

# Extremely non-complex $C(K)$ spaces <sup>☆</sup>

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## Abstract

We show that there exist infinite-dimensional extremely non-complex Banach spaces, i.e. spaces  $X$  such that the norm equality  $\|\text{Id} + T^2\| = 1 + \|T^2\|$  holds for every bounded linear operator  $T : X \rightarrow X$ . This answers in the positive Question 4.11 of [V. Kadets, M. Martín, J. Merí, Norm equalities for operators on Banach spaces, *Indiana Univ. Math. J.* 56 (2007) 2385–2411]. More concretely, we show that this is the case of some  $C(K)$  spaces with few operators constructed in [P. Koszmider, Banach spaces of continuous functions with few operators, *Math. Ann.* 330 (2004) 151–183] and [G. Plebanek, A construction of a Banach space  $C(K)$  with few operators, *Topology Appl.* 143 (2004) 217–239]. We also construct compact spaces  $K_1$  and  $K_2$  such that  $C(K_1)$  and  $C(K_2)$  are extremely non-complex,  $C(K_1)$  contains a complemented copy of  $C(2^\omega)$  and  $C(K_2)$  contains a (1-complemented) isometric copy of  $\ell_\infty$ .

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## 1. Introduction

Let  $X$  be a real Banach space. We write  $L(X)$  for the space of all bounded linear operators on  $X$  endowed with the supremum norm and  $W(X)$  for its subspace of all weakly compact operators on  $X$ .

The aim of this paper is to show that there are infinite-dimensional Banach spaces  $X$  for which the norm equality

$$\|\text{Id} + T^2\| = 1 + \|T^2\| \tag{sDE}$$

holds for every  $T \in L(X)$ . Actually, we will show that there are several different compact (Hausdorff) spaces  $K$  such that the corresponding real spaces  $C(K)$  satisfy this property. This answers positively Question 4.11 of the very recent paper [16] by V. Kadets and the second and third authors.

Let us motivate the interest of the question.

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Firstly, a good interpretation of the property we are dealing with is given by the so-called complex structures on real Banach spaces. We recall that a (real) Banach space  $X$  is said to have a *complex structure* if there exists  $T \in L(X)$  such that  $T^2 = -\text{Id}$ . This allows us to define on  $X$  a structure of vector space over  $\mathbb{C}$ , by setting

$$(\alpha + i\beta)x = \alpha x + \beta T(x) \quad (\alpha + i\beta \in \mathbb{C}, x \in X).$$

Moreover, by just defining

$$\|x\| = \max\{\|e^{i\theta}x\|: \theta \in [0, 2\pi]\} \quad (x \in X)$$

one gets a complex norm on  $X$  which is equivalent to the original one. Conversely, if  $X$  is the real space underlying a complex Banach space, then the bounded linear operator defined by  $T(x) = ix$  for every  $x \in X$ , satisfies that  $T^2 = -\text{Id}$ . In the finite-dimensional setting, complex structures appear if and only if the dimension of the space is even. In the infinite-dimensional setting, there are real Banach spaces admitting no complex structure. This is the case of the James' space  $\mathcal{J}$  (see [3, §3.4] for the definition), as it was shown by J. Dieudonné in 1952 [6]. More examples of this kind have been constructed over the years, including uniformly convex examples (S. Szarek [29]), the hereditary indecomposable space of T. Gowers and B. Maurey [13] or, more generally, any space such that every operator on it is a strictly singular perturbation of a multiple of the identity. Gowers also constructed a space of this kind with an unconditional basis [12,14]. We refer the reader to the very recent papers by V. Ferenczi and E. Medina Galego [9,10] and references therein for a discussion about complex structures on spaces and on their hyperplanes.

Let us comment that if Eq. (sDE) holds for all operators on a Banach space  $X$ , then  $X$  does not have complex structure in the strongest possible way, showing that, for every  $T \in L(X)$ , the distance from  $T^2$  to  $-\text{Id}$  is the biggest possible, namely  $1 + \|T^2\|$ . This observation justifies the following definition which we will use along the paper.

**Definition 1.1.** We say that  $X$  is *extremely non-complex* if the norm equality

$$\|\text{Id} + T^2\| = 1 + \|T^2\| \tag{sDE}$$

holds for every  $T \in L(X)$ .

Secondly, let us explain shortly the history leading to the appearance of (sDE) and the question of the existence of infinite-dimensional extremely non-complex spaces in the already cited paper [16]. The interest in norm equalities for operators goes back to the 1960s, when I. Daugavet [5] showed that each compact operator  $T$  on  $C[0, 1]$  satisfies the norm equality

$$\|\text{Id} + T\| = 1 + \|T\|. \tag{DE}$$

The above equation is nowadays referred to as *Daugavet equation*. This result has been extended to various classes of operators on some Banach spaces, including weakly compact operators on  $C(K)$  for perfect  $K$  and on  $L_1(\mu)$  for atomless  $\mu$  (see [30] for an elementary approach). In the late 1990s the study of the geometry of Banach spaces having the so-called Daugavet property was initiated. Following [17,18], we say that a Banach space  $X$  has the *Daugavet property* if every rank-one operator  $T \in L(X)$  satisfies (DE). In such a case, every operator on  $X$  not fixing a copy of  $\ell_1$  also satisfies (DE); in particular, this happens to every weakly compact operator on  $X$ . This property induces various isomorphic restrictions. For instance, a Banach space with the Daugavet property does not have the Radon–Nikodým property, it contains  $\ell_1$ , and it does not isomorphically embed into a Banach space with unconditional basis. We refer the reader to the books [1,2] and the papers [18,31] for background and more information.

The aim of the cited paper [16] was to study whether it is possible to define interesting isometric properties by requiring all rank-one operators on a Banach space to satisfy a suitable norm equality. Most of the results gotten there are occlusive, showing that the most natural attempts to introduce new properties by considering other norm equalities for operators (like  $\|g(T)\| = f(\|T\|)$  for some functions  $f$  and  $g$ ) lead in fact to the Daugavet property of the space. Nevertheless, there are some results in the paper valid in the complex case which are not known to be true for the real case. For instance, it is not known if a real Banach space  $X$  where every rank-one operator  $T \in L(X)$  satisfies

$$\|\text{Id} + T^2\| = 1 + \|T^2\|$$

has the Daugavet property (see also [21] for more information). Contrary to the Daugavet equation, the equation above could be satisfied by all bounded linear operators on a Banach space  $X$ . Actually, this holds in the simple case

$X = \mathbb{R}$ . During an informal discussion on these topics in May 2005, Gilles Godefroy asked the second and the third authors of this paper about the possibility of finding Banach spaces (of dimension greater than 1) having this property (i.e. finding extremely non-complex Banach spaces of dimension greater than one). Let us comment that, if a Banach space  $X$  is extremely non-complex, then it cannot contain a complemented subspace with complex structure (such as a square) and with summand  $\alpha$ -complemented with  $\alpha < 2$ . This can be seen by applying (sDE) to the operator  $T \in L(X)$  defined by  $Tx = 0$  on the summand, and by  $Tx = Jx$  with square of  $J$  equal to  $-\text{Id}$  on the complemented subspace with complex structure.

Our (successful) approach to this problem has been to consider Banach spaces  $C(K)$  with few operators in the sense introduced by the first author of this paper in [20]. Let us give two needed definitions.

**Definition 1.2.** Let  $K$  be a compact space and  $T \in L(C(K))$ . We say that  $T$  is a *weak multiplier* if  $T^* = g \text{Id} + S$  where  $g : K \rightarrow \mathbb{R}$  is a function which is integrable with respect to all Radon measures on  $K$  and  $S \in W(C(K)^*)$ . If one actually has  $T = g \text{Id} + S$  with  $g \in C(K)$  and  $S \in W(C(K))$ , we say that  $T$  is a *weak multiplication*.

In the literature, as far as now, there are several non-isomorphic types of  $C(K)$  spaces with few operators in the above sense (in ZFC): (1) of [20] for  $K$  totally disconnected such that  $C(K)$  is a subspace of  $\ell_\infty$  and all operators on  $C(K)$  are weak multipliers; (2) of [20] for  $K$  such that  $K \setminus F$  is connected for every finite  $F \subseteq K$ , such that  $C(K)$  is a subspace of  $\ell_\infty$  and all operators on  $C(K)$  are weak multipliers; these  $C(K)$ s, as shown in [20], are indecomposable Banach spaces, hence they are non-isomorphic to spaces of type (1); (3) of [23] for connected  $K$  such that all operators on  $C(K)$  are weak multiplications; these spaces are not subspaces of  $\ell_\infty$  and hence are non-isomorphic to spaces of type (1) nor (2). (It is still not known if such spaces can be subspaces of  $\ell_\infty$  without any special set-theoretic hypotheses; in [20] it is shown that the continuum hypothesis is sufficient to obtain such spaces.)

We will show in Section 2 that  $C(K)$  spaces on which every operator is a weak multiplication with  $K$  perfect give the very first examples of infinite-dimensional extremely non-complex Banach spaces. The argument is elementary. With a little bit more of work, we prove in Section 3 that also  $C(K)$  spaces on which every operator is a weak multiplier with  $K$  perfect are extremely non-complex.

One may think that the fact that a  $C(K)$  space is extremely non-complex implies some kind of fewness of operators. The aim of Section 4 is to give further examples to show that this is not the case. We construct compact spaces  $K_1$  and  $K_2$  such that  $C(K_1)$  and  $C(K_2)$  are extremely non-complex,  $C(K_1)$  contains a complemented copy of  $C(2^\omega)$  and  $C(K_2)$  contains a (1-complemented) isometric copy of  $\ell_\infty$ . One may use elementary arguments to show that the above  $C(K_1)$  and  $C(K_2)$  have many operators however let us quote a deeper recent result of I. Schlackow:

**Theorem 1.3.** (See [26, Theorem 4.6].) *Let  $K$  be a compact space. All operators on  $C(K)$  are weak multipliers if and only if the ring  $L(C(K))/W(C(K))$  is commutative.*

When  $C(\tilde{K})$  is isomorphic to a complemented subspace of  $C(K)$ , then the ring  $L(C(\tilde{K}))/W(C(\tilde{K}))$  can be canonically included in  $L(C(K))/W(C(K))$ . Hence, if a  $C(K)$  has a complemented isomorphic copy of a space  $C(\tilde{K})$ , then the corresponding ring is at least “as much non-commutative as” that of  $C(\tilde{K})$ . In particular if  $C(\tilde{K})$  has operators which are not weak multipliers, in other words, by the above theorem, the corresponding ring is non-commutative as in the case of  $\ell_\infty$  or  $C(2^\omega)$ , then  $C(K)$  has operators which are not weak multipliers.

To obtain  $K_1$  and  $K_2$  as above using the uniform language of Boolean algebras and their Stone spaces we will need  $C(K)$ ’s where all operators are weak multipliers which are a bit different than those of types (1)–(3) described above. Namely we will need  $K$  perfect and totally disconnected. Actually this type was the original construction of earlier versions of [20] which later was split into simpler type (1) which is not perfect and more complex type (2) which is perfect but not totally disconnected. In Section 4 we explain how to modify arguments of [20] to obtain the desired  $K$ .

We finish the introduction by commenting that the first attempt to find extremely non-complex Banach spaces could have been to check whether the known examples of spaces without complex structure work. Unfortunately, most of those examples are reflexive or quasireflexive spaces, and there are no extremely non-complex spaces in these classes (actually, the unit ball of an extremely non-complex space does not contain any strongly exposed point, see [21]). With respect to the family of spaces with few operators, it is readily checked that if a Banach space  $X$  has the Daugavet property and every operator  $T \in L(X)$  is a strictly singular perturbation of a multiple of the identity, then the space is extremely non-complex. Unfortunately, we have not been able to find any example of this kind. On

the one hand, if a Banach space is hereditarily indecomposable, then it cannot have the Daugavet property (which implies containment of  $\ell_1$ ). It is worth mentioning the existence of a hereditarily indecomposable space whose dual contains  $\ell_1$  [4, Remark 8.1]. On the other hand, most of the research about Banach spaces with few operators deals with the isomorphic structure and does not pay too much attention to isometric questions.

## 2. The first example: Weak multiplications

This short section is devoted to give a sufficient condition for a weak multiplication to satisfy the Daugavet equation, from which it will be straightforward to get the very first example of an extremely non-complex Banach space, namely, every  $C(K)$  space where the only bounded linear operators are the weak multiplications.

**Lemma 2.1.** *Let  $K$  be a perfect compact space. If an operator  $T \in L(C(K))$  has the form  $T = g \text{Id} + S$  where  $g \in C(K)$  is non-negative and  $S$  is weakly compact, then  $T$  satisfies the Daugavet equation.*

We need the following two old results. The first result goes back to the 1971 paper by J. Duncan, C. McGregor, J. Pryce and A. White [8, p. 483]. The second one was established in the sixties by I. Daugavet [5] for compact operators on  $C[0, 1]$  and extended to weakly compact operators by C. Foaş and I. Singer and to arbitrary perfect  $K$  by A. Pełczyński [11, p. 446]. Elementary proofs can be found in [30].

### Remark 2.2.

(a) For every compact space  $K$  and every  $T \in L(C(K))$ , one has

$$\max\{\|\text{Id} + T\|, \|\text{Id} - T\|\} = 1 + \|T\|.$$

(b) If  $K$  is a perfect compact space, then

$$\|\text{Id} + S\| = 1 + \|S\|$$

for every  $S \in W(C(K))$ .

**Proof of Lemma 2.1.** Since the set of those operators on  $C(K)$  satisfying the Daugavet equation is closed and stable by multiplication by positive scalars, we may suppose that  $\min_{t \in K} g(t) > 0$  and  $\|g\| \leq 1$ . Now, by using Remark 2.2(a) we have that

$$\max\{\|\text{Id} + g \text{Id} + S\|, \|\text{Id} - (g \text{Id} + S)\|\} = 1 + \|g \text{Id} + S\|.$$

So, we will be done by just proving that

$$\|\text{Id} - (g \text{Id} + S)\| < 1 + \|g \text{Id} + S\|.$$

On the one hand, it is easy to check that

$$\|\text{Id} - (g \text{Id} + S)\| \leq \|\text{Id} - g \text{Id}\| + \|S\| = 1 - \min_{t \in K} g(t) + \|S\|.$$

On the other hand, we observe that

$$\begin{aligned} \|g \text{Id} + S\| &= \|\text{Id} + S + (g \text{Id} - \text{Id})\| \geq \|\text{Id} + S\| - \|g \text{Id} - \text{Id}\| \\ &= 1 + \|S\| - \left(1 - \min_{t \in K} g(t)\right) = \|S\| + \min_{t \in K} g(t) \end{aligned}$$

where we used Remark 2.2(b). Since  $\min_{t \in K} g(t) > 0$ , it is clear that

$$\|\text{Id} - (g \text{Id} + S)\| < 1 + \|g \text{Id} + S\|. \quad \square$$

Suppose now that all the operators on a  $C(K)$  space are weak multiplications and  $K$  is perfect. Then, for every  $T \in L(C(K))$  one has  $T^2 = g \text{Id} + S$  where  $g \in C(K)$  is non-negative and  $S$  is weakly compact. The above lemma then yields the following result.

**Theorem 2.3.** *Let  $K$  be a perfect compact space such that every operator on  $C(K)$  is a weak multiplication. Then,  $C(K)$  is extremely non-complex.*

As we commented in the introduction, there are (even in ZFC) perfect compact spaces whose operators are weak multiplications [23]. Therefore, the above result really gives the existence of extremely non-complex infinite-dimensional Banach spaces.

**Corollary 2.4.** *There exist infinite-dimensional extremely non-complex Banach spaces.*

Let us finish the section showing that our requirement for the compact space to be perfect is not only methodological.

**Remark 2.5.** Let  $K$  be a compact space with at least two points. If  $C(K)$  is extremely non-complex, then  $K$  is perfect.

**Proof.** If there exists an isolated point  $t_0 \in K$ , we write  $K' = K \setminus \{t_0\}$ , we take  $t_1 \in K'$ , and define  $T \in L(C(K))$  by

$$[T(f)] = (3f(t_1) - f(t_0))(2\chi_{\{t_0\}} + \chi_{K'}) \quad (f \in C(K)).$$

It is readily checked that  $T^2 = T$ ,  $\|T\| = 8$ , and  $\|\text{Id} + T\| \leq 7 < 1 + \|T\|$ .  $\square$

### 3. More examples: Weak multipliers

Our aim in this section is to enlarge the class of extremely non-complex Banach spaces by adding those  $C(K)$  spaces with perfect  $K$  for which all operators are weak multipliers. Let us first fix some notation and preliminary results.

Given a compact space  $K$ , by the Riesz representation theorem, the dual of the Banach space  $C(K)$  is isometric to the space  $M(K)$  of Radon measures on  $K$ , i.e. signed, Borel, scalar-valued, countably additive and regular measures. More precisely, given  $\mu \in M(K)$  and  $f \in C(K)$ , the duality is given by

$$\mu(f) = \int f d\mu.$$

We introduce one more ingredient which will play a crucial role in our arguments of this section. Given an operator  $U \in L(M(K))$ , we consider an associated function  $g_U : K \rightarrow [-\|U\|, \|U\|]$  given by

$$g_U(x) = U(\delta_x)(\{x\}) \quad (x \in K).$$

This obviously extends to operators on  $C(K)$  by passing to the adjoint, that is, for  $T \in L(C(K))$  one can consider  $g_{T^*} : K \rightarrow [-\|T\|, \|T\|]$ . This tool was used in [30] under the name “stochastic kernel” to give an elementary approach to the Daugavet equation on  $C(K)$  spaces. One of the results in the aforementioned paper, which we state for the convenience of the reader, will be useful in the remainder of our exposition.

**Lemma 3.1.** (See [30, Lemma 3].) *Let  $K$  be a compact space and  $T \in L(C(K))$ . If the set  $\{x \in K : g_{T^*}(x) \geq 0\}$  is dense in  $K$ , then  $T$  satisfies the Daugavet equation.*

The next result tells us that for weakly compact operators on  $M(K)$  the associated functions take “few” values.

**Lemma 3.2.** *Let  $K$  be a compact space and  $U \in W(M(K))$ . Then, for every  $\varepsilon > 0$  the set  $\{x \in K : |g_U(x)| > \varepsilon\}$  is finite.*

**Proof.** If for some  $\varepsilon > 0$  the set  $\{x \in K : |g_U(x)| > \varepsilon\}$  is infinite, then there is an infinite sequence  $\{x_n\}_{n \in \mathbb{N}}$  of different points in  $K$  satisfying

$$g_U(x_n) = U(\delta_{x_n})(\{x_n\}) > \varepsilon$$

for every  $n \in \mathbb{N}$ . By using the regularity of the measures  $\{U(\delta_n)\}_{n \in \mathbb{N}}$  and passing to a subsequence of  $\{x_n\}_{n \in \mathbb{N}}$  if necessary, we can find a family of pairwise disjoint open sets  $\{V_n\}_{n \in \mathbb{N}}$  such that  $x_n \in V_n$  and

$$|U(\delta_{x_n})(V_n \setminus \{x_n\})| < \frac{\varepsilon}{2}$$

for every  $n \in \mathbb{N}$ , which implies

$$|U(\delta_{x_n})(V_n)| > \frac{\varepsilon}{2} \quad (n \in \mathbb{N}).$$

This, together with the Dieudonné–Grothendieck Theorem (see [7, VII.14]), tells us that the sequence of measures  $\{U(\delta_{x_n})\}_{n \in \mathbb{N}}$  is not relatively weakly compact and, therefore, that  $U$  is not weakly compact.  $\square$

As an immediate consequence we get the following corollary.

**Corollary 3.3.** *Let  $K$  be a compact space and  $U \in W(M(K))$ . Then, the set  $\{x \in K : g_U(x) \neq 0\}$  is at most countable.*

**Theorem 3.4.** *Let  $K$  be a perfect compact space and  $T \in L(C(K))$  an operator such that  $T^* = g \text{Id} + S$  where  $S \in W(M(K))$  and  $g$  is a Borel function satisfying  $g \geq 0$ . Then, the set*

$$\{x \in K : T^*(\delta_x)(\{x\}) \geq 0\}$$

*is dense in  $K$  and, therefore,  $T$  satisfies the Daugavet equation.*

**Proof.** It is clear that for every  $x \in K$  one has  $T^*(\delta_x)(\{x\}) \geq 0$  provided that  $S(\delta_x)(\{x\}) = 0$  and, therefore,

$$\{x \in K : T^*(\delta_x)(\{x\}) \geq 0\} \supset \{x \in K : S(\delta_x)(\{x\}) = 0\}.$$

Now, we observe that the last set is dense in  $K$  by Corollary 3.3 and the fact that non-empty open sets in perfect compact spaces are uncountable. (If  $U \subset K$  is open and countable we can find an open set  $V$  satisfying  $\bar{V} \subset U$  and, therefore,  $\bar{V}$  is a countable compact space and thus scattered by [27, Proposition 8.5.7]. So  $V$  has an isolated point which is also isolated in  $K$  since  $V$  is open, contradicting the perfectness of  $K$ .) Finally, the proof concludes with the use of Lemma 3.1.  $\square$

We can now state and prove the main result of the section.

**Theorem 3.5.** *Let  $K$  be a perfect compact space so that every operator on  $C(K)$  is a weak multiplier. Then,  $C(K)$  is extremely non-complex.*

**Proof.** Given  $T \in L(C(K))$ , there exist a bounded Borel function  $g$  and  $S \in W(M(K))$  with  $T^* = g \text{Id} + S$ , so

$$(T^2)^* = (T^*)^2 = (g \text{Id} + S)^2 = g^2 \text{Id} + S'$$

where  $S'$  is weakly compact, and we use the previous theorem.  $\square$

As we commented in the introduction, there are infinitely many non-isomorphic spaces  $C(K)$  on which every operator is a weak multiplier, providing infinitely many non-isomorphic extremely non-complex Banach spaces.

**Corollary 3.6.** *There exist infinitely many non-isomorphic infinite-dimensional extremely non-complex Banach spaces.*

**Remark 3.7.** It is clear that being extremely non-complex is not an isomorphic property. This is especially clear for spaces of continuous functions, since when a compact space  $K$  is infinite, then  $C(K)$  is isomorphic to  $C(K')$  where  $K'$  has an isolated point ( $K'$  is  $K$  with two points identified and one external point added). Anyhow, I. Schlackow has proved very recently [26, Theorem 4.12] that when  $K$  and  $L$  are perfect compact spaces,  $C(K)$  and  $C(L)$  are isomorphic and every operator on  $C(K)$  is a weak multiplier, then so is every operator on  $C(L)$ . We will see in the next section (see Remark 4.10) that the property of all operators being weak multipliers cannot be replaced by the property of  $C(K)$  being extremely non-complex.

#### 4. Further examples

In this section we will construct more examples of extremely non-complex spaces in the family of  $C(K)$  spaces. All the examples that we have exhibited so far share the characteristic of having few operators, thus it is natural to ask whether the property of being an extremely non-complex space is related to the “lack” of operators. This section is devoted to answer this question by presenting extremely non-complex  $C(K)$  spaces with many operators that are not weak multipliers. To construct such spaces we will consider a classical chain of inter-related structures: a Boolean algebra  $\mathcal{A}$ , its Stone space  $K$ , the Banach space  $C(K)$  and its dual  $M(K)$ . Let us fix some notation, terminology, and standard facts related to these structures.

Given a compact space  $K$ , the clopen subsets of  $K$  form a Boolean algebra which will be denoted by  $\text{Clop}(K)$ . It is well known that a compact space is totally disconnected if and only if it is zero-dimensional i.e., it has a basis of topology consisting of clopen sets [19, Theorem 7.5]. One can also recover all totally disconnected compact spaces as the Stone spaces of abstract Boolean algebras [19, Theorem 7.10]. If  $\mathcal{A}$  is a Boolean algebra, the Stone space  $K$  of  $\mathcal{A}$  is the set of all ultrafilters of  $\mathcal{A}$  endowed with the topology in which the basic sets are defined as  $[A] = \{u \in K : A \in u\}$  for any  $A \in \mathcal{A}$ . By the Stone duality (see [19, Theorem 8.2]) epimorphisms of Boolean algebras correspond to monomorphisms of their Stone spaces which in turn correspond to epimorphisms of their Banach spaces of continuous functions. In particular, we will be interested in homomorphisms of Boolean algebras which are the identity on their images. For background on Boolean algebras and Stone spaces we refer the reader to [15,19,27,28].

**Fact 4.1.** *Let  $P : \text{Clop}(K) \rightarrow \text{Clop}(K)$  be a homomorphism of Boolean algebras which is the identity on its image, then the mapping  $\tilde{P} : C(K) \rightarrow C(K)$  given by*

$$\tilde{P}(\chi_A) = \chi_{P(A)} \quad (A \in \text{Clop}(K))$$

*is a norm-one projection on  $C(K)$  (it extends to  $C(K)$  by linearity and the Stone–Weierstrass Theorem) whose image is isometric to  $C(L)$  where  $L$  is the Stone space of the Boolean algebra  $P(\text{Clop}(K))$ .*

**Proof.** By the Stone–Weierstrass Theorem, the functions of the form  $\sum_{i \leq n} a_i \chi_{A_i}$  span a dense subspace of  $C(K)$  where  $n \in \mathbb{N}$ ,  $a_i \in \mathbb{R}$  and  $A_i \in \text{Clop}(K)$  are pairwise disjoint. This, together with  $P^2 = P$  and  $\tilde{P}(\sum_{i \leq n} a_i \chi_{A_i}) = \sum_{i \leq n} a_i \chi_{P(A_i)}$ , implies that  $\tilde{P}^2 = \tilde{P}$ . On the other hand, we observe that

$$\left\| \sum_{i \leq n} a_i \chi_{A_i} \right\| = \sup\{|a_i| : i \leq n, A_i \neq \emptyset\}$$

for pairwise disjoint  $A_i \in \text{Clop}(K)$  and that  $P$  as a homomorphism of Boolean algebras preserves the disjointness, which allow us to deduce that  $\|\tilde{P}\| \leq 1$ .  $\square$

For a totally disconnected compact space  $K$ , the restriction of a Radon measure on  $K$  to the Boolean algebra  $\text{Clop}(K)$  is a finitely additive signed and bounded measure, that is,  $\mu(a \vee b) = \mu(a) + \mu(b)$  where  $a, b \in \text{Clop}(K)$  and  $\vee$  denotes the supremum in  $\text{Clop}(K)$ . Conversely, any bounded finitely additive signed measure on such a Boolean algebra extends uniquely to a Radon measure on Borel subsets of  $K$  (see [27, §18.7], for example). The following remark shows that pointwise convergence of measures on the Boolean algebra gives weak\* convergence of the corresponding Radon measures.

**Remark 4.2.** Suppose that  $\mu_n, \mu$  with  $n \in \mathbb{N}$  are uniformly bounded Radon measures on a totally disconnected compact space  $K$  and denote  $\nu = \mu|_{\text{Clop}(K)}$  and  $\nu_n = \mu_n|_{\text{Clop}(K)}$  the associated finitely additive measures on  $\text{Clop}(K)$ . Then,  $\{\nu_n(A)\}_{n \in \mathbb{N}}$  converges to  $\nu(A)$  for every clopen subset  $A \subseteq K$  if and only if  $\{\mu_n\}_{n \in \mathbb{N}}$  converges weakly\* to  $\mu$ .

**Proof.** It follows from the uniform boundedness of the sequence and the density of the span of the characteristic functions of clopen sets in  $C(K)$  which is an immediate consequence of the Stone–Weierstrass Theorem.  $\square$

Let us present the last elements we need: given a compact space  $K$  and  $K_i, K_j$  clopen subsets of  $K$ ,  $K_{-i}$  will stand for the set  $K \setminus K_i$  and  $P_j : C(K) \rightarrow C(K_j)$  and  $I_i : C(K_i) \rightarrow C(K)$  will denote the operators defined by

$$P_j(h) = h|_{K_j} \quad (h \in C(K)) \quad \text{and} \quad \begin{cases} [I_i(f)](x) = f(x) & \text{if } x \in K_i, \\ [I_i(f)](x) = 0 & \text{if } x \notin K_i \end{cases} \quad (f \in C(K_i)),$$

respectively. Moreover, we will also consider the operators  $P^i : M(K) \rightarrow M(K_i)$  and  $I^j : M(K_j) \rightarrow M(K)$  given by

$$[P^i(\mu)](L) = \mu(L) \quad (\mu \in M(K), L \subset K_i)$$

and

$$[I^j(\nu)](S) = \nu(S \cap K_j) \quad (\nu \in M(K_j), S \subset K).$$

Finally, all the operators above will be also used with index  $-i$ . The relationship between these operators is established in the following easy lemma.

**Lemma 4.3.** *Let  $K$  be a compact space, let  $K_i, K_j$  be clopen subsets of  $K$ , and consider the operators  $P_j$  and  $I_i$  defined above. Then,  $I_i^* = P^i$  and  $P_j^* = I^j$ . Moreover, given  $g$  a Borel function on  $K$ , the following holds:*

$$P^i(g \text{Id}_{M(K)})I^j = \begin{cases} 0 & \text{if } K_i \cap K_j = \emptyset, \\ g|_{K_i} \text{Id}_{M(K_i)} & \text{if } K_i = K_j. \end{cases}$$

**Proof.** For  $\mu \in M(K)$  and  $f \in C(K_i)$  we have

$$I_i^*(\mu)(f) = \int_K I_i(f) d\mu = \int_{K_i} f d(P^i(\mu))$$

which gives  $I_i^*(\mu) = P^i(\mu)$ . On the other hand, for  $\nu \in M(K_j)$  and  $f \in C(K)$  we can write

$$P_j^*(\nu)(f) = \int_{K_j} P_j(f) d\nu = \int_{K_j} (f|_{K_j}) d\nu = \int_K f d(I^j(\nu))$$

and, therefore,  $P_j^*(\nu) = I^j(\nu)$ . Finally, if  $K_i \cap K_j = \emptyset$ , given  $\nu \in M(K_j)$  and  $L \subset K_i$ , we observe that

$$[(g \text{Id}_{M(K)})I^j](\nu)(L) = \int_K g\chi_L dI^j(\nu)$$

so, since  $I^j(\nu)$  is a measure which assumes value zero on all sets disjoint from  $K_j$ , the same is true for  $g \text{Id}_{M(K)}I^j(\nu)$ , and so  $[P^i(g \text{Id})I^j](\nu) = 0$ .

If  $K_i = K_j$ , given  $\nu \in M(K_i)$  and  $L \subset K_i$ , we observe that

$$[P^i(g \text{Id})I^i](\nu)(L) = P^i[gI^i(\nu)](L) = [(g|_{K_i})I^i(\nu)](L) = \int g|_{K_i}\chi_L d\nu = g|_{K_i} \text{Id}_{M(K_i)}(\nu)(L)$$

from which it is immediate to deduce the moreover part.  $\square$

In the next two results we prove the existence of a family of  $C(K)$  spaces which are Grothendieck and so that there are few operators between any pair of them. We recall that a Banach space  $X$  is said to be Grothendieck if every weak\* convergent sequence in  $X^*$  is weak convergent and the fact that a  $C(K)$  space in which every operator is a weak multiplier is Grothendieck [20, Theorem 2.4].

**Proposition 4.4.** *There is a compact infinite totally disconnected and perfect space  $K$  such that all operators on  $C(K)$  are weak multipliers.*

**Proof.** We will describe a modification of the construction from Section 3 of [20]. As seen in Lemma 3.2 of [20], the only properties of points  $n^*$  for  $n \in \mathbb{N}$  of the constructed  $K$  which are needed to prove that every operator on  $C(K)$  is

a weak multiplier are those stated in Lemma 3.1 and the density in  $K$ . Thus, to prove the proposition it is enough to construct an atomless Boolean algebra  $\mathcal{A} \subseteq \wp(\mathbb{N})$  (the lack of atoms is equivalent to the fact that the Stone space  $K$  is perfect) and a countable dense subset  $\{q_n: n \in \mathbb{N}\}$  of its Stone space such that given

- (a) a sequence  $(A_n: n \in \mathbb{N})$  of pairwise disjoint elements of  $\mathcal{A}$ ,
- (b) a sequence  $(\ell_n: n \in \mathbb{N})$  of distinct natural numbers such that  $q_{\ell_n} \notin A_m$  for  $n, m \in \mathbb{N}$ ,

there is an infinite  $b \subseteq \mathbb{N}$  such that

- (c)  $\{A_m: m \in b\}$  has its supremum  $A$  in  $\mathcal{A}$ , and
- (d) the intersection of the sets  $\overline{\{q_{\ell_n}: n \in b\}}$  and  $\overline{\{q_{\ell_n}: n \notin b\}}$  in the Stone space  $K$  of  $\mathcal{A}$  is non-empty.

Such an algebra is constructed as in the proof of Lemma 3.1 of [20]. Indeed, we give an outline of the modification of that proof (observe that the notation used in [20] will be kept). The only complication is that the ultrafilters  $q_n$  are not absolutely defined as the ultrafilters  $n^*$ . This means that at each inductive step of the construction of the subalgebras  $\mathcal{A}_\alpha$  for  $\alpha \leq 2^\omega$  one needs to extend the ultrafilter  $q_n|_\alpha$  of  $\mathcal{A}_\alpha$  to an ultrafilter  $q_n|_{(\alpha+1)}$  of the new bigger algebra  $\mathcal{A}_{\alpha+1}$ , since at the limit stages the ultrafilters are determined by their intersections with the previous algebras.

This problem has been encountered in the connected construction of 5.1 of [20] and is resolved in the same way. Namely, if  $q_n|_\alpha$  has only one extension to an ultrafilter of  $\mathcal{A}_{\alpha+1}$ , then one puts it as  $q_n|_{(\alpha+1)}$ , and otherwise one needs to make some uniform choice, for example  $q_n|_{(\alpha+1)}$  is such an ultrafilter of  $\mathcal{A}_{\alpha+1}$  which extends  $q_n|_\alpha$  and does not contain  $A_\alpha$ , the generator of  $\mathcal{A}_{\alpha+1}$  over  $\mathcal{A}_\alpha$ .

Now, at stage  $\alpha < 2^\omega$  we are given  $\alpha$  premises of the form

$$\overline{\{q_n|_\alpha: n \in b_\beta\}} \cap \overline{\{q_n|_\alpha: n \in a_\beta - b_\beta\}} \neq \emptyset, \tag{1}$$

for  $\beta < \alpha$  where the closures are taken in the Stone space of the algebra  $\mathcal{A}_\alpha$ . We are also given an antichain  $(A_n: n \in \mathbb{N})$  in the algebra  $\mathcal{A}_\alpha$  and need to preserve the premises when passing to the algebra  $\mathcal{A}_{\alpha+1}$  generated over  $\mathcal{A}_\alpha$  by an element  $A_\alpha$  which is an infinite sum (in  $\wp(\mathbb{N})$ ) of some infinite subsequence of  $(A_n: n \in b)$  where  $b$  should be an arbitrary infinite subset of some  $a \subseteq \mathbb{N}$ . So we need to make a good choice of  $a \subseteq \mathbb{N}$ .

We may assume that all points of the intersection of the closures from (1) are from outside of the clopen sets  $[A_n]$ , since otherwise the premises are always preserved for any choice of the subsequence. Now we may find an infinite  $a \subseteq \mathbb{N}$  such that

$$\overline{\left\{q_n|_\alpha \notin \bigcup_{m \in a} [A_m]: n \in b_\beta\right\}} \cap \overline{\left\{q_n|_\alpha \notin \bigcup_{m \in a} [A_m]: n \in a_\beta - b_\beta\right\}} \neq \emptyset, \tag{2}$$

holds for every  $\beta < \alpha$ . Thus, if the extension is obtained from any infinite  $b \subseteq a$ , the points  $q_n|_\alpha \notin \bigcup_{m \in b} [A_m]$  all extend to ultrafilters which do not contain  $A_\alpha$  hence the preservation of the premises (1) can be easily proved and so the inductive construction can continue.

The method of finding  $a$  is already employed in 3.1 of [20]. One considers an almost disjoint family  $\{a^\theta: \theta < 2^\omega\}$  of size  $2^\omega$  of infinite subsets of  $\mathbb{N}$ , one chooses  $x_\beta$  from the intersections (1) and sees that for at most one choice of  $\theta$  the point  $x_\beta$  does not belong to  $\overline{\{q_n|_\alpha \notin \bigcup_{m \in a^\theta} [A_m]: n \in b_\beta\}}$ , the same holds for the other part of the premise. Thus, by the counting argument we have that there is  $\theta < 2^\omega$  such that (2) holds for  $a = a^\theta$  for all  $\beta < \alpha$ .

One is left with checking that the extensions of Boolean algebras which are used in 3.1 of [20] do not introduce atoms, but it is clear as we extend by adding an infinite union of elements of the previous algebra. Other arguments are as in 3.1 of [20].  $\square$

**Proposition 4.5.** *There is a family  $(K_i)_{i \in \mathbb{N}}$  of pairwise disjoint perfect and totally disconnected compact spaces such that every operator on  $C(K_i)$  is a weak multiplier (thus,  $C(K_i)$  is Grothendieck) and for  $i \neq j$  every operator  $T \in L(C(K_i), C(K_j))$  is weakly compact.*

**Proof.** Consider  $K$  perfect and totally disconnected so that every operator in  $C(K)$  is a weak multiplier, fix a family  $(K_i)_{i \in \mathbb{N}}$  of pairwise disjoint clopen subsets of  $K$ , and let us prove that this family satisfies the desired conditions. Fixed  $i, j \in \mathbb{N}$  and an operator  $T : C(K_i) \rightarrow C(K_j)$ , we define  $\tilde{T} : C(K) \rightarrow C(K)$  by

$$\tilde{T} = I_j T P_i$$

which is a weak multiplier by hypothesis and so there are  $g$  a Borel function on  $K$  and  $S \in W(M(K))$  so that  $\tilde{T}^* = g \text{Id} + S$ . Besides, it is clear that  $P_\ell I_\ell$  is the identity on  $C(K_\ell)$  for every  $\ell \in \mathbb{N}$ , so we have that  $T = P_j \tilde{T} I_i$  and, therefore,

$$T^* = I_i^* \tilde{T}^* P_j^* = I_i^* (g \text{Id} + S) P_j^* = I_i^* g \text{Id} P_j^* + I_i^* S P_j^*.$$

Finally, the operator  $I_i^* S P_j^*$  is weakly compact and Lemma 4.3 tells us that

$$I_i^* g \text{Id} P_j^* = \begin{cases} 0 & \text{for } i \neq j, \\ g|_{K_i} \text{Id}_{M(K_i)} & \text{for } i = j, \end{cases}$$

finishing thus the proof.  $\square$

In the following we will be considering some compactifications of disjoint unions of perfect compact spaces  $K_i$ , that is, compact spaces where  $\bigcup_{i \in \mathbb{N}} K_i$  is open and dense. The next result, which will be the cornerstone of our further discussion, gives a sufficient condition on such compactifications for obtaining that the associated space of continuous functions is extremely non-complex.

**Proposition 4.6.** *Let  $(K_i)_{i \in \mathbb{N}}$  be the family given in Proposition 4.5 and let  $K$  be a compactification of  $\bigcup_{i \in \mathbb{N}} K_i$  so that every operator  $T : C(K_i) \rightarrow C(K_{-i})$  is weakly compact or every operator  $T : C(K_{-i}) \rightarrow C(K_i)$  is weakly compact. Then,  $C(K)$  is an extremely non-complex space.*

**Proof.** Fixed  $T \in L(C(K))$  and  $i \in \mathbb{N}$ , we can write

$$P_i T^2 I_i = P_i T (I_i P_i + I_{-i} P_{-i}) T I_i = (P_i T I_i) (P_i T I_i) + (P_i T I_{-i}) (P_{-i} T I_i)$$

and, therefore,

$$(P_i T^2 I_i)^* = (P_i T I_i)^* (P_i T I_i)^* + [(P_i T I_{-i}) (P_{-i} T I_i)]^*.$$

We observe that the second summand is weakly compact by hypothesis since  $P_{-i} T I_i \in L(C(K_i), C(K_{-i}))$  and  $P_i T I_{-i} \in L(C(K_{-i}), C(K_i))$ . Besides,  $P_i T I_i$  is an operator on  $C(K_i)$  thus, there exist a bounded Borel function  $g$  and a weakly compact operator  $S \in L(M(K_i))$  so that  $(P_i T I_i)^* = g \text{Id} + S$  and hence we can deduce that

$$(P_i T^2 I_i)^* = (g \text{Id} + S)^2 + [(P_i T I_{-i}) (P_{-i} T I_i)]^* = g^2 \text{Id} + S'$$

where  $S'$  is a weakly compact operator on  $M(K_i)$ . Now, since  $K_i$  is perfect for every  $i \in \mathbb{N}$ , we can use Theorem 3.4 to get that the set

$$\{x \in K_i : (P_i T^2 I_i)^* (\delta_x)(\{x\}) \geq 0\}$$

is dense in  $K_i$ . Finally, we use that for  $x \in K_i$  one has

$$(P_i T^2 I_i)^* (\delta_x)(\{x\}) = (P^i (T^2)^* I^i) (\delta_x)(\{x\}) = (T^2)^* (\delta_x)(\{x\})$$

and the fact that  $\bigcup_{i \in \mathbb{N}} K_i$  is dense in  $K$  to deduce that the set

$$\{x \in K : (T^2)^* (\delta_x)(\{x\}) \geq 0\}$$

is dense in  $K$  which, by making use of Lemma 3.1, tells us that  $T^2$  satisfies the Daugavet equation.  $\square$

Our next aim is to construct compact spaces  $K$  in such a way that  $C(K)$  is an extremely non-complex space and so that there exist operators on  $C(K)$  which are not weak multipliers. To do so, we consider a suitable family of totally disconnected compact spaces  $(K_i)_{i \in \mathbb{N}}$  and we obtain our compact spaces as zero-dimensional compactifications of the disjoint union  $\bigcup_{i \in \mathbb{N}} K_i$ . This kind of spaces has been completely described (see [19, Proposition 8.8], for instance),

namely, they are the Stone spaces of the Boolean subalgebras of the cartesian product  $\mathcal{A} = \prod_{i \in \mathbb{N}} \text{Clop}(K_i)$  which contain the subalgebra given by

$$\prod_{i \in \mathbb{N}} \text{Clop}(K_i) = \{a \in \mathcal{A} : \{i \in \mathbb{N} : a_i \neq \emptyset\} \text{ is finite or } \{i \in \mathbb{N} : a_i \neq K_i\} \text{ is finite}\}.$$

Therefore, we will be interested in constructing such a type of Boolean algebras. Indeed, let  $\mathcal{B} \subset \wp(\mathbb{N})$  be a Boolean algebra containing all finite and cofinite subsets of  $\mathbb{N}$ , then  $\bigoplus_{i < \omega}^{\mathcal{B}} \text{Clop}(K_i)$  will denote the Boolean algebra isomorphic to the algebra of subsets of  $\bigcup_{i \in \mathbb{N}} K_i$  consisting of elements of the form

$$c = c(b, F, \{a_j : j \in F\}) = \bigcup_{i \in b} K_i \cup \{a_j : j \in F\}$$

where  $b \in \mathcal{B}$ ,  $F$  is a finite subset of  $\mathbb{N}$  and  $a_j \in \text{Clop}(K_j)$  for all  $j \in F$ . By the preceding observations, the Stone space  $K$  of  $\bigoplus_{i < \omega}^{\mathcal{B}} \text{Clop}(K_i)$  is a compactification of the disjoint union  $\bigcup_{i \in \mathbb{N}} K_i$ . We are ready to state and prove a result which includes great part of the difficulties of our first construction.

**Theorem 4.7.** *Let  $\mathcal{B} \subset \wp(\mathbb{N})$  be a Boolean algebra containing all finite and cofinite subsets of  $\mathbb{N}$ ,  $(K_i)_{i \in \mathbb{N}}$  a family of totally disconnected compact spaces, and  $K$  the Stone space of  $\bigoplus_{i < \omega}^{\mathcal{B}} \text{Clop}(K_i)$ . Suppose that  $C(K_i)$  are Grothendieck Banach spaces so that, for  $j \neq i$ , every operator in  $L(C(K_i), C(K_j))$  is weakly compact and suppose that  $B_{M(L)}$  is weak\*-sequentially compact where  $L$  is the Stone space of  $\mathcal{B}$ . Then, every operator from  $C(K_i)$  into  $C(K_{-i})$  is weakly compact.*

**Proof.** Suppose that there are  $i \in \mathbb{N}$  and a bounded operator  $T : C(K_i) \rightarrow C(K_{-i})$  which is not weakly compact. Then by Gantmacher’s Theorem its adjoint neither is weakly compact, which means by the Dieudonné–Grothendieck Theorem [7, VII.14] that there are a bounded sequence of measures  $\mu_n \in M(K_{-i})$ , pairwise disjoint clopen subsets  $A_n$  of  $K_i$ , and  $\varepsilon > 0$  such that

$$|T^*(\mu_n)(A_n)| > \varepsilon \quad (n \in \mathbb{N}).$$

For each  $j \neq i$ , we use Lemma 4.3 to write  $(P_j I_{-i} T)^* = T^* P^{-i} I^j$  and we observe that  $P_j I_{-i} T$  is an operator from  $C(K_i)$  into  $C(K_j)$  and thus it is weakly compact by hypothesis. Hence,  $\{T^* P^{-i} I^j (P^j I^{-i}(\mu_n))\}_{n \in \mathbb{N}}$  is relatively weakly compact and so

$$\{T^* P^{-i} I^j (P^j I^{-i}(\mu_n))(A_n)\}_{n \in \mathbb{N}} \rightarrow 0$$

for every  $j \neq i$ . Therefore, by a diagonalization process we may find subsequences of  $\{\mu_n\}_{n \in \mathbb{N}}$  and  $\{A_n\}_{n \in \mathbb{N}}$  (which we also call  $\{\mu_n\}_{n \in \mathbb{N}}$  and  $\{A_n\}_{n \in \mathbb{N}}$ ) so that

$$|T^* P^{-i} I^j (P^j I^{-i}(\mu_n))(A_n)| < \frac{\varepsilon}{2^{j+1}}$$

for every  $n \in \mathbb{N}$  and  $j \in \{1, \dots, n\} \setminus \{i\}$ . Now, we consider the family of measures on  $K_{-i}$  given by

$$\tilde{\mu}_n = \mu_n - \sum_{1 \leq j \leq n}^{j \neq i} P^{-i} I^j P^j I^{-i}(\mu_n)$$

which is bounded since the measures involved have disjoint supports (in fact  $|\tilde{\mu}_n| \leq 2|\mu_n|$ ) and satisfies that

$$|\tilde{\mu}_n|(K_j) = 0 \quad (j \in \{1, \dots, n\} \setminus \{i\}) \quad \text{and} \quad |T^*(\tilde{\mu}_n)(A_n)| > \frac{\varepsilon}{2}$$

for every  $n \in \mathbb{N}$ . Indeed, we can estimate as follows:

$$|T^*(\tilde{\mu}_n)(A_n)| \geq |T^*(\mu_n)(A_n)| - \sum_{1 \leq j \leq n}^{j \neq i} |T^* P^{-i} I^j (P^j I^{-i}(\mu_n))(A_n)| > \varepsilon - \frac{\varepsilon}{2}$$

which tells us that the sequence  $\{T^*(\tilde{\mu}_n)\}_{n \in \mathbb{N}}$  is not relatively weakly compact.

Besides, let  $\mathcal{B}'$  be the subalgebra of  $\text{Clop}(K)$  corresponding to the elements of the form  $\bigcup_{i \in b} K_i$  for  $b \in \mathcal{B}$ , which is clearly isomorphic to  $\mathcal{B}$ . For  $n \in \mathbb{N}$ , we consider the finitely additive measure on  $\mathcal{B}'$  given by  $\nu_n = I^{-i}(\tilde{\mu}_n)|_{\mathcal{B}'}$ . Since  $B_{M(L)}$  is weak\*-sequentially compact, we may and do assume without loss of generality that  $\{\nu_n\}_{n \in \mathbb{N}}$  is pointwise convergent to a finitely additive measure  $\nu$  on  $\mathcal{B}'$  which, in addition, satisfies that  $\nu(K_j) = 0$  for every  $j \in \mathbb{N}$ . Indeed, for  $j \in \mathbb{N}$  and  $n > j$ , we can write

$$I^{-i}(\tilde{\mu}_n)(K_j) = \tilde{\mu}_n(K_j \cap K_{-i}) = 0$$

and, therefore,  $\nu(K_j) = \lim I^{-i}(\tilde{\mu}_n)(K_j) = 0$ . Next, we extend  $\nu$  to a finitely additive measure  $\mu$  on  $\bigoplus_{i < \omega}^{\mathcal{B}} \text{Clop}(K_i)$  by putting

$$\mu(c(b, F, \{a_j: j \in F\})) = \nu(c(b, \emptyset, \emptyset))$$

for  $c(b, F, \{a_j: j \in F\}) \in \bigoplus_{i < \omega}^{\mathcal{B}} \text{Clop}(K_i)$  and we observe that

$$\begin{aligned} I^{-i}(\tilde{\mu}_n)(c(b, F, \{a_j: j \in F\})) &= I^{-i}(\tilde{\mu}_n)\left(\left(\bigcup_{\ell \in b} K_\ell\right) \cup \left(\bigcup_{j \in F \setminus b} a_j\right)\right) \\ &= \nu_n\left(\bigcup_{\ell \in b} K_\ell\right) + \tilde{\mu}_n\left(\bigcup_{j \in F \setminus (b \cup \{i\})} a_j\right) \\ &= \nu_n\left(\bigcup_{\ell \in b} K_\ell\right) \end{aligned}$$

where the last equality holds for every sufficiently large  $n$  since  $|\tilde{\mu}_n|(K_j) = 0$  for every  $j \in \{1, \dots, n\} \setminus \{i\}$ . So, we have that

$$\{I^{-i}(\tilde{\mu}_n)(c(b, F, \{a_j: j \in F\}))\}_{n \in \mathbb{N}} \rightarrow \mu(c(b, F, \{a_j: j \in F\}))$$

which, together with Remark 4.2, tells us that  $\{I^{-i}(\tilde{\mu}_n)\}_{n \in \mathbb{N}}$  converges in the weak\* topology to the unique extension of  $\mu$  to an element of  $M(K)$  that we also denote by  $\mu$ . Now, since  $P^{-i} = (I_{-i})^*$  is weak\*-weak\* continuous and  $\tilde{\mu}_n = P^{-i}I^{-i}(\tilde{\mu}_n)$  for  $n \in \mathbb{N}$ , we obtain that  $\{\tilde{\mu}_n\}_{n \in \mathbb{N}}$  weak\* converges to  $P^{-i}(\mu)$  and, therefore,  $\{T^*(\tilde{\mu}_n)\}_{n \in \mathbb{N}}$  is weak\* convergent. Finally, the hypothesis of  $C(K_i)$  being a Grothendieck space tells us that  $\{T^*(\tilde{\mu}_n)\}_{n \in \mathbb{N}}$  converges weakly, contradicting the fact that it is not relatively weakly compact and completing thus the proof of the theorem.  $\square$

**Theorem 4.8.** *There is a compact space  $K$  so that  $C(K)$  is extremely non-complex and contains a complemented isomorphic copy of  $C(2^\omega)$ .*

**Proof.** Let us first recall that a countable independent family in a Boolean algebra is a family  $\{a_n: n \in \mathbb{N}\}$  such that

$$\varepsilon_1 a_{n_1} \cap \dots \cap \varepsilon_k a_{n_k} \neq \emptyset$$

for any distinct choice of  $n_1, \dots, n_k$ ,  $k \in \mathbb{N}$  and  $\varepsilon = \pm 1$  where  $-a$  denotes the complement of  $a$ . We consider an independent family of subsets of  $\mathbb{N}$  and the Boolean algebra  $\mathcal{B}$  generated by it and the finite subsets of  $\mathbb{N}$ , we take  $(K_i)_{i \in \mathbb{N}}$  the family of perfect and totally disconnected compact spaces given by Proposition 4.5 and we define  $K$  as the Stone space of the Boolean algebra  $\bigoplus_{i < \omega}^{\mathcal{B}} \text{Clop}(K_i)$ . Let us check that  $K$  satisfies the desired conditions: since  $\mathcal{B}$  is countable, its Stone space  $L$  has a countable basis of topology and so it is metrizable, which implies that  $(B_{M(L)}, w^*)$  is metrizable and, therefore, sequentially compact. Henceforth, since every  $C(K_i)$  is Grothendieck, we can use Theorem 4.7 and Proposition 4.6 to get that  $C(K)$  is extremely non-complex. In order to prove that  $C(K)$  has a complemented copy of  $C(2^\omega)$ , we fix  $x_i \in K_i$  for  $i \in \mathbb{N}$ , we consider  $\mathcal{B}'$  the Boolean subalgebra of  $\bigoplus_{i < \omega}^{\mathcal{B}} \text{Clop}(K_i)$  formed by elements of the form  $\bigcup_{i \in b} K_i$  for  $b \in \mathcal{B}$  (which is obviously isomorphic to  $\mathcal{B}$ ), and we define a projection  $P: \bigoplus_{i < \omega}^{\mathcal{B}} \text{Clop}(K_i) \rightarrow \mathcal{B}'$  by

$$P(c(b, F, \{a_j: j \in F\})) = \bigcup_{i \in b} \{K_i: x_i \in c(b, F, \{a_j: j \in F\})\}$$

for  $c(b, F, \{a_j: j \in F\}) \in \bigoplus_{i < \omega}^{\mathcal{B}} \text{Clop}(K_i)$ . As we noted in Fact 4.1,  $P$  induces a norm-one projection from  $C(K)$  onto a subspace isometric to  $C(L)$ . Finally, since  $\mathcal{B}$  contains an infinite independent family, its Stone's space maps

onto  $2^\omega$  and hence  $L$  is an uncountable metric compact space and, therefore,  $C(L)$  is isomorphic to  $C(2^\omega)$  by Miljutin’s Theorem (see [3, Theorem 4.4.8], for instance).  $\square$

**Remark 4.9.** By Theorem 1.3 and the comments below it the above space has many operators which are not weak multipliers.

We recall that being extremely non-complex is a property stable under isomorphisms in the class of  $C(K)$  spaces where  $K$  is a perfect compact space so that every operator on  $C(K)$  is a weak multiplier (see Remark 3.7). The next remark shows that this is no longer true when one leaves this class even if one keeps perfectness.

**Remark 4.10.** There are perfect compact spaces  $K$  and  $L$  so that  $C(K)$  is isomorphic to  $C(L)$ ,  $C(K)$  is extremely non-complex, and  $C(L)$  fails to be extremely non-complex. Indeed, let  $C(K)$  be from Theorem 4.8 and recall that the proof of that result gives the existence of a subspace  $X$  of  $C(K)$  so that

$$C(K) \sim X \oplus C(2^\omega) \sim X \oplus C(2^\omega) \oplus C(2^\omega) \sim C(K) \oplus C(2^\omega) \sim C(K) \oplus_\infty C(2^\omega) \sim C(K \dot{\cup} 2^\omega).$$

The latter space contains a complemented subspace isometric to a square so that the projection on its complement is of norm one. Therefore, Remark 4.12 of [16] tells us that  $C(K \dot{\cup} 2^\omega)$  is not extremely non-complex.

Our next goal is to construct a compact space  $K$  so that  $C(K)$  is extremely non-complex and contains  $\ell_\infty$ . In order to use our machinery we have to define a suitable Boolean algebra: let  $(\mathcal{A}_i)_{i \in \mathbb{N}}$  be a family of Boolean algebras so that  $\mathcal{A}_i$  is isomorphic to an algebra of subsets of  $X_i$  for  $i \in \mathbb{N}$ , where  $(X_i)_{i \in \mathbb{N}}$  is a family of pairwise disjoint sets. Then,  $\otimes_{i < \omega} \mathcal{A}_i$  will denote the Boolean algebra isomorphic to the algebra of subsets of  $\bigcup_{i \in \mathbb{N}} X_i$  consisting of elements  $A \in \wp(\bigcup_{i \in \mathbb{N}} X_i)$  satisfying the condition  $A \cap X_i \in \mathcal{A}_i$  for every  $i \in \mathbb{N}$ .

**Theorem 4.11.** *Let  $(K_i)_{i \in \mathbb{N}}$  be a family of totally disconnected compact spaces so that  $C(K_i)$  does not include any copy of  $\ell_\infty$  and such that every operator from  $C(K_i)$  into  $C(K_j)$  is weakly compact for  $j \neq i$ , and let  $K$  be the Stone space of  $\otimes_{i < \omega} \text{Clopt}(K_i)$ . Then, every operator from  $C(K_{-i})$  into  $C(K_i)$  is weakly compact.*

**Proof.** Suppose that there are  $i \in \mathbb{N}$  and a bounded operator  $T : C(K_{-i}) \rightarrow C(K_i)$  which is not weakly compact. Then by Gantmacher’s Theorem its adjoint neither is weakly compact, which means by the Dieudonné–Grothendieck Theorem [7, VII.14] that there are a bounded sequence of measures  $\mu_n \in M(K_i)$ , pairwise disjoint clopen subsets  $A_n$  of  $K_{-i}$ , and  $\varepsilon > 0$  such that

$$|T^*(\mu_n)(A_n)| > \varepsilon \quad (n \in \mathbb{N}).$$

For each  $j \neq i$ , we use Lemma 4.3 to write  $(TP_{-i}I_j)^* = P^jI^{-i}T^*$  and we observe that  $TP_{-i}I_j$  is an operator from  $C(K_j)$  into  $C(K_i)$  and thus it is weakly compact by hypothesis. Hence,  $\{P^jI^{-i}T^*(\mu_n)\}_{n \in \mathbb{N}}$  is relatively weakly compact and so

$$\{P^jI^{-i}T^*(\mu_n)(A_n \cap K_j)\}_{n \in \mathbb{N}} \rightarrow 0$$

for every  $j \neq i$ , a fact which is obviously true for  $j = i$ . We also observe that for  $j \neq i$  and  $n \in \mathbb{N}$  we have

$$P^jI^{-i}T^*(\mu_n)(A_n \cap K_j) = I^{-i}T^*(\mu_n)(A_n \cap K_j) = T^*(\mu_n)(A_n \cap K_j \cap K_{-i}) = T^*(\mu_n)(A_n \cap K_j)$$

so, by passing to a convenient subsequence, we get that

$$|T^*(\mu_n)(A_n \cap K_j)| < \frac{\varepsilon}{2^{j+1}}$$

for every  $j \in \{1, \dots, n\}$ . Now, for  $n \in \mathbb{N}$ , we consider the clopen subset of  $K_{-i}$  given by  $\tilde{A}_n = A_n \setminus \bigcup_{j=1}^n K_j$  which satisfies

$$\begin{aligned} |T^*(\mu_n)(\tilde{A}_n)| &= \left| T^*(\mu_n)(A_n) - T^*(\mu_n)\left(A_n \cap \left(\bigcup_{j=1}^n K_j\right)\right) \right| \\ &\geq |T^*(\mu_n)(A_n)| - \sum_{j=1}^n |T^*(\mu_n)(A_n \cap K_j)| > \varepsilon - \frac{\varepsilon}{2}. \end{aligned} \tag{3}$$

Besides, for  $j \in \mathbb{N} \setminus \{i\}$  we define  $\mathcal{A}'_j$  as the finite subalgebra of  $\text{Clop}(K_j)$  generated by

$$\{\tilde{A}_n \cap K_j : n < j\}$$

and we observe that  $\tilde{A}_n$  belongs to  $\bigotimes_{j \neq i} \mathcal{A}'_j$  for every  $n \in \mathbb{N}$ . Moreover, it is not hard to check that  $\bigotimes_{j \neq i} \mathcal{A}'_j$  is a subalgebra of  $\bigotimes_{j \neq i} \text{Clop}(K_j)$  isomorphic to  $\wp(\mathbb{N})$ , so the corresponding subspace  $Y$  of  $C(K_{-i})$  is isomorphic to  $\ell_\infty$ . In fact,  $Y$  is the closure of the space spanned by the characteristic functions of clopen sets in the image of the projection  $P : \bigotimes_{j \neq i} \text{Clop}(K_j) \rightarrow \bigotimes_{j \neq i} \mathcal{A}'_j$  given by

$$P(A) = \bigcup \{a_{j,k} : x_{j,k} \in A\} \quad \left( A \in \bigotimes_{j \neq i} \text{Clop}(K_j) \right)$$

where  $\{a_{j,k} : k < k_j\}$  denotes the collection of all atoms of  $\mathcal{A}'_j$  for all  $j \in \mathbb{N} \setminus \{i\}$  and some  $k_j \in \mathbb{N}$ , and  $x_{j,k} \in a_{j,k}$  are some fixed points for  $j \in \mathbb{N} \setminus \{i\}$  and  $k < k_j$ . Since every element of  $\bigotimes_{j \neq i} \mathcal{A}'_j$  is contained in  $K_{-i}$  we get that  $Y \subset C(K_{-i})$ . Now, for  $n \in \mathbb{N}$ , we observe that

$$(T|_Y)^*(\mu_n)(\tilde{A}_n) = T^*(\mu_n)(\tilde{A}_n)$$

therefore, by using (3), we can deduce that  $T|_Y$  is an operator from an injective space of continuous functions into  $C(K_i)$  which is not weakly compact and so it fixes a copy of  $c_0$  (see [22, Section 4] or [25, Theorem 4.5]). Therefore, we can use [24, Proposition 1.2] to deduce that  $T|_Y$  fixes a copy of  $\ell_\infty$  since  $Y$  is injective. This gives a contradiction since  $C(K_i)$  does not contain any copy of  $\ell_\infty$  by the hypothesis.  $\square$

**Theorem 4.12.** *There is a compact space  $K$  so that  $C(K)$  is extremely non-complex and contains an isometric complemented copy of  $\ell_\infty$ .*

**Proof.** Let  $(K_i)_{i \in \mathbb{N}}$  be the family of perfect and totally disconnected compact spaces given in Proposition 4.5, then the Stone space  $K$  of  $\bigotimes_{i < \omega} \text{Clop}(K_i)$  satisfies the requested conditions. Indeed, we choose  $x_i \in K_i$  for every  $i \in \mathbb{N}$ , we consider the subalgebra  $\mathcal{B}'$  of  $\text{Clop}(K)$  consisting of the elements of the form  $\bigcup_{i \in b} K_i$  for  $b \subseteq \mathbb{N}$ , and we define a projection  $P : \bigotimes_{i < \omega} \text{Clop}(K_i) \rightarrow \mathcal{B}'$  by

$$P(A) = \bigcup_{i \in b_A} K_i \quad \left( A \in \bigotimes_{i < \omega} \text{Clop}(K_i) \right)$$

where  $b_A = \{i \in \mathbb{N} : x_i \in A\}$ . By Fact 4.1,  $P$  induces a norm-one projection from  $C(K)$  onto a subspace isometric to  $C(S(\wp(\mathbb{N})))$  where  $S(\wp(\mathbb{N}))$  is the Stone space of  $\wp(\mathbb{N})$ , i.e., the Stone–Čech compactification  $\beta\mathbb{N}$  of  $\mathbb{N}$ . Hence,  $C(S(\wp(\mathbb{N})))$  is isometric to  $\ell_\infty$ . To finish the proof we observe that  $C(K_i)$  does not contain any copy of  $\ell_\infty$ . Indeed, a  $C(K)$  space containing a (necessarily complemented) copy of  $\ell_\infty$  obviously has hyperplanes isomorphic to the entire space which is not the case for  $C(K_i)$  by [20, Theorem 2.4]. Therefore, we can use Theorem 4.11 and Proposition 4.6 to obtain that  $C(K)$  is extremely non-complex.  $\square$

**Remark 4.13.** By Theorem 1.3 and the comments below it the above space has many operators which are not weak multipliers.

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