on the cases above. Note that similar results can be proved provided (3.9) is satisfied and that functions in $BUC_0(\mathbb{R}^N)$ tend to zero at infinity.

From here, the order interval $[-\phi - \delta, \phi + \delta]$, with $\delta > 0$, is an absorbing interval in $BUC(\mathbb{R}^N)$. Hence, assuming $p \geq r$ in case (i) and using in an essential way that $\phi(x) \rightarrow 0$ as $|x| \rightarrow \infty$ in [4] the authors prove that the nonlinear semigroup is asymptotically compact in $BUC(\mathbb{R}^N)$. Thus, the existence of a global attractor follows, see Theorem 7.4 in [4].

Then we have the following result.

Theorem 3.7. *Under the hypotheses above, results in Theorem 1.1 hold in $X = BUC(\mathbb{R}^N)$.*

Proof. The result follows from the abstract Theorem 3.2 in [6] since $\phi + \delta \in BUC(\mathbb{R}^N)$, the order interval $[-\phi - \delta, \phi + \delta]$ is absorbing and the nonlinear semigroup is asymptotically compact. Notice that in that case, (1.2) holds with uniform convergence in \mathbb{R}^N . \square

Notice that now the proof is much easier than the case in which $X = H^{2\alpha,q}(\mathbb{R}^N)$ since constant functions belong to the base space $BUC(\mathbb{R}^N)$.

4. Minimal positive equilibria

Assume that f is as in Theorems 2.1, 2.3, 2.4 and 2.6. Also assume that $f(x, 0) \geq 0$ so that problem (1.1) preserves positivity. We deal now with the existence of minimal positive equilibria. In the simplest case, when $g(x) = f(x, 0) \geq 0$ is not identically zero, we have that 0 is a subsolution of (1.1). Then, the existence of a minimal equilibrium follows easily. In this case, $u(t, x; 0)$ is increasing, and relatively compact in $H_D^{2\alpha,q}(\Omega)$. Thus, the limit $\varphi_m = \lim_{t \rightarrow \infty} u(t; 0)$ exists in $H_D^{2\alpha,q}(\Omega)$. Then we conclude that φ_m is the minimal equilibrium. If, in addition, $g \in L^\sigma(\Omega)$ for some $\sigma > N/2$, with $p_0 \geq \sigma \geq q$, then the convergence above is uniform in Ω , see Remark 2.7.

A more interesting case is that in which 0 is a equilibrium. In such a case, since $g(x) = 0$, we can take $\sigma > N/2$ in Theorem 2.6 to obtain that the attractor \mathcal{A} satisfies $\mathcal{A} \subset H_D^{2\alpha,q}(\Omega) \cap BUC_{0,D}(\Omega)$. In particular, any equilibria belongs to $H_D^{2\alpha,q}(\Omega) \cap BUC_{0,D}(\Omega)$. Then, we have the following results. Notice that in the assumptions in the theorem below

we do not assume that Theorem 3.1 holds. In particular, we do not assume the existence of a maximal equilibrium. For this, restrictions on p, q and r are needed. Note however that when Theorem 3.1 holds, the maximal equilibria is positive since $\phi > 0$.

Theorem 4.1. *We consider problem (1.1) posed in $X = H_D^{2\alpha,q}(\Omega)$ for some $0 \leq \alpha < 1$. Suppose f as in Theorems 2.1, 2.3, 2.4 and 2.6. Assume that $g(x) = f(x, 0) = 0$. Also assume that there exists a positive equilibrium of (1.1), $0 < \varphi \in H_D^{2\alpha,q}(\Omega) \cap BUC_{0,D}(\Omega)$. Suppose that there exists $M \in L_U^p(\Omega)$, $p > N/2$, such that*

$$f(x, s) \geq M(x)s, \quad 0 \leq s \leq s_0, \quad (4.1)$$

for some $s_0 > 0$, and 0 is unstable for the linear problem

$$\begin{cases} v_t - \Delta v = M(x)v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ v(0) = u_0, \end{cases}$$

i.e., $\sigma(-\Delta - M) \cap \mathbb{R}^- \neq \emptyset$ where by $\sigma(-\Delta - M)$ we denote the spectrum of $-\Delta - M$.

Then, there exists a minimal positive equilibria, $0 < \varphi_m \in H_D^{2\alpha,q}(\Omega) \cap BUC_{0,D}(\Omega)$. Moreover, for all $0 \leq u_0 \in H_D^{2\alpha,q}(\Omega)$ not identically zero,

$$\liminf_{t \rightarrow \infty} u(t, x; u_0) \geq \varphi_m(x)$$

uniformly in Ω . In particular, φ_m is globally asymptotically stable from below for positive solutions, i.e., for all $u_0 \in H_D^{2\alpha,q}(\Omega)$, $0 \leq u_0 \leq \varphi_m$, $u_0 \not\equiv 0$ we have $\lim_{t \rightarrow \infty} u(t, x; u_0) = \varphi_m(x)$ in $H_D^{2\alpha,q}(\Omega)$ and uniformly in Ω .

Proof. From the hypotheses, if for all $R > 0$, λ_1^R denotes the first eigenvalue of $-\Delta - M$ in $\Omega_R = \Omega \cap B_R$ with Dirichlet boundary conditions, then for R large enough we have $\lambda_1^R < 0$.

Notice that $\varphi|_{\Omega_R}$ is a supersolution for the Dirichlet problem in Ω_R :

$$\begin{cases} u_t^R - \Delta u^R = f(x, u^R) & \text{in } \Omega_R, \\ u^R = 0 & \text{on } \partial\Omega_R, \\ u^R(0) = v_0. \end{cases} \quad (4.2)$$

Then the solution of (4.2) starting at $u^R(0) = \varphi|_{\Omega_R}$ is globally bounded since $0 \leq u^R(t, x; \varphi|_{\Omega_R}) \leq \varphi|_{\Omega_R}$. Thus, from Theorem 4.2 in [6] we have that the minimal positive equilibrium, φ_m^R , for (4.2) exists and satisfies $0 \leq \varphi_m^R \leq \varphi$ in Ω_R .

Moreover, φ_m^R is asymptotically stable from below for (4.2), i.e., for all nonzero $v_0 \in C_0(\overline{\Omega_R})$, $0 \leq v_0 \leq \varphi_m^R$ in Ω_R , we have $u^R(t, x; v_0) \rightarrow \varphi_m^R(x)$ uniformly in $x \in \Omega_R$ as $t \rightarrow \infty$.

Even more, for all $0 \leq v_0 \in C(\overline{\Omega_R})$, not identically zero,

$$\liminf_{t \rightarrow \infty} u^R(t, x; v_0) \geq \varphi_m^R(x), \quad \text{uniformly for } x \in \Omega_R. \quad (4.3)$$

Now, we want to solve (1.1) with initial data φ_m^R . For this, notice that since $\varphi \in L^\infty(\Omega)$ we can truncate f_0 in such a way that the truncated problem is well-posed in $L^q(\Omega)$ and solutions enter, for $t > 0$, in $H_D^{2\beta,q}(\Omega)$ for all $\beta < 1$ (in particular, we can take $\beta = \alpha$). Moreover, solutions of the truncated problem coincide with those of the original one as long as they remain below φ .

Also, the extension by zero to Ω of φ_m^R , that we denote the same, belongs to $L^q(\Omega) \cap C_0(\Omega)$ and is a subsolution for the elliptic problem associated to (1.1). Indeed, for any $0 \leq \eta \in \mathcal{D}(\Omega)$,

$$\int_{\Omega} \nabla \varphi_m^R \nabla \eta = \int_{\Omega_R} \nabla \varphi_m^R \nabla \eta = \int_{\Omega_R} -\eta \Delta \varphi_m^R + \int_{\partial\Omega_R} \eta \frac{\partial \varphi_m^R}{\partial n} \leq \int_{\Omega_R} f(x, \varphi_m^R) \eta = \int_{\Omega} f(x, \varphi_m^R) \eta,$$

where we have used that $\frac{\partial \varphi_m^R}{\partial n} \leq 0$ on $\partial\Omega_R$, $\eta \geq 0$ on $\partial\Omega_R$, $f(x, 0) = 0$ and $\varphi_m^R = 0$ out of Ω_R . From here, $u(t, x; \varphi_m^R)$ is monotonically increasing.

Thus, in particular,

$$0 \leq u(t, x; \varphi_m^R(x)) \leq u(t, x; \varphi(x)) = \varphi(x) \quad \text{for } x \in \Omega \text{ for all } t \geq 0.$$

Now, the nonlinear semigroup is relatively compact in $H_D^{2\alpha,q}(\Omega)$. So, the monotonic limit

$$\varphi_m(x) := \lim_{t \rightarrow \infty} u(t, x; \varphi_m^R) \leq \varphi(x), \quad x \in \Omega, \quad (4.4)$$

exists for $x \in \Omega$ and in $H_D^{2\alpha,q}(\Omega)$.

Now, since $g \equiv 0$ and $p_0 > N/2$ we have, from Theorem 2.6 and Remark 2.7, that the limit in (4.4) is uniform in compact sets of Ω . In addition, since $\varphi \in BUC_{0,D}(\Omega)$ then given $\varepsilon > 0$, out of a large enough ball we have $0 \leq u(t, x; \varphi_m^R) \leq \varphi(x) < \varepsilon$ for all $t > 0$. This plus the uniform convergence in compact sets allow us to conclude that the convergence in (4.4) is uniform for $x \in \Omega$.

We now show that φ_m is the minimal positive equilibrium. For this, given $u_0 \in C(\overline{\Omega}) \cap H_D^{2\alpha,q}(\Omega)$, $0 \leq u_0$, we set $v_0 = u_0|_{\Omega_R}$. Then, we have

$$0 \leq u^R(t, x; v_0) \leq u(t, x; u_0), \quad x \in \Omega_R, \quad (4.5)$$

and extending by zero u^R to Ω , (4.5) holds in Ω . By (4.3), taking limits as t goes to infinity, we have

$$\varphi_m^R(x) \leq \liminf_{t \rightarrow \infty} u^R(t, x; u_0|_{\Omega_R}) \leq \liminf_{t \rightarrow \infty} u(t, x; u_0), \quad x \in \Omega_R. \quad (4.6)$$

Let ψ be any equilibrium for (1.1). Then $\psi \in C(\overline{\Omega}) \cap H_D^{2\alpha,q}(\Omega)$ and from (4.6) with $u_0 = \psi$ we have

$$\varphi_m^R \leq \psi \quad \text{in } \Omega_R$$

and extending φ_m^R by zero to Ω the inequality holds in Ω . Letting act the nonlinear semigroup on both sides and taking limits as $t \rightarrow \infty$, by (4.4), we have

$$\varphi_m \leq \psi \quad \text{in } \Omega.$$

Thus, φ_m is a minimal equilibria for (1.1).

For the asymptotic stability, take first $u_0 \in C_0(\Omega) \cap H_D^{2\alpha,q}(\Omega)$ nonidentically zero and $0 \leq u_0 \leq \varphi_m$. Notice that we can assume that $u_0|_{\Omega_R}$ is positive (otherwise, it is enough to let evolve the solution $S(t)u_0$ at small time and take this as initial data). Consider the restriction to $u_0|_{\Omega_R}$ extended by zero to Ω . Now, notice that from (4.5) we have $u^R(t; u_0|_{\Omega_R}) \leq \varphi_m$ for all t and $u^R(t; u_0|_{\Omega_R}) \rightarrow \varphi_m^R$ as $t \rightarrow \infty$.

Then, using the continuity of the truncated problem in $L^q(\Omega)$, for $s > 0$,

$$\lim_{t \rightarrow \infty} S(s)u^R(t; u_0|_{\Omega_R}) = S(s) \lim_{t \rightarrow \infty} u^R(t; u_0|_{\Omega_R}) = S(s)\varphi_m^R \quad \text{in } \Omega.$$

Additionally, (4.5) implies

$$u(t+s, x; u_0) = S(s)u(t, x; u_0) \geq S(s)u^R(t, x; u_0|_{\Omega_R}) \quad \text{in } \Omega.$$

Now, taking limit as $t \rightarrow \infty$ we have

$$\liminf_{t \rightarrow \infty} u(t, x; u_0) \geq (S(s)\varphi_m^R)(x) = u(s, x; \varphi_m^R), \quad x \in \Omega.$$

Then, taking limit as $s \rightarrow \infty$ we have, by (4.4),

$$\liminf_{t \rightarrow \infty} u(t, x; u_0) \geq \varphi_m(x), \quad x \in \Omega. \quad (4.7)$$

Now, notice that given any $u_0 \in H_D^{2\alpha,q}(\Omega)$, $0 \leq u_0 \leq \varphi_m$, we have, $0 \leq u(t, x; u_0) \leq \varphi_m(x)$ and $u(t, x; u_0)$ is relatively compact in $H_D^{2\alpha,q}(\Omega)$ and in $H_D^{2\beta,\sigma}(\Omega)$ for some $\sigma > N/2$, see Theorem 2.6. Thus, the ω -limit set $\omega(u_0)$ exists and satisfies $\omega(u_0) \leq \varphi_m$. But from (4.7), $\omega(u_0) \geq \varphi_m$ and therefore $\omega(u_0) = \{\varphi_m\}$, that is, $u(t, x; u_0) \rightarrow \varphi_m(x)$ in $H_D^{2\alpha,q}(\Omega)$ as $t \rightarrow \infty$ and uniformly in compact sets of Ω . Using now that $\varphi(x) \rightarrow 0$ as $|x| \rightarrow \infty$ we obtain the uniform convergence in Ω . \square

A close look at the proof above shows that the assumption on the existence of the equilibrium $\varphi(x)$ is used several times to have a priori bounds on solutions and to control the tails of the solutions at infinity. More precisely, this is used right after (4.2), right after (4.3), right before and after (4.4) and finally in the last step of the proof above. Therefore it is not difficult to show that the existence of $\varphi(x)$ can be replaced by a time-dependent solution, suitable decaying at infinity, which leads to the following

Corollary 4.2. *Theorem 4.1 holds true provided there exists a global solution of (1.1), $0 < \varphi(t) \in H_D^{2\alpha,q}(\Omega) \cap BUC_{0,D}(\Omega)$ such that*

$$\lim_{|x| \rightarrow \infty} \varphi(t, x) = 0 \quad \text{uniformly in } t > 0.$$

Note that such solution, as well as $\varphi(x)$ in Theorem 4.1, exists provided Theorem 2.6 holds, since we can always take $\sigma > N/2$.

Concerning the behaviour of the minimal solutions constructed above, we have

Proposition 4.3. *Under the hypotheses of Theorem 4.1, we have*

$$\lim_{R \rightarrow \infty} \varphi_m^R(x) = \varphi_m(x) \quad \text{in } L^\sigma(\Omega), \quad W_{\text{loc}}^{2-\varepsilon,\sigma}(\Omega), \quad \text{and uniformly in } \Omega, \quad (4.8)$$

for all $q \leq \sigma < \infty$, for every $\varepsilon > 0$.

Proof. As we proved before, for all $R > 0$ large enough,

$$0 \leq \varphi_m^R \leq \varphi_m. \quad (4.9)$$

Furthermore, given $R_1 < R_2$, we have that $\varphi_m^{R_1}$ and $\varphi_m^{R_2}$ satisfy the same equation in Ω_{R_1} . Moreover, $\varphi_m^{R_2} > 0 = \varphi_m^{R_1}$ in $\partial\Omega_{R_1}$. Thus, $\varphi_m^{R_1} \leq \varphi_m^{R_2}$. So φ_m^R is increasing as $R \rightarrow \infty$.

Then, in particular, there exists the pointwise limit

$$\lim_{R \rightarrow \infty} \varphi_m^R(x) = \xi(x) \leq \varphi_m(x), \quad x \in \Omega. \quad (4.10)$$

Now, since $\varphi_m \in L^q(\Omega) \cap L^\infty(\Omega)$, by (4.9), we have $\xi \in L^q(\Omega) \cap L^\infty(\Omega)$ and, from the Dominated Convergence Theorem, $\varphi_m^R \rightarrow \xi$ in $L^\sigma(\Omega)$ as $R \rightarrow \infty$, for all $q \leq \sigma < \infty$.

Let $\eta \in C_c^\infty(\Omega)$ a function with compact support in Ω . Let $L > 0$ such that $\text{supp}(\eta) \subset B_L$. Then

$$\int_{B_L} -\eta \Delta \varphi_m^R = \int_{B_L} f(x, \varphi_m^R) \eta.$$

Integrating by parts, we get

$$\int_{\text{supp}(\eta)} -\varphi_m^R \Delta \eta = \int_{\text{supp}(\eta)} f(x, \varphi_m^R) \eta.$$

For the left-hand side in the equation we can apply the Dominated Convergence Theorem and we have

$$\lim_{R \rightarrow \infty} \int_{\text{supp}(\eta)} -\varphi_m^R \Delta \eta = \int_{\text{supp}(\eta)} -\xi \Delta \eta.$$

For the right-hand side, notice that, on the one hand, using that $0 \leq \varphi_m^R \leq \varphi_m \in L^\infty(\Omega)$ and f_0 is locally Lipschitz, we have

$$|f_0(x, \varphi_m^R)| \leq L_{f_0} |\varphi_m(x)| \quad (4.11)$$

where by L_{f_0} we denote the Lipschitz constant for f_0 in a ball of radius $\|\varphi_m\|_{L^\infty(\Omega)}$, the last term not depending on R . On the other hand,

$$|m(x) \varphi_m^R(x)| \leq |m(x)| |\varphi_m(x)| \in L_U^p(\Omega) \subset L_{\text{loc}}^p(\Omega). \quad (4.12)$$

So, we can pass to the limit by the Dominated Convergence Theorem and we have

$$\lim_{R \rightarrow \infty} \int_{\text{supp}(\eta)} f_0(x, \varphi_m^R) \eta = \int_{\text{supp}(\eta)} f_0(x, \xi) \eta.$$

Thus,

$$-\Delta \xi - m(x)\xi = f_0(x, \xi), \quad x \in \Omega,$$

in the sense of distributions. Furthermore, from (4.11) we have $f_0(\cdot, \xi) \in L^q(\Omega) \cap L^\infty(\Omega)$. Thus, by elliptic regularity, $\xi \in H_D^{2,q}(\Omega)$. In fact, $\xi \in H_D^{2,\sigma}(\Omega)$ for all $\sigma \leq p_0$. In particular, ξ is an equilibrium for (1.1). Since $0 \leq \xi \leq \varphi_m$ we have $\xi = \varphi_m$.

We now show that φ_m^R converges to φ_m in $W_{\text{loc}}^{2-\varepsilon,\sigma}(\Omega)$. Given $L > 0$, let $0 \leq \chi \in C_c^\infty(B_{2L})$ such that $\chi \equiv 1$ in B_L . Let $\eta = \varphi_m^R \chi$. Then, η solves

$$\begin{cases} -\Delta \eta - m(x)\eta = H_{R,L}(x) & \text{in } \Omega_{2L}, \\ \eta = 0 & \text{on } \partial\Omega_{2L} \end{cases}$$

with $H_{R,L}(x) = -2\nabla \varphi_m^R \nabla \chi + f_0(x, \varphi_m^R)\chi - \varphi_m^R \Delta \chi$. Now, since $0 \leq \varphi_m^R \leq \varphi_m \in L^q(\Omega) \cap L^\infty(\Omega)$, we have $\|\varphi_m^R\|_{L^\sigma(\Omega_{2L})} \leq C(L)$ for all $q \leq \sigma < \infty$, not depending on R . Thus, by (4.11),

$$\|H_{R,L}\|_{W^{-1,\sigma}(\Omega_{2L})} \leq C(L).$$

Then, by elliptic regularity $\eta \in W_0^{1,\sigma}(\Omega_{2L})$ and

$$\|\eta\|_{W_0^{1,\sigma}(\Omega_{2L})} \leq C(L)$$

for certain constant not depending on R . As a consequence $\{\varphi_m^R\}_R$ is a bounded set of $W_{\text{loc}}^{1,\sigma}(\Omega)$.

But we can repeat the argument above taking now into account that now, for all $q \leq \sigma < \infty$, $\|\nabla \varphi_m^R \nabla \chi\|_{L^\sigma(\Omega_{2L})} \leq C(L)$ not depending on R . Thus, $\|H_{R,L}\|_{L^\sigma(\Omega_{2L})} \leq C(L)$. Therefore, $\{\varphi_m^R\}_R$ is a bounded set of $W_{\text{loc}}^{2,\sigma}(\Omega)$. So, for every $\varepsilon > 0$

$$\lim_{R \rightarrow \infty} \varphi_m^R = \varphi_m \quad \text{in } W^{2-\varepsilon,\sigma}(\Omega_{2L}) \text{ for all } L > 0,$$

that is,

$$\lim_{R \rightarrow \infty} \varphi_m^R = \varphi_m \quad \text{in } W_{\text{loc}}^{2-\varepsilon,\sigma}(\Omega).$$

In particular, taking $\sigma > N/2$, $W^{2-\varepsilon,\sigma}(\Omega_{2L}) \subset C^\theta(\Omega_{2L})$ and the convergence holds in $C^\theta(\Omega_{2L})$ for some $\theta > 0$.

Now, since $\varphi_m(x) \rightarrow 0$ as $|x| \rightarrow \infty$, given $\varepsilon > 0$, we set L such that $\|\varphi_m\|_{L^\infty(\{|x|>L\})} < \varepsilon$. Applying the previous result we have

$$\lim_{R \rightarrow \infty} \varphi_m^R = \varphi_m \quad \text{uniformly in } \Omega. \quad \square$$

Remark 4.4. If $g \not\equiv 0$ then a similar argument can be carried out but now the set where σ belongs to depends on the regularity of g .

Observe that for the case $f(x, s) = s - s^3$ and $\Omega = \mathbb{R}^N$ the problem is not dissipative in the spaces we consider here. Moreover, from Proposition 2.6 in [4], we have that for any initial data $0 \leq u_0 \in C_0(\mathbb{R}^N)$, not identically zero,

$$u(t, x; u_0) \rightarrow 1 \quad \text{in } L_{\text{loc}}^\infty(\mathbb{R}^N).$$

In particular, taking $u_0 = \varphi_m^R$ we have that $u(t, x; \varphi_m^R) \rightarrow 1$ in $L_{\text{loc}}^\infty(\mathbb{R}^N)$. From here, the arguments above allow to conclude that in (4.8) we get $\lim_{R \rightarrow \infty} \varphi_m^R = 1$.

Note that in this case, 0 is an unstable equilibrium and there exists a positive bounded solution but there is not a minimal equilibrium. Of course the difference with Theorem 4.1 is that the globally defined semigroup by (1.1) is not asymptotically compact and then, we do not have an attractor for (1.1).

Remark 4.5 (Convergence of maximal equilibria). Note that when Theorem 3.1 or Theorem 3.3 are satisfied, using the results in [6,7], the maximal solutions in the bounded domains $\Omega_R = \Omega \cap B_R$, with Dirichlet boundary conditions, $\{\varphi_M^R\}_R$, also exist. Moreover they are increasing and bounded above by φ_M . Then with similar arguments we get that

they converge to some equilibria ξ , which is obviously below φ_M . Note that the convergence above is in the same norms as in Proposition 4.3.

It is an interesting problem then to determine whether or not the limit of φ_M^R is φ_M . Below we give several conditions which guarantee that

$$\lim_{R \rightarrow \infty} \varphi_M^R(x) = \varphi_M(x). \quad (4.13)$$

- (i) If the positive solutions are unique as in Section 5 below, then (4.13) holds true.
- (ii) Convergence in (4.13) is equivalent to the property that there exist a sequence of equilibria in the domains $\Omega_R = \Omega \cap B_R$, with Dirichlet boundary conditions, ξ^R such that

$$\lim_{R \rightarrow \infty} \xi^R = \varphi_M.$$

Note that the failure of this property implies that there exist equilibria in the unbounded domain and small at infinity, “not coming from the bounded domain approximation.”

Also, note that for problems under perturbation of the domain, in the case of bounded domains, the results in [3] allow to obtain that given a hyperbolic equilibrium of the limiting problem there exist an approximating sequence of equilibria in the approximating domains. In this case this would imply that if φ_M^R is stable then the sequence ξ^R above would exist. We are unaware however that this result applies in the case of unbounded domains, a question that will be studied elsewhere.

- (iii) Convergence in (4.13) is satisfied provided there exist an initial data $\eta \geq \varphi_M$ (which we can assume below ϕ in Theorem 3.1 or Theorem 3.3) such that

$$u^R(t, \eta) \rightarrow \varphi_M^R, \quad \text{as } t \rightarrow \infty$$

uniformly in R .

To see this note that given $\varepsilon > 0$ there exists $T > 0$, such that for all large R we have

$$\|u^R(T, \eta) - \varphi_M^R\| \leq \varepsilon, \quad \text{and} \quad \|u(T, \eta) - \varphi_M\| \leq \varepsilon$$

in any suitable norm. Moreover it is not difficult to show that also, for this fixed T and large enough R we have

$$\|u^R(T, \eta) - u(T, \eta)\| \leq \varepsilon$$

and the result follows.

- (iv) Convergence in (4.13) is equivalent to the existence of an initial data $\eta \geq \varphi_M$ (which we can assume below ϕ in Theorem 3.1 or Theorem 3.3) such that

$$u^R(t, \eta) \rightarrow u(t, \eta), \quad \text{uniformly in } t \text{ as } R \rightarrow \infty$$

in any suitable norm.

For the if part, note that given $\varepsilon > 0$ there exists $T > 0$, such that for all large R and $t \geq T$, we have

$$\|u^R(t, \eta) - u(t, \eta)\| \leq \varepsilon, \quad \text{and} \quad \|u(t, \eta) - \varphi_M\| \leq \varepsilon.$$

Now for any given large R there exist a time $\tau = \tau(R, \varepsilon) \geq T$ such that

$$\|u^R(\tau, \eta) - \varphi_M^R\| \leq \varepsilon$$

and we get the result.

For the only if part, note that taking $\eta = \varphi_M$ we have, for all $t \geq 0$

$$\varphi_M^R \leq u^R(t, \varphi_M) \leq u(t, \varphi_M) = \varphi_M$$

and since (4.13) is satisfied we have $u^R(t, \varphi_M^R) \rightarrow u(t, \varphi_M) = \varphi_M$ uniformly in t , as $R \rightarrow \infty$.

5. Uniqueness of positive equilibria

We will assume that $g(x) \equiv f(x, 0) \geq 0$. Then, Eq. (1.1) preserves the positivity.

Theorem 5.1. *Suppose Theorem 2.6 applies and $g \in L^\sigma(\Omega)$, for some $p_0 \geq \sigma \geq q$, with either $N \leq 3$ or $\sigma \geq 2N/(N+3)$. Assume that, for $u \geq 0$,*

$$\frac{f(x, u)}{u} \quad \text{is decreasing,}$$

strictly in a set of positive measure. Also assume that there exists a maximal or a minimal positive equilibrium of (1.1). Then, there exists a unique positive equilibrium.

Proof. We prove the result with a maximal equilibrium. The other one is analogous. Hence, denote the maximal equilibrium by φ_M and suppose there is another positive equilibrium ψ . By hypothesis, $\psi \leq \varphi_M$. Moreover,

$$-\Delta\psi = f(x, \psi) \quad \text{and} \quad -\Delta\varphi_M = f(x, \varphi_M).$$

Formally, multiplying the first equation by φ_M , the second one by ψ , subtracting and using that $\varphi(x), \varphi_M(x)$ are small at ∞ , we have

$$0 = \int_{\Omega} -(\varphi_M \Delta\psi - \psi \Delta\varphi_M) = \int_{\Omega} \left(\frac{f(x, \psi)}{\psi} - \frac{f(x, \varphi_M)}{\varphi_M} \right) \psi \varphi_M.$$

Thus,

$$\int_{\Omega} \left(\frac{f(x, \psi)}{\psi} - \frac{f(x, \varphi_M)}{\varphi_M} \right) \psi \varphi_M = 0.$$

But, $f(x, u)/u$ is decreasing, strictly in a set of positive measure, so we must have $\psi = 0$ or $\psi = \varphi_M$.

To justify this formal computation observe that, integrating by parts, we have

$$\int_{\partial\Omega_R} \varphi_M \frac{\partial\psi}{\partial n} - \psi \frac{\partial\varphi_M}{\partial n} = \int_{\Omega_R} \left(\frac{f(x, \psi)}{\psi} - \frac{f(x, \varphi_M)}{\varphi_M} \right) \psi \varphi_M \quad (5.1)$$

where $\Omega_R = \Omega \cap B_R$, $R > 0$.

Now, let u, v be two equilibria. Thus, from hypotheses on g and Theorem 2.6 we have $u, v \in \mathcal{A} \subset H_D^{2,\sigma}(\Omega)$ is bounded. Then we have $H_D^{2,\sigma}(\Omega) \subset L^s(\Omega) \cap W^{1,s'}(\Omega)$ for some $1 < s \leq \infty$ since, by Sobolev embeddings, this is equivalent to $2 - N/\sigma \geq -N/s \geq -1 - N + N/\sigma$ and the choice of s is possible provided $\sigma \geq 2N/(N+3)$. So, in particular, $v \in L^s(\Omega)$ and $u \in W^{1,s'}(\Omega)$. Then, for any $R_0 > 0$,

$$\left| \int_{\{|x|>R_0\} \cap \Omega} v |\nabla u| \right| \leq \int_{\{|x|>R_0\} \cap \Omega} |v| |\nabla u|.$$

By Hölder's inequality we have

$$\int_{\{|x|>R_0\} \cap \Omega} |v| |\nabla u| \leq \left(\int_{\{|x|>R_0\} \cap \Omega} |v|^s \right)^{1/s} \left(\int_{\{|x|>R_0\} \cap \Omega} |\nabla u|^{s'} \right)^{1/s'} < \infty.$$

Now notice that

$$\int_{R_0}^{\infty} \left[\int_{\partial\Omega_R} |v| \left| \frac{\partial u}{\partial n} \right| \right] dR \leq \int_{\{|x|>R_0\} \cap \Omega} |v| |\nabla u| < \infty.$$

Therefore, for some subsequence $\{R_n\}_n$, $R_n \rightarrow \infty$,

$$\left| \int_{\partial\Omega_{R_n}} v \frac{\partial u}{\partial n} \right| \rightarrow 0$$

as $n \rightarrow \infty$. Hence, in (5.1) the left-hand side converges to 0 as $R_n \rightarrow \infty$ and we get the result. \square

As a consequence we have the following result.

Corollary 5.2. *Assume Theorem 2.6 holds as well as the conditions for the existence of a maximal solution as in Section 3, the conditions for the existence of a minimal positive equilibria as in Section 4 and the hypotheses in Theorem 5.1.*

Then, the unique positive equilibrium for (1.1) is globally asymptotically stable for the nonnegative nontrivial solutions, i.e., for all $0 \leq u_0 \in H_D^{2\alpha,q}(\Omega)$ not identically zero

$$u(t; u_0) \rightarrow \varphi_M \quad \text{in } H_D^{2\alpha,q}(\Omega), \text{ and uniformly in } \Omega.$$

6. Logistic equations

In this section we apply the previous results to the class of logistics equations

$$\begin{cases} u_t - \Delta u = m(x)u - n(x)|u|^{\rho-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(0) = u_0 \end{cases} \quad (6.1)$$

with $0 \leq n \in L^\infty(\Omega)$ not identically zero and $m \in L_U^{p_0}(\Omega)$ for some $p_0 > N/2$.

Note that we can set this problem in $H_D^{2\alpha,q}(\Omega)$, since $f_0(x, s) = -n(x)|s|^{\rho-1}s$ which satisfies (2.6) with exponent ρ . Then Theorem 2.1 applies provided $p_0 \geq q$.

Also, note that the nonlinear term is quasi-monotone (with $L(x) = m(x)$), and then we have existence of global solutions in $L^q(\Omega)$.

Our aim is then to find conditions on $m(x)$ and $n(x)$ guaranteeing the existence of extremal equilibria. For this we will check whether $C(x)$ and $D(x)$ as in Theorem 3.3 can be obtained.

In such a case, since $n \not\equiv 0$ then $f(x, s)/s$, for $s \geq 0$, is decreasing, strictly in a set of positive measure. Then, the uniqueness result for positive equilibrium in Section 5 and Corollary 5.2 apply.

Note that if the semigroup generated by $\Delta + m$ has exponential decay then the attractor reduces to $\mathcal{A} = \{0\}$. Therefore we require $\sigma(-\Delta - m) \cap \mathbb{R}^- \neq \emptyset$.

In what follows we will prove that if $m(x)$ contains a good part, $m_1(x)$ such that $\Delta + m_1$ has exponential decay, a suitable balance between the bad part $m_2(x)$ and the absorption coefficient $n(x)$ makes it possible to apply the results in previous sections.

Note that, in the case of unbounded domains, a potential $V(x)$ such that $\Delta + V$ has exponential decay must be sufficiently positive at infinity, in some sense, see [1,2] for some characterisations.

We first have in fact the following

Proposition 6.1. *Assume $\sigma(-\Delta - m) \cap \mathbb{R}^- \neq \emptyset$ and that there exists a decomposition of $m(x)$ of the form*

$$m(x) = m_1(x) + m_2(x), \quad x \in \Omega,$$

with $m_1, m_2 \in L_U^{p_0}(\Omega)$ such that the semigroup generated by $\Delta + m_1$ has exponential decay and $m_2 \geq 0$. Let Ω_2 denote the support of m_2 .

Also assume that $n > 0$ a.e. in Ω_2 and

$$\frac{m_2}{n^{1/\rho}} \in L^a(\Omega_2) \cap L^b(\Omega_2) \quad \text{with } a > \frac{N}{2}, \quad q\rho' \geq b > \frac{Nq\rho'}{N+2q}.$$

Then, there exists extremal equilibria $\varphi_M = -\varphi_m$. Moreover φ_M is the unique positive equilibrium for (6.1) which is globally asymptotically stable for positive solutions in $H_D^{2\alpha,q}(\Omega)$.

In addition, the stability holds in the uniform norm.

Proof. Observe that in $\Omega \setminus \Omega_2$ we have that $m_2 \equiv 0$. Then, for $u \geq 0$ (for $u \leq 0$ the argument runs the same), $f(x, u) \leq m_1(x)u$. So, we can take $C(x) = m_1(x)$ and $D(x) = 0$. On the other hand, in Ω_2 we have

$$f(x, u) = m_1(x)u + (m_2(x)u - n(x)u^\rho).$$

Thus, since $n > 0$ a.e. in Ω_2 , using Young's inequality we have

$$f(x, u) \leq m_1(x)u + \frac{m_2^{\rho'}(x)}{n^{\rho'/\rho}(x)}.$$

Hence we set

$$C(x) = m_1(x), \quad x \in \Omega \quad \text{and} \quad D(x) = \begin{cases} \frac{m_2^{\rho'}(x)}{n^{\rho'/\rho}(x)} & \text{in } \Omega_2, \\ 0 & \text{in } \Omega \setminus \Omega_2. \end{cases}$$

Therefore, $C \in L_U^p(\Omega)$ with $p = p_0 > N/2$. Thus, to obtain the existence of an extremal (positive) equilibria from Theorem 3.3 it is enough to prove that $D \in L^r(\Omega) \cap L^s(\Omega)$ with $r > \frac{N}{2}(1 - \frac{1}{\rho})$ and $q \geq s > \frac{Nq}{N+2q}$ and $p \geq \min\{q, r\}$. Note that this last condition is satisfied since, from assumptions, $p = p_0 \geq q$ and, in particular, $p \geq \min\{q, r\}$.

Now notice that since $D \equiv 0$ in $\Omega \setminus \Omega_2$, it is enough to have $D \in L^r(\Omega_2) \cap L^s(\Omega_2)$, that is, $\frac{m_2}{n^{1/\rho}} \in L^a(\Omega_2) \cap L^b(\Omega_2)$ for some $a > \frac{N}{2}$ and $q\rho' \geq b > \frac{Nq\rho'}{N+2q}$.

Observe in addition that we can always assume $p = p_0 \geq r$ since the first number is larger than $N/2$ and the second one is only restricted by $r > \frac{N}{2}(1 - \frac{1}{\rho})$. Thus, the result follows from Theorem 3.3.

On the other hand, since $\sigma(-\Delta - m) \cap \mathbb{R}^- \neq \emptyset$ then for s_0 sufficiently small and $0 \leq s \leq s_0$,

$$f(x, s) = m(x)s - n(x)s^\rho \geq (m(x) - n(x)s_0^{\rho-1})s = M(x)s$$

and then $M \in L_U^p(\Omega)$ satisfies $\sigma(-\Delta - M) \cap \mathbb{R}^- \neq \emptyset$. Then the global asymptotic stability for positive solutions, of the unique positive equilibrium follows from Theorem 4.1 and Corollary 5.2. \square

Observe that if $n(x) \geq \delta > 0$ in the set Ω_2 above, the proposition applies under an integrability condition for $m_2(x)$ alone.

On the other hand, if the set Ω_2 above is bounded, the result above allows for certain simplifications, using that Lebesgue spaces in bounded domains are nested. In particular, we have

Corollary 6.2. Assume the support of $m_2(x)$, Ω_2 above, is a bounded set.

Then Proposition 6.1 holds provided

$$\frac{m_2}{n^{1/\rho}} \in L^a(\Omega_2) \quad \text{with} \quad \begin{cases} a > \frac{Nq}{N+2q}(1 - \frac{1}{\rho}) (> \frac{N}{2}), & \text{if } \rho < 1 + \frac{2q}{N}, \\ a > \frac{N}{2}, & \text{if } \rho \geq 1 + \frac{2q}{N}. \end{cases}$$

If moreover $n(x) \geq \delta > 0$ in Ω_2 then the above conditions hold provided $\rho \geq 1 + \frac{2q}{N}$, with no further assumptions on $m_2(x)$, or, $\rho < 1 + \frac{2q}{N}$ and $m_2 \in L^a(\Omega_2)$ with $a > \frac{Nq}{N+2q}(1 - \frac{1}{\rho}) (> \frac{N}{2})$.

Proof. Just note that, since Lebesgue spaces in bounded domains are nested, we just need to check the most restrictive integrability condition in Ω_2 in Proposition 6.1. For this then note that $\frac{N}{2} \geq \frac{Nq\rho'}{N+2q}$ iff $\rho \geq 1 + \frac{2q}{N}$.

On the other hand, if moreover $n(x) \geq \delta > 0$ in Ω_2 , note that we always have $m_2 \in L_U^{p_0}(\Omega_2) = L^{p_0}(\Omega_2)$ and then we can take $a = p_0 > N/2$ in case $\rho \geq 1 + \frac{2q}{N}$. \square

Remark 6.3.

- (i) To illustrate an example in which Ω_2 is bounded, assume that there exists a decomposition of $m(x)$ as $m(x) = M_0(x) + M_1(x)$ with $M_0, M_1 \in L_U^{p_0}(\Omega)$ such that the semigroup generated by $\Delta + M_1$ has exponential decay. Then, observe that if $M_0(x)$ is “small” at ∞ in the sense of $L_U^{p_0}(\Omega)$, that is,

$$\lim_{R \rightarrow \infty} \|(1 - \chi_{B_R})M_0\|_{L_U^{p_0}(\Omega)} = 0$$

then, for sufficiently large R , the semigroup generated by $\Delta + M_1 + (1 - \chi_{B_R})M_0$ has exponential decay also. In such a case we can take here $m_1(x) = M_1(x) + (1 - \chi_{B_R})M_0(x)$ and $m_2(x) = \chi_{B_R}M_0(x)$ which has bounded support. Hence the corollary above applies.

- (ii) Note that in the case Ω was a bounded domain, for any given potential $m(x)$ one can always take $m_1(x) = m(x) - \lambda$ and $m_2(x) = \lambda$, with λ a sufficiently large constant, see [6,7].

Then, the conditions in the proposition lead to some integrability of the inverse of $n(x)$. Here, this approach is not possible since $n(x)$ is bounded above and we are in an unbounded domain.

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