J. Math. Anal. Appl. 367 (2010) 389–400

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

An application for an implicit function theorem of Craven and Nashed: Continuum limits of lattice differential equations

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ARTICLE INFO

Article history:
Received 1 August 2009
Available online 28 January 2010
Submitted by Goong Chen

Keywords:
Lattice differential equations
Continuum limit
Implicit function theorem

ABSTRACT

We deal with the continuum limit of lattice differential equations and show how an implicit function theorem of Craven and Nashed can be used in order to continuate solutions of the resulting partial differential equation to solutions of the original spatially discrete system.

1. Introduction

In recent years, there has been considerable interest in travelling wavefront solutions of lattice differential equations (LDEs). Typically, an LDE has the form

$$\dot{u} = F(u) \tag{1.1}$$

with $u \in \mathcal{U}$ and $F : \mathcal{U} \to \mathcal{U}$. Here, $\mathcal{U} \subseteq B$ denotes a Banach space which, as a set, is contained in

$$B = \left(\mathbb{R}^n\right)^{\Lambda} = \left\{u \mid u : \Lambda \to \mathbb{R}^n\right\}, \tag{1.2}$$

the set of vectors u indexed by the set Λ. In most cases, Λ is a lattice (i.e., a discrete finite or infinite subset of $\mathbb{R}^d$ with some regular spatial structure). Typical examples are the d-dimensional integer lattice $\mathbb{Z}^d$, the hexagonal lattice in $\mathbb{R}^2$ and the crystallographic lattices in $\mathbb{R}^3$.

In many applications, LDEs are closely related to partial differential equations (PDEs). For example, LDEs might appear as spatial discretizations of PDEs or, the other way round, PDEs arise naturally as the continuum limit of LDEs if some discreteness parameter tends to zero. To become even more concrete, let us consider a classical example: The spatially discrete Nagumo equation is given by

$$\dot{u}(x,t) = d\big(u(x+1,t) - 2u(x,t) + u(x-1,t)\big) + f\big(u(x,t)\big) \tag{1.3}$$

with $d > 0$, $f(u) = u(u-a)(1-u)$ and $0 < a < \frac{1}{2}$. This model system is used, for example, to describe the conduction in myelinated nerve axons (cf. [1]). In [7, Theorem 1], there has been proved that for every sufficiently large d there exists a velocity $c \in \mathbb{R}$ such that this equation has travelling wavefront solutions $u(x,t) = u(x+ct)$ which satisfy $u(-\infty) = 0$ and $u(\infty) = 1$.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.01.043

In numerical calculations (using, for example, the methods presented in [3]), it can be observed that these solutions increase in width by a factor of $\sqrt{d}$ as $d \to \infty$. If we, therefore, rescale the spatial scale by a factor of $\frac{1}{\sqrt{d}}$, it turns out that the shape of a travelling wavefront solution changes only slightly if the parameter d is varied and, as a matter of fact, has a limit as $d \to \infty$.

In the following, we investigate this observation analytically. If we plug the travelling wave Ansatz $u(x,t) = u(x+ct)$ into Eq. (1.3), we get the travelling wave equation

$$T(u) = 0 \tag{1.4}$$

with

$$T(u) = -cu' + d\Delta^2(1,u) + f(u), \tag{1.5}$$

the discrete second derivative

$$\Delta^2(\tau, u; z) = \begin{cases} \frac{1}{\tau^2}\big(u(z+\tau) - 2u(z) + u(z-\tau)\big) & \text{for } \tau \neq 0, \\ u''(z) & \text{for } \tau = 0, \end{cases} \tag{1.6}$$

and the real variable $z = x+ct$. Using the stretching operator $S_\tau(u;z) = u(\tau z)$, we obtain the functional differential equation of mixed type

$$0 = S^{-1}_{\frac{1}{\sqrt{d}}} T S_{\frac{1}{\sqrt{d}}}(u) = -\frac{c}{\sqrt{d}}u' + \Delta^2\left(\frac{1}{\sqrt{d}}, u\right) + f(u). \tag{1.7}$$

Here, $\frac{1}{\sqrt{d}}$ serves as the discreteness parameter. Let $\tilde{c} = \frac{c}{\sqrt{d}}$. In the continuum limit $d \to \infty$, Eq. (1.7) becomes

$$0 = -\tilde{c}u'(z) + u''(z) + f\big(u(z)\big). \tag{1.8}$$

For $\tilde{c} = \frac{1-2a}{\sqrt{2}}$, this ordinary differential equation has a solution

$$u(z) = \frac{1}{2}\left(1 + \tanh\left(\frac{1}{\sqrt{8}}z\right)\right) \tag{1.9}$$

which satisfies $u(-\infty) = 0$ and $u(\infty) = 1$.

This paper deals with the question whether there exists for some $d < \infty$ a continuation of this solution of the continuum limit (1.8) to the original equation (1.4). The continuation of solutions of a (simpler) equation is a standard technique in the theory of LDEs (see, for example, [6]). But, as a matter of fact, this is done usually without rescaling the solution (i.e., the discreteness parameter is not touched).

The simple reason for this restriction can be seen if we have a close look at Eq. (1.4): If we rescale u, the parameter τ in the discrete second derivative $\Delta^2(\tau, u)$ becomes non-constant. Suppose we are dealing with two Banach spaces V and W (e.g. two weighted Sobolev spaces) such that $\Delta^2 : \mathbb{R} \times V \to W$. Then, standard tools as the implicit function theorem of Hildebrandt and Graves are not applicable since the operator Δ^2 has no continuous Frechet derivative. Nevertheless, it is still possible to use implicit function techniques also in this situation if we give up the uniqueness of the solution. One of the reasons for this loss of uniqueness lies in the fact that the general solution of the ordinary equation (1.8) has only two parameters whilst the solution space of the functional differential equation (1.4) is infinite dimensional. In general, this leads to a bunch of new effects such as pinning (also known as propagation failure).

In the following, we present an implicit function theorem of Craven and Nashed which gives up uniqueness in order to allow weaker differentiability properties (i.e., some kind of Hadamard differentiability). The main twist of this theorem is that it needs differentiability only in the direction where the solution actually lies. This direction – together with all estimates required – can be determined if the solution to be continuated has some additional smoothness (as in our case, where it is even analytic). This theorem is presented in Section 2. In Section 3, we verify these differentiability properties for a class of operators (which includes the discrete second derivative Δ^2).

In order to provide an example how the continuum approximation together with the above implicit function theorem can be utilized in order to give existence results for LDEs, we present in Section 4 an alternate proof for the following classical result on the spatially discrete Nagumo equation:

Theorem 1.1. *Let $0 < a < \frac{1}{2}$. Then, there exists a $d^* > 0$ such that for all $d > d^*$ there is a $c > 0$ such that the travelling wave equation (1.4) has a solution $u \in C^2(\mathbb{R})$ with $u(-\infty) = 0$ and $u(\infty) = 1$.*

It has to be stressed that the interest of this paper is not to prove this well-known result (for even more in depth results on the Nagumo equation see, for example, [4]) but to develop a technique which allows us to continuate travelling wave solutions of LDEs from the travelling wave solutions of a PDE, which can be treated as the limit of LDEs.

2. An implicit function theorem

In the following, we extensively deal with Hadamard differentials. The concept of Hadamard differentiability is one of the weakest which obey all the common rules of differentiability (including the chain rule).

Definition 2.1. Let X and Y be real Banach spaces. A function $G : X \to Y$ is *Hadamard differentiable* at $a \in X$ if there exists a linear mapping $M \in \mathcal{L}(X, Y)$ such that, for any continuous function $\omega : [0,1] \to X$ for which $\omega'(0^+)$ exists and $\omega(0) = a$, the function $F = G \circ \omega$ is differentiable at 0^+, with $F(0^+) = M\omega'(0^+)$ where M is the Hadamard derivative.

In addition, the function $G : X \to Y$ is *strongly Hadamard differentiable* at $a \in X$ if $F = G \circ \omega$ is strongly differentiable at 0^+. This is the case if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| G\bigl(\omega(u)\bigr) - G\bigl(\omega(v)\bigr) - M\omega'\bigl(0^+\bigr)(u - v)\right\| \leqslant \varepsilon |u - v| \tag{2.1}$$

whenever $|u - v|, |v| \leqslant \delta$.

The function $G : X \to Y$ is *restricted strongly Hadamard differentiable* at a if the strongly Hadamard differentiable property holds only for ω restricted to be strongly differentiable at 0^+ (i.e., if $\|\omega(u) - \omega(v) - \omega'(0^+)(u - v)\| \leqslant \varepsilon|u - v|$ whenever $|u - v|, |v| \leqslant \delta$).

Moreover, G is restricted strongly Hadamard differentiable at *a with respect to the initial slope $c \in X$* if we restrict ω further to functions with $\omega'(0^+) = c$.

In [2, Theorem 2], Craven and Nashed gave an inverse function theorem for Hadamard differentiable functions. In the following, we will use a reformulation of this theorem in the form of an implicit function theorem (for the equivalence of inverse and implicit function theorems see, for example, [5]):

Corollary 2.2. *Let V and W be real Banach spaces, $g_0 \in V$, $\mathcal{U} \subset \mathbb{R} \times V$ an open neighbourhood of $(0, g_0)$, $F : \mathcal{U} \to W$ be Hadamard differentiable at all $(\tau, g) \in \mathcal{U}$ with $\tau > 0$ as well as at $(0, g_0)$. Let $F(0, g_0) = 0$ and the Hadamard differential $M = F'(0, g_0) : \mathbb{R} \times V \to W$ be a bounded linear operator of the form $M(\bar\tau, \bar g) = M_1\bar\tau + M_2\bar g$.*

Suppose M_2 has a bounded right inverse $M_2^{-1} : W \to V$, $\bar g_0 = -M_2^{-1}M_1 1$ and F is restricted strongly Hadamard differentiable at $(0, g_0)$ with respect to the initial slope $(1, \bar g_0)$. Then, there exists a sufficiently small interval $I = [0, \varepsilon[$ and a solution $g : I \to V$ such that $g(0) = g_0$, $g'(0^+) = \bar g_0$ and $F(\tau, g(\tau)) = 0$ for all $\tau \in I$.

Proof. Let $X = \mathbb{R} \times V$, $Y = \mathbb{R} \times W$, $a = (0, g_0)$, $S = \mathbb{R} \times \{0\}$ and $G : X \to Y$ be given by $G(\tau, g) = (\tau, F(\tau, g))$. By construction, it holds $-b = -G(a) = (0, 0) \in S$. The Hadamard derivative

$$G'(a) = \begin{pmatrix} 1 & 0 \\ M_1 & M_2 \end{pmatrix} \tag{2.2}$$

exists, is bounded linear and has a bounded right inverse

$$B = \begin{pmatrix} 1 & 0 \\ -M_2^{-1}M_1 & M_2^{-1} \end{pmatrix}, \tag{2.3}$$

with bound Γ. In [2], there is made use of the notion of an approximative right inverse B_μ. Since we will assume the existence of a bounded right inverse B, let $B_\mu = B$, the bound function $\Gamma(\mu) = \Gamma$ and the weighted spaces $X_\mu = X$ be all independent of μ. Thus, all dependencies on the parameter μ can be neglected. For $c = \frac{1}{\|(1, \bar g_0)\|_X}(1, \bar g_0)$, it holds $-[G(a) + G'(a)c] = (\frac{1}{\|(1, \bar g_0)\|}, 0) \in S$. Thus, all prerequisites – with exception of the smoothness conditions – of [2, Theorem 2] are met.

A short inspection of the proof of [2, Theorem 2] shows that the required smoothness conditions can be weakened slightly: In fact, Hadamard differentiability is only needed twice: At the very beginning, there is made use of the *restricted strong Hadamard differentiability* of G at *a with respect to the initial slope c*. The more general *restricted strong Hadamard differentiability* without restriction to the initial slope is only needed if one does not want to take care about c.

Secondly, at the top of [2, p. 381], there is made use of the Hadamard derivative $H'(\omega(u)) = G'(\omega(u))$ for $u > 0$. Since $\omega(0^+) = a + tc$ with $t > 0$, it is guaranteed by construction, that there are only $\omega(u) = (\tau_u, g_u)$ under consideration with $\tau_u > 0$. Also, t can be chosen sufficiently small to give $\omega(u) \in \mathcal{U}$.

Thus, our requirements on F are sufficient for the proof of [2, Theorem 2] to go through. The resulting solution $x(t) = a + tc + \eta(t)$ is strongly differentiable at 0^+. Its τ-component has positive initial slope and, therefore, is strictly increasing for sufficiently small t. By a reparametrization, we obtain the final solution $g(\tau)$. $\square$

3. Hadamard differentiability of discrete derivatives

In this section, we verify that operators like Δ^2 satisfy all the differentiability properties required for the application of Corollary 2.2. First, we define some appropriate Banach spaces (equipped with exponential weights, which allow us an easy control of the asymptotic behavior of our final solution as $z \to \pm\infty$) and two basic operators: the shift operator A and the discrete (first) derivative Δ. More complex operators can be constructed afterwards by the combination of these two basic operators.

Definition 3.1. We denote by $X(m,\alpha)$ with $m \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$ the real linear space of functions $g : \mathbb{R} \to \mathbb{R}$ which are bounded under the weighted norm

$$\|g\|_{X(m,\alpha)} = \sum_{k=0}^{m} \sup_{z\in\mathbb{R}} \left| e^{\alpha|z|} \frac{d^k}{dz^k} g(z) \right| \tag{3.1}$$

and, in addition, satisfy that $e^{\alpha|z|} \frac{d^m}{dz^m} g(z)$ is uniformly continuous in $z \in \mathbb{R}$.

First, we check that $X(m,\alpha)$ is complete and, thus, becomes a Banach space: Given any sequence $g_n \in X(m,\alpha)$, some $g : \mathbb{R} \to \mathbb{R}$ which satisfies

$$\lim_{n\to\infty} \|g_n - g\|_{X(m,\alpha)} = 0 \tag{3.2}$$

and any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geqslant N$

$$\|g - g_n\|_{X(m,\alpha)} \leqslant \frac{\varepsilon}{3}. \tag{3.3}$$

Since g_N is uniformly continuous, there is a $\delta > 0$ such that whenever $|z_1 - z_2| \leqslant \delta$, we get

$$\left| e^{\alpha|z_1|} \frac{d^m}{dz_1^m} g_N(z_1) - e^{\alpha|z_2|} \frac{d^m}{dz_2^m} g_N(z_2) \right| \leqslant \frac{\varepsilon}{3}. \tag{3.4}$$

It holds

$$\begin{aligned} \left| e^{\alpha|z_1|} \frac{d^m}{dz_1^m} g(z_1) - e^{\alpha|z_2|} \frac{d^m}{dz_2^m} g(z_2) \right| &\leqslant \left| e^{\alpha|z_1|} \frac{d^m}{dz_1^m} g_N(z_1) - e^{\alpha|z_2|} \frac{d^m}{dz_2^m} g_N(z_2) \right| + \left| e^{\alpha|z_1|} \frac{d^m}{dz_1^m} \big(g(z_1) - g_N(z_1)\big) \right| \\ &\quad + \left| e^{\alpha|z_2|} \frac{d^m}{dz_2^m} \big(g(z_2) - g_N(z_2)\big) \right| \\ &\leqslant \frac{\varepsilon}{3} + 2\|g - g_N\|_{X(m,\alpha)} \\ &\leqslant \varepsilon. \end{aligned} \tag{3.5}$$

Therefore, $e^{\alpha|z|} \frac{d^m}{dz^m} g(z)$ is also uniformly continuous in z and we are finished.

Definition 3.2. Let $m \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$. By $A : \mathbb{R} \times X(m,\alpha) \to X(m,\alpha)$ we denote the *shift operator*

$$A(\tau, g; z) = g(z + \tau). \tag{3.6}$$

Furthermore, let $\Delta : \mathbb{R} \times X(m+1,\alpha) \to X(m,\alpha)$ denote the *discrete derivative*

$$\Delta(\tau, g) = \begin{cases} \frac{1}{\tau}(A(\tau,g) - g) & \text{for } \tau \neq 0, \\ g' & \text{for } \tau = 0. \end{cases} \tag{3.7}$$

Lemma 3.3. *Let $m \in \mathbb{N}_0$, $\alpha \in \mathbb{R}$ and $(\tau_0, g_0) \in \mathbb{R} \times X(m+1,\alpha)$. Then, the shift operator $A : \mathbb{R} \times X(m,\alpha) \to X(m,\alpha)$ is Hadamard differentiable at (τ_0, g_0) with the derivative*

$$M(\bar\tau, \bar g) = A\big(\tau_0, g_0'\big)\bar\tau + A(\tau_0, \bar g). \tag{3.8}$$

Proof. Let $\omega : [0,1] \to \mathbb{R} \times X(m,\alpha)$ with $\omega(0) = (\tau_0, g_0)$ and $\omega'(0^+) = (\bar\tau_0, \bar g_0) \in \mathbb{R} \times X(m,\alpha)$. For simplicity, we use the shortcut notation $(\tau_u, g_u) = \omega(u)$.

For any $\hat\varepsilon > 0$, we get from the differentiability of ω at 0^+ the existence of a $\hat\delta(\hat\varepsilon) > 0$ such that for all $u \in [0, \hat\delta(\hat\varepsilon)]$ it holds

$$|\tau_u - \tau_0 - \bar\tau_0 u| \leqslant \hat\varepsilon u \quad \text{and} \quad \|g_u - g_0 - \bar g_0 u\|_{X(m,\alpha)} \leqslant \hat\varepsilon u. \tag{3.9}$$

For $k = 0, 1, \ldots, m-1$, the functions $e^{\alpha|z|} g_0^{(k+2)}(z)$ and $e^{\alpha|z|} \bar g_0^{(k+1)}(z)$ are bounded. Therefore, $e^{\alpha|z|} g_0^{(k+1)}(z)$ and $e^{\alpha|z|} \bar g_0^{(k)}(z)$ are uniformly continuous. Together with the uniform continuity for $k = m$, we obtain a $\check\delta(\hat\varepsilon) > 0$ such that for all $k = 0, 1, \ldots, m$, $z \in \mathbb{R}$ and $\zeta \in [-\check\delta(\hat\varepsilon), \check\delta(\hat\varepsilon)]$

$$\left|e^{\alpha|z+\zeta|}g_0^{(k+1)}(z+\zeta)-e^{\alpha|z|}g_0^{(k+1)}(z)\right|\leqslant\frac{\hat{\varepsilon}}{|\bar{\tau}_0|},\tag{3.10}$$

$$\left|e^{\alpha|z+\zeta|}\bar{g}_0^{(k)}(z+\zeta)-e^{\alpha|z|}\bar{g}_0^{(k)}(z)\right|\leqslant\hat{\varepsilon}.\tag{3.11}$$

Now, let $\varepsilon>0$,

$$\hat{\varepsilon}=\frac{\varepsilon}{e^{|\alpha|(|\tau_0|+1)}(2m+4+2\|g_0\|_{X(m+1,\alpha)})},\tag{3.12}$$

$$\delta=\min\left\{\hat{\delta}(\hat{\varepsilon}),\frac{1}{|\bar{\tau}_0|+\hat{\varepsilon}},\frac{\check{\delta}(\hat{\varepsilon})}{|\bar{\tau}_0|+\hat{\varepsilon}},\frac{\ln(1+\frac{\hat{\varepsilon}}{\|g_0\|_{X(m+1,\alpha)}|\bar{\tau}_0|})}{|\alpha|(|\bar{\tau}_0|+\hat{\varepsilon})},\frac{\ln(1+\frac{\hat{\varepsilon}}{\|\bar{g}_0\|_{X(m,\alpha)}})}{|\alpha|(|\bar{\tau}_0|+\hat{\varepsilon})}\right\}\tag{3.13}$$

and $u\in[0,\delta]$. It holds

$$\begin{aligned}\left\|A(\tau_u,g_u)-A(\tau_0,g_0)-M(\bar{\tau}_0,\bar{g}_0)u\right\|_{X(m,\alpha)}&=\left\|A(\tau_u,g_u)-A(\tau_0,g_0)-\left(A\left(\tau_0,g_0'\right)\bar{\tau}_0+A(\tau_0,\bar{g}_0)\right)u\right\|_{X(m,\alpha)}\\&\leqslant\left\|A(\tau_u,g_u-g_0-\bar{g}_0u)\right\|_{X(m,\alpha)}\\&\quad+\left\|A(\tau_u,g_0)-A(\tau_0,g_0)-A\left(\tau_0,g_0'\right)\bar{\tau}_0u\right\|_{X(m,\alpha)}\\&\quad+\left\|A(\tau_u,\bar{g}_0)-A(\tau_0,\bar{g}_0)\right\|_{X(m,\alpha)}u.\end{aligned}\tag{3.14}$$

In the following, we give estimates for each of the above three terms. From Eq. (3.9) and our definition of δ, we get

$$|\tau_u|\leqslant|\tau_0|+\left(|\bar{\tau}_0|+\hat{\varepsilon}\right)u\leqslant|\tau_0|+1.\tag{3.15}$$

It follows

$$\begin{aligned}&\left\|A(\tau_u,g_u-g_0-\bar{g}_0u)\right\|_{X(m,\alpha)}\\&\quad=\sum_{k=0}^{m}\sup_{z\in\mathbb{R}}\left|e^{\alpha|z|}\frac{d^k}{dz^k}\left(g_u(z+\tau_u)-g_0(z+\tau_u)-\bar{g}_0(z+\tau_u)u\right)\right|\\&\quad=\sum_{k=0}^{m}\sup_{z\in\mathbb{R}}e^{\alpha(|z|-|z+\tau_u|)}\left|e^{\alpha|z+\tau_u|}\frac{d^k}{dz^k}\left(g_u(z+\tau_u)-g_0(z+\tau_u)-\bar{g}_0(z+\tau_u)u\right)\right|\\&\quad\leqslant e^{|\alpha\tau_u|}\|g_u-g_0-\bar{g}_0u\|_{X(m,\alpha)}\\&\quad\leqslant e^{|\alpha|(|\tau_0|+1)}\hat{\varepsilon}u.\end{aligned}\tag{3.16}$$

In the last line, we used Eqs. (3.15) and (3.9).

Now, we consider the second term in Eq. (3.14). For each $k=0,1,\dots,m$ and $z\in\mathbb{R}$, there is a ζ between τ_0 and τ_u such that

$$\begin{aligned}&\left|e^{\alpha|z|}\frac{d^k}{dz^k}\left(A(\tau_u,g_0;z)-A(\tau_0,g_0;z)-A\left(\tau_0,g_0';z\right)\bar{\tau}_0u\right)\right|\\&\quad=e^{\alpha|z|}\left|g_0^{(k)}(z+\tau_u)-g_0^{(k)}(z+\tau_0)-g_0^{(k+1)}(z+\tau_0)\bar{\tau}_0u\right|\\&\quad=e^{\alpha|z|}\left|g_0^{(k+1)}(z+\zeta)(\tau_u-\tau_0)-g_0^{(k+1)}(z+\tau_0)\bar{\tau}_0u\right|\\&\quad\leqslant e^{\alpha|z|}\left|g_0^{(k+1)}(z+\zeta)\right||\tau_u-\tau_0-\bar{\tau}_0u|+e^{\alpha|z|}\left|g_0^{(k+1)}(z+\zeta)-g_0^{(k+1)}(z+\tau_0)\right||\bar{\tau}_0|u\\&\quad\leqslant e^{|\alpha|(|\tau_0|+1)}\Big(\left|e^{\alpha|z+\zeta|}g_0^{(k+1)}(z+\zeta)\right|\hat{\varepsilon}u+\left|e^{\alpha|z+\zeta|}g_0^{(k+1)}(z+\zeta)-e^{\alpha|z+\tau_0|}g_0^{(k+1)}(z+\tau_0)\right||\bar{\tau}_0|u\\&\qquad+\left(e^{|\alpha(\zeta-\tau_0)|}-1\right)\left|e^{\alpha|z+\zeta|}g_0^{(k+1)}(z+\zeta)\right||\bar{\tau}_0|u\Big)\\&\quad\leqslant e^{|\alpha|(|\tau_0|+1)}\left(2\left|e^{\alpha|z+\zeta|}g_0^{(k+1)}(z+\zeta)\right|+1\right)\hat{\varepsilon}u.\end{aligned}\tag{3.17}$$

Here, we used Eqs. (3.15), (3.9) and (3.10). Therefore,

$$\left\|A(\tau_u,g_0)-A(\tau_0,g_0)-A\left(\tau_0,g_0'\right)\bar{\tau}_0u\right\|_{X(m,\alpha)}\leqslant e^{|\alpha|(|\tau_0|+1)}\left(2\|g_0\|_{X(m+1,\alpha)}+m+1\right)\hat{\varepsilon}u.\tag{3.18}$$

For the last term in Eq. (3.14), we get from (3.15) and (3.11)

$$\begin{aligned}\big\|A(\tau_u,\bar g_0)-A(\tau_0,\bar g_0)\big\|_{X(m,\alpha)} &= \sum_{k=0}^{m}\sup_{z\in\mathbb{R}}\big|e^{\alpha|z|}\big(\bar g_0^{(k)}(z+\tau_u)-\bar g_0^{(k)}(z+\tau_0)\big)\big|\\ &\leqslant e^{|\alpha|(|\tau_0|+1)}\sum_{k=0}^{m}\sup_{z\in\mathbb{R}}\big(\big|e^{\alpha|z+\tau_u|}\bar g_0^{(k)}(z+\tau_u)-e^{\alpha|z+\tau_0|}\bar g_0^{(k)}(z+\tau_0)\big|\\ &\quad+\big(e^{|\alpha(\tau_u-\tau_0)|}-1\big)\big|e^{\alpha|z+\tau_u|}\bar g_0^{(k)}(z+\tau_u)\big|\big)\\ &\leqslant e^{|\alpha|(|\tau_0|+1)}(m+2)\hat\varepsilon. \end{aligned}\tag{3.19}$$

Altogether, we obtain

$$\big\|A(\tau_u,g_u)-A(\tau_0,g_0)-M(\bar\tau_0,\bar g_0)u\big\|_{X(m,\alpha)}\leqslant\varepsilon \tag{3.20}$$

which is the stated result. □

Lemma 3.4. *Let $m\in\mathbb{N}_0$, $\alpha\in\mathbb{R}$, $\tau_0\in\mathbb{R}$, $g_0\in X(m+1,\alpha)$ if $\tau_0\neq 0$ and $g_0\in X(m+2,\alpha)$ if $\tau_0=0$ respectively. Then, the discrete derivative $\Delta:\mathbb{R}\times X(m+1,\alpha)\to X(m,\alpha)$ is Hadamard differentiable at (τ_0,g_0) with the derivative*

$$M(\bar\tau,\bar g)=\begin{cases}\frac{1}{\tau_0}(A(\tau_0,g_0')-\Delta(\tau_0,g_0))\bar\tau+\Delta(\tau_0,\bar g) & \text{for } \tau_0\neq 0,\\ \frac12 g_0''\bar\tau+\bar g' & \text{for } \tau_0=0.\end{cases}\tag{3.21}$$

Proof. Since $X(m+1,\alpha)\subset X(m,\alpha)$, the case $\tau_0\neq 0$ is an immediate consequence of Definition 3.2 and Lemma 3.3.

Now, let $\tau_0=0$ and $g_0\in X(m+2,\alpha)$. As in the proof of Lemma 3.3, we consider an $\omega:[0,1]\to\mathbb{R}\times X(m+1,\alpha)$ with $\omega(0)=(0,g_0)$ and $\omega'(0^+)=(\bar\tau_0,\bar g_0)\in\mathbb{R}\times X(m+1,\alpha)$. Again, we use the shortcut notation $(\tau_u,g_u)=\omega(u)$ and obtain from the differentiability of ω at 0^+ for any $\hat\varepsilon>0$ a $\hat\delta(\hat\varepsilon)>0$ such that for all $u\in[0,\hat\delta(\hat\varepsilon)]$

$$|\tau_u-\bar\tau_0u|\leqslant\hat\varepsilon u\quad\text{and}\quad\|g_u-g_0-\bar g_0u\|_{X(m+1,\alpha)}\leqslant\hat\varepsilon u. \tag{3.22}$$

Also, we get from uniform continuity a $\check\delta(\hat\varepsilon)>0$ such that for all $k=0,1,\dots,m$, $z\in\mathbb{R}$ and $\zeta\in[-\check\delta(\hat\varepsilon),\check\delta(\hat\varepsilon)]$

$$\big|e^{\alpha|z+\zeta|}g_0^{(k+2)}(z+\zeta)-e^{\alpha|z|}g_0^{(k+2)}(z)\big|\leqslant\frac{\hat\varepsilon}{|\bar\tau_0|+\hat\varepsilon}, \tag{3.23}$$

$$\big|e^{\alpha|z+\zeta|}\bar g_0^{(k+1)}(z+\zeta)-e^{\alpha|z|}\bar g_0^{(k+1)}(z)\big|\leqslant\hat\varepsilon. \tag{3.24}$$

Let $\varepsilon>0$,

$$\hat\varepsilon=\frac{\varepsilon}{6(m+1)+\frac12\|g_0\|_{X(m+2,\alpha)}}, \tag{3.25}$$

$$\delta=\min\left\{\hat\delta(\hat\varepsilon),\frac{\check\delta(\hat\varepsilon)}{|\bar\tau_0|+\hat\varepsilon},\frac{1}{|\alpha|(|\bar\tau_0|+\hat\varepsilon)},\frac{\ln(1+\frac{\hat\varepsilon}{(|\bar\tau_0|+\hat\varepsilon)\|g_0\|_{X(3,\alpha)}})}{|\alpha|(|\bar\tau_0|+\hat\varepsilon)},\frac{\ln(1+\frac{\hat\varepsilon}{\|\bar g_0\|_{X(2,\alpha)}})}{|\alpha|(|\bar\tau_0|+\hat\varepsilon)}\right\} \tag{3.26}$$

and $u\in[0,\delta]$. It holds

$$\begin{aligned}\big\|\Delta(\tau_u,g_u)-\Delta(\tau_0,g_0)-M(\bar\tau_0,\bar g_0)u\big\|_{X(m,\alpha)} &= \left\|\Delta(\tau_u,g_u)-g_0'-\left(\frac12 g_0''\bar\tau_0+\bar g_0'\right)u\right\|_{X(m,\alpha)}\\ &\leqslant\big\|\Delta(\tau_u,g_u-g_0-\bar g_0u)\big\|_{X(m,\alpha)}\\ &\quad+\left\|\Delta(\tau_u,g_0)-g_0'-\frac12 g_0''\bar\tau_0u\right\|_{X(m,\alpha)}\\ &\quad+\big\|\Delta(\tau_u,\bar g_0)-\bar g_0'\big\|_{X(m,\alpha)}|u|. \end{aligned}\tag{3.27}$$

As in Lemma 3.3, we give estimates for each of these three terms. From Eq. (3.22) and the definition of δ, we get

$$|\tau_u|\leqslant\big(|\bar\tau_0|+\hat\varepsilon\big)|u|\leqslant\frac{1}{|\alpha|}. \tag{3.28}$$

For $k=0,\dots,m$, there exists a $\zeta\in[-|\tau_u|,|\tau_u|]$ such that

$$
\begin{aligned}
e^{\alpha|z|}\left|\frac{d^k}{dz^k}\Delta(\tau_u, g_u - g_0 - \bar{g}_0 u; z)\right| &= e^{\alpha|z|}\left|g_u^{(k+1)}(z+\zeta) - g_0^{(k+1)}(z+\zeta) - \bar{g}_0^{(k+1)}(z+\zeta)u\right| \\
&\leqslant e^{|\alpha\zeta|}\hat{\varepsilon}u \\
&\leqslant e\hat{\varepsilon}u.
\end{aligned} \tag{3.29}
$$

In the second line, we used Eq. (3.22) and in the last line Eq. (3.28).

For $\tau_u = 0$, the last two terms in Eq. (3.27) become zero and we are finished. In the following, let $\tau_u \neq 0$ without loss of generality. Using Taylor's formula $g_0(z+v) = g_0(z) + g_0'(z)v + \int_0^v (v-\zeta)g_0''(z+\zeta)\,d\zeta$, we obtain by the use of (3.28) and (3.23) for $k = 0, \dots, m$ by the same kind of estimates as in the proof of Lemma 3.3

$$
\begin{aligned}
e^{\alpha|z|}\left|\frac{d^k}{dz^k}\left(\Delta(\tau_u, g_0; z) - g_0'(z) - \frac{1}{2}g_0''(z)\bar{\tau}_0 u\right)\right| &\leqslant e^{\alpha|z|}\left|\frac{1}{\tau_u}\int_0^{\tau_u}(\tau_u - \zeta)\big(g_0^{(k+2)}(z+\zeta) - g_0^{(k+2)}(z)\big)\,d\zeta\right| \\
&\quad + \frac{1}{2}\left|e^{\alpha|z|}g_0^{(k+2)}(z)\right||\tau_u - \bar{\tau}_0 u| \\
&\leqslant \left(1 + \frac{1}{2}\left|e^{\alpha|z|}g_0^{(k+2)}(z)\right|\right)\hat{\varepsilon}u.
\end{aligned} \tag{3.30}
$$

Considering the last term in Eq. (3.27), there is a $\zeta \in [-|\tau_u|, |\tau_u|]$ such that we get by the use of Eq. (3.24)

$$
\begin{aligned}
e^{\alpha|z|}\left|\frac{d^k}{dz^k}\big(\Delta(\tau_u, \bar{g}_0; z) - \bar{g}_0'(z)\big)\right| &= e^{\alpha|z|}\left|\bar{g}_0^{(k+1)}(z+\zeta) - \bar{g}_0^{(k+1)}(z)\right| \\
&\leqslant \left|e^{\alpha|z+\zeta|}\bar{g}_0^{(k+1)}(z+\zeta) - e^{\alpha|z|}\bar{g}_0^{(k+1)}(z)\right| \\
&\quad + \big(e^{|\alpha\zeta|} - 1\big)\left|e^{\alpha|z+\zeta|}\bar{g}_0^{(k+1)}(z+\zeta)\right| \\
&\leqslant \hat{\varepsilon} + \hat{\varepsilon}.
\end{aligned} \tag{3.31}
$$

Altogether, we have

$$
\big\|\Delta(\tau_u, g_u) - \Delta(\tau_0, g_0) - M(\bar{\tau}_0, \bar{g}_0)u\big\|_{X(m,\alpha)} \leqslant \varepsilon u \tag{3.32}
$$

which finishes the proof. □

Lemma 3.5. *Let $m \in \mathbb{N}_0$, $\alpha \in \mathbb{R}$, $g_0 \in X(m+2, \alpha)$, $\bar{\tau}_0 \neq 0$ and $\bar{g}_0 \in X(m+1, \alpha)$. Then, the discrete derivative $\Delta : \mathbb{R} \times X(m+1, \alpha) \to X(m, \alpha)$ is restricted strongly Hadamard differentiable at $(0, g_0)$ with respect to the initial slope $\omega'(0^+) = (\bar{\tau}_0, \bar{g}_0)$.*

Proof. Let $\omega : [0, 1] \to \mathbb{R} \times X(m+1, \alpha)$ be strongly differentiable with $\omega(0) = (0, g_0)$ and $\omega'(0^+) = (\bar{\tau}_0, \bar{g}_0)$. As before, we use the shortcut notation $(\tau_u, g_u) = \omega(u)$. For any $\hat{\varepsilon} > 0$, we get from the strong differentiability of ω at 0^+ the existence of a $\hat{\delta}(\hat{\varepsilon}) > 0$ such that for all $u, t \in [0, \hat{\delta}(\hat{\varepsilon})]$

$$
\big|\tau_u - \tau_t - \bar{\tau}_0(u-t)\big| \leqslant \hat{\varepsilon}|u-t| \quad \text{and} \quad \big\|g_u - g_t - \bar{g}_0(u-t)\big\|_{X(m+1,\alpha)} \leqslant \hat{\varepsilon}|u-t|. \tag{3.33}
$$

For $\hat{\varepsilon} < \frac{|\bar{\tau}_0|}{2}$, Eq. (3.33) implies

$$
\frac{|\bar{\tau}_0|}{2}|u-t| \leqslant |\tau_u - \tau_t| \leqslant \frac{3|\bar{\tau}_0|}{2}|u-t| \tag{3.34}
$$

which includes the special case

$$
\frac{|\bar{\tau}_0|}{2}u \leqslant |\tau_u| \leqslant \frac{3|\bar{\tau}_0|}{2}u. \tag{3.35}
$$

Again, we obtain from the uniform continuity of $e^{\alpha|z|}g_0^{(k+2)}(z)$ and $e^{\alpha|z|}\bar{g}_0^{(k+1)}(z)$ a $\check{\delta}(\hat{\varepsilon}) > 0$ such that for all $k = 0, 1, \dots, m$, $z \in \mathbb{R}$ and $\zeta \in [-\check{\delta}(\hat{\varepsilon}), \check{\delta}(\hat{\varepsilon})]$

$$
\left|e^{\alpha|z+\zeta|}g_0^{(k+2)}(z+\zeta) - e^{\alpha|z|}g_0^{(k+2)}(z)\right| \leqslant \hat{\varepsilon}, \tag{3.36}
$$

$$
\left|e^{\alpha|z+\zeta|}\bar{g}_0^{(k+1)}(z+\zeta) - e^{\alpha|z|}\bar{g}_0^{(k+1)}(z)\right| \leqslant \hat{\varepsilon}. \tag{3.37}
$$

Let $\varepsilon > 0$,

$$\hat{\varepsilon} = \min\left\{\frac{|\bar{\tau}_0|}{2}, \frac{\varepsilon}{(m+1)(20+7\mathrm{e}) + \frac{1}{2}\|g_0\|_{X(m+2,\alpha)}}\right\}, \tag{3.38}$$

$$\delta = \min\left\{\frac{1}{|2\alpha\bar{\tau}_0|}, \hat{\delta}(\hat{\varepsilon}), \check{\delta}(\hat{\varepsilon}), \frac{2\ln(1+\frac{\hat{\varepsilon}}{\|g_0\|_{X(m+2,\alpha)}})}{3|\alpha\bar{\tau}_0|}, \frac{2\ln(1+\frac{\hat{\varepsilon}}{\|\bar{g}_0\|_{X(m+1,\alpha)}})}{3|\alpha\bar{\tau}_0|}\right\} \tag{3.39}$$

and $u, t, |u-t| \in [0,\delta]$. The case $t=0$ follows already from Lemma 3.4. Also, the case $t=u$ is trivial. Therefore, let $0<t<u$. By our choice of $\hat{\varepsilon}$ and Eq. (3.33), this implies $0<|\tau_t|<|\tau_u|$. It holds

$$\begin{aligned}
&\big\|\Delta(\tau_u, g_u) - \Delta(\tau_t, g_t) - M(\bar{\tau}_0, \bar{g}_0)(u-t)\big\|_{X(m,\alpha)} \\
&\quad = \left\|\Delta(\tau_u, g_u) - \Delta(\tau_t, g_t) - \left(\frac{1}{2}g_0''\bar{\tau}_0 + \bar{g}_0'\right)(u-t)\right\|_{X(m,\alpha)} \\
&\quad \leqslant \big\|\Delta\big(\tau_t, g_u - g_t - \bar{g}_0(u-t)\big)\big\|_{X(m,\alpha)} + \big\|\Delta(\tau_u, g_u - g_0 - \bar{g}_0 u) - \Delta(\tau_t, g_u - g_0 - \bar{g}_0 u)\big\|_{X(m,\alpha)} \\
&\qquad + \left\|\Delta(\tau_u, g_0) - \Delta(\tau_t, g_0) - \frac{1}{2}g_0''\bar{\tau}_0(u-t)\right\|_{X(m,\alpha)} \\
&\qquad + \big\|\Delta(\tau_t, \bar{g}_0) - \bar{g}_0''\big\|_{X(m,\alpha)}|u-t| + \big\|\Delta(\tau_u, \bar{g}_0) - \Delta(\tau_t, \bar{g}_0)\big\|_{X(m,\alpha)} u.
\end{aligned} \tag{3.40}$$

From Eq. (3.35), we get $|\tau_t| < |\tau_u| < \frac{3|\bar{\tau}_0|}{2}u \leqslant \frac{1}{|\alpha|}$. Thus, for $k = 0, \dots, m$ there exists a $\zeta \in [-|\tau_t|, |\tau_t|]$ such that

$$\begin{aligned}
\mathrm{e}^{\alpha|z|}\left|\frac{\mathrm{d}^k}{\mathrm{d}z^k}\big(\Delta\big(\tau_t, g_u - g_t - \bar{g}_0(u-t); z\big)\big)\right| &= \mathrm{e}^{\alpha|z|}\big|g_u^{(k+1)}(z+\zeta) - g_t^{(k+1)}(z+\zeta) - \bar{g}_0^{(k+1)}(z+\zeta)(u-t)\big| \\
&\leqslant \mathrm{e}^{|\alpha|(|z|-|z+\zeta|)}\hat{\varepsilon}|u-t| \\
&\leqslant \mathrm{e}\hat{\varepsilon}|u-t|.
\end{aligned} \tag{3.41}$$

Here, we used Eq. (3.33).

Next, we consider the second term of Eq. (3.40): $\|\Delta(\tau_u, f) - \Delta(\tau_t, f)\|_{X(m,\alpha)}$ with $f(z) = g_u(z) - g_0(z) - \bar{g}_0(z)u$. From Eq. (3.33), we get $\|f\|_{X(m+1,\alpha)} \leqslant \hat{\varepsilon}u$. Using Taylor's formula $f(z+w) = f(z) + \int_0^w f'(z+w)\,\mathrm{d}\zeta$, we obtain

$$\begin{aligned}
\mathrm{e}^{\alpha|z|}\left|\frac{\mathrm{d}^k}{\mathrm{d}z^k}\big(\Delta(\tau_u, f; z) - \Delta(\tau_t, f; z)\big)\right| &= \mathrm{e}^{\alpha|z|}\left|\frac{1}{\tau_u}\int_0^{\tau_u} f^{(k+1)}(z+\zeta)\,\mathrm{d}\zeta - \frac{1}{\tau_t}\int_0^{\tau_t} f^{(k+1)}(z+\zeta)\,\mathrm{d}\zeta\right| \\
&\leqslant \mathrm{e}^{\alpha|z|}\left(\left|\frac{1}{\tau_u}\int_{\tau_t}^{\tau_u} f^{(k+1)}(z+\zeta)\,\mathrm{d}\zeta\right| + \left|\left(\frac{1}{\tau_u} - \frac{1}{\tau_t}\right)\int_0^{\tau_t} f^{(k+1)}(z+\zeta)\,\mathrm{d}\zeta\right|\right) \\
&\leqslant 6\mathrm{e}\hat{\varepsilon}(u-t).
\end{aligned} \tag{3.42}$$

Here, we used Eqs. (3.34) and (3.35).

For the third term of (3.40), Taylor's formula – together with the same kind of estimates as above – leads to

$$\mathrm{e}^{\alpha|z|}\left|\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\Delta(\tau_u, g_0; z) - \Delta(\tau_t, g_0; z) - \frac{1}{2}g_0''(z)\bar{\tau}_0(u-t)\right)\right| \leqslant \left(6 + \frac{1}{2}\big|\mathrm{e}^{\alpha|z|}g_0^{(k+2)}(z)\big|\right)\hat{\varepsilon}(u-t). \tag{3.43}$$

In the same fashion, we get for the remaining terms of (3.40)

$$\mathrm{e}^{\alpha|z|}\left|\frac{\mathrm{d}^k}{\mathrm{d}z^k}\big(\Delta(\tau_t, \bar{g}_0; z) - \bar{g}_0'(z)\big)\right| \leqslant 2\hat{\varepsilon} \tag{3.44}$$

and

$$\mathrm{e}^{\alpha|z|}\left|\frac{\mathrm{d}^k}{\mathrm{d}z^k}\big(\Delta(\tau_u, \bar{g}_0; z) - \Delta(\tau_t, \bar{g}_0; z)\big)\right|u \leqslant 12\hat{\varepsilon}(u-t). \tag{3.45}$$

Thus, we end up with

$$\big\|\Delta(\tau_u, g_u) - \Delta(\tau_t, g_t) - M(\bar{\tau}_0, \bar{g}_0)(u-t)\big\|_{X(m,\alpha)} \leqslant \varepsilon|u-t| \tag{3.46}$$

which is the stated result. □

Lemma 3.6. *Let $m \in \mathbb{N}_0$, $\alpha \in \mathbb{R}$, $(\tau_0, g_0) \in \mathbb{R} \times X(m+2,\alpha)$, $\bar{\tau}_0 \neq 0$ and $\bar{g}_0 \in X(m+1,\alpha)$. Then, the shift operator $A : \mathbb{R} \times X(m+1,\alpha) \to X(m,\alpha)$ and the discrete derivative $\Delta : \mathbb{R} \times X(m+1,\alpha) \to X(m,\alpha)$ are restricted strongly Hadamard differentiable at (τ_0, g_0) with respect to the initial slope $\omega'(0^+) = (\bar{\tau}_0, \bar{g}_0)$.*

Proof. The shift operator A can be written in the form

$$A(\tau, g) = (\tau - \tau_0)\Delta\bigl(\tau - \tau_0, A(\tau_0, g)\bigr) + A(\tau_0, g). \tag{3.47}$$

Since $A(\tau_0, g)$ does not depend on τ and, therefore, is linear in (τ, g), the restricted strongly Hadamard differentiability of $A(\tau, g)$ at (τ_0, g_0) with respect to the initial slope $(\bar{\tau}_0, \bar{g}_0)$ follows immediately from Lemma 3.5 and the common rules of differentiation.

Moreover, we get by Definition 3.2 the restricted strongly Hadamard differentiability of $\Delta(\tau, g)$ at (τ_0, g_0) with respect to the initial slope $(\bar{\tau}_0, \bar{g}_0)$ for $\tau_0 \neq 0$. The case $\tau_0 = 0$ has already been proved in Lemma 3.5. □

Corollary 3.7. *Let $m \in \mathbb{N}_0$, $\alpha \in \mathbb{R}$, $\tau_0 \in \mathbb{R}$, $g_0 \in X(m+2,\alpha)$ if $\tau_0 \neq 0$ and $g_0 \in X(m+3,\alpha)$ if $\tau_0 = 0$ respectively. Then, the* discrete second derivative *$\Delta^2 : \mathbb{R} \times X(m+2,\alpha) \to X(m,\alpha)$ given by*

$$\Delta^2(\tau, g) = \begin{cases} \frac{1}{\tau^2}\bigl(A(\tau, g) - 2g + A(-\tau, g)\bigr) & \text{for } \tau \neq 0, \\ g'' & \text{for } \tau = 0 \end{cases} \tag{3.48}$$

is Hadamard differentiable at (τ_0, g_0) with the derivative

$$M(\bar{\tau}, \bar{g}) = \begin{cases} \bigl(\frac{A(\tau_0, g_0') - A(-\tau_0, g_0')}{\tau_0^2} - \frac{2}{\tau_0}\Delta^2(\tau_0, g_0)\bigr)\bar{\tau} + \Delta^2(\tau_0, \bar{g}) & \text{for } \tau_0 \neq 0, \\ \bar{g}'' & \text{for } \tau_0 = 0. \end{cases} \tag{3.49}$$

Furthermore for $\bar{\tau}_0 \neq 0$ and $\bar{g}_0 \in X(m+2,\alpha)$, it is restricted strongly Hadamard differentiable at $(\tau_0, g_0) \in \mathbb{R} \times X(m+3,\alpha)$ with respect to the initial slope $\omega'(0^+) = (\bar{\tau}_0, \bar{g}_0)$.

Proof. Since

$$\Delta^2(\tau, g) = \Delta\bigl(\tau, \Delta(-\tau, g)\bigr), \tag{3.50}$$

this corollary is an immediate consequence of Lemma 3.6 and the chain rule. □

4. Existence of travelling wavefronts for the spatially discrete Nagumo equation

After these more general preparations, we are now able to prove Theorem 1.1. First, we give a lemma on the right invertibility of the relevant derivative. This result will be the main ingredient in the subsequent proof of the theorem.

Lemma 4.1. *Let $0 < b < 1$. Then, the linear bounded operator $L_b : \mathbb{R} \times X(2,1) \to X(0,1)$ given by*

$$L_b(c, u; z) = -4\bigl(1 - \tanh^2(z)\bigr)c + u''(z) - 2bu'(z) - 2\bigl(3\tanh^2(z) + 2b\tanh(z) - 1\bigr)u(z) \tag{4.1}$$

has a bounded right inverse.

Proof. Let $0 < b < 1$ and $f \in X(0,1)$. The inhomogeneous equation $L_b(c, u) = f$ is equivalent to $\hat{L}_b(u) = \hat{f}(c)$ with

$$\hat{L}_b(u; z) = u''(z) - 2bu'(z) - 2\bigl(3\tanh^2(z) + 2b\tanh(z) - 1\bigr)u(z) \tag{4.2}$$

and

$$\hat{f}(c; z) = f(z) + 4\bigl(1 - \tanh^2(z)\bigr)c. \tag{4.3}$$

The homogeneous linear differential equation $\hat{L}_b(u) = 0$ has the two linearly independent solutions

$$u_1(z) = 1 - \tanh^2(z), \tag{4.4}$$

$$\begin{aligned} u_2(z) = e^{2bz}\frac{1}{1 - \tanh^2(z)}\bigl(&3\tanh^4(z) - 6b\tanh^3(z) + 6\bigl(b^2 - 1\bigr)\tanh^2(z) \\ &- 2b\bigl(2b^2 - 5\bigr)\tanh(z) + \bigl(2b^4 - 8b^2 + 3\bigr)\bigr) \end{aligned} \tag{4.5}$$

and the Wronskian

$$W(z) = 4b\big(b^2-1\big)\big(b^2-4\big)e^{2bz}. \tag{4.6}$$

Let us choose

$$c = -\frac{\int_{-\infty}^{\infty} \frac{u_1(\zeta)}{W(\zeta)} f(\zeta)\,d\zeta}{4\int_{-\infty}^{\infty} \frac{u_1(\zeta)}{W(\zeta)}(1-\tanh^2(\zeta))\,d\zeta}. \tag{4.7}$$

Both integrals exist since the integrands go exponentially to zero for $z \to \pm\infty$. Also, the denominator is nonzero since the integrand is positive. Then, $\hat{f}$ satisfies

$$\int_{-\infty}^{\infty} \frac{u_1(\zeta)}{W(\zeta)} \hat{f}(c;\zeta)\,d\zeta = 0. \tag{4.8}$$

A solution of the inhomogeneous linear differential equation $\hat{L}_b(u) = \hat{f}(c)$ is given by the variation of constants formula

$$u(z) = -u_1(z)\int_0^z \frac{u_2(\zeta)}{W(\zeta)}\hat{f}(c;\zeta)\,d\zeta - u_2(z)\int_z^{\infty}\frac{u_1(\zeta)}{W(\zeta)}\hat{f}(c;\zeta)\,d\zeta. \tag{4.9}$$

Solving the inhomogeneous equation $\hat{L}_b(u) = \hat{f}(c)$ for u'', we obtain

$$u''(z) = \hat{f}(c;z) + 2bu'(z) + 2\big(3\tanh^2(z) + 2b\tanh(z) - 1\big)u(z). \tag{4.10}$$

Since $e^{|z|}\hat{f}(c;z)$ is uniformly continuous, the same becomes true for $e^{|z|}u''(z)$ if $e^{|z|}u(z)$ and $e^{|z|}u'(z)$ have a bounded derivative. Thus, it is only left to prove the boundedness of u and its first two derivatives in our weighted norm.

It holds

$$\begin{aligned}
&\big|u_1(z)\big| \leqslant 4e^{-2|z|}, \qquad \big|u_1'(z)\big| =\leqslant 8e^{-2|z|},\\
&\big|u_2(z)\big| \leqslant 38e^{2bz}e^{2|z|}, \qquad \big|u_2'(z)\big| \leqslant 96e^{2bz}e^{2|z|},\\
&\frac{1}{|W(z)|} \leqslant \frac{1}{4b(b-1)}e^{-2bz} < \infty.
\end{aligned} \tag{4.11}$$

In the case $z \geqslant 0$, this leads to

$$\begin{aligned}
\big|u(z)\big| &\leqslant \big|u_1(z)\big|\int_0^z \frac{|u_2(\zeta)|}{|W(\zeta)|}\big|\hat{f}(c;\zeta)\big|\,d\zeta + \big|u_2(z)\big|\int_z^{\infty}\frac{|u_1(\zeta)|}{|W(\zeta)|}\big|\hat{f}(c;\zeta)\big|\,d\zeta\\
&\leqslant 4e^{-2z}\int_0^z \frac{19}{2b(b-1)}e^{\zeta}\,\big\|\hat{f}(c)\big\|_{X(0,1)}\,d\zeta + 38e^{2(b+1)z}\int_z^{\infty}\frac{1}{b(b-1)}e^{-(3+2b)\zeta}\,\big\|\hat{f}(c)\big\|_{X(0,1)}\,d\zeta\\
&\leqslant \frac{1}{b(b-1)}\left(38\big(e^{-z}-e^{-2z}\big) + \frac{38}{3+2b}e^{-z}\right)\big\|\hat{f}(c)\big\|_{X(0,1)}\\
&\leqslant \frac{51}{b(b-1)}e^{-|z|}\big\|\hat{f}(c)\big\|_{X(0,1)}
\end{aligned} \tag{4.12}$$

and in the same fashion

$$\big|u'(z)\big| \leqslant \frac{108}{b(b-1)}e^{-|z|}\big\|\hat{f}(c)\big\|_{X(0,1)}. \tag{4.13}$$

In the case $z < 0$, we get by Eq. (4.8)

$$\begin{aligned}
\big|u(z)\big| &\leqslant \big|u_1(z)\big|\int_z^0 \frac{|u_2(\zeta)|}{|W(\zeta)|}\big|\hat{f}(c;\zeta)\big|\,d\zeta + \big|u_2(z)\big|\int_{-\infty}^{z}\frac{|u_1(\zeta)|}{|W(\zeta)|}\big|\hat{f}(c;\zeta)\big|\,d\zeta\\
&\leqslant \frac{76}{b(b-1)}e^{-|z|}\big\|\hat{f}(c)\big\|_{X(0,1)},
\end{aligned} \tag{4.14}$$

$$\big|u'(z)\big| \leqslant \frac{172}{b(b-1)}e^{-|z|}\big\|\hat{f}(c)\big\|_{X(0,1)}. \tag{4.15}$$

Using once more Eq. (4.10), we obtain for all $z \in \mathbb{R}$

$$\begin{aligned}\left|u''(z)\right| &= \left|\hat{f}(c;z) + 2bu'(z) + 2\big(3\tanh^2(z) + 2b\tanh(z) - 1\big)u(z)\right| \\ &\leqslant \mathrm{e}^{-|z|}\big(\left\|\hat{f}(c)\right\|_{X(0,1)} + 2\left\|u'\right\|_{X(0,1)} + 12\|u\|_{X(0,1)}\big) \\ &\leqslant \frac{1257}{b(b-1)}\mathrm{e}^{-|z|}\left\|\hat{f}(c)\right\|_{X(0,1)}. \end{aligned} \tag{4.16}$$

Thus, the linear bounded operator $L_b : \mathbb{R} \times X(2,1) \to X(0,1)$ has a bounded right inverse. $\square$

Proof of Theorem 1.1. Let $0 < a < \frac{1}{2}$ and $u_0(z) = \frac{1}{2}(1 + \tanh(z))$. We describe the asymptotic transition $d \to \infty$ by letting $\tau \to 0^+$ in

$$d(\tau) = \frac{1}{8\tau^2}, \tag{4.17}$$

$$c(\tau) = \frac{1-2a}{4\tau} + c_1(\tau), \tag{4.18}$$

$$u(\tau) = u_0 + \tau u_1(\tau) \tag{4.19}$$

with $u_1(\tau) \in X(2,1)$. Let $\tau > 0$ be sufficiently small. Using the stretching operator $S_\tau(u;z) = u(\tau z)$, we get

$$\begin{aligned}\hat{T}\big(\tau, c_1(\tau), u_1(\tau)\big) &= \frac{1}{\tau}S_\tau^{-1}TS_\tau\big(u(\tau)\big) \\ &= -c(\tau)u'(\tau) + \tau d(\tau)\Delta^2\big(\tau, u(\tau)\big) + \frac{1}{\tau}f\big(u(\tau)\big) \\ &= -\frac{1-2a}{4}u_1'(\tau) + \frac{1}{8}\Delta^2\big(\tau, u_1(\tau)\big) - u_0'c_1(\tau) + \hat{u}_0(\tau) + \hat{f}\big(\tau, u_1(\tau)\big) \end{aligned} \tag{4.20}$$

with

$$\begin{aligned}\hat{u}_0(\tau;z) &= \frac{1}{\tau}\left(-\frac{1-2a}{4}u_0'(z) + \frac{1}{8}\Delta^2(\tau, u_0; z) + f\big(u_0(z)\big)\right) \\ &= \frac{1}{8\tau}\left(\frac{\tanh^2(\tau)}{\tau^2(1-\tanh^2(\tau)\tanh^2(z))} - 1\right)\tanh(z)\big(\tanh^2(z) - 1\big), \end{aligned} \tag{4.21}$$

$$\begin{aligned}\hat{f}\big(\tau, u_1(\tau)\big) &= \frac{1}{\tau}\Big(f\big(u_0(z) + \tau u_1(\tau;z)\big) - f\big(u_0(z)\big)\Big) \\ &= -\tau^2 u_1^3(\tau;z) - \frac{\tau}{2}\big((1-2a) - 3\tanh(z)\big)u_1^2(\tau;z) \\ &\quad - \frac{1}{4}\big(3\tanh^2(z) + 2(1-2a)\tanh(z) - 1\big)u_1(\tau;z). \end{aligned} \tag{4.22}$$

Since

$$\lim_{\tau \to 0^+} \hat{u}_0(\tau;z) = 0, \tag{4.23}$$

the operator $\hat{T}$ is well defined in the limit $\tau \to 0^+$. Then, it becomes

$$\begin{aligned}\hat{T}(0, c_1, u_1; z) &= -\frac{1-2a}{4}u_1'(z) + \frac{1}{8}u_1''(z) - \frac{1}{2}\big(1 - \tanh^2(z)\big)c_1 \\ &\quad - \frac{1}{4}\big(3\tanh^2(z) + 2(1-2a)\tanh(z) - 1\big)u_1(z). \end{aligned} \tag{4.24}$$

From $\hat{u}_0(\tau) \in X(0,1)$, it follows that $\hat{T}$ maps $\mathbb{R}_0^+ \times \mathbb{R} \times X(2,1) \to X(0,1)$. By Corollary 3.7 and the common rules of differentiation, it is Hadamard differentiable at all $(\tau, c_1, u_1) \in \mathbb{R}_0^+ \times \mathbb{R} \times X(2,1)$ with $\tau \neq 0$ as well as at the origin. Also, it is restricted strongly Hadamard differentiable at the origin with respect to any initial slope $(1, \bar{c}_1, \bar{u}_1) \in \{1\} \times \mathbb{R} \times X(2,1)$. For $\tau = 0$, $\hat{T}$ is already a linear bounded operator. Therefore, $M_2 = D_{(c_1,u_1)}\hat{T}(0,0,0) : \mathbb{R} \times X(2,1) \to X(0,1)$ is given by the right-hand side of Eq. (4.24). If we multiply M_2 by 8 and set $b = 1 - 2a$, we get the linear bounded operator $L_b : \mathbb{R} \times X(2,1) \to X(0,1)$ given by

$$L_b(c_1, u_1; z) = -4\big(1 - \tanh^2(z)\big)c_1 + u_1''(z) - 2bu_1'(z) - 2\big(3\tanh^2(z) + 2b\tanh(z) - 1\big)u_1(z). \tag{4.25}$$

By Lemma 4.1, it has a bounded right inverse.

Now, we are able to apply Corollary 2.2. For sufficiently small $\tau \geqslant 0$, we obtain the existence of a family of functions $(c_1(\tau), u_1(\tau)) \in \mathbb{R} \times X(2,1)$ which satisfy $(c_1(0), u_1(0)) = (0,0)$, $\lim_{\tau \to 0^+}(c_1(\tau), u_1(\tau)) = (0,0)$ and

$$\hat{T}\big(\tau, c_1(\tau), u_1(\tau)\big) = 0. \tag{4.26}$$

The backward transformation leads to a solution

$$u(z) = u_0(\tau z) + \tau u_1(\tau; \tau z) \tag{4.27}$$

of the travelling wave equation (1.4) for $d = \frac{1}{8\tau^2}$ and $c = \frac{1-2a}{4\tau} + c_1(\tau)$. By construction, it holds $u \in X(2,0) \subset C^2(\mathbb{R})$, $u(-\infty) = 0$ and $u(\infty) = 1$. Furthermore, $c > 0$ if $0 \leqslant \tau$ is sufficiently small. $\square$

References

[1] J. Bell, Some threshold results for models of myelinated nerves, Math. Biosci. 54 (1981) 181–190.
[2] B.D. Craven, M.Z. Nashed, Generalized implicit function theorems when the derivative has no bounded inverse, Nonlinear Anal. 6 (1982) 375–387.
[3] C.E. Elmer, E.S. van Vleck, A variant of Newton's method for the computation of traveling waves of bistable differential–difference equations, J. Dynam. Differential Equations 14 (2002) 493–517.
[4] C.E. Elmer, E.S. van Vleck, Dynamics of monotone travelling fronts for discretizations of Nagumo PDEs, Nonlinearity 18 (2005) 1605–1628.
[5] S.G. Krantz, H.R. Parks, The Implicit Function Theorem: History, Theory, and Applications, Birkhäuser, Boston, 2002.
[6] J. Mallet-Paret, The global structure of traveling waves in spatially discrete dynamical systems, J. Dynam. Differential Equations 11 (1999) 49–127.
[7] B. Zinner, Existence of traveling wavefront solutions for the discrete Nagumo equation, J. Differential Equations 96 (1992) 1–27.