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Cascades of Hopf bifurcations from boundary delay ☆

José M. Arrieta [a,*], Neus Cónsul [b], Sergio M. Oliva [c]

[a] Dept. Mat. Aplicada, Facultad de Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain
[b] Dept. Matemática Aplicada I, ETSEIB, Universitat Politècnica de Catalunya, 08028 Barcelona, Spain
[c] Dept. Mat. Aplicada, IME, USP, Rua do Matão, 1.010, CEP 05508-900, São Paulo, SP, Brazil

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ABSTRACT

We consider a 1-dimensional reaction–diffusion equation with nonlinear boundary conditions of logistic type with delay. We deal with non-negative solutions and analyze the stability behavior of its unique positive equilibrium solution, which is given by the constant function $u \equiv 1$. We show that if the delay is small, this equilibrium solution is asymptotically stable, similar as in the case without delay. We also show that, as the delay goes to infinity, this equilibrium becomes unstable and undergoes a cascade of Hopf bifurcations. The structure of this cascade will depend on the parameters appearing in the equation. This equation shows some dynamical behavior that differs from the case where the nonlinearity with delay is in the interior of the domain.

1. Introduction

We consider the following 1-dimensional reaction–diffusion equation with nonlinear boundary conditions of logistic type with delay

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\xi) = u_{\xi\xi}(t,\xi), & (t,\xi) \in \mathbb{R}^+ \times (0,1), \\ \dfrac{\partial u}{\partial n}(t,\xi) = \alpha u(t,\xi)\big(1-u(t-r,\xi)\big), & \xi = 0,1,\ t \in \mathbb{R}^+, \\ u(t,\xi) = \varphi(t,\xi) \geqslant 0, & (t,\xi) \in [-r,0] \times [0,1], \end{cases} \tag{1.1}$$

where $\alpha > 0$ and $r \geqslant 0$. We consider non-negative initial conditions and study the asymptotic behavior of the solutions depending on the two parameters α and r.

In recent years there has been a lot of work dealing with reaction–diffusion equations with delays, see for instance [22,13] and references therein. Particularly, the logistic reaction term $f(u,v) = \alpha u(1-v)$ has been used in many applied models, first in ordinary differential equations of the type $\dot{x} = f(x,x)$ (the so-called logistic equation) and its extension to retarded differential equations $\dot{x} = f(x(t), x(t-r))$, called Hutchinson's equation. Recently, the retarded partial differential equation $\dot{u} = \Delta u + f(u(t), u(t-r))$ has been extensively studied and a nice survey can be found in [22]. A common feature to these equations is the appearance of oscillations, a fact which is very important in the model problems coming

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* Corresponding author.
E-mail addresses: arrieta@mat.ucm.es (J.M. Arrieta), neus.consul@upc.edu (N. Cónsul), smo@ime.usp.br (S.M. Oliva).

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from biology, chemical processes and others. The appearance of these oscillations has been studied by many authors like [7,18,16,8,11,9], where finally it was shown the existence of a cascade of Hopf bifurcations and, moreover, that the principal bifurcation is always of dimension two.

In this work we analyze Eq. (1.1), which has a nonlinear logistic term with delay on the boundary. Reaction–diffusion equations with nonlinear boundary conditions with delays have been used to describe phenomena related to collision-dominated plasma (see [21]).

When there is no delay present in the equation, that is $r = 0$, this equation generates a well-defined nonlinear semigroup in $X = \{u \in H^1(0,1);\ u \geqslant 0\}$. For this case the system has only two equilibrium solutions $u \equiv 0$ and $u \equiv 1$, the first is unstable and the second one is always asymptotically stable. The system is dissipative and gradient and the dynamics is well understood. For any initial condition $\varphi \geqslant 0$, $\varphi \neq 0$, its solution converges in $H^1(0,1)$ and even in stronger norms to the equilibrium solution $u \equiv 1$, see [3] for general results on parabolic equations with nonlinear boundary conditions.

When $r > 0$ the equation generates also a well-defined nonlinear semigroup in $Y = C([-r,0], X)$, where X is defined above. Again, it has only two equilibrium solutions $u \equiv 0$ and $u \equiv 1$. The trivial equilibrium solution is always unstable but the stability of $u \equiv 1$ is not determined a priori. We will see that for r small, this equilibrium is asymptotically stable but as r increases it will loose its stability and it will undergo a sequence of Hopf bifurcations as the parameter r increases from 0 to ∞. The structure of this cascade of Hopf bifurcations will depend on the parameter α.

Observe that the existence of cascades of Hopf bifurcations for these delay problems is in some sense expected. Note that if for fixed $\alpha > 0$, $\psi(t,\xi)$ is a periodic orbit of period T_0 of (1.1) for the value of the delay $r = r_0 > 0$, then ψ is also a periodic orbit for (1.1) for the sequence of delays $r_0 + kT_0$ for all $k \in Z$ such that $r_0 + kT_0 > 0$. This result is obtained just by noting that if $\psi(t,\xi)$ satisfies the first two equations of (1.1) for the delay r_0 then $\psi(t,\xi)$ also satisfies the same equations for the delay $r_0 + kT_0$. This follows just by noting that $\psi(t - (r_0 + kT_0), \xi) = \psi(t - kT_0 - r_0, \xi) = \psi(t - r_0, \xi)$. In particular, if for $r = r_0$, the equilibrium $u \equiv 1$ undergoes a Hopf bifurcation of periodic orbits, that is, we have continuous curves of delays $r(\mu)$ and periodic orbits $\psi_\mu(t,\xi)$ of period $T(\mu)$, $\mu \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ small such that $r(0) = r_0$ and $T(0) = T_0$, then the points $r_k = r_0 + kT_0$, $k \in Z$ such that $r_k > 0$ are also points where a Hopf bifurcation occurs, with delay curves $r_k(\mu) = r(\mu) + kT(\mu)$ and periodic orbits $\psi_\mu(t,\xi)$ of period $T(\mu)$.

We summarize the results of this paper in the two following results.

Theorem 1.1 *(Case $0 < \alpha \leqslant 2$). For fixed $\alpha \in (0,2]$ there exists a delay $r_0 > 0$, such that the equilibrium solution $u \equiv 1$ is asymptotically stable for $0 < r < r_0$ and unstable for $r_0 < r$. Moreover, there exists a $T_0 > r_0$, such that the equilibrium $u \equiv 1$, undergoes a Hopf bifurcation at the points $r_k = r_0 + kT_0$ for $k = 0, 1, 2, \ldots$ and these are the only values of r for which there is a bifurcation of the equilibrium solution $u \equiv 1$.*

Moreover, if for $r = r_0$, the bifurcation curves are given by the continuous functions $r_0(\mu)$, with $r(0) = r_0$, with periodic orbits ψ_μ of period $T(\mu)$ for $\mu \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, then these functions are all analytic, $T(0) = T_0$ and the bifurcation curves at $r = r_k$ are given by the functions $r_k(\mu) = r_0(\mu) + kT(\mu)$ with periodic orbits ψ_μ of period $T(\mu)$.

The periodic orbits bifurcating at $r = r_k$ for $k \geqslant 1$ are all unstable.

Theorem 1.2 *(Case $\alpha > 2$). For any fixed $\alpha > 2$ there exist $0 < r_0 < T_0$ and $0 < \tilde{r}_0 < \tilde{T}_0$ with the property that either $r_0 \neq \tilde{r}_0$ or $T_0 \neq \tilde{T}_0$, such that the equilibrium solution $u \equiv 1$ is asymptotically stable for $0 < r < \min\{r_0, \tilde{r}_0\}$ and unstable for $\min\{r_0, \tilde{r}_0\} < r$. Moreover, there exists a discrete set $I \subset (2,\infty)$ which is either a finite set or a sequence $= \{\alpha_j\}_{j=1}^\infty$ with $\alpha_j \to 2$, such that for all $\alpha \in (2,\infty) \setminus I$, the equilibrium $u \equiv 1$ undergoes a double cascade of Hopf bifurcations, at the points $r_k = r_0 + kT_0$ and $\tilde{r}_k = \tilde{r}_0 + k\tilde{T}_0$, $k = 0, 1, \ldots$ and these are the only values of r for which there is a bifurcation of the equilibrium solution $u \equiv 1$.*

Moreover, as in Theorem 1.1, if for $r = r_0, \tilde{r}_0$, the bifurcation curves are given by the continuous functions $r_0(\mu), \tilde{r}_0(\mu)$, with $r(0) = r_0$, $\tilde{r}(0) = \tilde{r}_0$, with periodic orbits $\psi_\mu, \tilde{\psi}_\mu$ of period $T(\mu), \tilde{T}(\mu)$ for $\mu \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, then all these functions are analytic, $T(0) = T_0$, $\tilde{T}(0) = \tilde{T}_0$ and the bifurcation curves at $r = r_k, \tilde{r}_k$ are given by the functions $r_k(\mu) = r_0(\mu) + kT(\mu)$, $\tilde{r}_k(\mu) = \tilde{r}_0(\mu) + k\tilde{T}(\mu)$ with periodic orbits ψ_μ, $\tilde{\psi}_\mu$ of period $T(\mu)$, $\tilde{T}(\mu)$. The periodic orbits bifurcating from $u \equiv 1$ at $r = r_k, \tilde{r}_k$ for $k \geqslant 1$ are all unstable.

Remark 1.3. The set I in Theorem 1.2 consists of the cases where either $\{r_k\}_{k=0}^\infty \subset \{\tilde{r}_k\}_{k=0}^\infty$ or $\{\tilde{r}_k\}_{k=0}^\infty \subset \{r_k\}_{k=0}^\infty$. In particular we are not excluding the case where $r_0 = \tilde{r}_0$ but $\{r_k\}_{k=0}^\infty \not\subset \{\tilde{r}_k\}_{k=0}^\infty$ or $\{\tilde{r}_k\}_{k=0}^\infty \not\subset \{r_k\}_{k=0}^\infty$. In this case we have two curves of periodic solutions bifurcating from the equilibrium solution $u \equiv 1$. The minimal periods of these two periodic solutions are different, and for values of the parameter r near $r_0 = \tilde{r}_0$ the two periodic solutions live in a 4-dimensional center stable manifold. Nevertheless the dynamics of these two periodic orbits is not clear.

This case is new and it does not occur when the nonlinearity with the delay is set in the interior of the domain (see [11]).

Remark 1.4. In terms of the first Hopf bifurcation ($r = r_0$) we conjecture that it is supercritical and the periodic orbit is globally stable, see [5] for some numerical evidence in this respect. This issue will be addressed in a future work.

As we mentioned above, when there is no delay present, the dynamics generated by Eq. (1.1) is a simple one, that is, it is a gradient system with just two equilibria and the evolution of any non-negative and nontrivial initial condition will approach the constant equilibria $u \equiv 1$. Also, in terms of the linearization around an equilibrium $u = u_0$, we see that the linearized operator is a nice self-adjoint operator (the laplacian with some Robin boundary conditions) and the spectrum consists of a sequence of simple real eigenvalues converging to $-\infty$ and the associated sequence of eigenfunctions form a nice sequence of orthonormal eigenfunctions which form a base in $L^2(0,1)$ for instance. We refer to [4] for a general reference for the dynamics of reaction–diffusion equations in several spaces dimensions with nonlinear boundary conditions.

On the other hand, notice that by turning on the delay in Eq. (1.1), Theorems 1.1 and 1.2 tell us that, even in a 1-dimensional problem like (1.1), we are producing some dynamic behavior which is more complex than its counterpart equation without delay. As a matter of fact, these theorems account for a cascade of Hopf bifurcations as the delay increases to infinity and this type of bifurcation is never present in a gradient system. Also, as it is stated in Theorem 1.2, for some values of the parameters of the equation, it is possible by increasing the delay to destabilize the constant equilibrium solution $u \equiv 1$ and make that 4-complex eigenvalues of the form $\pm i w_1$, $\pm i w_2$, cross the imaginary axis from the stable part to the unstable part of the complex plane. This is saying that at this value of the delay, we have a 4-dimensional center manifold around the equilibria $u \equiv 1$ and the dynamics in this center manifold is by no means a trivial one.

This phenomena is related to other results from the literature. As a matter of fact in [10] the authors compare the dynamics of a scalar 1-d reaction–diffusion equation of the type $u_t = u_{xx} + f(x,u)$ in $(0,1)$ with either Dirichlet or Neumann boundary conditions with the dynamics of the same equation where a linear term of the form $c(x)\int_0^1 \nu(x)u(x)\,dx$ is added to the equation. They show that by choosing appropriately the functions f, c and ν, the dynamics of this new nonlocal equation is much more complex than the original one. Actually, they are able to show that for appropriate f, c and ν the linearized equation around $u = 0$ has $2m$-complex eigenvalues of the form $\pm i w_1, \ldots, \pm i w_m$ where the numbers $w_1, \ldots, w_m$ and m are chosen a priori. Moreover the rest of the eigenvalues live in the stable part of the complex plane. If this is the case, a center manifold theorem can be applied which gives us the existence of an exponentially attracting center manifold. Observe also that if we choose $w_1, \ldots, w_m$ rationally independent we have that the linearized equation reduced to the linear $2m$-dimensional space is formed by quasiperiodic orbits. Moreover, they are able to show, that the function f can be chosen in such a way that the reduced vector field to the center manifold can have a prescribed Taylor expansion of an order as large as we want. In this way it is possible to see how very complicated dynamics may be generated by the addition of linear nonlocal term as described above. Several other generalizations of this ideas, applied to equations with non-x-dependent nonlinearities, are also obtained in [17]. Moreover, other complicated dynamics is also address in [6] where a combination of Center Manifold techniques and KAM theory is applied to show the persistence of complicated dynamics, persistence in the sense that is stable under high order term perturbations.

With respect to delays, we would like to mention also the work [11] where the author analyzes the bifurcations occurring in a reaction–diffusion equation when the delay acts in the interior of the domain (not on the boundary). He analyzed in detail the eigenvalue of the linearized problem and showed that it is possible to choose appropriate nonlinearities such that for some value of the delay we have again $2m$ purely imaginary eigenvalues of the form $\pm i w_1, \ldots, \pm i w_m$ where $w_1, \ldots, w_m$ and m can be chosen a priori. This is another indication of the possibility of obtaining complex dynamics by introducing a delay in a reaction–diffusion equation. One important remark should be made. In the case where $m \geqslant 2$, the possible complex dynamics obtained in [11] is necessarily unstable because in this case there is always some positive eigenvalues that makes the center manifold unstable and therefore the dynamics is not stable, see Theorems 3.2 and 3.3 of [11]. In this respect, our setting contemplates the possibility of having 4-complex eigenvalues crossing the imaginary axis (see Remark 1.3) and therefore, we may have some complex dynamics while the rest of the eigenvalues have negative real parts (they are stable) and therefore this complex dynamics maybe stable. In the present paper we do not deal with this case but it will be addressed in a future work. It maybe possible that the methods and techniques from [10,17,6,11] may give some light for this case.

We describe now the contents of the paper.

In Section 2 we reformulate problem (1.1) as an evolutionary problem in the space $C([-r,0], H^1(0,1))$. We indicate its main properties and obtain the linearized equation around the equilibrium $u \equiv 1$. We also introduce some notation that will be used hereafter.

In Section 3 we study the linearized problem around the equilibrium solution $u \equiv 1$. We will determine the behavior of the eigenvalues as functions of the parameters α and the delay, r. Here we will see that as the delay goes to infinity, we will have pairs of eigenvalues crossing transversally the imaginary axis and this produce Hopf bifurcations.

In Section 4 we study the stability of the equilibrium point $u \equiv 1$, obtain the cascades of Hopf bifurcations and provide a proof of the main results.

In Appendix A we provide a proof of the Hopf bifurcation theorem for the case we are studying. For this we follow the work of [7] and [18]. This theorem is used in Section 4.

2. Abstract setting and linearization

In this section we rewrite Eq. (1.1) as an abstract evolutionary equation in appropriate functional spaces. We start out by setting some notation that will be used through out the rest of the paper.

Let $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ be the unbounded linear operator, $A\varphi = -\varphi_{xx}$, with domain $D(A) = \{\varphi \in H^2(0,1):$ $\varphi_x(0) = \varphi_x(1) = 0\}$. Following [1,2], we know that this operator has an associated scale of Hilbert spaces X^β, $\beta \in \mathbb{R}$, which are obtained through interpolation–extrapolation procedures and that, since we are working in a Hilbert setting, they coincide, for $\beta \in [0,1]$, with the scale of fractional power spaces, that is, $X^\beta = D(A^\beta)$, and for $\beta \in [-1,0]$, $X^{-\beta}$ are the dual spaces of X^β.

Moreover, the operator A, or more properly speaking, the realization of the operator A in X^β, is an unbounded operator in X^β with domain $X^{1+\beta}$. The operator A generates an analytic semigroup in X^β for all $\beta \in \mathbb{R}$ and the following regularizing estimate holds

$$\left\| e^{At} u \right\|_{X^\gamma} \leqslant M t^{\beta-\gamma} \|u\|_{X^\beta}, \quad \gamma \leqslant \beta.$$

Moreover, the constant M can be chosen uniform for all $-1 \leqslant \gamma \leqslant \beta \leqslant 1$.

In particular, we are interested in the operator $A_{-1/2} : D(A_{-1/2}) \subset H^{-1}(0,1) \to H^{-1}(0,1)$, where $D(A_{-1/2}) = H^1(0,1)$. Given $r > 0$ and $\beta \in \mathbb{R}$, we define $C_\beta = C([-r,0], X^\beta)$, the Banach space of all continuous functions from $[-r,0]$ to X^β with sup-norm, and similarly $C^\beta = C([0,r], X^\beta)$.

We will denote by $H^{-1}(0,1)$ the dual space of $H^1(0,1)$ (notice that this notation is usually reserved for the dual of H_0^1) and we consider the duality product $\langle\cdot,\cdot\rangle$, between $H^1(0,1)$ and $H^{-1}(0,1)$ which is obtained as the extension of the standard L^2-product. That is, if $\psi \in H^1(0,1)$ and $f \in L^2(0,1) \subset H^{-1}(0,1)$ then

$$\langle \psi, f \rangle = \int_0^1 \psi(x) f(x)\, dx.$$

Let us consider, for $\alpha \in \mathbb{R}$, the linear operator $\mathcal{L} : H^1(0,1) \to H^{-1}(0,1)$, given by $\mathcal{L}(\psi)(\phi) = -\alpha(\psi(0)\phi(0) + \psi(1)\phi(1))$, for all $\phi \in H^1(0,1)$. Now we define some operators related with our Eq. (1.1). Moreover, given $a > 0$, we define the linear operator $L_a : C([-a,0], H^1(0,1)) \to H^{-1}(0,1)$, by $L_a(\phi) = \mathcal{L}(\phi(-a))$, for all $\phi \in C([-a,0], H^1(0,1))$. We will also consider L_{-a}.

Let $A_U : C_{-1/2} \to C_{-1/2}$ be the linear operator with domain

$$D(A_U) = \left\{ \varphi \in C_{1/2},\ \text{such that,}\ \dot\varphi \in C_{1/2},\ \varphi(0) \in H^1(0,1),\ \dot\varphi^-(0) = -A_{-1/2}\varphi(0) + L_r(\varphi) \right\},$$

and defined $(A_U\varphi)(\theta) = \dot\varphi(\theta)$, for all $\varphi \in D(A_U)$ and $\theta \in [-r,0]$.

Consider now, for a given $\alpha > 0$, the nonlinearity $g : H^1(0,1) \times H^1(0,1) \to H^{-1}(0,1)$, defined by $g(\phi,\Psi) = -\mathcal{L}(\phi(1-\Psi))$, for all $\phi,\Psi \in H^1(0,1)$. And, finally, given $\alpha, r > 0$, we define $G : C_{1/2} \to H^{-1}(0,1)$, as $G(\phi) = g(\phi(0), \phi(-r)) - L_r(\phi)$.

With this definition, Eq. (1.1) can be written as

$$\begin{cases} \dot u(t) + A_U u(t) = G(u_t), & t > 0, \\ u(t) = \varphi(t), & t \in [-r,0], \end{cases} \tag{2.1}$$

where $u_t(s) = u(t+s)$ for $s \in [-r,0]$. Following [14,15,19,20,22], we get that the solutions are in $C([-r,0], H^1(0,1))$ and, if the initial condition is positive, their are positive for all times.

We will need to analyze the stability properties of the equilibrium solution $u \equiv 1$ of Eq. (1.1) for $\alpha > 0$ and $r > 0$. This stability properties are given by the stability properties of the zero solution of the linearization around $u \equiv 1$. This linearized equation is given by

$$\begin{cases} \dfrac{dv}{dt} = v_{\xi\xi} & \text{in } (0,1) \times \mathbb{R}^+, \\ \dfrac{\partial v}{\partial n} = -\alpha v(t-r) & \text{in } \big\{\{0\} \cup \{1\}\big\} \times \mathbb{R}^+. \end{cases} \tag{2.2}$$

And, finally, using the definition of the operator $\mathcal{L}$ we can rewrite (2.2) in the abstract form

$$\begin{cases} \dot v(t) + A_U v(t) = \mathcal{L} v_t, & t > 0, \\ v(t) = \varphi(t), & t \in [-r,0]. \end{cases} \tag{2.3}$$

3. Eigenvalue behavior

The analysis of the stability properties of the equilibrium solution $u \equiv 1$ and of its possible bifurcations is based on the study of the numbers $\lambda \in \mathbb{C}$ for which there exists a nontrivial solution of the problem

$$\begin{cases} \varphi_{\xi\xi} = \lambda\varphi & \text{in } (0,1), \\ \dfrac{\partial\varphi}{\partial n} = -\alpha e^{-\lambda r}\varphi & \text{in } \{0\} \cup \{1\}. \end{cases} \tag{3.1}$$

Although properly speaking problem (3.1) is not an eigenvalue problem, we will call the numbers λ, eigenvalues and their corresponding solutions of (3.1) the eigenfunctions φ.

This section is devoted to the study of problem (3.1) and to the analysis of the dependence of the eigenvalues on the parameters $\alpha > 0$ and $r \geqslant 0$.

Notice first that $\lambda = 0$ is not an eigenvalue for any $\alpha > 0$, $r \geqslant 0$. This is due to the fact that the unique solution of the problem

$$\begin{cases} \varphi_{\xi\xi} = 0 & \text{in } (0,1), \\ \dfrac{\partial \varphi}{\partial n} = -\alpha\varphi & \text{in } \{0\} \cup \{1\} \end{cases} \tag{3.2}$$

is $\varphi = 0$.

Notice also that λ is an eigenvalue of (3.1) with eigenfunction φ if and only if $\bar{\lambda}$ is also an eigenvalue with eigenfunction $\bar{\varphi}$. To see this we just take complex conjugates in (3.1). In particular we just need to study the eigenvalues with non-negative imaginary part. From now on in this section we will consider only the case of $\lambda \in \mathbb{C}$ with $\mathrm{Im}(\lambda) \geqslant 0$.

Moreover, if we take $r = 0$ the eigenvalues of (3.1) are well determined. They all are negative real numbers, they can be explicitly calculated and they are all simple eigenvalues. We will denote them by $0 > \lambda_1 > \lambda_2 > \cdots$.

Let us set $\lambda = \omega^2$. Obviously, under this transformation the set $\{\lambda \in \mathbb{C},\ \mathrm{Im}(\lambda) \geqslant 0\}$ is mapped one to one to the first quadrant of the complex plane $\{w \in \mathbb{C},\ \mathrm{Re}(w) \geqslant 0,\ \mathrm{Im}(w) \geqslant 0\}$. Let us look for solutions φ of (3.1) of the form $\varphi(\xi) = c_1 \exp(-\omega\xi) + c_2 \exp(\omega\xi)$, for c_1 and c_2 not both being zero. After some simple calculations we obtain that w must satisfy that the following system of linear equations in (c_1, c_2) must have nontrivial solutions:

$$\begin{pmatrix} e^{-w}(\alpha e^{-w^2 r} - w) & e^{w}(\alpha e^{-w^2 r} + w) \\ \alpha e^{-w^2 r} + w & \alpha e^{-w^2 r} - w \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.3}$$

The fact that the rank of the matrix of the system above is never zero, for any $\alpha > 0$, $r \in \mathbb{R}$ and $w \in \mathbb{C}$ tells us that all eigenvalues of (3.1) are geometrically simple, that is they have only one independent eigenfunction.

If we denote by $\mathcal{F}(w, r)$ the determinant of the matrix of the system (3.3), the solutions of the equation $\mathcal{F}(w, r) = 0$ gives us the values of w so that $\lambda = w^2$ is an eigenvalue of (3.1). The function $\mathcal{F}$ is given by

$$\mathcal{F}(w, r) = e^{-\omega}\big(\alpha e^{-\omega^2 r} - \omega\big)^2 - e^{\omega}\big(\alpha e^{-\omega^2 r} + \omega\big)^2.$$

For fixed $r \in \mathbb{R}$ the function $\mathcal{F}(\cdot, r) : \mathbb{C} \to \mathbb{C}$ is a holomorphic function, which obviously is not identically zero. Therefore the roots of $\mathcal{F}(\cdot, r)$ form a discrete set in the complex plane with no accumulation point. Moreover, since the function $\mathcal{F}$ is continuous in both w and r we will have that if (w_k, r_k) are roots of $\mathcal{F}$ and $(w_k, r_k) \to (w_\infty, r_\infty) \in \mathbb{C} \times \mathbb{R}$ then (w_∞, r_∞) is also a root of $\mathcal{F}$. We also know that if $\mathcal{F}(w_0, r_0) = 0$ for some $(w_0, r_0) \in \mathbb{C} \times \mathbb{R}$ and $\mathcal{F}'(w_0, r_0) \neq 0$, where $'$ stands for d/dw, then by the implicit function theorem, there exists a neighborhood $B(w_0, \delta) \times (r_0 - \varepsilon, r_0 + \varepsilon)$ such that for all $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$ there exists a unique $w(r) \in B(w_0, \delta)$ such that $\mathcal{F}(w(r), r) = 0$. Moreover since the function $(w, r) \to \mathcal{F}(w, r)$ is analytic then the function $r \to w(r)$ is real analytic. We also have that, if for fixed r_0, the complex number w_0 is a root of multiplicity k of $\mathcal{F}(\cdot, r_0)$, that is $\mathcal{F}(w, r_0) = (w - w_0)^k G(w, r_0)$ for some analytic function $G(\cdot, r_0)$ with $G(w_0, r_0) \neq 0$, then in a neighborhood of r_0 we have exactly k continuous branches of roots $w_1(r), w_2(r), \ldots, w_k(r)$, maybe coinciding some of them.

Notice that F can be decomposed as $\mathcal{F}(w, r) = e^{-w} F(w, r) \cdot \tilde{F}(w, r)$ where

$$F(w, r) = \omega\big(e^{\omega} - 1\big) + \alpha e^{-\omega^2 r}\big(e^{\omega} + 1\big),$$
$$\tilde{F}(w, r) = \omega\big(e^{\omega} + 1\big) + \alpha e^{-\omega^2 r}\big(e^{\omega} - 1\big).$$

The following result states that F and $\tilde{F}$ cannot have common roots.

Lemma 3.1. *A pair (w, r), cannot be a simultaneous root of both F and $\tilde{F}$.*

Proof. If $F(w, r) = \mathcal{F}(w, r) = 0$ then $e^w \neq 1$, since if $e^w = 1$ from the first equation we obtain $F(w, r) = 2\alpha e^{-w^2 r} \neq 0$. With a similar argument we can also prove that $e^w \neq -1$.

Obtaining the value of $\alpha e^{-w^2 r}$ from the first equation and plugging it into the second equation we obtain

$$\frac{4we^w}{e^w + 1} = 0$$

which is impossible since $e^w \neq 1$ and therefore $w \neq 0$. $\square$

Lemma 3.2. *If $w \in \mathbb{C}^+ \setminus \{0\}$ is a root of $F(\cdot, r)$ (resp. $\tilde{F}(\cdot, r)$) for some $r \in \mathbb{R}$ and $w \notin (1+i)\mathbb{R}$ then there is not other $s \in \mathbb{R}$, $s \neq r$ such that w is a root of $F(\cdot, s)$ (resp. $\tilde{F}(\cdot, s)$).*

Proof. If $F(w, r) = F(w, s) = 0$ then we have that $\alpha(e^{-w^2 r} - e^{-w^2 s})(e^w + 1) = 0$. As we did in the previous lemma, $e^w + 1 \neq 0$ which implies that $e^{-w^2 r} = e^{-w^2 s}$. But this is only possible if there exists a $k \in N$ such that $w^2 r = w^2 s + 2k\pi i$, or equivalently $r = s + 2k\pi i / w^2$. But if $w \notin (1+i)\mathbb{R}$ then $w^2 \notin \mathbb{R}$. This means that either r or s are not real numbers. $\square$

As an immediate consequence of this lemma, we have:

Corollary 3.3. *If $w \in \mathbb{C}^+$ with $w \notin (1+i)\mathbb{R}$ is a root of $\mathcal{F}(\cdot, r)$ for some $r \in \mathbb{R}$ then there exists at most another $s \in \mathbb{R}$, $s \neq r$ so that w is a root of $\mathcal{F}(\cdot, s)$.*

In the following lemma, we will show that if $\lambda = w^2$, with $w = x + iy \in \mathbb{C}^+$ is an eigenvalue of (3.1) for some $r > 0$ then we have several restrictions on the places where w can lie. In fact, we can divide the complex plane in regions where the eigenvalues can lie and these regions will give us an insight on the dependence of the eigenvalues on r.

Lemma 3.4. *Let $\lambda = \omega^2$, $w = x + iy \in \mathbb{C}^+ \setminus \{0\}$ be an eigenvalue of* (3.1), *associated to the eigenfunction φ. Then the following hold*:

(i) *If $\mathtt{Im}(\lambda) \neq 0$, then there exists $k \in \mathbb{Z}$, $k \geqslant 0$, such that*

$$2xyr \in \big(2k\pi, (2k+1)\pi\big).$$

(ii) *If, moreover, $\mathtt{Re}(\lambda) \geqslant 0$ then*

$$2xyr \in \left[\frac{(4k+1)\pi}{2}, (2k+1)\pi\right).$$

Proof. Multiplying Eq. (3.1) by the conjugate of φ, integrating by parts and getting real and imaginary parts, we get that the following system must be satisfied

$$\begin{cases} -\displaystyle\int_0^1 \overline{\nabla\varphi}\nabla\varphi\, d\xi - \alpha \exp\big(-(x^2-y^2)r\big)\cos(2xyr)\big(\overline{\varphi(0)}\varphi(0) + \overline{\varphi(1)}\varphi(1)\big) = \big(x^2-y^2\big)\displaystyle\int_0^1 \overline{\varphi}\varphi\, d\xi, \\ \alpha \exp\big(-(x^2-y^2)r\big)\sin(2xyr)\big(\overline{\varphi(0)}\varphi(0) + \overline{\varphi(1)}\varphi(1)\big) = 2xy\displaystyle\int_0^1 \overline{\varphi}\varphi\, dx. \end{cases}$$

From this, and keeping in mind that $x, y \geqslant 0$, (i) and (ii) follow immediately from the restrictions imposed to the sign of $\sin(2xyr)$ and $\cos(2xyr)$. $\square$

We also have the following important result.

Lemma 3.5. *For any $R > 0$ and for any $a > 0$ there exists $b > 0$ such that there are no eigenvalues λ of* (3.1), *for any $0 \leqslant r \leqslant R$, in the region $\{\lambda \in \mathbb{C},\ \mathtt{Re}(\lambda) > -a,\ |\mathtt{Im}(\lambda)| > b\}$.*

Proof. As usual it is enough to prove that there are no eigenvalues in the region $\{\lambda \in \mathbb{C},\ \mathtt{Re}(\lambda) > -a,\ \mathtt{Im}(\lambda) > b\}$. This region is transformed by the map $\lambda = w^2$ into the region $\{w = x + iy \in \mathbb{C},\ x, y \geqslant 0,\ x^2 - y^2 > -a,\ 2xy > b\}$. This region is delimited by the intersection of the two hyperbolas $x^2 - y^2 = -a$ and $2xy = b$ in the first quadrant and it can be easily seen that $x^2 + y^2 \geqslant b$ for any $w = x + iy$ in the region. In particular we have that $2x^2 \geqslant b - a$ or equivalently $x \geqslant ((b-a)/2)^{\frac{1}{2}}$.

If $\lambda = w^2$ with $w = x + iy$ is an eigenvalue of (3.1) then w satisfies $F(w, r) = 0$ or $\tilde{F}(w, r) = 0$.

Consider first that $F(w, r) = 0$. Then $\omega(e^\omega - 1) + \alpha e^{-\omega^2 r}(e^\omega + 1)$ which implies that $\alpha e^{-\omega^2 r} = -\omega\frac{(e^\omega - 1)}{(e^\omega + 1)}$ and taking modulus, we obtain

$$\alpha e^{-(x^2-y^2)r} = \big(x^2 + y^2\big)^{\frac{1}{2}} \frac{|e^{x+iy} - 1|}{|e^{x+iy} + 1|} \geqslant \big(x^2 + y^2\big)^{\frac{1}{2}} \frac{|e^x - 1|}{|e^x + 1|}$$

which implies that

$$b^{\frac{1}{2}} \leqslant e^{aR} \frac{|1 + e^{-((b-a)/2)^{\frac{1}{2}}}|}{|1 - e^{-((b-a)/2)^{\frac{1}{2}}}|}.$$

But this last inequality cannot be satisfied for large b since the right-hand side is uniformly bounded as $b \to \infty$ and the left-hand side goes to infinity.

With a similar argument for $\tilde{F}$ we prove the result. $\square$

Lemma 3.6. *For any $R > 0$ there exists $a > 0$ such that there are no eigenvalues λ of* (3.1), *for any $0 \leqslant r \leqslant R$, in the region $\{\lambda \in \mathbb{C},\ \mathtt{Re}(\lambda) \geqslant a\}$.*

Proof. As before we need to consider only the case $\mathrm{Im}(\lambda) > 0$. The region $\{\lambda \in \mathbb{C},\ \mathrm{Re}(\lambda) > a,\ \mathrm{Im}(\lambda) > 0\}$ is transformed by the map $\lambda = w^2$ into the region of the first quadrant given by $\{w = x + iy\colon\ x, y \geqslant 0,\ x^2 - y^2 \geqslant a\}$ and in this region we have $(x^2 + y^2)^{\frac{1}{2}} \geqslant x \geqslant \sqrt{a}$.

If $\lambda = w^2$ with $w = x + iy$ is an eigenvalue of (3.1) then w satisfies $F(w, r) = 0$ or $\tilde{F}(w, r) = 0$.

Consider first that $F(w, r) = 0$. Then, as we did in the previous lemma $\alpha e^{-\omega^2 r} = -\omega \frac{(e^{\omega}-1)}{(e^{\omega}+1)}$ and taking modulus, we obtain

$$\alpha \geqslant \alpha e^{-(x^2-y^2)r} = \left(x^2 + y^2\right)^{\frac{1}{2}} \frac{|e^{x+iy} - 1|}{|e^{x+iy} + 1|} \geqslant x \frac{|e^x - 1|}{|e^x + 1|}.$$

Again, this inequality does not have any solution for large x since the right-hand side goes to infinity and the left-hand side is bounded.

With a similar argument for $\tilde{F}$ we prove the result. $\square$

Let us use this bounds to try to say something more about the eigenvalues.

Lemma 3.7. *There exist $r^* > 0$ and $a^* > 0$ such that for any r with $0 \leqslant r < r^*$ all eigenvalues λ of* (3.1) *satisfy $\mathrm{Re}(\lambda) \leqslant -a^*$.*

Proof. Consider a value $a^* > 0$ such that for $r = 0$ there are no eigenvalues in the region $\{\lambda;\ \mathrm{Re}(\lambda) \geqslant -a^*\}$. This is always possible since for $r = 0$ all eigenvalues are real negative and uniformly bounded away from zero.

Applying the previous lemma to $R = 1$ and a^* we obtain that there are positive numbers $a, b > 0$ such that for any $0 \leqslant r \leqslant 1$ if there is an eigenvalue with $\mathrm{Re}(\lambda) \geqslant -a^*$ then it must lie in the compact region $\{\lambda;\ -a^* \leqslant \mathrm{Re}(\lambda) \leqslant a,\ |\mathrm{Im}(\lambda)| \leqslant b\}$.

If the statement of the lemma would not be true, there would exist a sequence r_k with $r_k \overset{k\to\infty}{\longrightarrow} 0^+$ and corresponding eigenvalues λ_k lying in the region above. By compactness there would exist a subsequence of λ_k converging to λ_∞ and by the continuity of the branches of eigenvalues we would have that λ_∞ is an eigenvalue for (3.1) for $r = 0$. But this is impossible since $\mathrm{Re}(\lambda_\infty) \geqslant -a^*$. $\square$

From Lemmas 3.6 and 3.7 we easily see that if for some $r > 0$ there is an eigenvalue of (3.1) with positive real part, then this eigenvalue has crossed the imaginary axis at some point. Hence, we look now for eigenvalues λ that cross the imaginary axis, or equivalently for branches $w(r)$, roots of F, that cross the line $(1 + i)\mathbb{R}^+$, which is the diagonal of the first quadrant. Therefore we will look first for roots of F of the type $w = x + ix$.

We start with the following technical lemma that will be used in several points below.

Lemma 3.8. *For any $w = x + ix$ with $x > 0$ we have*

$$\mathrm{Re}\left(1 \pm \frac{2w}{e^w - e^{-w}}\right) > 0 \tag{3.4}$$

or equivalently if we define $\eta(x) = e^{2x} + e^{-2x} - 2\cos(2x)$ and $\gamma(x) = 2x((e^x - e^{-x})\cos(x) + (e^x + e^{-x})\sin(x))$ then

$$\eta(x) \pm \gamma(x) > 0. \tag{3.5}$$

Proof. Notice that

$$\mathrm{Re}\left(\frac{2w}{e^w - e^{-w}}\right) = \frac{1}{|e^w - e^{-w}|^2}\mathrm{Re}\left(2w\left(e^{\overline{w}} - e^{-\overline{w}}\right)\right),$$

but $|e^w - e^{-w}|^2 = e^{2x} + e^{-2x} - 2\cos(2x)$, and

$$\mathrm{Re}\left(2w\left(e^{\overline{w}} - e^{-\overline{w}}\right)\right) = 2x\left(\left(e^x - e^{-x}\right)\cos(x) + \left(e^x + e^{-x}\right)\sin(x)\right)$$

which shows the equivalence between both statements (3.4) and (3.5). We show this last inequality.

Using the expression $\cos(x) = (e^{ix} + e^{-ix})/2$ and $\sin(x) = (e^{ix} - e^{-ix})/2i$ we obtain that

$$\begin{aligned}\gamma(x) &= 2x\left[\left(e^x - e^{-x}\right)\cos(x) + \left(e^x + e^{-x}\right)\sin(x)\right] \\ &= x\left(e^{x+ix}(1 - i) + e^{-x+ix}(-1 - i) + e^{x-ix}(1 + i) + e^{-x-ix}(-1 + i)\right)\end{aligned}$$

and by a power series expansion we obtain

$$\gamma(x) = \sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+1)!} 2^{2k+3} (-1)^k.$$

Similarly,

$$\eta(x) = e^{2x} + e^{-2x} - 2\cos(2x) = \sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+2)!} 2^{4k+4}.$$

Therefore,

$$\begin{aligned}\eta(x) \pm \gamma(x) &= \sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+2)!} 2^{4k+4} \pm \sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+1)!} 2^{2k+3}(-1)^k \\ &= \sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+2)!} 2^{2k+3}\big(2^{2k+1} \pm (4k+2)(-1)^k\big).\end{aligned}$$

But the real numbers $a_\pm^k = 2^{2k+1} \pm (4k+2)(-1)^k$ are all positive except for $a_-^0 = 0$. This shows that $\eta(x) \pm \gamma(x) > 0$ for any $x > 0$. □

We have decomposed F as $\mathcal{F}(w,r) = e^{-w}\tilde{F}(w,r)F(w,r)$ where

$$\begin{aligned}F(\omega, r) &= \omega\big(e^{\omega} - 1\big) + \alpha e^{-\omega^2 r}\big(e^{\omega} + 1\big),\\ \tilde{F}(\omega, r) &= \omega\big(e^{\omega} + 1\big) + \alpha e^{-\omega^2 r}\big(e^{\omega} - 1\big).\end{aligned} \tag{3.6}$$

Lemma 3.9.

(i) *For any $\alpha > 0$ fixed, the value $w = x + ix$, $x > 0$, is a root of $F(\cdot, r)$ (resp. $\tilde{F}(\cdot, r)$) for some $r > 0$ if and only if $h(x) = 0$ (resp. $\tilde{h}(x) = 0$) is satisfied, where*

$$\begin{aligned}h(x) &= \big(2x^2 - \alpha^2\big)\big(e^x + e^{-x}\big) - 2\cos(x)\big(2x^2 + \alpha^2\big),\\ \tilde{h}(x) &= \big(2x^2 - \alpha^2\big)\big(e^x + e^{-x}\big) + 2\cos(x)\big(2x^2 + \alpha^2\big).\end{aligned}$$

(ii) *If $x > 0$ is a root of h and if we define $\Theta = \frac{5\pi}{4} + \operatorname{arctg}(\frac{2\sin(x)}{e^x - e^{-x}}) \in (\pi, 3\pi/2)$, $r_0 = \frac{2\pi - \Theta}{2x^2} > 0$ and $T_0 = \frac{\pi}{x^2}$ then we have $0 < r_0 < T_0$ and $(x + ix, r_k)$ are roots of F for all $r_k = r_0 + kT_0$ for all $k = 0, 1, 2, \ldots$. Moreover, these are the only values of r for which $w = x + ix$ is a root of F.*

(iii) *If $\tilde{x} > 0$ is a root of $\tilde{h}$ and if we define $\tilde{\Theta} = \frac{5\pi}{4} - \operatorname{arctg}(\frac{2\sin(\tilde{x})}{e^{\tilde{x}} - e^{-\tilde{x}}}) \in (\pi, 3\pi/2)$, $\tilde{r}_0 = \frac{2\pi - \tilde{\Theta}}{2\tilde{x}^2} > 0$, and $\tilde{T}_0 = \frac{\pi}{\tilde{x}^2}$ then we have $0 < \tilde{r}_0 < \tilde{T}_0$ and $(\tilde{x} + i\tilde{x}, \tilde{r}_k)$ are roots of $\tilde{F}$ for all $\tilde{r}_k = \tilde{r}_0 + k\tilde{T}_0$ for all $k = 0, 1, 2, \ldots$. Moreover, these are the only values of r for which $\tilde{w} = \tilde{x} + i\tilde{x}$ is a root of $\tilde{F}$.*

Proof. (i) If the value $w = x + ix$, $x > 0$, is a root of F or $\tilde{F}$, then

$$\big|w\big(e^w \mp 1\big)\big|^2 = \big|\alpha e^{-w^2 r}\big(e^w \pm 1\big)\big|^2 = \alpha^2\big|\big(e^w \pm 1\big)\big|^2.$$

Evaluating this expression in $w = x + ix$ we obtain the equations $h_\mp(x) = 0$.

(ii) If $x > 0$ is a root of h and we denote by $w = x + ix$, then we have

$$\big|w\big(e^w - 1\big)\big| = \alpha\big|e^w + 1\big|.$$

Hence, if we denote by $\Theta = Arg(\frac{w(e^w - 1)}{-\alpha(e^w + 1)}) \in [0, 2\pi)$ we have that $w(e^w - 1) = -e^{(\Theta + 2k\pi)i}\alpha(e^w + 1)$, $k \in Z$. In particular, since $w^2 = 2x^2 i$, we have that for all values $r = -\frac{\Theta + 2k\pi}{2x^2}$ we obtain

$$w\big(e^w - 1\big) = -e^{-w^2 r}\alpha\big(e^w + 1\big)$$

which is equivalent to $F(w, r) = 0$. A simple computation shows now that $\Theta = \frac{5\pi}{4} + \operatorname{arctg}(\frac{2\sin(x)}{e^x - e^{-x}}) \in (\pi, 3\pi/2)$ and that if $r_0 = \frac{2\pi - \Theta}{2x^2}$ then all the values $r > 0$ can be rewritten as $r_k = r_0 + kT_0$ with $T_0 = \frac{\pi}{x^2}$. □

We also have the following

Lemma 3.10.

(i) *For any $\alpha > 0$ the equation $h(x) = 0$ has only one positive root $x = x(\alpha)$. Moreover $x(\alpha)$ is an increasing function of α.*

(ii) *For any $\alpha \in (0, 2]$ the equation $\tilde{h}(x) = 0$ has no positive roots. Moreover for any $\alpha > 2$ the equation $\tilde{h}(x) = 0$ has only one positive root $\tilde{x} = \tilde{x}(\alpha)$. Moreover $\tilde{x}(\alpha)$ is an increasing function of α.*

Proof. The equations $h(x) = 0$, $\tilde{h}(x) = 0$ can be written as $g(x) = \alpha^2/2$, $\tilde{g}(x) = \alpha^2/2$, respectively, where

$$g(x) = x^2 \frac{e^x + e^{-x} - 2\cos(x)}{e^x + e^{-x} + 2\cos(x)}, \tag{3.7}$$

$$\tilde{g}(x) = x^2 \frac{e^x + e^{-x} + 2\cos(x)}{e^x + e^{-x} - 2\cos(x)}. \tag{3.8}$$

But the derivatives of g, $\tilde{g}$ satisfy

$$g'(x) = 2x\big(\eta(x) - \gamma(x)\big)/\big(e^x + e^{-x} + 2\cos(x)\big)^2 > 0,$$
$$\tilde{g}'(x) = 2x\big(\eta(x) + \gamma(x)\big)/\big(e^x + e^{-x} - 2\cos(x)\big)^2 > 0,$$

for any $x > 0$ by (3.5).

With this result, by the fact that $g(x), \tilde{g}(x) \to +\infty$ as $x \to +\infty$, and the fact that $g(0) = 0$ and $\tilde{g}(0) = 2$ we conclude the proof of the lemma. □

Lemma 3.11. *If $w = x + ix$ with $x > 0$ is a root of $F(\cdot, r)$ (resp. $\tilde{F}(\cdot, r)$) for some positive r then it is a simple root of $F(\cdot, r)$ (resp. $\tilde{F}(\cdot, r)$).*

Proof. Assume $w = x + ix$ is a root of $F(\cdot, r)$ such that $F'(w, r) = 0$. After some simple calculations we show that w must satisfy the equation

$$\frac{2w}{e^w - e^{-w}} = -1 - 2w^2 r.$$

But if $w = x + ix$ then $-1 - 2w^2 r = -1 - 2x^2 ri$ and taking real parts above, we obtain

$$\mathrm{Re}\left(1 + \frac{2w}{e^w - e^{-w}}\right) = 0$$

which is impossible by Lemma 3.8. With a similar argument we prove it for $\tilde{F}$. □

Lemma 3.12.

(i) *For any $\alpha > 0$, if $w^* = x^* + ix^*$, with $x^* > 0$, is a root of $F(\cdot, r^*)$ then there exist $\varepsilon^* > 0$ and an analytic function $w : (r^* - \varepsilon^*, r^* + \varepsilon^*) \to \mathbb{C}$ with $w(r^*) = w^*$ such that $F(w(r), r) = 0$ and these are the unique roots of F in a neighborhood of (w^*, r^*).*
(ii) *The branch $w(r)$ satisfies $w'(r^*) = a + bi$ with $a > b$, that is, the branch crosses transversally the diagonal $\{z = x + ix;\ x > 0\}$ and always in the same direction, from $\{z = x + iy;\ x < y\}$ towards $\{z = x + iy;\ x > y\}$. Moreover, $w(\cdot)$ is defined for all $r \geqslant r^*$, $w(r) \in \{z = x + iy;\ x > y > 0\}$ for all $r > r^*$ and $w(r) \to 0$ as $r \to +\infty$.*
(iii) *If $w_1(\cdot)$ and $w_2(\cdot)$ are two branches passing by w^* for r_1^* and r_2^* respectively with $r_1^* \neq r_2^*$, then both branches are different in the sense that $w_1(r) \neq w_2(s)$ for any $r \neq r_1^*$ and $s \neq r_2^*$.*
(iv) *Similar statements are obtained for $\tilde{F}$.*

Proof. (i) The existence of the branch and its analyticity follows from the simplicity of the roots given by the lemma above.

(ii) In order to study the crossing with the diagonal, let us compute $w'(r^*)$. By implicit differentiation and using that w^* is a root of $F(\cdot, r^*)$ we have that

$$w'\big(r^*\big) = \frac{(w^*)^3}{-1 - 2(w^*)^2 r^* + \frac{2w^*}{e^{-w^*} - e^{w^*}}}.$$

If we denote by

$$A = \frac{\sqrt{2}^2 x^3}{|-1 - 2(w^*)^2 r^* + \frac{2w^*}{e^{-w^*} - e^{w^*}}|^2} > 0,$$

then we can write

$$w'(r) = A\left(-2x^2 r(1+i) + (1-i)\left(1 + \frac{2\overline{w}^*}{e^{\overline{w}^*} - e^{-\overline{w}^*}}\right)\right).$$

If we express $1 + \frac{2\overline{w}^*}{e^{\overline{w}^*} - e^{-\overline{w}^*}} = a + bi$ then $w'(r) = A(-2x^2 r(1+i) + (1-i)(a+bi)) = A(-2x^2 r(1+i) + (a+b) + i(b-a))$ and $w(r)$ will cross transversally in the direction stated if and only if $a + b > b - a$ or equivalently if and only if $a > 0$. But $a = \mathrm{Re}(1 + \frac{2\overline{w}^*}{e^{\overline{w}^*} - e^{-\overline{w}^*}}) = \mathrm{Re}(1 + \frac{2w^*}{e^{w^*} - e^{-w^*}}) > 0$ by Lemma 3.8.

Assume that the branch $w(\cdot)$ is defined in $[r^*, r_\infty)$. Since the crossings with the diagonal are only in one direction then $w(r) \notin \{z = x+iy;\ x = y > 0\}$ for any $r \in (r^*, r_\infty)$. If there exists $r_0 \in (r^*, r_\infty)$ with $w(r_0) \in [0,\infty) \subset \mathbb{R}$ and we assume this is the first $r_0 > r^*$ with this property, it would mean that by continuity of the eigenvalues $\lambda(r_0) = w(r_0)^2 \in [0,\infty)$ is an eigenvalue of (3.1). But this is impossible. Therefore $w(r) \in \{z = x+iy;\ x > y > 0\}$ for all $r \in (r^*, r_\infty)$.

Since $F(w,r) = 0$ can be written as $\alpha e^{-\omega^2 r} = -\omega \frac{(e^\omega - 1)}{(e^\omega + 1)}$ and taking modulus, we obtain for $w(r) = x + iy$,

$$\alpha e^{-(x^2-y^2)r} = \left(x^2+y^2\right)^{\frac{1}{2}} \frac{|e^{x+iy}-1|}{|e^{x+iy}+1|} \geqslant \left(x^2+y^2\right)^{\frac{1}{2}} \frac{|e^{x}-1|}{|e^{x}+1|}. \tag{3.9}$$

Since $x > y$ the left-hand side is bounded by α for all $r > r^*$. This implies that $|w(r)|^2 = x^2 + y^2$ is uniformly bounded for all $r \in (r^*, r_\infty)$. This implies that $r_\infty = +\infty$. By the bounds from Lemma 3.4 we have that there exists a constant C such that $2xyr \leqslant C$ and therefore any accumulation point of $w(r)$ as $r \to \infty$ must lie in the set $\{x+iy;\ y = 0,\ x \geqslant 0\}$. But passing to the limit as $r \to \infty$ in (3.9) we obtain that necessarily $w(r) \to 0$ as $r \to \infty$.

(iii) That the two branches are different follows from Lemma 3.10. □

Remark 3.13. Notice that not only the two branches are different but we can show that they cross the diagonal with different inclination. For this we will show that $w_1'(r_1^*)/w_2'(r_2^*) \notin \mathbb{R}$. But,

$$\frac{w_1'(r_1^*)}{w_2'(r_2^*)} = 1 - \frac{2(w^*)^2(r_1^* - r_2^*)}{-1 - 2(w^*)^2 r_2^* + \frac{2w^*}{e^{-w^*} - e^{w^*}}} = 1 + \frac{2x^2 i (r_1^* - r_2^*)}{1 + 2x^2 r_2^* i + \frac{2w^*}{e^{w^*} - e^{-w^*}}}.$$

This implies that $w_1'(r_1^*)/w_2'(r_2^*) \in \mathbb{R}$ if and only if

$$\mathrm{Re}\left(1 + \frac{2w^*}{e^{w^*} - e^{-w^*}}\right) = 0$$

which is impossible by Lemma 3.8.

Summarizing the results of this section and rephrasing them for the original eigenvalue problem (3.1) we obtain:

Proposition 3.14. *We have the following results*:

(i) *If $0 < \alpha \leqslant 2$ there exists only one value $\lambda_0 = bi$, with $b > 0$, such that λ_0 is an eigenvalue of* (3.1) *for some $r > 0$. Moreover, $b = 2x(\alpha)^2$, where $x(\alpha)$ is the unique root of $h(x)$, where h is given by Lemma* 3.9.

(i1) *If we consider r_0 defined by Lemma* 3.9 *and $T_0 = \frac{2\pi}{b} = \frac{\pi}{x(\alpha)^2}$ then $0 < r_0 < T_0$ and all the values of r for which λ_0 is an eigenvalue of* (3.1) *are given by $r_k = r_0 + kT_0$, $k = 0, 1, \ldots$. Moreover, for all these values r_k, $k = 0, 1, \ldots$, the corresponding eigenfunction is always the same, which is given by the unique nonzero solution, φ_0 (up to a multiplicative constant) of the problem* (3.1) *with $r = r_0$ and $\lambda = \lambda_0$.*

(i2) *For any $0 < r < r_0$ there exists $a(r) > 0$ such that all eigenvalues of* (3.1) *satisfy $\mathrm{Re}(\lambda) \leqslant -a(r)$.*

(i3) *For any $k = 0, 1, \ldots$, there exist $\varepsilon_k > 0$ and an analytic function $\lambda_k : (r_k - \varepsilon_k, \infty) \to \mathbb{C}$ such that $\lambda_k(r)$ and its complex conjugate $\bar{\lambda}_k(r)$ are eigenvalues for* (3.1) *for the value of the delay r. Moreover, $\mathrm{Re}(\lambda_k(r)) < 0$ for $r < r_k$, $\lambda_k(r_k) = bi$, $\mathrm{Re}(\lambda_k(r)) > 0$ for $r > r_k$ and $\mathrm{Re}(\lambda'(r_k)) > 0$. We also have $\lambda_k(r) \to 0$ as $r \to \infty$. In particular, for $r \in (r_{k-1}, r_k)$ we have exactly $2k$ eigenvalues of* (3.1) *with positive real part.*

(i4) *Associated to the branches of eigenvalues $\lambda_k(r)$ we have the branches of eigenfunctions, which are given by analytic functions $\chi_k : (r_k - \varepsilon_k, \infty) \to H^1(0,1)$ where $\chi_k(r_k) = \varphi_0$.*

(ii) *If $\alpha > 2$ there exist only two values $\lambda_0 = bi$, $\tilde{\lambda}_0 = \tilde{b}i$ with $b, \tilde{b} > 0$, such that λ_0, $\tilde{\lambda}_0$ are eigenvalues of* (3.1) *for some $r > 0$. Moreover, $b = 2x(\alpha)^2$, $\tilde{b} = 2\tilde{x}(\alpha)^2$ where $x(\alpha)$ and $\tilde{x}(\alpha)$ are the unique roots of $h(x)$ and $\tilde{h}(x)$ respectively, where h and $\tilde{h}$ are given by Lemma* 3.9.

(ii1) *If we consider r_0, $\tilde{r}_0$ defined by Lemma* 3.9 *and $T_0 = \frac{2\pi}{b} = \frac{\pi}{x(\alpha)^2}$, $\tilde{T}_0 = \frac{2\pi}{\tilde{b}} = \frac{\pi}{\tilde{x}(\alpha)^2}$ then $0 < r_0 < T_0$, $0 < \tilde{r}_0 < \tilde{T}_0$ and either $r_0 \neq \tilde{r}_0$ or $T_0 \neq \tilde{T}_0$. Moreover, all the values of r for which λ_0 is an eigenvalue of* (3.1) *are given by $r_k = r_0 + kT_0$, $k = 0, 1, \ldots$ and all the values of r for which $\tilde{\lambda}_0$ is an eigenvalue of* (3.1) *are given by $\tilde{r}_k = \tilde{r}_0 + k\tilde{T}_0$, $k = 0, 1, \ldots$.*

(ii2) *For any $0 < r < \min\{r_0, \tilde{r}_0\}$ there exists $a(r) > 0$ such that all eigenvalues of* (3.1) *satisfy $\mathrm{Re}(\lambda) \leqslant -a(r)$.*

(ii3) *For any $k = 0, 1, \ldots$, there exist $\varepsilon_k > 0$ and an analytic functions $\lambda_k : (r_k - \varepsilon_k, \infty) \to \mathbb{C}$, $\tilde{\lambda}_k : (\tilde{r}_k - \varepsilon_k, \infty) \to \mathbb{C}$ such that $\mathrm{Re}(\lambda_k(r)) < 0$ for $r < r_k$, $\mathrm{Re}(\tilde{\lambda}_k(r)) < 0$ for $r < \tilde{r}_k$, $\lambda_k(r_k) = bi$, $\tilde{\lambda}_k(\tilde{r}_k) = \tilde{b}i$, $\mathrm{Re}(\lambda_k(r)) > 0$ for $r > r_k$, $\mathrm{Re}(\tilde{\lambda}_k(r)) > 0$ for $r > \tilde{r}_k$, $\mathrm{Re}(\lambda'(r_k)) > 0$ and $\mathrm{Re}(\tilde{\lambda}'(\tilde{r}_k)) > 0$. Moreover $\lambda_k(r), \tilde{\lambda}_k(r) \to 0$ as $r \to \infty$.*

4. Cascades of Hopf bifurcations. Proofs of the main results

In this section we analyze the stability properties of the equilibrium solution $u \equiv 1$ and show the existence of cascades of Hopf bifurcations.

We keep the notation from previous sections, in particular the meaning of λ_0, $\tilde{\lambda}_0$, r_0, $\tilde{r}_0$, T_0, $\tilde{T}_0$, from Proposition 3.14. We will show now several results that will conclude with a proof of the main results.

Proposition 4.1.

(i) *If for $0 < \alpha \leqslant 2$ we have $0 < r < r_0$ or for $\alpha > 2$ we have $0 < r < \min\{r_0, \tilde{r}_0\}$ then the equilibrium point $u \equiv 1$ is asymptotically stable.*

(ii) *If for $0 < \alpha \leqslant 2$ we have $r > r_0$ or for $\alpha > 2$ we have $r > \min\{r_0, \tilde{r}_0\}$ then the equilibrium point $u \equiv 1$ is unstable.*

Proof. (i) We just need to realize that, in both cases, from Proposition 3.14(i2) and (ii2), we have that there exists a function $a(r) > 0$ such that all the eigenvalues λ of (3.1) have $\mathrm{Re}(\lambda) < -a(r)$.

(ii) In both cases, there is at least one eigenvalue of (3.1) with positive real part by Proposition 3.14(i3) and (ii3). $\square$

Our basic result on Hopf bifurcations is the following

Proposition 4.2. *Assume $\alpha > 0$ and let $r = \rho > 0$ be a value of the delay for which problem (3.1) has an eigenvalue $\lambda = \beta i$, $\beta > 0$, with eigenfunction φ (and obviously it also has the eigenvalue $-\beta i$ with eigenfunction $\overline{\varphi}$). Assume also that none of the values $n\beta i$, $n = 2, 3, \ldots$ is an eigenvalue of (3.1) with $r = \rho$. Then, there exist $\varepsilon > 0$ and three analytic functions $r : (-\varepsilon, \varepsilon) \to \mathbb{R}$, $T : (-\varepsilon, \varepsilon) \to \mathbb{R}$ and $\Psi : (-\varepsilon, \varepsilon) \to C^1_{2\pi}(\mathbb{R}, H^1(0, 1))$ such that $r(0) = \rho$, $T(0) = \frac{2\pi}{b}$, $\Psi(0) = 1$ and for all $\mu \in (-\varepsilon, \varepsilon)$ the function*

$$\chi_\mu(t, \xi) = \Psi(\mu)\left(t\frac{2\pi}{T(\mu)}, \xi\right)$$

is a $T(\mu)$-periodic solution of problem (1.1) for the value of the delay $r = r(\mu)$ and $\chi_\mu \not\equiv 1$. Moreover these are the only periodic solutions near the equilibrium $u \equiv 1$ of period near $\frac{2\pi}{b}$ for r near ρ.

The proof of the proposition follows the lines of the Hopf bifurcation theorem given in [7] and [18], uses the results in Proposition 3.14 and the fact that $\mathrm{Re}(\lambda'(0)) > 0$. The proof is long, very technical and needs some extra notation and lemmas, so we will give it in Appendix A.

We can provide now a proof of the main results.

Proof of Theorem 1.1. Observe that from Proposition 3.14, if $0 < \alpha \leqslant 2$, we have that for all values of the delay $r_0 + kT_0$, $k = 0, 1, \ldots$, we can apply Proposition 4.2 and obtain the existence of a Hopf bifurcation at these points. Notice that for any of these values $r_0 + kT_0$, the linearization around $u \equiv 1$ has only two eigenvalues, $\pm bi$, $b > 0$, in the imaginary axis. Moreover, these are the only values of the delay for which there exists a bifurcation, since by Proposition 3.14, these are the unique values for which the eigenvalues of the linearization around the equilibrium point $u \equiv 1$, cross the imaginary axis. Moreover, if the bifurcating curves from r_0 are given by $r_0(\mu)$ with $r_0(0) = r_0$ and the periodic orbits are ψ_μ with period $T(\mu)$ with $T(0) = T_0$, then the bifurcating curves for $r_0 + kT_0$ have to be $r_k(\mu) = r_0(\mu) + kT(\mu)$ with periodic orbits ψ_μ and period $T(\mu)$. This is obtained just by noting that, as it was pointed out in the introduction, if ψ_μ is a periodic orbit with period $T(\mu)$ the value of the delay $r_0(\mu)$, then ψ_μ is also a periodic orbit for the value of the delay $r_0(\mu) + kT(\mu)$ for all $k = 0, 1, \ldots$. Moreover, from the uniqueness of the curves of the Hopf bifurcation we obtain that the branches bifurcating from $r_0 + kT_0$ are necessarily given by $r_0(\mu) + kT(\mu)$. The fact that the periodic orbits obtained from the Hopf bifurcation for $k \geqslant 1$ are all unstable are due to the fact that for $k \geqslant 1$, the linearization of the equilibrium point has at least two eigenvalues with positive real part. $\square$

Proof of Theorem 1.2. For the case $\alpha > 2$ we have to set of values of the delays which are candidates for Hopf bifurcating points, $r_k = r_0 + kT_0$, $\tilde{r}_k = \tilde{r}_0 + k\tilde{T}_0$. It is clear that if for some $k_0 = 0, 1, \ldots$, we have that $r_{k_0} \notin \{\tilde{r}_j\}_{j=0}^\infty$, then at $r = r_{k_0}$ there will be only two eigenvalues $\pm bi$ crossing the imaginary axis and, therefore, for this value of the delay we will be able to apply Proposition 4.2 and will obtain a Hopf bifurcation. If we denote by $r_{k_0}(\mu)$, ψ_μ, $T(\mu)$ the curves of delays, periodic orbits and periods, respectively, bifurcating from r_{k_0}, then for any other r_k, $k = 0, 1, \ldots$, we will have that the curves $r_k(\mu) = r_{k_0} + (k - k_0)T(\mu)$, ψ_μ, $T(\mu)$ are curves of delays, periodic orbits and periods, respectively, bifurcating from r_k. Moreover, from the uniqueness obtained in the Hopf bifurcation theorem, we deduce that these are the unique curves bifurcating from r_k. In a similar way we can argue if there exists $\tilde{r}_{k_0} \notin \{r_k\}_{k=0}^\infty$. For this case, we will obtain bifurcation curves emanating from $\tilde{r}_k$ for all $k = 0, 1, \ldots$.

Hence we will have a double cascades of Hopf bifurcations if we can show that $\{r_k\}_{k=0}^\infty \setminus \{\tilde{r}_k\}_{k=0}^\infty \neq \emptyset$ and $\{\tilde{r}_k\}_{k=0}^\infty \setminus \{r_k\}_{k=0}^\infty \neq \emptyset$. We will be able to show this relation for all values of $\alpha > 2$ except for a sequence of $\alpha_j \to 2$. In order to prove this, we proceed as follows.

Notice first that if we have $\{r_k\}_{k=0}^\infty \subset \{\tilde{r}_k\}_{k=0}^\infty$ then $r_0 = \tilde{r}_0 + k_0\tilde{T}_0$ and $r_0 + T_0 = \tilde{r}_0 + k_0\tilde{T}_0 + n\tilde{T}_0$ for some $k_0 \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$. From here we obtain that necessarily $T_0 = n\tilde{T}_0$ for some $j \in \mathbb{N}$. Similarly, if $\{\tilde{r}_k\}_{k=0}^\infty \subset \{r_k\}_{k=0}^\infty$ we will obtain that $\tilde{T}_0 = mT_0$ for some $m \in \mathbb{N}$. But we know that $T_0 = \frac{\pi}{x(\alpha)^2}$, $\tilde{T}_0 = \frac{\pi}{\tilde{x}(\alpha)^2}$, where $x(\alpha)$, $\tilde{x}(\alpha)$ are the unique roots of g and $\tilde{g}$ defined in (3.7), (3.8), respectively. Hence, we are looking for values of $\alpha > 2$ for which either $\tilde{x}(\alpha) = \sqrt{n}x(\alpha)$ or $x(\alpha) = \sqrt{m}\tilde{x}(\alpha)$ for some $n, m \in \mathbb{N}$.

Let us analyze first the case $n = m = 1$, that is, the case $x(\alpha) = \tilde{x}(\alpha)$. From the expression of the functions g and $\tilde{g}$, we have that $x(\alpha) = \tilde{x}(\alpha)$ if and only if $\cos(x(\alpha)) = 0$, which implies $x(\alpha) = \frac{\pi}{2} + k\pi$, $k = 0, 1, \ldots$ and $\alpha = \sqrt{2}(\frac{\pi}{2} + k\pi)$, $k = 0, 1, \ldots$. But, from Lemma 3.9, we have that $r_0 = \frac{2\pi - \Theta}{2x^2}$ with $\Theta = \frac{5\pi}{4} + \operatorname{arctg}(\frac{2\sin(x)}{e^x - e^{-x}})$ and $\tilde{r}_0 = \frac{2\pi - \tilde{\Theta}}{2x^2}$ with $\tilde{\Theta} = \frac{5\pi}{4} - \operatorname{arctg}(\frac{2\sin(x)}{e^x - e^{-x}})$. Since $\sin(x) = \pm 1$, we get that $r_0 \neq \tilde{r}_0$ and also $|r_0 - \tilde{r}_0| = \frac{1}{2x^2}|\Theta - \tilde{\Theta}| = \frac{1}{2x^2}2arctg(\frac{2}{e^x - e^{-x}}) < \frac{\pi}{x^2} = T_0 = \tilde{T}_0$. From here we easily get not only that $r_0 \neq \tilde{r}_0$ but that $\{r_k\} \cap \{\tilde{r}_k\} = \emptyset$. This implies that for this case we have always a double cascade of Hopf bifurcations.

We look now to the case where n or $m \geqslant 2$. Observe that, as it was proved in Lemma 3.10, the functions g and $\tilde{g}$ are strictly increasing. Moreover, it is obvious to see that $g(\frac{\pi}{2} + k\pi) = \tilde{g}(\frac{\pi}{2} + k\pi)$ for all $k = 0, 1, \ldots$. This implies that necessarily, for any $\alpha > 2$, if $x(\alpha) \in [\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi]$ then $\tilde{x}(\alpha) \in [\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi]$ also. Hence, we always have that for any $\alpha > 2$, $|x(\alpha) - \tilde{x}(\alpha)| \leqslant \pi$. Define now, for $\beta > 2$, the set

$$D_\beta = \bigl\{\alpha \geqslant \beta\colon \text{there exists } n \in \mathbb{N},\ n \geqslant 2 \text{ with } x(\alpha) = \sqrt{n}\tilde{x}(\alpha),\ \text{or } \tilde{x}(\alpha) = \sqrt{n}x(\alpha)\bigr\}.$$

Let us prove the following two statements:

(i) $D_\beta \subset D_\gamma$ if $\beta < \gamma$ and there exists a $\beta_0 > 2$ such that $D_{\beta_0} = \emptyset$.
(ii) For any $\beta > 2$, D_β is always a finite set.

That $D_\beta \subset D_\gamma$ if $\beta < \gamma$, is a direct consequence of the definition of D_β. Moreover, since $x(\alpha), \tilde{x}(\alpha) \to \infty$ as $\alpha \to \infty$, we can choose β_0 such that for any $\alpha \geqslant \beta_0$ we have $(\sqrt{2} - 1)\min\{x(\alpha), \tilde{x}(\alpha)\} > \pi$. This implies that $|\sqrt{n}x(\alpha) - \tilde{x}(\alpha)| \geqslant (\sqrt{n} - 1)\min\{x(\alpha), \tilde{x}(\alpha)\} - \pi > 0$, for any $n \geqslant 2$, which proves (i).

In order to prove (ii), notice that, since for $\alpha \geqslant \beta > 2$, we have $x(\alpha), \tilde{x}(\alpha) \geqslant d \equiv \min\{x(\beta), \tilde{x}(\beta)\} > 0$, then, if $\sqrt{n_0} > 1 + \frac{\pi}{d}$ we have $|\sqrt{n_0}x(\alpha) - \tilde{x}(\alpha)| \geqslant (\sqrt{n_0} - 1)|x(\alpha)| - |x(\alpha) - \tilde{x}(\alpha)| \geqslant (\sqrt{n_0} - 1)d - \pi > 0$. This means that

$$D_\beta = \bigl\{\alpha \in [\beta, \beta_0]\colon \text{there exists } n = 2, \ldots, n_0 \text{ with } x(\alpha) = \sqrt{n}\tilde{x}(\alpha),\ \text{or } \tilde{x}(\alpha) = \sqrt{n}x(\alpha)\bigr\}.$$

But then

$$D_\beta = \bigcup_{n=1}^{n_0}\bigl\{\alpha \in [\beta, \beta_0]\colon x(\alpha) = \sqrt{n}\tilde{x}(\alpha),\ \text{or } \tilde{x}(\alpha) = \sqrt{n}x(\alpha)\bigr\}.$$

But, g and $\tilde{g}$ are real analytic functions, strictly increasing for $x > 0$. Hence, $g^{-1}, \tilde{g}^{-1} : [\beta, \beta_0] \to \mathbb{R}$ are real analytic. In particular, for each n, the number of roots of the equation $g^{-1} - \sqrt{n}\tilde{g}^{-1}$ and $\tilde{g}^{-1} - \sqrt{n}g^{-1}$ in $[\beta, \beta_0]$ is finite for all n. This proves (ii).

If we consider now a decreasing sequence of $\beta_m \to 2$, then, since $D_{\beta_m} \subset D_{\beta_n}$ for $m \geqslant n$ and each of them is finite, if we define

$$D = \bigcup_{m=1}^{\infty} D_{\beta_m}$$

then D is either a finite set or a sequence converging to 2. Moreover, for all values of $\alpha > 2$ with $\alpha \notin I$, we have that $\{r_k\}_{k=1}^\infty \setminus \{\tilde{r}_k\}_{k=1}^\infty \neq \emptyset$ and $\{\tilde{r}_k\}_{k=1}^\infty \setminus \{r_k\}_{k=1}^\infty \neq \emptyset$ and a double cascade of Hopf bifurcations will occur. This concludes the proof of the theorem. $\square$

Appendix A. Hopf bifurcation

In this section we prove the existence of the Hopf bifurcation, where we adapt the results from [7,12,18] to our situation. This adaptation is not straightforward since a careful study of the functional setting and the inequalities involved in the proof must be done with care.

We show the Hopf bifurcation as a perturbation result, as was introduced by [12]. In order to do this, it is convenient to consider our equation and the corresponding eigenvalue problem from a different point of view. This new point of view will express the perturbation more clearly.

First of all, given $r, \alpha > 0$, we define the pairing $(\cdot,\cdot) : C^{1/2} \times C_{1/2} \to \mathbb{R}$, by

$$(\psi, \varphi) = \bigl\langle \psi(0), \varphi(0)\bigr\rangle + \int_{-r}^{0} \mathcal{L}\bigl(\psi(s+r)\bigr)\varphi(s)\, ds,$$

for all $\psi \in C^{1/2}$ and $\varphi \in C_{1/2}$.

This pairing will induce the transpose equation of the linear equation, that is the solutions of

$$\begin{cases} \dfrac{dv}{dt} = v_{\xi\xi} & \text{in } (0,1)\times\mathbb{R}^-, \\ \dfrac{\partial v}{\partial n} = -\alpha v(t+r) & \text{in } \big\{\{0\}\cup\{1\}\big\}\times\mathbb{R}^-. \end{cases} \tag{A.10}$$

In the same way as in Section 2 we can define it in an abstract form. This is done by defining the transpose, $A_U^T : C^{-1/2} \to C^{-1/2}$, to be the linear operator with domain

$$D\big(A_U^T\big) = \big\{\Psi \in C^{1/2},\ \text{such that},\ \dot{\Psi} \in C^{1/2},\ \Psi(0) \in H^1(0,1),\ -\dot{\Psi}^+(0) = -A_{-1/2}\Psi(0) + L_{-r}(\Psi)\big\},$$

and define $(A_U^T\Psi)(\theta) = -\dot{\Psi}(\theta)$, for all $\Psi \in D(A_U^T)$ and $\theta \in [0,r]$. And, in the same way, the solution exists, is positive (for positive initial data) and belongs to $C([0,r], H^1(0,1))$ for all time.

Remark A.1. For all $\varphi \in D(A_U)$ and $\Psi \in D(A_U^T)$, we have $(\Psi, A_U\varphi) = (A_U^T\Psi, \varphi)$.

Proof. Let be $\varphi \in D(A_U)$ and $\Psi \in D(A_U^T)$, then we have

$$\begin{aligned}
(\Psi, A_U\varphi) &= \big\langle \Psi(0), A_U\varphi(0)\big\rangle + \int_{-r}^{0} \mathcal{L}\big(\Psi(s+r)\big)A_U\varphi(s)\,ds \\
&= \big\langle \Psi(0), \dot{\varphi}(0)\big\rangle + \int_{-r}^{0} \mathcal{L}\big(\Psi(s+r)\big)\dot{\varphi}(s)\,ds \\
&= \big\langle \Psi(0), -A_{-1/2}\varphi(0)\big\rangle + \big\langle \Psi(0), L_r(\varphi)\big\rangle + \mathcal{L}\big(\Psi(s+r)\big)\varphi(s)\big|_{-r}^{0} - \int_{-r}^{0} \mathcal{L}\big(\dot{\Psi}(s+r)\big)\varphi(s)\,ds \\
&= \big\langle -A_{-1/2}^T\Psi(0), \varphi(0)\big\rangle + \mathcal{L}\big(\Psi(r)\big)\varphi(0) + \int_{-r}^{0} \mathcal{L}\big(A_U^T\Psi(s+r)\big)\varphi(s)\,ds \\
&= \big(A_U^T\Psi, \varphi\big). \qquad \square
\end{aligned}$$

Let us start by rewriting the eigenvalue problem to get the bifurcation equation. For each $\lambda \in \mathbb{C}$ and $r > 0$, define the linear operator $\Delta(\lambda, r) : H^{-1}(0,1) \to H^{-1}(0,1)$, by

$$\Delta(\lambda, r)u = -A_{-1/2}u + e^{-\lambda r}L_r u - \lambda u,$$

for all $0 \leqslant u \in H^1(0,1)$.

Remark A.2. We can now look the eigenvalues and eigenfunctions in a different perspective, compared to the previous section, that is, the operator A_U has an eigenvalue $\lambda = \omega^2 = ib = 2ix^2$, $x, b \neq 0$, for some $r > 0$, if and only if,

$$-A_{-1/2}\varphi + e^{-\lambda\theta}L_r\varphi - ib\varphi = 0$$

is solvable for some $b > 0$ and $\theta = (2\pi - \Theta) \in [0, 2\pi]$, where Θ was defined in Lemma 3.9. If this is the case, for a pair (b, θ) and φ, then for all $n = 0, 1, 2, \ldots$, we have

$$\Delta(ib, r_n)\varphi = 0,$$

where $r_n = \frac{\theta + 2n\pi}{b}$. We have shown in this article the existence of such sequence. So we are going to fix through out this section b, λ, θ, r_n and φ as above. In order to get the notation more simple, we will still denote the linear operators as A_U and A_U^T despite its dependence on r_n.

Consider

$$\tilde{\Phi}(s) = [\,\varphi e^{ibs} \quad \overline{\varphi} e^{-ibs}\,],$$

for all $-r_n \leqslant s \leqslant 0$ and

$$\tilde{\Psi}(s) = \begin{bmatrix} B_n \varphi e^{-ibs} \\ \overline{B_n} \overline{\varphi} e^{ibs} \end{bmatrix},$$

for all $0 \leqslant s \leqslant r_n$, where $B_n \in \mathbb{C}$ is such that $B_n[\int_0^1 \varphi^2(x)\,dx + r_n \mathcal{L}(\varphi^2) e^{-ibr_n}] = 1$. And, consider also the real basis related to $\tilde{\Phi}$ and $\tilde{\Psi}$,

$$\begin{aligned} \Phi(s) &= [\,\Phi_1(s) \quad \Phi_2(s)\,] = [\,\mathtt{Re}(\varphi e^{ibs}) \quad \mathtt{Im}(\varphi e^{ibs})\,] \\ &= [\,\tfrac{\varphi e^{ibs} + \overline{\varphi} e^{-ibs}}{2} \quad \tfrac{-i\varphi e^{ibs} + i\overline{\varphi} e^{-ibs}}{2}\,], \end{aligned}$$

for all $-r_n \leqslant s \leqslant 0$ and

$$\Psi(s) = \begin{bmatrix} \Psi_1(s) \\ \Psi_2(s) \end{bmatrix} = \begin{bmatrix} 2\,\mathtt{Re}(B_n \varphi e^{-ibs}) \\ -2\,\mathtt{Im}(B_n \varphi e^{-ibs}) \end{bmatrix} = \begin{bmatrix} B_n \varphi e^{-ibs} + \overline{B_n} \overline{\varphi} e^{ibs} \\ iB_n \varphi e^{-ibs} + -i\overline{B_n} \overline{\varphi} e^{ibs} \end{bmatrix},$$

for all $0 \leqslant s \leqslant r_n$.

We will denote, for a given r_n, by Λ the eigenfunction space of A_U corresponding to the eigenvalues $\lambda = \pm ib$, it is easy to see that Φ is a real basis for Λ and Ψ is a real basis of the eigenfunction space of A_U^T of A_U corresponding to $\lambda = \pm ib$. One can also check that

$$(\tilde{\Psi}, \tilde{\Phi}) = \begin{pmatrix} (\tilde{\Psi}_1, \tilde{\Phi}_1) & (\tilde{\Psi}_1, \tilde{\Phi}_2) \\ (\tilde{\Psi}_2, \tilde{\Phi}_1) & (\tilde{\Psi}_2, \tilde{\Phi}_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This follows from the definition of B_n and from the following equality: $ib(\varphi e^{-ib}, \overline{\varphi} e^{-ib}) = (A_U^* \varphi e^{-ib}, \overline{\varphi} e^{-ib}) = (\varphi e^{-ib}, A_U \overline{\varphi} e^{-ib}) = -ib(\varphi e^{-ib}, \overline{\varphi} e^{-ib}) = 0$.

Define now the projection $\pi_\Lambda : C_{1/2} \to \Lambda$ by $\pi_\Lambda(\vartheta) = \Phi(\Psi, \vartheta)$, for all $\vartheta \in C_{1/2}$, and, given $T_0 > 0$, let $\mathcal{P}_{T_0} = \{g \in C(\mathbb{R},\ H^{-1}(0,1))\colon\ g(t + T_0) = g(t),\ t \in \mathbb{R}\}$, with the norm $\|\cdot\|_{\mathcal{P}_{T_0}}$ defined by $\|g\|_{\mathcal{P}_{T_0}} = \sup_{t \in [0,T_0]} \|g(t)\|_{L^2(0,1)}$. With this, for each r_n and $T_0 > 0$, consider the linear operator $\mathcal{I} : \mathcal{P}_{T_0} \to \mathbb{R}^2$ defined, for any $g \in \mathcal{P}_{T_0}$ by

$$\mathcal{I}g = \int_0^{T_0} \langle g(t), \Psi(t) \rangle\,dt \triangleq \begin{bmatrix} \int_0^{T_0} \langle g(t), \Psi_1(t) \rangle\,dt \\ \int_0^{T_0} \langle g(t), \Psi_2(t) \rangle\,dt \end{bmatrix}.$$

Now, summarizing the results in [22,19,20,15,14] we get that:

Facts A.3.

1. *The linear equation* (2.2) *can be written in abstract form as*

$$\dot{u}(t) = -A_{-1/2} u(t) + L_r(u_t), \quad t \geqslant 0, \tag{A.11}$$

which has an analytic semigroup $U(t)$ whose infinitesimal generator is A_U.

2. *The nonlinear equation* (1.1) *can be written in abstract form as*

$$\dot{u}(t) = -A_{-1/2} u(t) + L_r(u_t) + G(u_t), \quad t \geqslant 0. \tag{A.12}$$

From now on n and α will be fixed. Following the usual techniques to study Hopf bifurcation, one should introduce a change of variables in order to have a perturbation of the origin. To do this, first, we introduce v and ρ such that $u = v + 1$ and $\rho = r - r_n$, and substituting this in (A.12) we get

$$\dot{v}(t) = -A_{-1/2} v(t) + L_{(\rho + r_n)}(v_t) + f\big(v(t) + 1, v(t - \rho - r_n) + 1\big) - L_{(\rho + r_n)}(v_t), \quad t \geqslant 0. \tag{A.13}$$

The next step is to fix also the period, therefore let $T_0 = \frac{2\pi}{b}$ and $w(t) = v(t(1 + \beta))$. With this, v is a $T_0(1 + \beta)$-periodic solution of (A.13) if and only if w is a T_0-periodic solution of

$$\begin{aligned} \dot{w}(t) &= -A_{-1/2} w(t) + L_{r_n}(w_t) \\ &\quad + \left\{ -\beta A_{-1/2} w(t) - L_{r_n}(w_t) + (1 + \beta) f\left(w(t) + 1, w\left(t - \frac{\rho + r_n}{1 + \beta} \right) + 1 \right) \right\}, \quad t \geqslant 0. \end{aligned} \tag{A.14}$$

Thus, we will define

$$\tilde{G}(\rho, \beta, w_t) = \left\{ -\beta A_{-1/2} w(t) - L_{r_n}(w_t) + (1 + \beta) f\left(w(t) + 1, w\left(t - \frac{\rho + r_n}{1 + \beta} \right) + 1 \right) \right\}. \tag{A.15}$$

Remark A.4. In fact, one should look for this equation, for each t, in the unknown $u_t(\cdot) \in C_{1/2}$ and, respectively, $v_t(\cdot)$ and $w_t(\cdot)$.

Following [12] and [22], we get that, for all $g \in \mathcal{P}_{T_0}$, the equation

$$\frac{dw}{dt} = A_U w + g \tag{A.16}$$

has a T_0-periodic solution if and only if $g \in \mathcal{N}(\mathcal{I})$.

With this, one can define the linear operator $\mathcal{K} : \mathcal{N}(\mathcal{I}) \to \mathcal{P}_{T_0}$ such that $\mathcal{K}(g)$ is the T_0-periodic solution of (A.16) satisfying $\pi_A(\mathcal{K}(g)) = 0$ or $(\Psi(\cdot), (\mathcal{K}(g))_0(\cdot)) = 0$. If we apply this to our case, we get that (A.14) has a T_0-periodic solution $w(t)$ if and only if there is a constant c such that

$$\mathcal{I}\tilde{G}(\rho, \beta, w_t) = 0, \quad \text{and} \tag{A.17}$$

$$w(t) = c\Phi_1(t) + \big[\mathcal{K}\tilde{G}(\rho, \beta, w_t)\big](t), \tag{A.18}$$

for all $t \in \mathbb{R}$.

Remark A.5. By applying the implicit function theorem for c, ρ and β sufficiently small, we can solve (A.18). Let $w(t) = w(c, \rho, \beta)(t)$ be this solution, then $w(c, 0, 0) - c\Phi_1 = o(|c|)$, as $|c| \to 0$. Moreover, since $w(c, \rho, \beta)$ satisfies (A.18) and (A.16), it is continuously differentiable in c, ρ, β and t.

The strategy is to expand all in terms of c. To do this, let $\rho = c\mu$, $\beta = c\delta$ and $w(c, c\mu, c\delta)(t) = c\Phi_1(t) + c^2 W(t)$, noting that, from the remark above, $W \in \mathcal{P}_{T_0}$ and cW is $O(|c|)$, as $|c| \to 0$. Thus we can rewrite (A.15) as

$$\begin{aligned}
\tilde{G}(c\mu, c\delta, w_t) &= c^2 \bigg\{ -\delta A_{-1/2}\Phi_1(t) - \delta c A_{-1/2} W(t) - \frac{1}{c} L_{r_n}\big((\Phi_1)_t\big) - L_{r_n}(W_t) \\
&\quad + \frac{1+c\delta}{c^2} f\bigg(1 + c\Phi_1(t) + c^2 W(t), 1 + c\Phi_1\bigg(t - \bigg(\frac{c\mu + r_n}{1+c\delta}\bigg)\bigg) + c^2 W\bigg(t - \bigg(\frac{c\mu + r_n}{1+c\delta}\bigg)\bigg)\bigg)\bigg\} \\
&= c^2 \bigg\{ -\delta A_{-1/2}\Phi_1(t) - \delta c A_{-1/2} W(t) - \frac{1}{c}\Big(L_{r_n}\big((\Phi_1)_t\big) - L_{(\frac{c\mu + r_n}{1+c\delta})}\big((\Phi_1)_t\big)\Big) \\
&\quad - \Big(L_{r_n}(W_t) - L_{(\frac{c\mu + r_n}{1+c\delta})}(W_t)\Big) + \delta L_{(\frac{c\mu + r_n}{1+c\delta})}\big((\Phi_1)_t\big) + c\delta L_{(\frac{c\mu + r_n}{1+c\delta})}(W_t) \\
&\quad + (1 + c\delta)\mathcal{L}\bigg(\big(\Phi_1(t) + cW(t)\big)\bigg(\Phi_1\bigg(t - \bigg(\frac{c\mu + r_n}{1+c\delta}\bigg)\bigg) + cW\bigg(t - \bigg(\frac{c\mu + r_n}{1+c\delta}\bigg)\bigg)\bigg)\bigg)\bigg\} \\
&= c^2 N(c, \mu, \delta, W_t).
\end{aligned} \tag{A.19}$$

So, Eqs. (A.17), (A.18) become equivalent to

$$\mathcal{I}N(c, \mu, \delta, W_t) = 0, \quad \text{and} \tag{A.20}$$

$$W(t) = \big[\mathcal{K}N(c, \mu, \delta, W_t)\big](t), \tag{A.21}$$

for all $t \in \mathbb{R}$, which is the bifurcation equation that we have to solve to observe the oscillatory behavior of Eq. (1.1) near the positive constant equilibrium.

Lemma A.6.

$$\lim_{c \to 0} N(c, \mu, \delta, W_t) = -\delta A_{-1/2}\Phi_1(t) + (\delta r_n - \mu) L_{r_n}\big((\dot{\Phi}_1)_t\big) + \delta L_{r_n}\big((\Phi_1)_t\big) + \mathcal{L}\big(\Phi_1(t)\Phi_1(t - r_n)\big).$$

Proof. The proof follows easily if we take the limit as c goes to zero in (A.19), taking into account Remark A.5. □

Lemma A.7. $\mathcal{I}N(0, 0, 0, W_t) = 0$.

Proof. Observe that

$$\mathcal{I}N(0, 0, 0, W_t) = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \int_0^{T_0} \langle B_n \varphi e^{-ibt}, N(0, 0, 0, W_t)\rangle\, dt \\ \int_0^{T_0} \langle \overline{B_n \varphi e^{-ibt}}, N(0, 0, 0, W_t)\rangle\, dt \end{pmatrix}.$$

Thus it is sufficient to show that $\int_0^{T_0}\langle\varphi e^{-ibt}, N(0,0,0,W_t)\rangle\,dt = 0$. In fact,

$$\int_0^{T_0}\big\langle\varphi e^{-ibt}, N(0,0,0,W_t)\big\rangle dt = \mathcal{L}\left(\varphi\left\{\int_0^{T_0} e^{-ibt}\Phi_1(t)\Phi_1(t-r_n)\,dt\right\}\right)$$

and the result follows easily if one observes that

$$e^{-ibs}\Phi_1(s)\Phi_1(s-r_n) = \frac{1}{4}\big(\varphi^2 e^{ib(s-r_n)} + \varphi\overline{\varphi}e^{-ib(s-r_n)} + \overline{\varphi}\varphi e^{-ib(s+r_n)} + \overline{\varphi}^2 e^{-ib(3s-r_n)}\big), \tag{A.22}$$

using that $\Phi_1(s) = (1/2)(\varphi e^{ibs} + \overline{\varphi}e^{-ibs})$. Finally, noting that $T_0 = \frac{2\pi}{b}$, we have the result. □

Following [18], since a periodic solution of (A.14) is a C^1 function, one can restrict its attention in (A.20), (A.21) to $W \in \mathcal{P}^1_{T_0} = \{g \in \mathcal{P}_{T_0},$ such that $\dot{g} \in \mathcal{P}_{T_0}\}$, with $\|g\|_{\mathcal{P}^1_{T_0}} = \|g\|_{\mathcal{P}_{T_0}} + \|\dot{g}\|_{\mathcal{P}_{T_0}}$. Thus it is not difficult to see that $\mathcal{I}N : I_{\bar{c}} \times \mathbb{R} \times I_{\bar{c}} \times \mathcal{P}^1_{T_0} \to \mathbb{R}$ is a continuously differentiable function. From Lemma A.7, we have that $N(0,0,0,W_t) \in \mathcal{N}(\mathcal{I})$. We have that, if we denote by $W^\sharp = \mathcal{K}[N(0,0,0,W_t)]$, using the decomposition (A.22) it is easy to see that

$$W^\sharp(t) = Ye^{2ibt} + \overline{Y}e^{-2ibt} + Z + \Phi(t)o \tag{A.23}$$

where $Y = -\frac{1}{4}e^{-i\theta}(-A_{-1/2} + \mathcal{L}e^{-2i\theta} - 2ib)^{-1}\mathcal{L}(\varphi^2)$, $Z = -\frac{1}{2}\cos(\theta)(-A_{-1/2} + \mathcal{L})^{-1}\mathcal{L}(\varphi\overline{\varphi})$, and $o = -(\Psi(\cdot), Ye^{2ib\cdot} + \overline{Y}e^{-2ib\cdot} + Z)$.

On the other hand, as shown before, there exists a continuous branch of eigenvalues $\lambda(r)$, such that $\mathrm{Re}(\dot{\lambda}(r_n)) > 0$. But we can say a little more now, observing that

$$\Delta(\lambda, r)\varphi = \big[-A_{-1/2} + e^{-\lambda r}\mathcal{L} - \lambda I\big]\varphi = 0, \tag{A.24}$$

we can differentiate with respect to r, and applying $\langle\varphi,\cdot\rangle$, we get that

$$\begin{aligned} 0 &= \left\langle\varphi, \frac{d}{dr}\big(\Delta(\lambda,r)\varphi\big)\right\rangle = \left\langle\varphi, \Delta\frac{d\varphi}{dr}\right\rangle - \left\langle\varphi, \frac{d\lambda}{dr}\big(re^{-\lambda r}\mathcal{L} + I\big)\varphi\right\rangle - \big\langle\varphi, \lambda e^{-\lambda r}\mathcal{L}\varphi\big\rangle \\ &= \left\langle\big(-A_{-1/2} + e^{-\lambda r}\mathcal{L} - \lambda I\big)\varphi, \frac{d\varphi}{dr}\right\rangle - \left\langle\varphi, \frac{d\lambda}{dr}\big(re^{-\lambda r}\mathcal{L} + I\big)\varphi\right\rangle - \big\langle\varphi, \lambda e^{-\lambda r}\mathcal{L}\varphi\big\rangle. \end{aligned} \tag{A.25}$$

Therefore, using the definition of B_n, we get

$$\dot{\lambda} = -B_n\lambda e^{-\lambda r}\mathcal{L}\big(\varphi^2\big). \tag{A.26}$$

Lemma A.8.

$$\frac{\partial\mathcal{I}N}{(\partial\mu, \partial\delta)}(0,0,0,W_t) = T_0\begin{pmatrix} \mathrm{Re}(\dot{\lambda}) & 0 \\ -\mathrm{Im}(\dot{\lambda}) & -\mathrm{Im}(\lambda)\end{pmatrix}.$$

Proof. First of all, we have that $\frac{\partial N}{\partial\mu}(0,0,0,W_t) = -L_{r_n}((\dot{\Phi}_1)_t)$ and $\frac{\partial N}{\partial\delta}(0,0,0,W_t) = -A_{-1/2}\Phi_1(t) + L_{r_n}((\Phi_1)_t) + r_nL_{r_n}((\dot{\Phi}_1)_t)$.

Moreover we can write $\Phi = \tilde{\Phi}H$ and $\Psi = H^{-1}\tilde{\Psi}$, where $H = \frac{1}{2}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}$.

Thus, in order to evaluate $\frac{\partial\mathcal{I}N}{\partial\mu}(0,0,0,W_t)$ we will start with

$$\begin{aligned} -\int_0^{T_0}\big\langle\Psi(t), L_{r_n}\big((\dot{\Phi}_1)_t\big)\big\rangle dt &= -H^{-1}\int_0^{T_0}\big\langle\tilde{\Psi}(t), L_{r_n}\big((\dot{\tilde{\Phi}})_t\big)\big\rangle dt\, H\begin{pmatrix}1\\0\end{pmatrix} \\ &= -H^{-1}\int_0^{T_0}\begin{pmatrix} \langle B_n\varphi e^{-\lambda t}, \mathcal{L}(\lambda\varphi e^{\lambda(t-r_n)})\rangle & \langle B_n\varphi e^{-\lambda t}, \mathcal{L}(-\lambda\overline{\varphi}e^{-\lambda(t-r_n)})\rangle \\ \langle\overline{B_n}\overline{\varphi}e^{\lambda t}, \mathcal{L}(\lambda\varphi e^{\lambda(t-r_n)})\rangle & \langle\overline{B_n}\overline{\varphi}e^{\lambda t}, \mathcal{L}(-\lambda\overline{\varphi}e^{-\lambda(t-r_n)})\rangle\end{pmatrix} dt\, H\begin{pmatrix}1\\0\end{pmatrix}. \end{aligned} \tag{A.27}$$

First of all, using the definition of distributional derivatives to commute derivatives and traces (recall that φ is as smooth as one wishes), following the same argument as to show that $(\tilde{\Psi}, \tilde{\Phi}) = I$, we have that the matrix inside the integral is diagonal, thus, having in mind (A.26), one has only to evaluate

$$\int_0^{T_0}\big\langle B_n\varphi e^{-\lambda(t)}, \mathcal{L}\big(\lambda\varphi e^{\lambda(t-r_n)}\big)\big\rangle dt = \int_0^{T_0} B_n\lambda e^{-\lambda r_n}\mathcal{L}\big(\varphi^2\big)dt = T_0B_n\lambda e^{-\lambda r_n}\mathcal{L}\big(\varphi^2\big) = -T_0\dot{\lambda}.$$

In order to evaluate $\frac{\partial \mathcal{I}N}{\partial \delta}(0,0,0,W_t)$, we proceed as above and we have

$$\int_0^{T_0} \langle \Psi(t), -A_{-1/2}\Phi_1(t) + L_{r_n}\big((\Phi_1)_t\big) + r_n L_{r_n}\big((\dot{\Phi}_1)_t\big)\rangle\, dt = T_0 \begin{pmatrix} 0 \\ -\mathtt{Im}(\lambda) \end{pmatrix}. \qquad \square$$

Lemma A.9.

$$\frac{\partial \mathcal{I}N}{\partial c}\big(0,0,0,W_t^\sharp\big) = T_0 \begin{pmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \end{pmatrix},$$

where $\mathcal{G}_1 = T_0 \mathtt{Re}(\mathcal{L}(B_n \varphi^2 Z(1+e^{-\lambda r_n}) + B_n \varphi \overline{\varphi} Y(e^{\lambda r_n} + e^{2\lambda r_n})))$.

Proof. First of all, we have that

$$\frac{\partial N}{\partial c}\big(0,0,0,W_t^\sharp\big) = \mathcal{L}\big(\Phi_1(t) W^\sharp(t-r_n) + \Phi_1(t-r_n) W^\sharp(t)\big).$$

Since we have $W^\sharp(t) = Ye^{2\lambda t} + \overline{Y}e^{-2\lambda t} + Z + \Phi(t)b = \tilde{W}(t) + \Phi(t)o$, we have that

$$\begin{aligned} \frac{\partial \mathcal{I}N}{\partial c}\big(0,0,0,W_t^\sharp\big) &= \int_0^{T_0} \left\langle \Psi, \frac{\partial N}{\partial c}\big(0,0,0,W_t^\sharp\big)\right\rangle dt \\ &= \int_0^{T_0} \big\langle \Psi, \mathcal{L}\big(\Phi_1(t)\tilde{W}(t-r_n) + \Phi_1(t-r_n)\tilde{W}(t)\big)\big\rangle\, dt, \end{aligned} \tag{A.28}$$

where we have used $\int_0^{T_0} \langle \Psi, \mathcal{L}(\Phi_1(t)\Phi(t-r_n)o + \Phi_1(t-r_n)\Phi(t)o)\rangle\, dt = 0$. Finally we finish the proof using the definition of Ψ, $\tilde{W}$ and T_0. $\square$

Lemma A.10. $\mathcal{G}_1 \neq 0$.

Proof. One can note that φ, Y and Z are symmetric around $x = \frac{1}{2}$, and we can choose φ in such a way that $\varphi(0) = \varphi(1) = 1$. Let us remind that $\lambda = ib$, θ is such that the bifurcation points are $r_n = \frac{\theta + 2n\pi}{b}$, $T_0 = \frac{2\pi}{b}$, φ satisfies $\varphi''(x) = \lambda\varphi(x)$, for $x \in (0,1)$, $-\varphi'(0) = -\alpha e^{-\lambda r}$ and $\varphi'(1) = -\alpha e^{-\lambda r}$, Y satisfies $Y''(x) = 2\lambda Y(x)$, for $x \in (0,1)$, $-Y'(0) = -\alpha(e^{-2i\theta}Y(0) + \frac{1}{4}e^{-i\theta})$ and $Y'(1) = -\alpha(e^{-2i\theta}Y(1) + \frac{1}{4}e^{-i\theta})$ and, finally $Z(x) \equiv -\frac{1}{2}\cos(\theta)$. With this, we must show that

$$\mathtt{Re}\left(B_n\left(Y(0)\big(e^{\lambda r_n} + e^{-2\lambda r_n}\big) - \frac{1}{2}\cos(\theta)\big(1 + e^{-\lambda r_n}\big)\right)\right) \neq 0.$$

Let be $(Y(0)(e^{\lambda r_n} + e^{-2\lambda r_n}) - \frac{1}{2}\cos(\theta)(1+e^{-\lambda r_n}))$. We have that $\frac{\mathtt{Re}(B_n)}{\mathtt{Im}(B_n)}$ does not depend on n, and using the definition of B_n we get

$$\frac{\mathtt{Re}(B_n)}{\mathtt{Im}(B_n)} = -\frac{\mathtt{Re}(\int_0^1 \varphi^2(x)\,dx - 2\alpha r_n \varphi^2(0)e^{-\lambda r_n})}{\mathtt{Im}(\int_0^1 \varphi^2(x)\,dx - 2\alpha r_n \varphi^2(0)e^{-\lambda r_n})} \xrightarrow{n\to\infty} \frac{\cos(\theta)}{\sin(\theta)}.$$

Therefore, if $\mathtt{Re}(B_n\zeta) = 0$, for all n and $\zeta \neq 0$, then

$$\frac{\mathtt{Re}(\int_0^1 \varphi^2(x)\,dx)}{\mathtt{Im}(\int_0^1 \varphi^2(x)\,dx)} = -\frac{\cos(\theta)}{\sin(\theta)},$$

thus, $\mathtt{Im}(e^{i\theta}\int_0^1 \varphi^2(x)\,dx) = 0$. Using that φ is an eigenfunction, we get that

$$0 = \mathtt{Im}\left(e^{i\theta}\int_0^1 \varphi^2(x)\,dx\right) = \alpha\left(-\frac{1}{\lambda} + \frac{2e^{\sqrt{\lambda}}}{\sqrt{\lambda}(1-e^{2\sqrt{\lambda}})}\right).$$

Therefore, λ must satisfy $\mathtt{Im}(\frac{2}{\sqrt{\lambda}(e^{-\sqrt{\lambda}} - e^{\sqrt{\lambda}})}) = -\frac{1}{b}$, or $\mathtt{Re}(\frac{2\sqrt{\lambda}}{e^{-\sqrt{\lambda}} - e^{\sqrt{\lambda}}}) = -1$, which contradicts (3.4) in Lemma 3.8.

It is left to prove that $\zeta \neq 0$. Once again, we suppose that $\zeta = 0$, that is $Y(0) = \frac{1}{2}\cos(\theta)\frac{1+e^{-i\theta}}{e^{i\theta}+e^{-2i\theta}} = \frac{\cos(\theta)}{2(2\cos(\theta)-1)} > 0$.

From the eigenvalue equation and the equation for Y, we have that

$$\begin{cases} \sqrt{2\lambda}\dfrac{1-e^{\sqrt{2\lambda}}}{1+e^{\sqrt{2\lambda}}}Y(0)=\alpha\left(e^{-2i\theta}Y(0)+\dfrac{e^{-i\theta}}{4}\right), \\ \sqrt{\lambda}\dfrac{1-e^{\sqrt{\lambda}}}{1+e^{\sqrt{\lambda}}}=\alpha e^{-i\theta}. \end{cases}$$

Since $\alpha e^{-i\theta}\neq 0$, we can divide the two equations and get

$$\sqrt{2}\frac{1+e^{\sqrt{\lambda}}-e^{\sqrt{2\lambda}}-e^{\sqrt{\lambda}+\sqrt{2\lambda}}}{1-e^{\sqrt{\lambda}+\sqrt{2\lambda}}-e^{\sqrt{\lambda}}+e^{\sqrt{2\lambda}}}=e^{-i\theta}+\frac{1}{4Y(0)}=\frac{2(\cos(\theta))^2+2\cos(\theta)-1}{2\cos(\theta)}-i\sin(\theta).$$

Expanding the left-hand side and collecting real and imaginary parts, one gets $\frac{R_b+iI_b}{D_b}$, where

$$\begin{aligned} D_b&=1-2e^{\sqrt{b/2}+\sqrt{b}}\cos(\sqrt{b/2}+\sqrt{b})-2e^{\sqrt{b/2}}\cos(\sqrt{b/2})+2e^{\sqrt{b}}\cos(\sqrt{b})+e^{2(\sqrt{b/2}+\sqrt{b})} \\ &\quad+2e^{2\sqrt{b/2}+\sqrt{b}}\cos(\sqrt{b})-2e^{\sqrt{b/2}+2\sqrt{b}}\cos(\sqrt{b/2})+e^{2\sqrt{b/2}}-2e^{\sqrt{b/2}+\sqrt{b}}\cos(\sqrt{b/2}-\sqrt{b})+e^{2\sqrt{b}}, \\ R_b&=\sqrt{2}\big(1-2e^{\sqrt{b/2}+\sqrt{b}}\cos(\sqrt{b/2}+\sqrt{b})-e^{2\sqrt{b/2}}+2e^{\sqrt{b/2}+\sqrt{b}}\cos(\sqrt{b/2}-\sqrt{b})-e^{2\sqrt{b}}+e^{2(\sqrt{b/2}+\sqrt{b})}\big), \quad \text{and} \\ I_b&=\sqrt{2}\big(2e^{\sqrt{b/2}}\sin(\sqrt{b/2})-2e^{\sqrt{b}}\sin(\sqrt{b})+2e^{2\sqrt{b/2}+\sqrt{b}}\sin(\sqrt{b})-2e^{\sqrt{b/2}+2\sqrt{b}}\sin\sqrt{b/2}\big). \end{aligned}$$

Therefore, we must have

$$\begin{cases} \dfrac{I_b}{D_b}=-\sin(\theta), \\ \dfrac{R_b}{D_b}=\dfrac{2(\cos(\theta))^2+2\cos(\theta)-1}{2\cos(\theta)}, \end{cases}$$

and from the second equation, since $\cos(\theta)<0$, we must have

$$\cos(\theta)=\frac{(2\frac{R_b}{D_b}-2)-\sqrt{(2\frac{R_b}{D_b}-2)^2+8}}{4},$$

and since $\sin^2\theta+\cos^2\theta=1$, we get

$$\frac{1}{D_b^2}\left(I_b^2+\frac{(R_b-D_b)^2}{2}-\frac{(R_b-D_b)\sqrt{4(R_b-D_b)^2+8D_b^2}}{4}\right)=\frac{1}{2}.$$

However, one can check that the left-hand side is always negative, which is a contradiction. □

Proof of Proposition 4.2. We know that $\frac{d\lambda}{dr}(r_n)>0$, $n=0,1,2,\dots$ and from Lemma A.7, $\mathcal{IN}(0,0,0,W_t)\equiv 0$. It follows from Lemma A.8 that we can take c_0, a neighborhood $B\subseteq\mathbb{R}$ of the origin, a neighborhood $V_0\subseteq\mathcal{P}^1_{T_0}$ of $W^\sharp$ and continuously differentiable functions $\mu,\delta:[-c_0,c_0]\times V_0\to B$, so that $\mu(0,W^\sharp)=0=\delta(0,W^\sharp)$, and for each $(c,W)\in[-c_0,c_0]\times V_0$, $(\mu,\delta)\in B\times B$, $\mathcal{IN}(c,\mu,\delta,W)=0$ if and only if $\mu=\mu(c,W)$, $\delta=\delta(c,W)$. Then we can define a differentiable map $\Omega:[-c_0,c_0]\times V_0\to\mathcal{P}^1_{T_0}$ by

$$\Omega(c,W)=W-\big[\mathcal{K}N\big(c,\mu(c,W),\delta(c,W),W_t\big)\big](t), \tag{A.29}$$

satisfying $\Omega(0,W^\sharp)=W^\sharp-[\mathcal{K}N(0,0,0,W^\sharp_t)](t)=W^\sharp-\mathcal{K}\mathcal{L}\Phi_1(t)\Phi_1(t-r_n)=0$. Once again, from Lemma A.7, we have $\frac{\partial}{\partial W}\mathcal{I}N(0,0,0,W)=0$, and differentiating (A.29) with respect to W at $c=0$, we can see that $\frac{\partial}{\partial W}\Omega(0,W^\sharp)=I$. Hence, $\frac{\partial}{\partial W}\Omega(0,W^\sharp):\mathcal{P}^1_{T_0}\to\mathcal{P}^1_{T_0}$ is bijective. We can now apply the implicit function theorem to solve Eq. (A.29). Specifically, there are a constant $c_1\in(0,c_0]$, and a neighborhood $V_1\subseteq V_0$ of $W^\sharp$, and a function $W^*:[-c-1,c-1]\to V_1$ such that $W^*(0)=W^\sharp$, and for each $(c,W)\in[-c_1,c-1]\times V_1$, $\Omega(c,W)=0$ if and only if $W=W^*(c)$. Therefore (A.13) has a T_0 periodic solution $W(t)$ near zero for ρ,β sufficiently small, if and only if $W(t)=c\Phi_1(t)+W(c,t)$, $\rho=c\mu(c,W^*(c))$, $\beta=c\delta(c,W^*(c))$, for some value of $c\in[-c_1,c_1]$, where $W^*(c,t)=(W^*(c))(t)$, for all $t\in\mathbb{R}$, then $\mathcal{IN}(c,\mu^*(c),\delta^*(c),W^*(c))\equiv 0$, for all $c\in[-c_1,c_1]$, where $\mu^*(c)=\mu(c,W^*(c)),\delta^*(c)=\delta(c,W^*(c))$. Since $\frac{d\lambda}{dr}(r_n)>0$, $\frac{\partial\mathcal{IN}}{\partial(\mu,\delta)}(0,0,0,W)$ is invertible. Differentiating $\mathcal{IN}$ with respect to c at $c=0$, and applying Lemmas A.9, A.10 for c small enough, we arrive at, for some real $\hbar\neq 0$,

$$\frac{d\mu^*}{dc}(0)=\hbar\Gamma(n,\xi)c^2+O\big(c^3\big)\neq 0, \quad \text{where } \Gamma(k,\xi)=\frac{\mathcal{G}_1}{Real(\dot{\lambda})}.$$

This implies that for each $n = 0, 1, 2, \ldots,$ the Hopf bifurcation arising from the constant positive equilibrium occurs as the delay r passes monotonically through each r_n, and thus proves the proposition. □

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