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Nonlinear equations for the recurrence coefficients of discrete orthogonal polynomials

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ABSTRACT

We study polynomials orthogonal on a uniform grid. We show that each weight function gives two potentials and each potential leads to a structure relation (lowering operator). These results are applied to derive second order difference equations satisfied by the orthogonal polynomials and nonlinear difference equations satisfied by the recursion coefficients in the three-term recurrence relations.

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1. Introduction

It has been realized that polynomials orthogonal with respect to exponential weights $\exp(-\pi(x))$ on $\mathbb{R}$ for polynomial $\pi(x)$ satisfy structure relations of the form

$$P_n'(x) = A_n(x)P_{n-1}(x) + B_n(x)P_n(x) \tag{1.1}$$

[12, Chapter 3]. Such structure relation and the three-term recurrence relation show that the orthogonal polynomials satisfy a second order differential equation. In the 1990's it was realized that the recurrence coefficients of orthogonal polynomials satisfy nonlinear difference equations which became known as the Freud equations because Freud [9] was the first to observe this phenomenon. They also appeared in physical models [16,7,8,17]. For $\pi(x) = x^4 - cx^2$ and $\pi(x) = x^6$ the difference equations turned out to be discrete analogues of the Painlevé transcendents [10,18,20]. The developments of the structure relations seems to have been initiated by [5]. There is extensive literature on orthogonal polynomials on the unit circle and discrete Painlevé equations. The interested reader may consult [2] and the references therein. More discrete Painlevé results are in the recent papers [3] and [4].

In this work we study discrete analogues of the structure relation (1.1) for polynomials orthogonal on a finite or infinite set of equidistant points. We also derive second order difference equations satisfied by the orthogonal polynomials. We give examples where the recursion coefficients satisfy nonlinear difference equations. We also give a plausible guess for the asymptotics of the recursion coefficients.

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2. Preliminaries and main results

Let a and $h > 0$ be fixed and define $a_l = a + lh$ for all integers l. We consider a set of monic orthogonal polynomials $\{P_n\}$ with discrete orthogonality relation

$$\sum_{l=s}^{t} P_m(a_l) P_n(a_l) w(a_l) = \zeta_m \delta_{m,n}. \tag{2.1}$$

Here $t > 0$ is finite or ∞, $s < t$ is finite or $-\infty$, and w is a positive weight function with finite moments that satisfies the initial conditions

$$w(a_{s-1}) = w(a_{t+1}) = 0. \tag{2.2}$$

The monic orthogonal polynomials satisfy the three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x). \tag{2.3}$$

In particular,

$$\begin{aligned} &P_0(x) = 1, \qquad P_1(x) = x - \alpha_0, \\ &P_2(x) = (x - \alpha_1)(x - \alpha_0) - \beta_1. \end{aligned} \tag{2.4}$$

The three-term recurrence relation (2.3) implies

$$\zeta_n = \zeta_0 \beta_1 \beta_2 \cdots \beta_n. \tag{2.5}$$

The polynomials P_n also satisfy the Christoffel–Darboux identity [19,11]

$$\sum_{r=0}^{n} P_r(x) P_r(y)/\zeta_r = \frac{P_{n+1}(x) P_n(y) - P_n(x) P_{n+1}(y)}{(x - y)\zeta_n}. \tag{2.6}$$

By the orthogonality relation (2.1) for every polynomial $q(x)$ of degree at most n and every fixed integer d,

$$\sum_{l=s}^{t} \frac{q(x) P_n(a_l)}{x - a_{l+d}} w(a_l) = \sum_{l=s}^{t} \frac{q(a_{l+d}) P_n(a_l)}{x - a_{l+d}} w(a_l) \tag{2.7}$$

since $(q(x) - q(y))/(x - y)$ is a polynomial of y of degree less than n and $a_{l+d} = a_l + dh$.

We set $\Delta_h f(x) = f(x + h) - f(x)$ and then we define the external fields:

$$u(x + h) = -\Delta_h w(x)/w(x + h) \tag{2.8}$$

and

$$v(x) = -\Delta_{-h} w(x)/w(x - h). \tag{2.9}$$

The main results of this work are the following five theorems:

Theorem 2.1. *The polynomials $\{P_n\}$ satisfy*

$$\Delta_h P_n(x) = A_n(x) P_{n-1}(x) - B_n(x) P_n(x) \tag{2.10}$$

where

$$A_n(x) = \frac{1}{\zeta_{n-1}} \sum_{l=s}^{t} \left(\frac{u(x + h) - u(a_l)}{x + h - a_l} \right) P_n(a_l) P_n(a_{l-1}) w(a_l) + \frac{P_n(a_{t+1}) P_n(a_t) w(a_t)}{\zeta_{n-1}(a_t - x)} \tag{2.11}$$

and

$$B_n(x) = \frac{1}{\zeta_{n-1}} \sum_{l=s}^{t} \left(\frac{u(x + h) - u(a_l)}{x + h - a_l} \right) P_n(a_l) P_{n-1}(a_{l-1}) w(a_l) + \frac{P_n(a_{t+1}) P_{n-1}(a_t) w(a_t)}{\zeta_{n-1}(a_t - x)}. \tag{2.12}$$

Theorem 2.2. *The polynomials $\{P_n\}$ satisfy*

$$\Delta_{-h} P_n(x) = C_n(x) P_{n-1}(x) - D_n(x) P_n(x) \tag{2.13}$$

where

$$C_n(x) = \frac{1}{\zeta_{n-1}} \sum_{l=s}^{t} \left(\frac{v(a_{l+1}) - v(x)}{a_{l+1} - x} \right) P_n(a_l) P_n(a_{l+1}) w(a_l) + \frac{P_n(a_s) P_n(a_{s-1}) w(a_s)}{\zeta_{n-1}(a_s - x)} \tag{2.14}$$

and

$$D_n(x) = \frac{1}{\zeta_{n-1}} \sum_{l=s}^{t} \left(\frac{v(a_{l+1}) - v(x)}{a_{l+1} - x} \right) P_n(a_l) P_{n-1}(a_{l+1}) w(a_l) + \frac{P_{n-1}(a_s) P_n(a_{s-1}) w(a_s)}{\zeta_{n-1}(a_s - x)}. \tag{2.15}$$

Furthermore,

Theorem 2.3. *The following (supplementary) relations hold*:

$$B_{n+1}(x) + B_n(x) = \frac{(x - \alpha_n) A_n(x)}{\beta_n} + h \sum_{j=0}^{n} \frac{A_j(x)}{\beta_j} - u(x+h) + P_n(a_t) P_n(a_{t+1}) w(a_t)/\zeta_n, \tag{2.16}$$

$$\begin{aligned}(x - \alpha_n) B_{n+1}(x) - (x + h - \alpha_n) B_n(x) \\ = -h + A_{n+1}(x) - \frac{\beta_n}{\beta_{n-1}} A_{n-1}(x) - \big(P_{n+1}(a_{t+1}) P_n(a_t) - \beta_n P_n(a_{t+1}) P_{n-1}(a_t) \big) w(a_t)/\zeta_n,\end{aligned} \tag{2.17}$$

$$D_{n+1}(x) + D_n(x) = \frac{(x - \alpha_n) C_n(x)}{\beta_n} - h \sum_{j=0}^{n} \frac{C_j(x)}{\beta_j} - v(x) + P_n(a_{s-1}) P_n(a_s) w(a_s)/\zeta_n, \tag{2.18}$$

and

$$\begin{aligned}(x - \alpha_n) D_{n+1}(x) - (x - h - \alpha_n) D_n(x) \\ = -h + C_{n+1}(x) - \frac{\beta_n}{\beta_{n-1}} C_{n-1}(x) - \big(P_{n+1}(a_{s-1}) P_n(a_s) - \beta_n P_n(a_{s-1}) P_{n-1}(a_s) \big) w(a_s)/\zeta_n.\end{aligned} \tag{2.19}$$

The four relations above can be used to derive recurrence relations for the recurrence coefficients in the three-term recurrence relation (2.3) for the orthogonal polynomials when the external fields $u(x)$ are $v(x)$ are polynomials or rational functions.

The action of the forward shift operator only with step $h = 1$ and associated external field $u(x)$ on discrete orthogonal polynomials over uniform equidistant point sets was studied in [13], and analogues of Theorems 2.1 and 2.3 were established. Similarly, the action of the forward q-difference operator with associated external field $u(x)$ on discrete q-orthogonal polynomials over q-lattices was first investigated in [6]. This work was recently extended in [14] by considering the action of pairs of q and $1/q$-difference operators with external fields $u(x)$ and $v(x)$ on discrete q-orthogonal polynomials. Second order q-difference equations satisfied by the q-orthogonal polynomials and certain functions of the second kind were established, as well as q-analogues of Theorems 2.1–2.5. The present work explores the role of the function v which was not known to us at the time when we wrote [13] with Nikolova.

It is possible to obtain other more direct relations between A_n and C_n, or B_n and D_n. This approach was employed earlier in [14]. It is based on calculating a determinant involving the polynomials P_n and the above mentioned functions of the second kind Q_n.

The functions of the second kind Q_n are defined by

$$Q_n(x) = \frac{1}{w(x)} \sum_{l=s}^{t} \frac{P_n(a_l)}{a_l - x} w(a_l), \quad x \notin \{a_l\}_{l=s}^{t}. \tag{2.20}$$

Theorem 2.4. *For every $n \geqslant 1$, the function of the second kind Q_n satisfies relations* (2.3), (2.10), *and* (2.13).

We define the Casorati type determinant

$$\Delta_n(x) = \begin{vmatrix} P_n(x) & Q_n(x) \\ P_n(x+h) & Q_n(x+h) \end{vmatrix}. \tag{2.21}$$

Theorem 2.5. *The Casorati determinant $\Delta_n(x)$ satisfies*

$$\Delta_n(x) = -\frac{\zeta_{n-1}}{w(x)} A_n(x) \tag{2.22}$$

and

$$\Delta_n(x) = \frac{\zeta_{n-1}}{w(x+h)} C_n(x+h). \tag{2.23}$$

In particular $A_n(x)$ *and* $C_n(x)$ *are related via*

$$C_n(x+h) = -w(x+h)A_n(x)/w(x). \tag{2.24}$$

Similar results were obtained in [14] for the q-difference operator defined by $D_q f(x) = (f(qx) - f(x))/((q-1)x)$.

3. Proofs of Theorems 2.1–2.5

In what follows we shall use the inner product defined by

$$(f, g) = \sum_{l=s}^{t} f(a_l) g(a_l) w(a_l). \tag{3.1}$$

In this notation the summation by parts rule is

$$(f, \Delta_h g) = (g, \Delta_{-h} f) + \big(f, v(\cdot + h) g(\cdot + h)\big) \tag{3.2}$$

and

$$(f, \Delta_h g) = (g, \Delta_{-h} f) + \big(f(\cdot - h), ug\big). \tag{3.3}$$

Proof of Theorem 2.1. Let $\Delta_h P_n = \sum_{r=0}^{n-1} c_{n,r} P_r$. By (2.1), (2.2), and (2.8),

$$\begin{aligned}
\zeta_r c_{n,r} = (\Delta_h P_n, P_r) &= \sum_{l=s}^{t} P_n(a_l)\big(P_r(a_{l-1}) w(a_{l-1}) - P_r(a_l) w(a_l)\big) + P_n(a_{t+1}) P_r(a_t) w(a_t) \\
&= \sum_{l=s}^{t} P_n(a_l)\big(-\Delta_h P_r(a_{l-1}) w(a_l) - P_r(a_{l-1}) \Delta_h w(a_{l-1})\big) + P_n(a_{t+1}) P_r(a_t) w(a_t) \\
&= \sum_{l=s}^{t} P_n(a_l) P_r(a_{l-1}) u(a_l) w(a_l) + P_n(a_{t+1}) P_r(a_t) w(a_t) \\
&= \big(\big(u - u(x+h)\big) P_n, P_r(\cdot - h)\big) + P_n(a_{t+1}) P_r(a_t) w(a_t).
\end{aligned} \tag{3.4}$$

Then,

$$\begin{aligned}
\Delta_h P_n(x) &= \left(\big(u - u(x+h)\big) P_n, \sum_{r=0}^{n-1} P_r(x) P_r(\cdot - h)/\zeta_r\right) + P_n(a_{t+1}) \sum_{r=0}^{n-1} P_r(a_t) P_r(x) w(a_t)/\zeta_r \\
&= \left(\left(\frac{(u - u(x+h)) P_n}{\cdot - x - h}, P_n(\cdot - h)\right) + \frac{P_n(a_{t+1}) P_n(a_t) w(a_t)}{a_t - x}\right) \frac{P_{n-1}(x)}{\zeta_{n-1}} \\
&\quad - \left(\left(\frac{(u - u(x+h)) P_n}{\cdot - x - h}, P_{n-1}(\cdot - h)\right) + \frac{P_n(a_{t+1}) P_{n-1}(a_t) w(a_t)}{a_t - x}\right) \frac{P_n(x)}{\zeta_{n-1}}
\end{aligned} \tag{3.5}$$

where we used (2.6). Note that (3.5) is equivalent to (2.10)–(2.12). □

Proof of Theorem 2.2. Let $\Delta_{-h} P_n = \sum_{r=0}^{n-1} \tilde{c}_{n,r} P_r$. By (2.1), (2.2), and (2.9),

$$\begin{aligned}
\zeta_r \tilde{c}_{n,r} = (\Delta_{-h} P_n, P_r) &= \sum_{l=s}^{t} P_n(a_l)\big(P_r(a_{l+1}) w(a_{l+1}) - P_r(a_l) w(a_l)\big) + P_n(a_{s-1}) P_r(a_s) w(a_s) \\
&= \sum_{l=s}^{t} P_n(a_l)\big(-\Delta_{-h} P_r(a_{l+1}) w(a_l) - P_r(a_{l+1}) \Delta_{-h} w(a_{l+1})\big) + P_n(a_{s-1}) P_r(a_s) w(a_s) \\
&= \sum_{l=s}^{t} P_n(a_l) P_r(a_{l+1}) v(a_{l+1}) w(a_l) + P_n(a_{s-1}) P_r(a_s) w(a_s) \\
&= \big(\big(v(\cdot + h) - v(x)\big) P_n, P_r(\cdot + h)\big) + P_n(a_{s-1}) P_r(a_s) w(a_s).
\end{aligned} \tag{3.6}$$

Then,

$$\begin{aligned}\Delta_{-h}P_n(x) &= \left(\big(v(\cdot+h)-v(x)\big)P_n, \sum_{r=0}^{n-1} P_r(\cdot+h)P_r(x)/\zeta_r\right) + P_n(a_{s-1})\sum_{r=0}^{n-1} P_r(x)P_r(a_s)w(a_s)/\zeta_r\\ &= \left(\left(\frac{(v(\cdot+h)-v(x))P_n}{\cdot+h-x}, P_n(\cdot+h)\right) + \frac{P_n(a_{s-1})P_n(a_s)w(a_s)}{a_s-x}\right)\frac{P_{n-1}(x)}{\zeta_{n-1}}\\ &\quad - \left(\left(\frac{(v(\cdot+h)-v(x))P_n}{\cdot+h-x}, P_{n-1}(\cdot+h)\right) + \frac{P_n(a_{s-1})P_{n-1}(a_s)w(a_s)}{a_s-x}\right)\frac{P_n(x)}{\zeta_{n-1}} \end{aligned} \tag{3.7}$$

where we used (2.6). Clearly (3.7) implies (2.13)–(2.15). □

Proof of Theorem 2.3. First we verify (2.16). From (2.12) and (2.3) we have

$$\begin{aligned}&\zeta_n\big(B_{n+1}(x)+B_n(x)\big)\\ &\quad= \big(P_{n+1}(a_{t+1})P_n(a_t)+\beta_n P_n(a_{t+1})P_{n-1}(a_t)\big)w(a_t)/(a_t-x)\\ &\qquad+\sum_{l=s}^{t}\frac{u(x+h)-u(a_l)}{x+h-a_l}\big(P_{n+1}(a_l)P_n(a_{l-1})+\beta_n P_n(a_l)P_{n-1}(a_{l-1})\big)w(a_l)\\ &\quad= \big((a_t-\alpha_n)P_n(a_{t+1})P_n(a_t)+P_{n+1}(a_{t+1})P_n(a_t)-P_n(a_{t+1})P_{n+1}(a_t)\big)\frac{w(a_t)}{a_t-x}+I_1+I_2.\end{aligned} \tag{3.8}$$

The sum I_1 is

$$\begin{aligned}I_1 &= \sum_{l=s}^{t}\frac{u(x+h)-u(a_l)}{x+h-a_l}(a_{l-1}-\alpha_n)P_n(a_l)P_n(a_{l-1})w(a_l)\\ &= (x-\alpha_n)\big(\zeta_{n-1}A_n(x)-P_n(a_{t+1})P_n(a_t)w(a_t)/(a_t-x)\big)-\big(\big[u(x+h)-u\big]P_n, P_n(\cdot-h)\big)\\ &= (x-\alpha_n)\big(\zeta_{n-1}A_n(x)-P_n(a_{t+1})P_n(a_t)w(a_t)/(a_t-x)\big)-\zeta_n u(x+h),\end{aligned} \tag{3.9}$$

since $(P_n, P_n(\cdot-h)) = (P_n, \Delta_{-h}P_n)+(P_n, P_n) = \zeta_n$, and by (3.3) with $f=g=P_n$,

$$\big(uP_n, P_n(\cdot-h)\big)=0. \tag{3.10}$$

The sum I_2 is

$$\begin{aligned}I_2 &= \sum_{l=s}^{t}\frac{u(x+h)-u(a_l)}{x+h-a_l}\big(P_{n+1}(a_l)P_n(a_{l-1})-P_n(a_l)P_{n+1}(a_{l-1})\big)w(a_l)\\ &= h\zeta_n\sum_{l=s}^{t}\frac{u(x+h)-u(a_l)}{x+h-a_l}\sum_{j=0}^{n}P_j(a_l)P_j(a_{l-1})w(a_l)/\zeta_j\\ &= h\zeta_n\sum_{j=0}^{n}\frac{\zeta_{j-1}}{\zeta_j}\left(A_j(x)-\frac{P_j(a_{t+1})P_j(a_t)w(a_t)}{(a_t-x)\zeta_{j-1}}\right)\\ &= h\zeta_n\sum_{j=0}^{n}\frac{A_j(x)}{\beta_j}-\frac{(P_{n+1}(a_{t+1})P_n(a_t)-P_n(a_{t+1})P_{n+1}(a_t))w(a_t)}{(a_t-x)},\end{aligned} \tag{3.11}$$

where (2.6) was used. Combining (3.8), (3.9), and (3.11) we obtain (2.16).

Now we verify (2.17). From (2.12) we have

$$\begin{aligned}&\zeta_n\big((x-\alpha_n)B_{n+1}(x)-(x+h-\alpha_n)B_n(x)\big)\\ &\quad=\sum_{l=s}^{t}\frac{u(x+h)-u(a_l)}{x+h-a_l}\big((x-\alpha_n)P_{n+1}(a_l)P_n(a_{l-1})-(x+h-\alpha_n)\beta_n P_n(a_l)P_{n-1}(a_{l-1})\big)w(a_l)\\ &\qquad+\big((x-\alpha_n)P_{n+1}(a_{t+1})P_n(a_t)-(x+h-\alpha_n)\beta_n P_n(a_{t+1})P_{n-1}(a_t)\big)w(a_t)/(a_t-x)\end{aligned}$$

$$= \sum_{l=s}^{t} \big(u(x+h) - u(a_l)\big) \left\{ \left(1 + \frac{a_l - h - \alpha_n}{x+h-a_l}\right) P_{n+1}(a_l) P_n(a_{l-1}) \right.$$
$$\left. - \beta_n \left(1 + \frac{a_l - \alpha_n}{x+h-a_l}\right) P_n(a_l) P_{n-1}(a_{l-1}) \right\} w(a_l) + I, \tag{3.12}$$

where I denotes the expression in line three of (3.12). From (3.3) and (2.1) we have

$$\big(u P_{n+1},\, P_n(\cdot - h)\big) = (P_n,\, \Delta_h P_{n+1}) - (P_{n+1},\, \Delta_{-h} P_n) = (n+1) h \zeta_n \tag{3.13}$$

since $\Delta_h P_{n+1} - (n+1) h P_n$ has degree less than n. From (3.12), (3.13), (2.1), and (2.3) we obtain

$$\begin{aligned}
&\zeta_n \big((x-\alpha_n) B_{n+1}(x) - (x+h-\alpha_n) B_n(x)\big) \\
&= -(n+1) h \zeta_n + n h \zeta_n + \sum_{l=s}^{t} \frac{u(x+h) - u(a_l)}{x+h-a_l} \\
&\quad \times \big[(a_l - h - \alpha_n) P_{n+1}(a_l) P_n(a_{l-1}) - \beta_n (a_l - \alpha_n) P_n(a_l) P_{n-1}(a_{l-1})\big] w(a_l) + I \\
&= -h\zeta_n + \sum_{l=s}^{t} \frac{u(x+h) - u(a_l)}{x+h-a_l} \\
&\quad \times \big[\big(P_{n+1}(a_{l-1}) + \beta_n P_{n-1}(a_{l-1})\big) P_{n+1}(a_l) - \beta_n \big(P_{n+1}(a_l) + \beta_n P_{n-1}(a_l)\big) P_{n-1}(a_{l-1})\big] w(a_l) + I \\
&= -h\zeta_n + \big(\zeta_n A_{n+1}(x) - P_{n+1}(a_{t+1}) P_{n+1}(a_t) w(t)/(a_t - x)\big) \\
&\quad - \beta_n^2 \big(\zeta_{n-2} A_{n-1}(x) - P_{n-1}(a_{t+1}) P_{n-1}(a_t) w(t)/(a_t - x)\big) + I \\
&= \zeta_n \big(-h + A_{n+1}(x) - \beta_n A_{n-1}(x)/\beta_{n-1}\big) \\
&\quad + \big[\big((x-\alpha_n) P_n(a_t) - P_{n+1}(a_t)\big) P_{n+1}(a_{t+1}) \\
&\quad - \beta_n \big((x+h-\alpha_n) P_n(a_{t+1}) - \beta_n P_{n-1}(a_{t+1})\big) P_{n-1}(a_t)\big] w(a_t)/(a_t - x).
\end{aligned} \tag{3.14}$$

Using (2.3) we can write the expression in the last two lines of (3.14) as

$$\begin{aligned}
&\big[\big((x-a_t) P_n(a_t) + \beta_n P_{n-1}(a_t)\big) P_{n+1}(a_{t+1}) - \beta_n \big((x-a_t) P_n(a_{t+1}) + P_{n+1}(a_{t+1})\big) P_{n-1}(a_t)\big] w(a_t)/(a_t - x) \\
&= -\big(P_n(a_t) P_{n+1}(a_{t+1}) - \beta_n P_n(a_{t+1}) P_{n-1}(a_t)\big) w(a_t).
\end{aligned} \tag{3.15}$$

Formula (2.17) follows from (3.14) and (3.15).

Next, we verify (2.18). From (2.15) and (2.3) we have

$$\begin{aligned}
&\zeta_n \big(D_{n+1}(x) + D_n(x)\big) \\
&= \big(P_{n+1}(a_{s-1}) P_n(a_s) + \beta_n P_n(a_{s-1}) P_{n-1}(a_s)\big) w(a_s)/(a_s - x) \\
&\quad + \sum_{l=s}^{t} \frac{v(a_{l+1}) - v(x)}{a_{l+1} - x} \big(P_{n+1}(a_l) P_n(a_{l+1}) + \beta_n P_n(a_l) P_{n-1}(a_{l+1})\big) w(a_l) \\
&= \big((a_s - \alpha_n) P_n(a_s) P_n(a_{s-1}) + P_{n+1}(a_{s-1}) P_n(a_s) - P_n(a_{s-1}) P_{n+1}(a_s)\big) \frac{w(a_s)}{(a_s - x)} + \tilde{I}_1 + \tilde{I}_2.
\end{aligned} \tag{3.16}$$

The sum $\tilde{I}_1$ is

$$\begin{aligned}
\tilde{I}_1 &= \sum_{l=s}^{t} \frac{v(a_{l+1}) - v(x)}{a_{l+1} - x} (a_{l+1} - \alpha_n) P_n(a_l) P_n(a_{l+1}) w(a_l) \\
&= (x - \alpha_n) \big(\zeta_{n-1} C_n(x) - P_n(a_s) P_n(a_{s-1}) w(a_s)/(a_s - x)\big) + \big(\big(v(\cdot + h) - v(x)\big) P_n,\, P_n(\cdot + h)\big) \\
&= (x - \alpha_n) \big(\zeta_{n-1} C_n(x) - P_n(a_s) P_n(a_{s-1}) w(a_s)/(a_s - x)\big) - \zeta_n v(x),
\end{aligned} \tag{3.17}$$

since $(P_n, P_n(\cdot + h)) = (P_n, \Delta_h P_n) + (P_n, P_n) = \zeta_n$, and by (3.2) with $f = g = P_n$, $(v(\cdot + h) P_n, P_n(\cdot + h)) = 0$.

The sum $\tilde{I}_2$ is

$$\begin{aligned}
\tilde{I}_2 &= \sum_{l=s}^{t} \frac{v(a_{l+1}) - v(x)}{a_{l+1} - x} \big(P_{n+1}(a_l)P_n(a_{l+1}) - P_n(a_l)P_{n+1}(a_{l+1})\big) w(a_l) \\
&= -h\zeta_n \sum_{l=s}^{t} \frac{v(a_{l+1}) - v(x)}{a_{l+1} - x} \sum_{j=0}^{n} P_j(a_l)P_j(a_{l+1}) w(a_l)/\zeta_j \\
&= -h\zeta_n \sum_{j=0}^{n} \frac{\zeta_{j-1}}{\zeta_j} \left(C_j(x) - \frac{P_j(a_s)P_j(a_{s-1})w(a_s)}{(a_s - x)\zeta_{j-1}} \right) \\
&= -h\zeta_n \sum_{j=0}^{n} \frac{C_j(x)}{\beta_j} + \frac{(P_{n+1}(a_s)P_n(a_{s-1}) - P_n(a_s)P_{n+1}(a_{s-1}))w(a_s)}{(a_s - x)},
\end{aligned} \tag{3.18}$$

where we used (2.6). Now (2.18) follows from (3.16), (3.17), and (3.18).

Finally, we prove (2.19). From (2.15) we have

$$\begin{aligned}
&\zeta_n\big((x-\alpha_n)D_{n+1}(x) - (x-h-\alpha_n)D_n(x)\big) \\
&= \sum_{l=s}^{t} \frac{v(a_{l+1}) - v(x)}{a_{l+1} - x} \big((x-\alpha_n)P_{n+1}(a_l)P_n(a_{l+1}) - (x-h-\alpha_n)\beta_n P_n(a_l)P_{n-1}(a_{l+1})\big) w(a_l) \\
&\quad + \big((x-\alpha_n)P_n(a_s)P_{n+1}(a_{s-1}) - (x-h-\alpha_n)\beta_n P_{n-1}(a_s)P_n(a_{s-1})\big) w(a_s)/(a_s - x) \\
&= -\sum_{l=s}^{t} \big(v(a_{l+1}) - v(x)\big) \left\{ \left(1 + \frac{a_{l+1} - \alpha_n}{x - a_{l+1}}\right) P_{n+1}(a_l)P_n(a_{l+1}) \right. \\
&\quad \left. - \beta_n \left(1 + \frac{a_l - \alpha_n}{x - a_{l+1}}\right) P_n(a_l)P_{n-1}(a_{l+1}) \right\} w(a_l) + \tilde{I},
\end{aligned} \tag{3.19}$$

where $\tilde{I}$ denotes the expression in line three of (3.19). From (3.2) and (2.1) we have

$$\big(v(\cdot + h)P_n(\cdot + h),\, P_{n+1}\big) = (P_{n+1},\, \Delta_h P_n) - (P_n,\, \Delta_{-h} P_{n+1}) = (n+1)h\zeta_n \tag{3.20}$$

since $\Delta_{-h}P_{n+1} + (n+1)hP_n$ has degree less than n. Thus, from (3.19), (3.20), (2.1), and (2.3) we obtain

$$\begin{aligned}
&\zeta_n\big((x-\alpha_n)D_{n+1}(x) - (x-h-\alpha_n)D_n(x)\big) \\
&= -(n+1)h\zeta_n + \beta_n nh\zeta_{n-1} - \sum_{l=s}^{t} \frac{v(a_{l+1}) - v(x)}{x - a_{l+1}} \\
&\quad \times \big[(a_{l+1} - \alpha_n)P_n(a_{l+1})P_{n+1}(a_l) - \beta_n(a_l - \alpha_n)P_n(a_l)P_{n-1}(a_{l+1})\big] w(a_l) + \tilde{I} \\
&= -h\zeta_n + \sum_{l=s}^{t} \frac{v(a_{l+1}) - v(x)}{a_{l+1} - x} \\
&\quad \times \big[\big(P_{n+1}(a_{l+1}) + \beta_n P_{n-1}(a_{l+1})\big)P_{n+1}(a_l) - \beta_n\big(P_{n+1}(a_l) + \beta_n P_{n-1}(a_l)\big)P_{n-1}(a_{l+1})\big] w(a_l) + \tilde{I} \\
&= -h\zeta_n + \big(\zeta_n C_{n+1}(x) - P_{n+1}(a_s)P_{n+1}(a_{s-1})w(a_s)/(a_s - x)\big) \\
&\quad - \beta_n^2\big(\zeta_{n-2}C_{n-1}(x) - P_{n-1}(a_s)P_{n-1}(a_{s-1})w(a_s)/(a_s - x)\big) + \tilde{I} \\
&= \zeta_n\big(-h + C_{n+1}(x) - \beta_n C_{n-1}(x)/\beta_{n-1}\big) \\
&\quad + \big[\big((x-\alpha_n)P_n(a_s) - P_{n+1}(a_s)\big)P_{n+1}(a_{s-1}) \\
&\quad - \beta_n\big((x-h-\alpha_n)P_n(a_{s-1}) - \beta_n P_{n-1}(a_{s-1})\big)P_{n-1}(a_s)\big] w(a_s)/(a_s - x).
\end{aligned} \tag{3.21}$$

Using (2.3) we can write the expression in the last two lines of (3.21) as

$$\begin{aligned}
&\big[\big((x-a_s)P_n(a_s) + \beta_n P_{n-1}(a_s)\big)P_{n+1}(a_{s-1}) - \beta_n\big((x-a_s)P_n(a_{s-1}) + P_{n+1}(a_{s-1})\big)P_{n-1}(a_s)\big] w(a_s)/(a_s - x) \\
&\quad = -\big(P_n(a_s)P_{n+1}(a_{s-1}) - \beta_n P_n(a_{s-1})P_{n-1}(a_s)\big) w(a_s).
\end{aligned} \tag{3.22}$$

Formula (2.19) follows from (3.21) and (3.22). □

Since we will use Theorem 2.5 in the proof of Theorem 2.4, we prove Theorem 2.5 first.

Proof of Theorem 2.5. From (2.20), (2.21), (2.7), and (2.2) we get

$$\begin{aligned}\Delta_n(x) &= \frac{1}{w(x+h)}\sum_{l=s}^{t}\frac{P_n(x)P_n(a_l)}{a_l-x-h}w(a_l) - \frac{1}{w(x)}\sum_{l=s}^{t}\frac{P_n(x+h)P_n(a_l)}{a_l-x}w(a_l)\\ &= \frac{1}{w(x+h)}\sum_{l=s}^{t}\frac{P_n(a_{l-1})P_n(a_l)}{a_{l-1}-x}w(a_l) - \frac{1}{w(x)}\sum_{l=s}^{t}\frac{P_n(a_{l+1})P_n(a_l)}{a_l-x}w(a_l)\\ &= \frac{1}{w(x)w(x+h)}\sum_{\nu=s-1}^{t}\frac{[w(x)w(a_{\nu+1})-w(x+h)w(a_\nu)]}{a_\nu-x}P_n(a_{\nu+1})P_n(a_\nu). \end{aligned} \tag{3.23}$$

Using (2.8), for $s-1\leqslant\nu\leqslant t-1$ we can write

$$\begin{aligned}w(x)w(a_{\nu+1})-w(x+h)w(a_\nu) &= -w(a_{\nu+1})\Delta_h w(x) + w(x+h)\Delta_h w(a_\nu)\\ &= w(x+h)w(a_{\nu+1})\big(u(x+h)-u(a_{\nu+1})\big)\end{aligned}$$

and (3.23) becomes

$$\begin{aligned}\Delta_n(x) &= -\frac{1}{w(x)}\sum_{\nu=s-1}^{t-1}\frac{(u(x+h)-u(a_{\nu+1}))}{x-a_\nu}P_n(a_{\nu+1})P_n(a_\nu)w(a_{\nu+1}) - \frac{P_n(a_{t+1})P_n(a_t)w(a_t)}{w(x)(a_t-x)}\\ &= -\frac{\zeta_{n-1}}{w(x)}A_n(x).\end{aligned} \tag{3.24}$$

Similarly, using (2.9), for $s\leqslant\nu\leqslant t$ we can write

$$\begin{aligned}w(x)w(a_{\nu+1})-w(x+h)w(a_\nu) &= w(a_\nu)\Delta_{-h}\big(w(x+h)\big) - w(x)\Delta_{-h}\big(w(a_{\nu+1})\big)\\ &= w(x)w(a_\nu)\big(v(a_{\nu+1})-v(x+h)\big)\end{aligned}$$

and (3.23) takes the form

$$\begin{aligned}\Delta_n(x) &= \frac{1}{w(x+h)}\sum_{\nu=s}^{t}\frac{(v(a_{\nu+1})-v(x+h))}{a_\nu-x}P_n(a_{\nu+1})P_n(a_\nu)w(a_\nu) + \frac{P_n(a_{s-1})P_n(a_s)w(a_s)}{w(x+h)(a_{s-1}-x)}\\ &= \frac{\zeta_{n-1}}{w(x+h)}C_n(x+h).\end{aligned} \tag{3.25}$$

This completes the proof of Theorem 2.5. □

Proof of Theorem 2.4. From (2.20), (2.7) and (2.3) it follows immediately that Q_n satisfies (2.3). Next, we show that

$$\Delta_h Q_n(x) = A_n(x)Q_{n-1}(x) - B_n(x)Q_n(x). \tag{3.26}$$

By (2.20) and (2.8)

$$\begin{aligned}\Delta_h Q_n(x) &= \frac{1}{w(x)w(x+h)}\sum_{l=s}^{t}\frac{[(a_l-x)w(x)-(a_l-x-h)w(x+h)]}{(a_l-x)(a_l-x-h)}P_n(a_l)w(a_l)\\ &= \frac{1}{w(x)}\sum_{l=s}^{t}\frac{((a_l-x)u(x+h)+h)}{(a_l-x)(a_l-x-h)}P_n(a_l)w(a_l)\\ &= \frac{1}{w(x)}\sum_{l=s}^{t}\frac{(u(x+h)-u(a_l))}{a_l-x-h}P_n(a_l)w(a_l) + \frac{1}{w(x)}\sum_{l=s}^{t}\frac{((a_l-x)u(a_l)+h)}{(a_l-x)(a_l-x-h)}P_n(a_l)w(a_l).\end{aligned} \tag{3.27}$$

The right-hand side of (3.26) times $w(x)$ is

$$
\begin{aligned}
&w(x)\big(A_n(x)Q_{n-1}(x)-B_n(x)Q_n(x)\big)\\
&=\sum_{\nu=s}^{t}\frac{u(x+h)-u(a_\nu)}{x+h-a_\nu}\sum_{l=s}^{t}\frac{(P_n(a_{\nu-1})P_{n-1}(a_l)-P_{n-1}(a_{\nu-1})P_n(a_l))}{\zeta_{n-1}(a_l-x)}w(a_l)P_n(a_\nu)w(a_\nu)\\
&\quad+\frac{P_n(a_{t+1})w(a_t)}{a_t-x}\sum_{l=s}^{t}\frac{(P_n(a_t)P_{n-1}(a_l)-P_{n-1}(a_t)P_n(a_l))}{\zeta_{n-1}(a_l-x)}w(a_l)\\
&=\sum_{j=0}^{n-1}\frac{1}{\zeta_j}\sum_{\nu=s}^{t}\frac{(u(x+h)-u(a_\nu))}{x+h-a_\nu}P_j(a_{\nu-1})P_n(a_\nu)\sum_{l=s}^{t}\frac{(a_{\nu-1}-a_l)}{a_l-x}P_j(a_l)w(a_l)w(a_\nu)+J_1,
\end{aligned}
\tag{3.28}
$$

where we used (2.11), (2.12), (2.20), and (2.6), and J_1 denotes the expression in line three of (3.28). Applying the decomposition

$$
\frac{a_l-a_{\nu-1}}{(x+h-a_\nu)(x-a_l)}=\frac{1}{x-a_l}-\frac{1}{x+h-a_\nu}
$$

we get

$$
\begin{aligned}
&w(x)\big(A_n(x)Q_{n-1}(x)-B_n(x)Q_n(x)\big)\\
&=-\sum_{j=0}^{n-1}\frac{1}{\zeta_j}\sum_{\nu=s}^{t}\frac{(u(x+h)-u(a_\nu))}{x+h-a_\nu}P_j(a_{\nu-1})P_n(a_\nu)w(a_\nu)\sum_{l=s}^{t}P_j(a_l)w(a_l)\\
&\quad+\sum_{j=0}^{n-1}\frac{1}{\zeta_j}\sum_{l=s}^{t}\frac{P_j(a_l)w(a_l)}{x-a_l}\sum_{\nu=s}^{t}\big(u(x+h)-u(a_\nu)\big)P_j(a_{\nu-1})P_n(a_\nu)w(a_\nu)+J_1\\
&=-\sum_{\nu=s}^{t}\frac{(u(x+h)-u(a_\nu))}{x+h-a_\nu}P_n(a_\nu)w(a_\nu)+J_2+J_1,
\end{aligned}
\tag{3.29}
$$

where we used (2.1) and J_2 denotes the triple sum in line three of (3.29). In J_2 we replace $u(x+h)-u(a_\nu)$ by $u(x+h)-u(a_{l+1})+u(a_{l+1})-u(a_\nu)$. We obtain

$$
\begin{aligned}
J_2&=\sum_{l=s}^{t}\frac{u(x+h)-u(a_{l+1})}{x+h-a_{l+1}}\sum_{j=0}^{n-1}\frac{1}{\zeta_j}P_j(a_l)w(a_l)\big(P_j(\cdot-h),P_n\big)\\
&\quad+\sum_{l=s}^{t}\frac{w(a_l)}{x-a_l}\sum_{\nu=s}^{t}\big(u(a_{l+1})-u(a_\nu)\big)P_n(a_\nu)w(a_\nu)\sum_{j=0}^{n-1}\frac{1}{\zeta_j}P_j(a_l)P_j(a_{\nu-1})\\
&=\sum_{l=s}^{t}\frac{w(a_l)}{x-a_l}\sum_{\nu=s}^{t}\big(u(a_{l+1})-u(a_\nu)\big)\frac{[P_n(a_l)P_{n-1}(a_{\nu-1})-P_n(a_{\nu-1})P_{n-1}(a_l)]}{\zeta_{n-1}(a_l-a_{\nu-1})}P_n(a_\nu)w(a_\nu)
\end{aligned}
\tag{3.30}
$$

by (2.1) and (2.6). Thus, using (2.12) and (2.11) we get

$$
\begin{aligned}
J_1+J_2&=\sum_{l=s}^{t}\frac{w(a_l)}{x-a_l}\Bigg\{B_n(a_l)P_n(a_l)-A_n(a_l)P_{n-1}(a_l)+\frac{P_n(a_{t+1})w(a_t)}{\zeta_{n-1}}\\
&\qquad\times\big(P_n(a_t)P_{n-1}(a_l)-P_{n-1}(a_t)P_n(a_l)\big)\left[\frac{1}{x-a_t}+\frac{1}{a_t-a_l}\right]\Bigg\}\\
&=-\sum_{l=s}^{t}\frac{w(a_l)}{x-a_l}\Delta_h P_n(a_l)+\frac{P_n(a_{t+1})w(a_t)}{x-a_t}\sum_{l=s}^{t}\sum_{j=0}^{n-1}\frac{P_j(a_t)P_j(a_l)}{\zeta_j}w(a_l)\\
&=\sum_{l=s}^{t}\frac{w(a_l)}{a_l-x}\big(P_n(a_{l+1})-P_n(a_l)\big)+\frac{P_n(a_{t+1})w(a_t)}{x-a_t}\sum_{j=0}^{n-1}P_j(a_t)(P_j,P_0)/\zeta_j\\
&=\sum_{\mu=s}^{t}P_n(a_\mu)\left[\frac{w(a_{\mu-1})}{a_{\mu-1}-x}-\frac{w(a_\mu)}{a_\mu-x}\right]
\end{aligned}
$$

$$= \sum_{\mu=s}^{t} \frac{(-(a_\mu - x)\Delta_h w(a_{\mu-1}) + h w(a_\mu))}{(a_\mu - x)(a_{\mu-1} - x)} P_n(a_\mu)$$

$$= \sum_{\mu=s}^{t} \frac{((a_\mu - x)u(a_\mu) + h)}{(a_\mu - x)(a_{\mu-1} - x)} P_n(a_\mu) w(a_\mu) \tag{3.31}$$

where we also used (2.10), (2.6), (2.1), and (2.8). Now (3.26) follows from (3.27), (3.29), and (3.31).

By Theorems 2.1 and 2.2, $y = P_n$ satisfies the (second order) difference equation

$$(\Delta_h + B_n) y/A_n = (\Delta_{-h} + D_n) y/C_n \tag{3.32}$$

which can be written as

$$y(x+h) + (B_n - 1)y(x) = A_n(D_n - 1)y(x)/C_n + A_n y(x-h)/C_n$$

or as

$$y(x+2h) + \left(B_n(x+h) - 1 - \big(D_n(x+h) - 1\big)\frac{A_n(x+h)}{C_n(x+h)} \right) y(x+h) - \frac{A_n(x+h)}{C_n(x+h)} y(x) = 0. \tag{3.33}$$

Thus, if $\{y_1, y_2\}$ is a solution basis of (3.33), it follows that the corresponding Casorati determinant

$$W_n(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1(x+h) & y_2(x+h) \end{vmatrix}$$

satisfies

$$W_n(x+h) = -\big[A_n(x+h)/C_n(x+h)\big] W_n(x). \tag{3.34}$$

This can be seen by adding to the second row of $W_n(x+h)$, the middle coefficient of (3.33) times the first row of $W_n(x+h)$ and then applying (3.33).

On the other hand, if (3.34) holds and if y_1 satisfies (3.33), then y_2 also satisfies (3.33). Indeed, let $\lambda_n(x) = A_n(x)/C_n(x)$ and let $\mu_n(x+h)$ denote the coefficient in front of $y(x+h)$ in (3.33). Then (3.34) can be written as

$$\begin{vmatrix} y_1(x+h) & y_2(x+h) \\ y_1(x+2h) - \lambda_n(x+h) y_1(x) & y_2(x+2h) - \lambda_n(x+h) y_2(x) \end{vmatrix} = 0. \tag{3.35}$$

By (3.33), the first entry in the second row equals $-\mu_n(x+h) y_1(x+h)$. Thus (3.35) becomes

$$y_1(x+h)\big[y_2(x+2h) - \lambda_n(x+h) y_2(x) + \mu_n(x+h) y_2(x+h)\big] = 0,$$

hence y_2 satisfies (3.33).

By Theorem 2.5, we have

$$\Delta_n(x+h) = -\zeta_{n-1} \frac{A_n(x+h)}{w(x+h)} = -\frac{A_n(x+h)}{C_n(x+h)} \Delta_n(x).$$

Since $P_n(x)$ satisfies (3.33), the above argument shows that $Q_n(x)$ also satisfies (3.33), and therefore (3.32). From (3.32) and (3.26) it follows that $Q_n(x)$ satisfies

$$\Delta_{-h} Q_n(x) = C_n(x) Q_{n-1}(x) - D_n(x) Q_n(x). \tag{3.36}$$

Theorem 2.4 is proved. □

4. Some examples

In this section we consider several weights for which we obtain recurrence relations for the recurrence coefficients α_n and β_n in (2.3).

We begin with a brief discussion about the coefficients of the orthogonal polynomials. With

$$P_n(x) = x^n + \sum_{j=0}^{n-1} c_{j,n} x^j \tag{4.1}$$

from (2.3) we have the following relations

$$c_{n,n+1} = -\alpha_n + c_{n-1,n}, \tag{4.2}$$

$$c_{j,n+1} = c_{j-1,n} - \alpha_n c_{j,n} - \beta_n c_{j,n-1}, \quad 1 \leqslant j \leqslant n-1, \tag{4.3}$$

and

$$c_{0,n+1} = -\alpha_n c_{0,n} - \beta_n c_{0,n-1}. \tag{4.4}$$

From (2.4) and (4.2) it follows that

$$c_{n,n+1} = -\sum_{i=0}^{n} \alpha_i. \tag{4.5}$$

Next, from (4.3) we have

$$c_{n-1,n+1} - c_{n-2,n} = -\alpha_n c_{n-1,n} - \beta_n$$

and by (2.4) and (4.5), $c_{0,2} = -(\alpha_1 c_{0,1} + \beta_1)$. Therefore,

$$c_{n-1,n+1} = c_{0,2} + \sum_{i=2}^{n} (c_{i-1,i+1} - c_{i-2,i}) = -\sum_{i=1}^{n} (\alpha_i c_{i-1,i} + \beta_i). \tag{4.6}$$

We shall consider the following two types of examples:

$$\text{Type I.} \quad w(x) = 1/\Gamma(kx), \quad x > 0, \tag{4.7}$$

$$\text{Type II.} \quad w(x) = \frac{\Gamma(x+b)^M \lambda^x}{\Gamma(kx)^N}, \quad x > 0. \tag{4.8}$$

In both types we shall assume that $h = 1$ and k is a positive integer. In Type II we assume that $b > 0$, $\lambda > 0$ and that the integers $k > 0$ and $N \geqslant M \geqslant 0$ are such that $w(x)$ has finite moments. When $M = 0$ and $k = 1$, $w(x)$ is the generalized Charlier weight (see [20]). If h is an integer, from (2.8) and (4.8) it follows that

$$u(x+h) = \frac{w(x)}{w(x+h)} - 1 = \lambda^{-h} \frac{(kx)_{kh}^N}{(x+b)_h^M} - 1 \tag{4.9}$$

is a rational function. Here we consider only the cases $N = M = 1$, $h = 1$, and $k = 1$, $k = 2$, or $k = 3$ with $a_l = l$, $l = 0, 1, \ldots, \infty$.

Example 1. Let $w(x) = 1/\Gamma(x)$. Note that this case is a limiting case of the Meixner polynomials [15,12]. Indeed, we get this $w(x)$ if in the Meixner polynomials we take $c = 1/\beta$ and let $\beta \to \infty$. The Charlier polynomials are orthogonal with respect to the weight function

$$w_C(x) = a^x/\Gamma(x+1), \quad x = 0, 1, 2, \ldots, \tag{4.10}$$

[12, §6.1]. Thus we are considering a special case to illustrate our technique.

In this case $u(x+1) = x - 1$ and from (2.11), (2.12), and (2.1) we get

$$A_n(x) = \frac{1}{\zeta_{n-1}} \big(P_n(l),\, P_n(l-1)\big) = \frac{\zeta_n}{\zeta_{n-1}} = \beta_n$$

and

$$B_n(x) = \frac{1}{\zeta_{n-1}} \big(P_n(l),\, P_{n-1}(l-1)\big) = 0.$$

From (2.16) we obtain $\alpha_n = n + 2$, and from (2.17), $\beta_{n+1} = \beta_n + 1$ which implies $\beta_{n+1} = \beta_1 + n$. Since

$$\beta_1 = \frac{\zeta_1}{\zeta_0} = \frac{\sum_{l=1}^{\infty} (l-2)^2/\Gamma(l)}{\sum_{l=1}^{\infty} 1/\Gamma(l)} = 1,$$

it follows that $\beta_n = n$. The recurrence relation (2.3) in this case is

$$(x - n - 2) P_n(x) = P_{n+1}(x) + n P_{n-1}(x).$$

Example 2. Let $w(x) = 1/\Gamma(2x)$. Before we proceed with the calculations it is worth pointing out the difference between the orthogonal polynomials in this case and the Charlier polynomials. If we replace x by $2x - 1$ in the orthogonality relation of the Charlier polynomials we get orthogonality with respect to $a^{2x}/\Gamma(2x)$ but on $\{1/2, 1, 3/2, \ldots\}$, which is different from the case we are about to consider. The polynomials in our case are very different from the Charlier polynomials.

In this case $u(x+1) = 4x^2 + 2x - 1$. From (2.11) we obtain

$$\begin{aligned} A_n(x) &= \frac{1}{\zeta_{n-1}}\big((4x+4l-2)P_n(l), P_n(l-1)\big) \\ &= \big((4x-2)+4(\alpha_n - n)\big)\beta_n \end{aligned}$$

using the identity

$$\begin{aligned} \big(lP_n(l), P_n(l\pm 1)\big) &= \sum_{l=0}^{\infty}\big(P_{n+1}(l) + \alpha_n P_n(l) + \beta_n P_{n-1}(l)\big)\big(P_n(l) \pm nP_{n-1}(l) + q_{n-2}(l)\big)w(l) \\ &= (\alpha_n \pm n)\zeta_n, \end{aligned} \tag{4.11}$$

which follows from (2.3) and (2.1), and q_{n-2} denotes a polynomials of degree $n-2$. From (2.12) we get

$$B_n(x) = \frac{1}{\zeta_{n-1}}\big((4x+4l-2)P_n(l), P_{n-1}(l-1)\big) = 4\beta_n.$$

Now we substitute in (2.16) and compare the constant terms to obtain

$$\beta_{n+1} + \beta_n = -\alpha_n^2 + (n+1/2)\alpha_n + \sum_{j=0}^{n}\alpha_j - (n+1)^2/2 + 1/4. \tag{4.12}$$

Therefore,

$$\beta_{n+1} - \beta_{n-1} = (n+3/2-\alpha_n)\alpha_n - (n-1/2-\alpha_{n-1})\alpha_{n-1} - n - 1/2. \tag{4.13}$$

A similar substitution for A_n and B_n in (2.17) and comparison of the constant terms yields

$$\big(\alpha_{n+1} + \alpha_n - (n+1) - 1/2\big)\beta_{n+1} = (\alpha_n + \alpha_{n-1} - n - 1/2)\beta_n + 1/4.$$

Iterating the last equation we obtain

$$\big(\alpha_{n+1} + \alpha_n - (n+1) - 1/2\big)\beta_{n+1} = (\alpha_1 + \alpha_0 - 3/2)\beta_1 + n/4. \tag{4.14}$$

With $\alpha_n = u_n + (n+1)/2$ Eq. (4.13) becomes

$$\beta_{n+1} - \beta_{n-1} = (u_n + u_{n-1})(-u_n + u_{n-1} + 1/2) \tag{4.15}$$

and Eq. (4.14) takes the form

$$(u_{n+1} + u_n)\beta_{n+1} = (u_1 + u_0)\beta_1 + n/4. \tag{4.16}$$

From (4.15) and (4.16) with $d_0 = (u_1 + u_0)\beta_1$ we get

$$\frac{d_0 + n/4}{u_{n+1} + u_n} - \frac{d_0 + (n-2)/4}{u_{n-1} + u_{n-2}} = (u_n + u_{n-1})(-u_n + u_{n-1} + 1/2) \tag{4.17}$$

which is a nonlinear recurrence relation for the recurrence coefficients. Moreover, if we assume that u_n has a limit then (4.17) indicates that $u_n \to 1/2$. Thus, we have the Ansatz:

$$\alpha_n = \frac{n}{2} + 1 + o(1), \qquad \beta_n = \big(1/4 + o(1)\big)(n-1) + (u_0 + u_1)\beta_1 + o(1). \tag{4.18}$$

Some simple evaluations show that $\zeta_0 = \sinh(1)$, $2\alpha_0\zeta_0 = e$,

$$u_0 = \coth(1)/2, \qquad u_1 = -\frac{1}{2} + \frac{2 - \alpha_0 + 2(1-\alpha_0)^2\coth(1)}{4\beta_1},$$

and

$$\beta_1 = \frac{\sinh(2) - 2}{4(\cosh(2) - 1)}.$$

Example 3. Consider the weight $w(x) = 1/\Gamma(3x)$. In this case $u(x+1) = (3x)_3 - 1$ and by (2.12) and (2.1) we get

$$\begin{aligned} B_n(x) &= \frac{3}{\zeta_{n-1}}\big(\big[9\big(x^2 + xl + l(l-1)\big) + 2\big]P_n(l), P_{n-1}(l-1)\big) \\ &= \frac{27}{\zeta_{n-1}}\big[x\big(lP_n(l), P_{n-1}(l-1)\big) + \big(lP_n(l), (l-1)P_{n-1}(l-1)\big)\big]. \end{aligned}$$

To evaluate the last two sums we apply (2.3) to $lP_n(l)$ and to $(l-1)P_{n-1}(l-1)$ and use the orthogonality relation. Since the orthogonal polynomials are monic, then $P_{n-1}(l-1)-P_{n-1}(l)$ and $P_n(l-1)-P_n(l)+nP_{n-1}(l)$ are polynomials of degree $n-2$. In particular,

$$\big(P_{n-1}(l),\, P_n(l-1)\big) = -n\zeta_{n-1}. \tag{4.19}$$

Thus, we obtain

$$B_n(x) = 27\beta_n(x+\alpha_n+\alpha_{n-1}-n). \tag{4.20}$$

Next, from (2.11), (2.1), and (4.11) we have

$$\begin{aligned}\frac{\zeta_{n-1}}{3}A_n(x) &= \big(\big[9\big(x^2+xl+l(l-1)\big)+2\big]P_n(l),\, P_n(l-1)\big)\\ &= \big(9x^2+2\big)\zeta_n + 9x(\alpha_n-n)\zeta_n + 9\big(lP_n(l),\,(l-1)P_n(l-1)\big).\end{aligned} \tag{4.21}$$

Applying (2.3) to $lP_n(l)$ and to $(l-1)P_n(l-1)$ we obtain

$$\begin{aligned}\big(lP_n(l),\,(l-1)P_n(l-1)\big) &= \sum_{l=0}^{\infty}\big(P_{n+1}(l)+\alpha_nP_n(l)+\beta_nP_{n-1}(l)\big)\big(P_{n+1}(l-1)+\alpha_nP_n(l-1)+\beta_nP_{n-1}(l-1)\big)w(l)\\ &= \zeta_{n+1}-\alpha_n(n+1)\zeta_n+\alpha_n^2\zeta_n+\beta_nJ_n-\beta_n\alpha_nn\zeta_{n-1}+\beta_n^2\zeta_{n-1},\end{aligned} \tag{4.22}$$

where we used (2.1) and (4.19), and we defined

$$J_n = \big(P_{n-1}(l),\, P_{n+1}(l-1)\big). \tag{4.23}$$

On the other hand from (2.3) and (4.11) we get

$$\begin{aligned}\big(lP_n(l),\,(l-1)P_n(l-1)\big) &= \big(l^2P_n(l),\,P_n(l-1)\big)-\big(lP_n(l),\,P_n(l-1)\big)\\ &= \sum_{l=0}^{\infty}l\big(P_{n+1}(l)+\alpha_nP_n(l)+\beta_nP_{n-1}(l)\big)P_n(l-1)w(l)-(\alpha_n-n)\zeta_n\\ &= \sum_{l=0}^{\infty}\big[P_{n+2}(l)+(\alpha_{n+1}+\alpha_n)P_{n+1}(l)+\big(\beta_{n+1}+\alpha_n^2+\beta_n\big)P_n(l)\\ &\quad+(\alpha_n\beta_n+\beta_n\alpha_{n-1})P_{n-1}(l)+\beta_n\beta_{n-1}P_{n-2}(l)\big]P_n(l-1)w(l)-(\alpha_n-n)\zeta_n.\end{aligned} \tag{4.24}$$

From (4.24), (2.1), (4.19), and (4.23) it follows that

$$\big(lP_n(l),\,(l-1)P_n(l-1)\big) = \big(\beta_{n+1}+\alpha_n^2+\beta_n\big)\zeta_n-\beta_n(\alpha_n+\alpha_{n-1})n\zeta_{n-1}+\beta_n\beta_{n-1}J_{n-1}-(\alpha_n-n)\zeta_n. \tag{4.25}$$

Setting equal the right-hand sides of (4.22) and (4.25) and applying (2.5) yields a recurrence relation for J_n:

$$J_n = n(\alpha_n-\alpha_{n-1}+1)\zeta_{n-1}+\beta_{n-1}J_{n-1}. \tag{4.26}$$

With $K_n=\beta_nJ_n-n\alpha_n\zeta_n$, (4.26) becomes

$$K_n = \beta_nK_{n-1}-(\alpha_{n-1}-n)\zeta_n$$

and after $n-1$ iterations using (2.5) we obtain

$$K_n = (\zeta_n/\zeta_1)K_1-\zeta_n\sum_{i=1}^{n-1}(\alpha_i-i-1).$$

From (2.4) we have $P_2(l-1)=P_2(l)-2P_1(l)+\alpha_1-\alpha_0+1$ and (4.23) and (2.1) yield $J_1=(\alpha_1-\alpha_0+1)\zeta_0$. Thus,

$$K_1 = \beta_1J_1-\alpha_1\zeta_1 = -(\alpha_0-1)\zeta_1$$

and then

$$K_n = -\zeta_n\sum_{i=0}^{n-1}(\alpha_i-i-1),$$

and by the definition of K_n and (4.5) it follows that

$$J_n = \big(n\alpha_n+c_{n-1,n}+n(n+1)/2\big)\zeta_{n-1}. \tag{4.27}$$

Eqs. (4.21), (4.22), (4.27), and (4.5) yield

$$A_n(x) = 27\beta_n\big(x^2 + (\alpha_n - n)x + \beta_{n+1} + \beta_n + \alpha_n^2 - n\alpha_n + c_{n,n+1} + n(n+1)/2 + 2/9\big). \tag{4.28}$$

Let $a_{0,n}$ and $b_{0,n}$ denote the constant terms in $A_n(x)/(27\beta_n)$ and $B_n(x)/(27\beta_n)$, respectively. Comparing the coefficients of x^{n-1} in (2.10) and using (4.20), (4.28), (4.1), (4.3), and (4.2) we get

$$\begin{aligned} n &= 27\beta_n\big(c_{n-3,n-1} + (\alpha_n - n)c_{n-2,n-1} + a_{0,n} - b_{0,n}c_{n-1,n} - c_{n-2,n}\big) \\ &= 27\beta_n\big(\alpha_{n-1}c_{n-2,n-1} + \beta_{n-1} + (\alpha_n - n)c_{n-2,n-1} + a_{0,n} - b_{0,n}c_{n-1,n}\big) \\ &= 27\beta_n(b_{0,n}c_{n-2,n-1} - b_{0,n}c_{n-1,n} + \beta_{n-1} + a_{0,n}) \\ &= 27\beta_n(\alpha_{n-1}b_{0,n} + \beta_{n-1} + a_{0,n}). \end{aligned}$$

Then,

$$a_{0,n} = n/(27\beta_n) - \beta_{n-1} - \alpha_{n-1}(\alpha_{n-1} + \alpha_n - n). \tag{4.29}$$

Evaluating $a_{0,n} - a_{0,n-1}$, first using (4.29) and then (4.28) we obtain the nonlinear recurrence relation

$$\begin{aligned} &\frac{n}{27\beta_n} - \frac{n-1}{27\beta_{n-1}} - \beta_{n+1} + \beta_{n-2} \\ &\quad = \alpha_n^2 + \alpha_n\alpha_{n-1} - \alpha_{n-1}\alpha_{n-2} - \alpha_{n-2}^2 - n(\alpha_n - \alpha_{n-2}) - \alpha_n - \alpha_{n-1} - \alpha_{n-2} + n. \end{aligned} \tag{4.30}$$

The supplementary relations in Theorem 2.3 can be used to obtain other nonlinear recurrence relations involving α_n and β_n but they are very complicated. Here we give one such relation that follows immediately from (2.17), (4.28) and (4.20):

$$\begin{aligned} &\alpha_n\beta_{n+1}(\alpha_{n+1} + \alpha_n - n - 1) - (\alpha_n - 1)\beta_n(\alpha_n + \alpha_{n-1} - n) \\ &\quad = 1/27 - \beta_{n+1}\big[\beta_{n+2} + \beta_{n+1} + \alpha_{n+1}^2 - (n+1)\alpha_{n+1} + c_{n+1,n+2} + (n+1)(n+2)/2 + 2/9\big] \\ &\qquad + \beta_n\big[\beta_n + \beta_{n-1} + \alpha_{n-1}^2 - (n-1)\alpha_{n-1} + c_{n-1,n} + (n-1)n/2 + 2/9\big]. \end{aligned} \tag{4.31}$$

The next two examples are of Type II, as defined in (4.8).

Example 4. Consider the weight $w(x) = \Gamma(x+b)\lambda^x/\Gamma(x)$ with $\lambda \in (0,1)$. In this case by (4.9), $u(x+1) = x/(\lambda(x+b)) - 1$ and

$$\frac{u(x+1) - u(l)}{x+1-l} = -\frac{u(l) + 1 - 1/\lambda}{x+b}.$$

From (2.11) using (3.10) and (2.1) we get

$$A_n(x) = \frac{-1}{(x+b)\zeta_{n-1}}\big(\big(u(l) + 1 - 1/\lambda\big)P_n(l),\, P_n(l-1)\big) = \frac{(1-\lambda)\beta_n}{\lambda(x+b)} \tag{4.32}$$

and from (2.12), (3.13), and (2.1) we get

$$B_n(x) = \frac{-1}{(x+b)\zeta_{n-1}}\big(\big(u(l) + 1 - 1/\lambda\big)P_n(l),\, P_{n-1}(l-1)\big) = \frac{-n}{x+b}. \tag{4.33}$$

Substituting (4.32) and (4.33) into (2.16) and multiplying by $-\lambda(x+b)$ yields

$$(2n+1)\lambda = (\lambda - 1)(x - \alpha_n) + (\lambda - 1)(n+1) - (\lambda - 1)(x+b) - b.$$

Comparing the constant terms and solving for α_n we obtain

$$\alpha_n = \frac{(\lambda+1)n + b\lambda + 1}{1-\lambda}. \tag{4.34}$$

Next, substituting (4.32) and (4.33) into (2.17) and multiplying by $-\lambda(x+b)$ we get

$$\lambda(x - \alpha_n)(n+1) - \lambda(x+1-\alpha_n)n = \lambda(x+b) + (\lambda - 1)(\beta_{n+1} - \beta_n).$$

Comparison of the constant terms yields

$$\beta_{n+1} - \beta_n = \frac{\lambda(\alpha_n + n + b)}{1-\lambda} = \frac{\lambda(2n+1+b)}{(1-\lambda)^2}$$

where (4.34) was used. Thus

$$\beta_n = \beta_1 + \sum_{j=1}^{n-1}(\beta_{j+1} - \beta_j) = \beta_1 + \frac{\lambda(n-1)(n+1+b)}{(1-\lambda)^2} \tag{4.35}$$

where $\beta_1 = \zeta_1/\zeta_0$. Next we evaluate ζ_0 and ζ_1. By the binomial theorem [12] we get

$$\zeta_0 = \sum_{l=1}^{\infty} w(l) = \sum_{l=1}^{\infty} \Gamma(l+b)\lambda^l/\Gamma(l) = \Gamma(b+1)\lambda(1-\lambda)^{-b-1}. \tag{4.36}$$

By (2.4) and (2.1) we have

$$\zeta_1 = \sum_{l=1}^{\infty}(l-\alpha_0)^2 w(l) = \sum_{l=1}^{\infty} l^2 w(l) - \alpha_0^2\zeta_0. \tag{4.37}$$

Since $\lambda dw(l)/d\lambda = lw(l)$, from (2.3), (2.1), and (4.36) we get

$$\alpha_0\zeta_0 = \sum_{l=1}^{\infty} lw(l) = \lambda\frac{d\zeta_0}{d\lambda} = \Gamma(b+1)\lambda(1+\lambda b)(1-\lambda)^{-b-2}. \tag{4.38}$$

Since $\lambda^2 d^2 w(l)/d\lambda^2 = l(l-1)w(l)$, from (4.38) and (4.36) it follows that

$$\sum_{l=1}^{\infty} l^2 w(l) = \alpha_0\zeta_0 + \lambda^2\frac{d^2\zeta_0}{d\lambda^2} = \alpha_0\zeta_0 + \Gamma(b+2)\lambda^2(2+\lambda b)(1-\lambda)^{-b-3}$$

which combined with (4.37), (4.34), and (4.38) yields

$$\zeta_1 = \Gamma(b+2)\lambda^2(1-\lambda)^{-b-3}. \tag{4.39}$$

Hence $\beta_1 = \zeta_1/\zeta_0 = (b+1)\lambda/(1-\lambda)^2$ and from (4.35) we get

$$\beta_n = \frac{\lambda n(n+b)}{(1-\lambda)^2}. \tag{4.40}$$

The polynomials considered in this example are the Meixner polynomials, see [1,12,15].

Example 5. In our last example let $w(x) = \Gamma(x+b)\lambda^x/\Gamma(2x)$. Now

$$u(x+1) = \frac{2x(2x+1)}{\lambda(x+b)} - 1$$

by (4.9), and

$$\frac{u(x+1)-u(l)}{x+1-l} = \frac{4}{\lambda} - \frac{(u(l)+1+2(2b-2l+1)/\lambda)}{x+b}.$$

Then, from (2.11), (3.10), (2.1), and (4.11) it follows that

$$\begin{aligned} A_n(x) &= \frac{1}{\zeta_{n-1}}\left[\frac{4\zeta_n}{\lambda} - \frac{(\lambda+4b+2)\zeta_n - 4(lP_n(l),\, P_n(l-1))}{\lambda(x+b)}\right] \\ &= \frac{\beta_n(4x-\lambda-2+4\alpha_n-4n)}{\lambda(x+b)}. \end{aligned} \tag{4.41}$$

The function $B_n(x)$ is similarly calculated using (2.12), (2.1), and (3.13):

$$\begin{aligned} B_n(x) &= \frac{-[(u(l)P_n(l),\, P_{n-1}(l-1)) - 4(lP_n(l),\, P_{n-1}(l-1))/\lambda]}{(x+b)\zeta_{n-1}} \\ &= \frac{4\beta_n - \lambda n}{\lambda(x+b)}. \end{aligned} \tag{4.42}$$

First we substitute (4.41) and (4.42) into (2.17), multiply by $\lambda(x+b)$ and equate the constant terms to obtain the recurrence relation

$$(4\alpha_{n+1} + 4\alpha_n - 4n - \lambda - 6)\beta_{n+1} = \big(4\alpha_n + 4\alpha_{n-1} - 4(n-1) - \lambda - 6\big)\beta_n + (\alpha_n + n + b)\lambda. \tag{4.43}$$

A second recurrence relation is obtained similarly by substituting (4.41) and (4.42) into (2.16):

$$4(\beta_{n+1}+\beta_n) = -\alpha_n(4\alpha_n - 4n - \lambda - 2) - 4c_{n,n+1} - 2n^2 + (\lambda-4)n + \lambda b - 2, \tag{4.44}$$

where $c_{n,n+1}$ is given by (4.5). Recurrence relations (4.43) and (4.44) form a linear system with respect to β_{n+1} and β_n. By solving that system one can derive a nonlinear recurrence relation for α_n.

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